



Hodge theory of degenerations, (I): consequences of the decomposition theorem

Matt Kerr¹ · Radu Laza² · Morihiko Saito³

Accepted: 30 May 2021 / Published online: 22 July 2021
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract

We use the Decomposition Theorem to derive several generalizations of the Clemens–Schmid sequence, relating asymptotic Hodge theory of a degeneration to the mixed Hodge theory of its singular fiber(s).

Mathematics Subject Classification 14D06 · 14D07

1 Introduction

This paper initiates a series of articles on the relationship between the asymptotic Hodge theory of a degeneration and the mixed Hodge theory of its singular fiber(s), motivated by the study of compactifications of moduli spaces. In this first installment, we concentrate on what may be derived from the Decomposition Theorem (DT) of [8] in the setting of mixed Hodge modules [68], including several variants of the Clemens–Schmid (C–S) exact sequence [14] (also see [32,39]) and basic results on the vanishing cohomology. In a forthcoming sequel [41], referred to henceforth as

During its writing, MK was supported by NSF Grant DMS-1361147; RL was supported by NSF Grant DMS-1802128 and DMS-1254812.

Matt Kerr and Radu Laza, with an Appendix by Morihiko Saito.

✉ Matt Kerr
matkerr@math.wustl.edu

Radu Laza
rlaza@math.stonybrook.edu

Morihiko Saito
msaito@kurims.kyoto-u.ac.jp

¹ Department of Mathematics and Statistics, Washington University in St. Louis, St. Louis, MO 63130-4899, USA

² Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA

³ RIMS Kyoto University, Kyoto 606-8502, Japan

Part II, we investigate the vanishing cohomology in further detail, and give several applications to geometric degenerations.

The period map is the main tool for studying the moduli spaces of abelian varieties, $K3$ surfaces [57, 74], and related objects such as hyper-Kähler manifolds [34, 43] and cubic threefolds and fourfolds [2, 52, 58, 80]. What these “classical” examples have in common is a “strong global Torelli” property, to the effect that the period map embeds each moduli space as an open subset of a locally symmetric variety. This facilitates the comparison, or even an explicit birational correspondence, between Hodge-theoretic (i.e. toroidal [6] and Baily-Borel) compactifications and geometric ones (such as KSBA or GIT compactifications); see for example the series of papers [53–56]. A program led by Griffiths, with contributions of many people (including the authors), aims to extend the use of period maps in studying moduli to the “non-classical” case, especially surfaces of general type with $p_g \geq 2$ and Calabi–Yau threefolds, with the premise that this strong connection between compactifications should remain. In particular, the geometric boundary (suitably blown up) carries variations of limiting mixed Hodge structures (LMHS) on its strata, which in principle yield period maps to Hodge-theoretic boundary components.¹ The challenge is thus to compute these LMHS, and their associated monodromies, as well as possible from the geometry of the (singular) fibers over the geometric boundary.

There are two main parts to this challenge. *The first is to compute the MHS on the singular fibers and relate this to the invariant cycles in the LMHS.* For the ideal topological set-up, that of a semistable degeneration $\mathcal{X} \xrightarrow{f} \Delta$ over a disk (centered at the origin) with singular fiber X_0 , a piece of the Clemens–Schmid sequence says that

$$H_{X_0}^k(\mathcal{X}) \xrightarrow{\mu} H^k(X_0) \xrightarrow{\text{SD}} H_{\text{inv}}^k(X_t) \rightarrow 0 \quad (1.1)$$

is an exact sequence of MHS, with $\text{im}(\mu)$ pure of weight k (and level $\leq k - 2$). While this is a very strong statement, the natural degenerations occurring (say) in GIT (Geometric Invariant Theory [61]) or KSBA (Kollár–Shepherd-Barron–Alexeev [4, 50]) compactifications are rarely semistable, and difficult to put in this form via semistable reduction. Indeed, the philosophy of the minimal model program (MMP) is that, for sufficiently “mild” singularities on $\mathcal{X}$ and X_0 , we need *not* carry out semistable reduction, as illustrated by papers from [73–75] to [43, 52].

In accord with this principle, we have largely focused this paper on various generalizations of Clemens–Schmid, starting with the simple observation (cf. Theorem 5.3 and (6.2)) that (1.1) remains valid for smooth $\mathcal{X}$ and projective f , regardless of unipotence of monodromy or singularities of X_0 . Specifically, we have:

Theorem 1.1 *Let $f: \mathcal{X} \rightarrow \Delta$ be a flat projective family of varieties over the disk, which is the restriction of an algebraic family over a curve, such that f smooth over Δ^* . If $\mathcal{X}$ is smooth, then we have exact sequences of MHS*

$$0 \rightarrow H_{\text{lim}}^{k-2}(X_t)_T(-1) \xrightarrow{\text{SD}^\vee} H_{2n-k+2}(X_0)(-n-1) \xrightarrow{\text{qy}} H^k(X_0) \xrightarrow{\text{SD}} H_{\text{lim}}^k(X_t)^T \rightarrow 0 \quad (1.2)$$

¹ Suitable compactifications are known for locally symmetric subvarieties of Hodge-theoretic classifying spaces, and are expected to exist (in horizontal directions) in general.

for every $k \in \mathbb{Z}$, where the outer terms are the coinvariants resp. invariants of the monodromy operator T on the LMHS.

We are interested especially in versions of Clemens–Schmid for 1-parameter families arising in the study of KSBA compactifications. In this direction, we obtain the following result (cf. Theorem 9.3 and Corollary 9.9), which in particular gives that the frontier Hodge numbers (i.e. $h^{p,q}$ with $p \cdot q = 0$) are preserved for such degenerations. Weaker versions of our result (cf. [73,78]) proved to be very useful for the study of degenerations of $K3$ surfaces and hyper-Kähler manifolds (e.g. [43,54,74]).

Theorem 1.2 *Let $f : \mathcal{X} \rightarrow \Delta$ be as in the first sentence of Theorem 1.1 (in particular, $\mathcal{X} \setminus X_0$ is smooth). Suppose that $\mathcal{X}$ is normal and $\mathbb{Q}$ -Gorenstein, and that the special fiber X_0 is reduced.*

(i) *If X_0 is semi-log-canonical (slc), then*

$$\mathrm{Gr}_F^0 H^k(X_0) \cong \mathrm{Gr}_F^0 H_{\lim}^k(X_t) \cong \mathrm{Gr}_F^0 H_{\lim}^k(X_t)^{T^{ss}} \quad (\forall k),$$

where $T = T^{un} T^{ss}$ is the Jordan decomposition of the monodromy into unipotent and (finite) semisimple parts.

(ii) *If X_0 is log-terminal, then additionally $(\forall k)$*

$$W_{k-1} \mathrm{Gr}_F^0 H_{\lim}^k(X_t) = \{0\}.$$

Remark 1.3 Under the assumption of X_0 having du Bois singularities, the first isomorphism of item (i) above is due to Steenbrink [78]. Kollár–Kovács [40] (see also [45, §6.2]) proved that slc singularities are du Bois, recovering our version above.

Under stronger assumptions (especially smoothness for the total space $\mathcal{X}$), we are able to go deeper into the Hodge filtration (Theorem 9.11). We expect that this result (which to our knowledge is new) will play an important role in the study of degenerations of Calabi–Yau threefolds with canonical singularities, and respectively surfaces of general type with $p_g = 2$. (Several related questions about these two geometric cases are currently under investigation by the authors and their collaborators under the aegis of Griffiths’s program.)

Theorem 1.4 (= Theorem 9.11) *Let $f : \mathcal{X} \rightarrow \Delta$ be as in Theorem 1.2. Assume that the total space $\mathcal{X}$ is smooth and the special fiber X_0 is log-terminal (or more generally, has rational singularities). Then*

$$\mathrm{Gr}_F^1 H^k(X_0) \cong \mathrm{Gr}_F^1 (H_{\lim}^k(X_t))^{T^{ss}}.$$

Remark 1.5 The general philosophy of Theorems 1.2 and 1.4 is that the milder the singularities, the closer the relationship between the Hodge structure on the central fiber X_0 and the limit Hodge structure is. In **Part II** of our paper, we will give some further versions based on the concept of k -log-canonicity of Mustata–Popa [63] (see also [35] for some more recent developments). In the opposite direction, one can ask what happens if X_0 is not log canonical. This leads to questions on the Hodge structure

of the “tail” (e.g. see [33] and [54, Sect. 6]) occurring in a KSBA stable replacement. While some examples are discussed here, we revisit the topic in a more systematic way in **Part II**.

Remark 1.6 Versions of Theorems 1.2 and 1.4 (under somewhat weaker, but less geometric assumptions) are the subject of a forthcoming paper joint with Saito [42].

For singular total spaces, there are “clean” versions of Clemens–Schmid only for semisimple perverse sheaves (5.2) (including intersection cohomology (5.7)). For us, the importance of semisimplicity with respect to the perverse t-structure was driven home by [9], and we explain in Example 7.1 how this typically fails for $\mathbb{Q}_{\mathcal{X}}[d_{\mathcal{X}}]$ even when it is perverse. So the versions for usual cohomology with $\mathcal{X}$ singular are necessarily more partial, as seen in the context of base-change and log-resolutions (8.4), quotient singularities (8.8), and MMP-type singularities (results in §9). Finally, in Theorem 10.3 we arrive at an analogue of Clemens–Schmid for the simplest kinds of multiparameter degenerations (smooth total space, snc discriminant divisor), including for instance those termed semistable by [3].

The second main aspect to determining the LMHS of a 1-parameter degeneration (without applying semistable reduction) is to tease out of the geometry of X_0 those aspects which are invisible to $H^(X_0)$.* Here the main tool (for $\mathcal{X} \rightarrow \Delta$) is the exact sequence

$$\cdots \rightarrow H^k(X_0) \rightarrow H_{\lim}^k(X_t) \rightarrow H_{\text{van}}^k(X_t) \rightarrow H^{k+1}(X_0) \rightarrow \cdots \quad (1.3)$$

where $H_{\text{van}}^k(X_t)$ denotes $\mathbb{H}^{k+d_{\mathcal{X}}-1}$ of the vanishing cycle sheaf ${}^p\phi_f \mathbb{Q}_{\mathcal{X}}[d_{\mathcal{X}}]$ on X_0 , promoted to a MHS by Saito’s realization of ${}^p\phi_f \mathbb{Q}_{\mathcal{X}}[d_{\mathcal{X}}]$ as a mixed Hodge module (MHM) in [67]. We shall refer to (1.3) as the *vanishing cycle sequence*. Basic results on the *vanishing cohomology* $H_{\text{van}}^k(X_t)$ are proved in Propositions 5.5 and 6.3 and Theorem 6.4 here; for instance, in the case of an isolated singularity, its underlying $\mathbb{Q}$ -vector space is the reduced cohomology of the Milnor fiber [62]. These are but a small taste of what will be the main topic in **Part II** of our study, in which tools such as mixed spectra and the motivic Milnor fiber are used to compute H_{van}^* for various singularities arising in GIT and MMP.

Of course, there is a vast literature on the subject of relating the cohomology and singularity theory of X_0 with the limit cohomology (e.g. [14, 18, 25, 27, 40, 51, 68, 73, 77]). Our purpose in this series is to survey, adapt, and (where possible) improve this for degenerations that occur naturally in the geometric context. Beyond relating geometric and Hodge-theoretic compactifications of moduli, we anticipate applications to the classification of singularities and KSBA (or semistable) replacements of singular fibers occurring in GIT, as well as to limits of normal functions in the general context of [16].

Structure of the paper

In Sects. 2 and 3, we start with a review of the Decomposition Theorem and make some preliminary considerations for our situation. The following three sections discuss the

case of the Decomposition Theorem over a curve (with an eye towards one-parameter degenerations). First, in Sect. 4, we introduce the vanishing and nearby cycles, and the *vanishing cycle triangle* relating them (see (4.3)), followed by general forms of vanishing, limiting, and “phantom” cohomology. These preliminaries allow us to begin Sect. 5 with a very general form of the Clemens–Schmid exact sequence, which is eventually specialized to the more recognizable form (Theorem 5.3) under the assumption of smooth total space. The fact that there is a close connection between the Clemens–Schmid sequence and the Decomposition Theorem is well known to experts (e.g. Remark 5.1(ii)). In an “Appendix” to our paper, Saito proves a general (suitable) equivalence between Clemens–Schmid sequence, the local invariant cycle Theorem, and the Decomposition Theorem over a curve.

Some concrete geometric examples are then discussed in Sect. 6. These range from the very classical, e.g. families of elliptic curves with various types of Kodaira fibers (Example 6.1), to examples (Example 6.5) that we encountered in the study of degenerations of $K3$ surfaces (see especially [54]), to the more exotic example of Katz [37] of a family of surfaces ($p_g = 2$) with G_2 monodromy (whose treatment uses the most general form of Clemens–Schmid). These examples serve both to illustrate the C–S and vanishing-cycle exact sequences in well-known settings, and to show the efficacy of the methods developed here in some less familiar situations. While our examples are not new per se, we believe the discussion of Sect. 6 gives a deeper and more conceptual understanding of them. In **Part II**, further tools are developed, which will allow us to give further examples and applications.

While there is a suitable general theory (and we only touch on MHM work of Saito), the focus of our paper is on specializing these results to concrete situations relevant for geometric questions (esp. compactifications). We start this discussion with the case of isolated singularities (Sect. 7), and their relationship to the failure of the DT for non-semisimple perverse sheaves. We then discuss (Sect. 8) another common geometric scenario—that of finite base changes and quotient singularities. While some of the discussion here might seem very special from the perspective of the general theory, in concrete geometric situations (including those considered in **Part II**) subtle issues arise. We hope that our discussion clarifies some of those issues, and we expect that further applications will be obtained in the future. Some examples (including some that we encountered in our previous work) are included along the way.

The most novel aspects of our work occur in the last two sections. First, in Sect. 9, we discuss the situation of one parameter degenerations of KSBA type. Among other things, we obtain Theorems 1.2 and 1.4 discussed above. In the final Sect. 10, we start a discussion of the Hodge theoretic behavior of degenerations over multi-dimensional bases. To our knowledge, very little in this direction exists in the current literature. We expect that the study of degenerations over multi-dimensional bases will play a more prominent role in the future—especially due to the fast progress on multi-dimensional semistable reduction theorems (Abramovich, Temkin and others, e.g. [5], improving on Abramovich–Karu [3]). A concrete geometric example where multi-dimensional bases occur and our methods might be relevant, we mention the case of cubic threefolds. In [15], a study of the degenerations of intermediate Jacobians in the classical setup of normal crossing discriminants is done; while in [59], it is essential to study the degenerations of intermediate Jacobians without blowing up the discriminant to

normal crossings. The method used for both of these studies is reduction to curves via Mumford's Prym construction for the intermediate Jacobian. It would be interesting to study the degenerations of intermediate Jacobians directly in terms of cubics (see [12] for a step in this direction).

2 Motivation: why the decomposition theorem?

For any projective map $f : \mathcal{X} \rightarrow \mathcal{S}$ of quasi-projective varieties over $\mathbb{C}$ and $\mathcal{K}^\bullet \in D_c^b(\mathcal{X}^{\text{an}})$, the equality of functors²

$$R\Gamma_{\mathcal{S}} \circ Rf_* = R\Gamma_{\mathcal{X}} : D_c^b(\mathcal{X}^{\text{an}}) \rightarrow D_c^b(\mathcal{S}^{\text{an}})$$

produces the *Leray spectral sequence*

$$E_2^{p,q} = \mathbb{H}^p(\mathcal{S}, R^q f_* \mathcal{K}^\bullet) \implies \mathbb{H}^{p+q}(\mathcal{X}, \mathcal{K}^\bullet). \quad (2.1)$$

The accompanying *Leray filtration*

$$\mathcal{L}^\rho \mathbb{H}^k(\mathcal{X}, \mathcal{K}^\bullet) := \text{im} \left\{ \mathbb{H}^k(\mathcal{S}, \tau_{\leq k-\rho} Rf_* \mathcal{K}^\bullet) \rightarrow \mathbb{H}^k(\mathcal{S}, Rf_* \mathcal{K}^\bullet) \right\} \quad (2.2)$$

(with $\text{Gr}_{\mathcal{L}}^\rho \cong E_\infty^{\rho, k-\rho}$) may be described in terms of kernels of restrictions to (special) subvarieties of $\mathcal{S}$ [7]. Hence when $\mathcal{K}^\bullet$ has the structure of a MHM, $\mathcal{L}^\bullet$ is a filtration by sub-MHS.

However, we would prefer to have more than just a filtration. Recall the following classical result of Deligne [21]:

Theorem 2.1 *If $\mathcal{X}$ and $\mathcal{S}$ are smooth, f is smooth projective (of relative dimension n), and $\mathcal{K}^\bullet = \mathbb{Q}_{\mathcal{X}}$, then (2.1) degenerates at E_2 .*

Proof See [64, Proposition 1.33]. □

As an immediate corollary, this produces a noncanonical decomposition

$$H^k(\mathcal{X}, \mathbb{Q}) \cong \bigoplus_{\text{nc}} H^p(\mathcal{S}, R^{k-p} f_* \mathbb{Q}) \quad (2.3)$$

into MHS, which includes an easy case of the global invariant cycle theorem. Neither the description of the graded pieces of $\mathcal{L}^\bullet$ nor its splitting in (2.3) may be valid when $\mathcal{X}$, $\mathcal{S}$, or f is not smooth.

Example 2.2 Let $\mathcal{Y} \rightarrow \mathbb{P}^1$ be an extremal (smooth, minimal) rational elliptic surface with (zero-)section σ . By Noether's formula,

$$h_{\mathcal{Y}}^{1,1} = 10 + 10h_{\mathcal{Y}}^{2,0} - 8h_{\mathcal{Y}}^{1,0} - K_{\mathcal{Y}}^2 = 10;$$

² As we continue to work in the analytic topology, the superscript “an” will be dropped for brevity.

and we let $\hat{\mathcal{Y}} \xrightarrow{\beta} \mathcal{Y}$ be the blow-up at a nontorsion point p on a smooth fiber Y_0 , with $P = \beta^{-1}(p) \cong \mathbb{P}^1$. Contracting the proper transform $\hat{Y}_0$ of Y_0 yields an elliptic surface $\mathcal{X} \xrightarrow{f} \mathbb{P}^1$ with isolated $\tilde{E}_8$ singularity $q \in X_0(\cong P) = f^{-1}(0)$, since $\hat{Y}_0^2 = -1$.

First consider the Leray spectral sequence for $\pi: \hat{\mathcal{Y}} \rightarrow \mathcal{X}$. This has E_2 -page

$$\begin{array}{ccccc} H^2(\hat{Y}_0) & & & & \\ H^1(\hat{Y}_0) & \xrightarrow{d_2} & H^2(\mathcal{X}) & 0 & H^4(\mathcal{X}) \\ H^0(\mathcal{X}) & 0 & & & \end{array}$$

with d_2 injective, and $H^2(\hat{\mathcal{Y}}) \cong H^2(\hat{Y}_0) \oplus H^2(\mathcal{X})/\text{im}(d_2)$. (Note that $H^2(\mathcal{X})/\text{im}(d_2)$ and $H^2(\mathcal{Y})$ both have Hodge numbers $(0, 10, 0)$.) So degeneration at E_2 fails.

On the other hand, the Leray spectral sequence for $f: \mathcal{X} \rightarrow \mathbb{P}^1$ takes the form

$$\begin{array}{ccccc} H^0(R^2 f_* \mathbb{Q}) & 0 & \mathbb{Q}(-2) \\ 0 & H^1(R^1 f_* \mathbb{Q}) & 0 \\ \mathbb{Q} & 0 & H^2(R^0 f_* \mathbb{Q}) \end{array}$$

with d_2 zero. However, the resulting Leray filtration on $H^2(\mathcal{X})$ is non-split in the category of MHS. To see this, remark that:

- $H^2(R^0 f_* \mathbb{Q}) \cong \mathbb{Q}(-1)$ is generated by the class of a smooth fiber;
- $H^1(R^1 f_* \mathbb{Q}) \cong H^1(Y_0)$ with Y_0 a smooth elliptic curve; and
- $H^0(R^0 f_* \mathbb{Q}) \cong \mathbb{Q}(-1)^{\oplus 9}$ is generated by 8 components of fibers other than X_0 , and one cocycle $\kappa \in \text{Cone}(C^\bullet(\hat{\mathcal{Y}}) \rightarrow C^\bullet(\hat{Y}_0))$ given by $(P - \sigma, \wp)$, where $\wp$ is a path on $\hat{Y}_0$ from 0 to p .

Writing $\Omega^1(\hat{Y}_0) = \mathbb{C}\langle\omega_0\rangle$, $\int_{\wp} \omega_0 = \text{AJ}(p - 0)$ gives the (nontorsion) extension class of $[\kappa]$ by $H^1(Y_0)$ in $H^2(\mathcal{X})$, and hence of $\text{Gr}_{\mathcal{L}}^0$ by $\text{Gr}_{\mathcal{L}}^1$. Of course, Poincaré duality also fails for $H^2(\mathcal{X})$.

As we shall see, one gets better behavior on all fronts by using perverse Leray filtrations and intersection complexes.

3 Perverse Leray

We begin by stating the Decomposition Theorem (DT) for a projective morphism $f: \mathcal{X} \rightarrow \mathcal{S}$ of complex algebraic varieties (of relative dimension $n = d_{\mathcal{X}} - d_{\mathcal{S}}$). Let $K^\bullet \in D_c^b(\mathcal{X}^{(\text{an})})$ [resp. $D^b\text{MHM}(\mathcal{X})$] be a complex of sheaves of abelian groups [resp. mixed Hodge modules] which is constructible with respect to some stratification $\mathcal{S}$. Assume that $K^\bullet$ is **semisimple** in the sense of being a direct sum of shifts of (semi)simple perverse sheaves $\text{IC}_{\mathcal{Z}}^\bullet(\mathbb{L})$ [resp. polarizable Hodge modules $\text{MH}_{\mathcal{Z}}(\mathcal{L})$].³

³ Here $\mathbb{L}$ [resp. $\mathcal{L}$] is a (semi)simple local system [resp. polarizable VHS] on a Zariski open in $\mathcal{Z}$. For uniformity of notation we shall use the notation $\text{IC}_{\mathcal{Z}}^\bullet(\mathbb{L})$ to refer to both perverse sheaves and Hodge modules.

References for the following statement are [8, Theorem 6.2.5], [18, Theorem 2.1.1], and [17] for the perverse sheaf version, resp. [68, Thms. 0.1-0.3] and [67, (4.5.4)] for the MHM version.

Theorem 3.1 (Decomposition Theorem)

(a) Writing ${}^p R^j f_*$ for ${}^p \mathcal{H}^j Rf_*$, we have

$$Rf_* \mathcal{K}^\bullet \underset{\text{nc}}{\simeq} \bigoplus_j ({}^p R^j f_* \mathcal{K}^\bullet)[-j] \simeq \bigoplus_{j,d} \text{IC}_{Z_d}^\bullet(\mathbb{V}_d^j(\mathcal{K}^\bullet)[d])[-j] \quad (3.1)$$

as (up to shift) perverse sheaves [resp. polarizable Hodge modules], for some local systems [resp. polarizable VHS] $\mathbb{V}_d^j(\mathcal{K}^\bullet)$ on $Z_d \setminus Z_{d-1}$ (smooth of dimension d). Moreover, the ${}^p R^j f_* \mathcal{K}^\bullet$ are **semisimple** perverse.

(b) If h is the class of a relatively ample line bundle on $\mathcal{X}$ and $\mathcal{K}^\bullet$ is perverse, then multiplication by h^j induces an isomorphism

$${}^p R^{-j} f_* \mathcal{K}^\bullet(-j) \xrightarrow{\simeq} {}^p R^j f_* \mathcal{K}^\bullet \quad (3.2)$$

for each $j \geq 0$.

- Remark 3.2** (i) If $\mathcal{K}^\bullet$ is a (pure) Hodge module of weight w , the ${}^p R^j f_* \mathcal{K}^\bullet$ [resp. $Rf_* \mathcal{K}^\bullet$] is pure of weight $w + j$ [resp. w], and $\mathbb{V}_d^j(\mathcal{K}^\bullet)$ underlies a VHS of weight $w + j - d$.
- (ii) In the key special case where $\mathcal{K}^\bullet = \text{IC}_{\mathcal{X}}^\bullet$ (which is $\mathbb{Q}_{\mathcal{X}}[d_{\mathcal{X}}]$ if $\mathcal{X}$ is smooth), we write $\mathbb{V}_d^j(\mathcal{K}^\bullet) =: \mathbb{V}_d^j$. In view of [68], we still have (3.1) in this case when we relax the hypotheses on $f : \mathcal{X} \rightarrow \mathcal{S}$ to: f proper, $\mathcal{X}$ Fujiki class C (dominated by Kähler).
- (iii) Although $\mathbb{Q}_{\mathcal{X}}[d_{\mathcal{X}}]$ is perverse as long as $\mathcal{X}$ has local complete intersection singularities, it may not be semisimple (and the DT may not apply). See Example 7.1 below.
- (iv) When restricting f to an open analytic subset of the base such as a polydisk Δ^r (as we shall do below), (3.1)–(3.2) still hold though the $\mathbb{V}_d^j(\mathcal{K}^\bullet)$ may not be semisimple on Δ^r . For this reason (only), one should not call the ${}^p R^j f_* \mathcal{K}^\bullet$ “semisimple” in this setting. Instead we shall say that they *decompose*.
- (v) Over a disk, a weak form of the DT (first $\simeq$ of (3.1)) holds without the semisimplicity constraint on $\mathcal{K}^\bullet$; see the “Appendix” by Saito.

Remark 3.3 When $\mathcal{X} \rightarrow \mathcal{S} = \mathbb{P}H^0(X, \mathcal{O}(L^{\otimes m}))$ ($L^{\otimes m}$ very ample) is the universal hypersurface section of a smooth $2D$ -fold X , the *perverse weak Lefschetz theorem* [10, Theorem 5.2] says that $\mathbb{V}_d^j = 0$ unless $j = 0$ or $d = d_{\mathcal{S}}$. Moreover, $\mathbb{V}_{d_{\mathcal{S}}}^j$ is constant $\equiv H^{j+(2D-1)}(X)$ if $j \neq 0$, and $\mathbb{V}_{d_{\mathcal{S}}-1}^0 = 0$ for $m \gg 0$. This plays a key rôle in producing singularities in normal functions associated to D -dimensional cycles on X .

Taking hypercohomology of (3.1) yields a decomposition

$$\mathbb{H}^k(\mathcal{X}, \mathcal{K}^\bullet) \cong \bigoplus_i \mathbb{H}^i(\mathcal{S}, {}^p R^{k-i} f_* \mathcal{K}^\bullet) \cong \bigoplus_{i,d} \mathbb{H}^{d+i}(Z_d, \mathbb{V}_d^{k-i}(\mathcal{K}^\bullet)) \quad (3.3)$$

on the level of mixed Hodge structures. The *perverse Leray filtration* induced by

$${}^p\mathcal{L}^{-\alpha} Rf_*\mathcal{K}^\bullet := \bigoplus_{j \leq \alpha} ({}^p R^j f_*\mathcal{K}^\bullet)[-j] \quad (3.4)$$

is simply

$${}^p\mathcal{L}^\beta \mathbb{H}^k(\mathcal{X}, \mathcal{K}^\bullet) := \mathbb{H}^k(\mathcal{S}, {}^p\mathcal{L}^{\beta-k} Rf_*\mathcal{K}^\bullet) \cong \bigoplus_{i \geq \beta} \mathbb{H}^i(\mathcal{S}, {}^p R^{k-i} f_*\mathcal{K}^\bullet). \quad (3.5)$$

That is, under our hypotheses (of semisimplicity for $\mathcal{K}^\bullet$ and projectivity for f), the *perverse Leray spectral sequence* $\mathbb{H}^i(\mathcal{S}, {}^p R^j f_*\mathcal{K}^\bullet) \implies \mathbb{H}^*(\mathcal{X}, \mathcal{K}^\bullet)$ converges at E_2 , and the resulting (perverse Leray) filtration on $\mathbb{H}^*(\mathcal{X}, \mathcal{K}^\bullet)$ is split in the category of MHS. This stands in marked contrast to the scenario in Example 2.2.

Example 3.4 For any variety $\mathfrak{X}$, the morphism $\mathbb{Q}_{\mathfrak{X}}[d_{\mathfrak{X}}] \rightarrow \mathrm{IC}_{\mathfrak{X}}^\bullet$ in $D^b\mathrm{MHM}(\mathfrak{X})$ induces a MHS map $H^k(\mathfrak{X}) \rightarrow \mathrm{IH}^k(\mathfrak{X})$, and for a projective resolution of singularities $\tilde{\mathfrak{X}} \xrightarrow{\pi} \mathfrak{X}$ with compact exceptional divisor, $\frac{H^k(\mathfrak{X})}{W_{k-1}} \xrightarrow{\pi^*} H^k(\tilde{\mathfrak{X}})$ (by the same proof as [64, Theorem 5.41]). Theorem 3.1 guarantees that $\mathrm{IC}_{\mathfrak{X}}^\bullet$ is a direct summand of $R\pi_*\mathbb{Q}_{\tilde{\mathfrak{X}}}[d_{\mathfrak{X}}]$, so that $\frac{H^k(\mathfrak{X})}{W_{k-1}} \hookrightarrow \mathrm{IH}^k(\mathfrak{X}) \hookrightarrow H^k(\tilde{\mathfrak{X}})$. For $\mathfrak{X}$ compact, the first term becomes $\mathrm{Gr}_k^W H^k(\mathfrak{X})$, and the first injection is [19, Theorem 3.2.1].

In particular, for $\pi: \hat{\mathcal{Y}} \rightarrow \mathcal{X}$ as in Example 2.2, we have $\mathrm{Gr}_2^W H^2(\mathcal{X}) \cong \mathrm{IH}^2(\mathcal{X}) \cong \mathbb{Q}(-1)^{\oplus 10}$ and $H^2(\hat{\mathcal{Y}}) \cong \mathbb{Q}(-1)^{\oplus 11}$. Writing $f^{\mathrm{sm}}: f^{-1}(U) \rightarrow U$ for the smooth part of f and $\mathcal{H}^i := Rf_*^{\mathrm{sm}}\mathbb{Q}$, Theorem 3.1 applies to:

- $\mathcal{K}^\bullet = \mathbb{Q}_{\hat{\mathcal{Y}}}[2]$ and $\pi \implies R\pi_*\mathcal{K}^\bullet \simeq {}^p R^0 \pi_*\mathcal{K}^\bullet \simeq \mathrm{IC}_{\mathcal{X}}^\bullet \oplus i_*^q \mathbb{Q}(-1)$ (cf. (7.1)–(7.2)), so that ${}^p\mathcal{L}_\pi^\bullet$ on $H^2(\hat{\mathcal{Y}})$ is trivial; and
- $\mathcal{K}^\bullet = \mathrm{IC}_{\mathcal{X}}^\bullet$ and $f \implies Rf_*\mathcal{K}^\bullet \simeq \bigoplus_{j=-1}^1 ({}^p R^j f_*\mathcal{K}^\bullet)[-j]$ with ${}^p R^j f_*\mathcal{K}^\bullet \simeq \overline{\mathrm{IC}_{\mathbb{P}^1}^\bullet(\mathcal{H}^{j+1}[1])}$ ($j = \pm 1$) and ${}^p R^0 f_*\mathcal{K}^\bullet \simeq \mathrm{IC}_{\mathbb{P}^1}^\bullet(\mathcal{H}^1[1]) \oplus \bigoplus_{\sigma \in \mathbb{P}^1 \setminus U} \mathrm{IH}_{\mathrm{ph}, \sigma}^2$ (cf. (5.5)). The graded pieces of ${}^p\mathcal{L}_f^\bullet$ on $\mathrm{IH}^2(\mathcal{X})$ (cf. (5.4) and (5.9)) are then $\mathrm{Gr}_{p\mathcal{L}}^{-1} \cong H^0(U, \mathcal{H}^2) \cong \mathbb{Q}(-1)$ (class of a section), $\mathrm{Gr}_{p\mathcal{L}}^1 \cong H_c^2(U, \mathcal{H}^0) \cong \mathbb{Q}(-1)$ (class of a smooth fiber), and $\mathrm{Gr}_{p\mathcal{L}}^0 \cong \bigoplus_{\sigma} \mathrm{IH}_{\mathrm{ph}, \sigma}^2 \cong \mathbb{Q}(-1)^{\oplus 8}$ (from singular fibers; $\mathrm{IH}^1(\mathbb{P}^1, \mathcal{H}^1)$ vanishes).

Theorem 3.1 does *not* apply to $\mathcal{K}^\bullet = \mathbb{Q}_{\mathcal{X}}[2]$ and f ; see Example 7.1.

We now look more systematically at immediate consequences of the DT for families over a curve and resolutions of isolated singularities.

4 Decomposition theorem over a curve (1): nearby and vanishing cycles

Consider the scenario

$$\begin{array}{ccccc} \mathcal{X}_U & \xhookrightarrow{\mathcal{J}} & \mathcal{X} & \xleftarrow{\mathcal{I}} & \mathcal{X}_\Sigma & \xlongequal{\quad} & \bigcup_{\sigma \in \Sigma} X_\sigma \\ \downarrow f^s & & \downarrow f & & \downarrow & & \\ S \setminus \Sigma & \xlongequal{\quad} & U & \xhookrightarrow{\mathcal{J}} & S & \xleftarrow{\mathcal{I}} & \Sigma \end{array} \quad (4.1)$$

where $d_S = 1$ ($\implies d_{\mathcal{X}} = n + 1$), S is smooth, $\mathcal{X}_U$ and $S|_{\mathcal{X}_U}$ are topologically locally constant (e.g. equisingular) over U , and our semisimple $\mathcal{K}^\bullet$ belongs to $\text{MHM}(\mathcal{X})$ (i.e. its underlying complex is perverse). For each j , we have

$${}^p R^j f_* \mathcal{K}^\bullet = j_* \mathbb{V}^j(\mathcal{K}^\bullet)[1] \oplus \bigoplus_{\sigma \in \Sigma} \iota_{\sigma*} W_{\sigma}^j(\mathcal{K}^\bullet), \quad (4.2)$$

where $\mathbb{V}^j(\mathcal{K}^\bullet) = R^{j-1} f_*^s \mathcal{K}^\bullet|_{\mathcal{X}_U}$ are local systems/VMHS and $W_{\sigma}^j(\mathcal{K}^\bullet)$ are vector spaces/MHS. Note that by X_{σ} (and later, X_0) we always mean the *reduced* special fiber, since MHM live on a complex analytic space.⁴

Writing t for (the composition of f with) a local coordinate on a small disk $\Delta_{\sigma} \subset S$ about σ , the associated nearby and vanishing cycle functors sit in a dual pair of distinguished *vanishing cycle triangles*⁵

$$\begin{array}{ccc} \mathcal{I}_{\sigma}^* & \xrightarrow{\text{sp}} & \psi_t \\ \delta \swarrow +1 & & \searrow \text{can} \\ & \phi_t & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}_{\sigma}^! & \xrightarrow{\delta^{\vee}} & {}^p \phi_t \\ +1 \swarrow \text{sp}^{\vee} & & \searrow \widetilde{\text{var}} \\ & {}^p \psi_t(-1) & \end{array} \quad (4.3)$$

and satisfy $R\Gamma \mathcal{I}_{\sigma}^! = \iota_{\sigma}^! Rf_*$, $R\Gamma \mathcal{I}_{\sigma}^* = \iota_{\sigma}^* Rf_*$, $R\Gamma \psi_t = \psi_t Rf_*$, and $R\Gamma \phi_t = \phi_t Rf_*$ [60]. Applied to $\mathcal{K}^\bullet$, each morphism in the triangles yields a morphism of MHM, with the exception of $\widetilde{\text{var}}$:⁶ here one needs to break $\psi_t \mathcal{K}^\bullet$ and $\phi_t \mathcal{K}^\bullet$ into unipotent and non-unipotent parts for the action of T_{σ} , whereupon $\widetilde{\text{var}}^u : \phi_t^u \rightarrow \psi_t^u(-1)$ and $\widetilde{\text{var}}^n : \phi_t^n \rightarrow \psi_t^n$ induce MHM maps. Here $\widetilde{\text{var}}^u$ [resp. $\widetilde{\text{var}}^n$] is the morphism Var in [66, 3.4.10] [resp. a canonical isomorphism], and $(-)^u = \ker(T^{\text{ss}} - I)$ [resp. $(-)^n = \text{im}(T^{\text{ss}} - I)$] is written $(-)_1$ [resp. $(-)_{\neq 1}$] in the notation of *op. cit.* (see also [72, §§8-9]).

Next, setting $\mathbb{V}_{\text{lim}}^{\ell}(\mathcal{K}^\bullet) := \psi_{t*} \mathbb{V}_{\sigma}^{\ell}(\mathcal{K}^\bullet)$, we have the monodromy invariants $\mathbb{V}_{\text{lim}}^{\ell}(\mathcal{K}^\bullet)^{T_{\sigma}} := \ker(T_{\sigma} - I)$ and coinvariants $\mathbb{V}_{\text{lim}}^{\ell}(\mathcal{K}^\bullet)_{T_{\sigma}} := \text{coker}(T_{\sigma} - I)$. By the DT, we compute

$$\begin{aligned} H^{\ell}(X_{\sigma}, \mathcal{K}^\bullet) &:= \mathbb{H}^{\ell}(X_{\sigma}, \mathcal{I}_{\sigma}^*[-1]\mathcal{K}^\bullet) \cong H^{\ell-1}(\iota_{\sigma}^* Rf_* \mathcal{K}^\bullet) \\ &= \bigoplus_j H^{\ell-j-1}(\iota_{\sigma}^* {}^p R^j f_* \mathcal{K}^\bullet) \\ &\cong \mathbb{V}_{(\text{lim})}^{\ell}(\mathcal{K}^\bullet)^{T_{\sigma}} \oplus W_{\sigma}^{\ell-1}(\mathcal{K}^\bullet) \end{aligned} \quad (4.4)$$

⁴ Warning: the (“nonreduced”) components of X_{σ} along which a local coordinate t has order > 1 are philosophically part of the singularity locus of X_{σ} , e.g. when considering support of ${}^p \phi_t \mathbb{Q}_{\mathcal{X}}$. See Proposition 5.5 below.

⁵ See [23] for the first and [66, 5.2.1] for the second in the form used here. Note that ${}^p \phi_t := \phi_t[-1]$ and ${}^p \psi_t := \psi_t[-1]$ send $\text{MHM}(\mathcal{X})$ to $\text{MHM}(X_{\sigma})$.

⁶ The tilde reflects the fact that, while related, $\widetilde{\text{var}}$ is not the standard var in the theory of perverse sheaves, because we do not have $\widetilde{\text{var}} \circ \text{can} = T - I$ (see below).

for the *special fiber cohomology* and

$$\begin{aligned} H_{X_\sigma}^\ell(\mathcal{X}, \mathcal{K}^\bullet) &:= \mathbb{H}^\ell(X_\sigma, \mathcal{I}_\sigma^1[-1]\mathcal{K}^\bullet) \cong H^{\ell-1}(\iota_\sigma^1 Rf_* \mathcal{K}^\bullet) \\ &= \mathbb{V}_{(\lim)}^{\ell-2}(\mathcal{K}^\bullet)_{T_\sigma}(-1) \oplus W_\sigma^{\ell-1}(\mathcal{K}^\bullet) \end{aligned} \quad (4.5)$$

for the *special fiber “homology”*, where we used $\iota_\sigma^* J_* \mathbb{V} = H^0(\Delta_\sigma, J_* \mathbb{V}) = \mathbb{V}^{T_\sigma}$ and $\iota_\sigma^! J_* \mathbb{V} = H_c^2(\Delta_\sigma, J_* \mathbb{V})[-2] \cong H^1(\Delta_\sigma^*, \mathbb{V})[-2] = \mathbb{V}_{T_\sigma}(-1)[-2]$. We also write

$$\begin{aligned} H_{\lim, \sigma}^\ell(\mathcal{K}^\bullet) &:= \mathbb{H}^\ell(X_\sigma, {}^p\psi_t \mathcal{K}^\bullet) \cong H^{\ell-1}(\psi_t Rf_* \mathcal{K}^\bullet) \\ &= \mathbb{V}_{\lim}^\ell(\mathcal{K}^\bullet) \end{aligned} \quad (4.6)$$

for the *limiting cohomology* and

$$H_{\text{van}, \sigma}^\ell(\mathcal{K}^\bullet) := \mathbb{H}^\ell(X_\sigma, {}^p\phi_t \mathcal{K}^\bullet) \cong H^{\ell-1}(\phi_t Rf_* \mathcal{K}^\bullet) \quad (4.7)$$

for the *vanishing cohomology*. These spaces carry natural MHSs with morphisms induced by the MHM-maps above; we can either break $\widetilde{\text{var}} : H_{\text{van}, \sigma}^\ell(\mathcal{K}^\bullet) \rightarrow H_{\lim, \sigma}^\ell(\mathcal{K}^\bullet)$ into unipotent and non-unipotent parts, or regard it as a map of $\mathbb{Q}$ -vector spaces — one whose composition $\widetilde{\text{var}} \circ \text{can} \in \text{End}_{\mathbb{Q}}(H_{\lim, \sigma}^\ell(\mathcal{K}^\bullet))$ with can yields $N_{(H_{\lim}^\ell)^u} \oplus \text{Id}_{(H_{\lim}^\ell)^n}$. While not a morphism of MHS (since N is a $(-1, -1)$ morphism on $(H_{\lim}^\ell)^u$), the kernel [resp. cokernel] of the latter (which is the same as the kernel [resp. cokernel] of $T_\sigma - I$) is a sub- [resp. quotient-] MHS of $H_{\lim, \sigma}^\ell(\mathcal{K}^\bullet)$.

It remains to better understand $H_{\text{van}, \sigma}^\ell(\mathcal{K}^\bullet)$ and $W_\sigma^j(\mathcal{K}^\bullet)$. For *any* (not necessarily semisimple) perverse sheaf $\mathcal{P}^\bullet$ on $\mathcal{S}$, sub- resp. quotient- objects of $\mathcal{P}^\bullet$ supported on $\{\sigma\}$ correspond to $\ker(\widetilde{\text{var}})$ resp. $\text{coker}(\text{can})$ on ${}^p\phi_t \mathcal{P}^\bullet$ [72]. So for $\mathcal{P}^\bullet$ semisimple, we have ${}^p\phi_t \mathcal{P}^\bullet = \ker(\widetilde{\text{var}}) \oplus \text{im}(\text{can})$, which (together with ${}^p\psi_t \xrightarrow{\text{can}} {}^p\phi_t \rightarrow \iota_\sigma^* \xrightarrow{+1}_{\text{sp}}$ and $\widetilde{\text{var}} \circ \text{can} = N \oplus I$) yields identifications

$$\begin{cases} \text{im}(\text{can}) \cong \text{coim}(\widetilde{\text{var}}) \cong {}^p\psi_t \mathcal{P}^\bullet / \ker(T_\sigma - I) \\ \ker(\widetilde{\text{var}}) \cong \text{coker}(\text{can}) \cong \ker\{\text{sp} : \iota_\sigma^* \rightarrow {}^p\psi_t \mathcal{P}^\bullet[1]\}. \end{cases} \quad (4.8)$$

If $\mathcal{P}^\bullet = {}^p R^j f_* \mathcal{K}^\bullet$, then $\text{im}(\text{can}) \cong \mathbb{V}_{\lim}^j(\mathcal{K}^\bullet) / \ker(T_\sigma - I)$ while sp maps $\mathbb{V}_{\lim}^j(\mathcal{K}^\bullet)_{T_\sigma}[1] \oplus W_\sigma^j(\mathcal{K}^\bullet) \rightarrow \mathbb{V}_{\lim}^j(\mathcal{K}^\bullet)[1]$; we conclude that

$$\begin{aligned} H_{\text{van}, \sigma}^\ell(\mathcal{K}^\bullet) &= \oplus_j H^{\ell-j}({}^p\phi_t {}^p R^j f_* \mathcal{K}^\bullet) \\ &= \frac{\mathbb{V}_{\lim}^\ell(\mathcal{K}^\bullet)}{\ker(T_\sigma - I)} \oplus W_\sigma^\ell(\mathcal{K}^\bullet). \end{aligned} \quad (4.9)$$

Remark 4.1 To put this more simply, a perverse sheaf $\mathbf{P}$ on Δ decomposes (à la Remark 3.2(iv)) $\iff {}^p\phi_t \mathbf{P} = \ker(\widetilde{\text{var}}) \oplus \text{im}(\text{can}) \iff \mathbf{P}$ takes the form $\iota_* W \oplus J_* \mathbb{V}[1]$
 $\iff$ the corresponding quiver representation ${}^p\psi_t \mathbf{P} \xrightleftharpoons[\widetilde{\text{var}}]{\text{can}} {}^p\phi_t \mathbf{P}$ takes the form $V \xrightleftharpoons[(N \oplus I, 0)]{(\eta, 0)}$

$\frac{V}{\sqrt{T}} \oplus W$. The decomposition (4.9) is precisely this statement for $P = ({}^p R^j f_* \mathcal{K}^\bullet)|_\Delta$, which decomposes by Theorem 3.1.

Finally, consider the composition

$$H_{X_\sigma}^\ell(\mathcal{X}, \mathcal{K}^\bullet) \xrightarrow{\delta^\vee} H_{\text{van}, \sigma}^{\ell-1}(\mathcal{K}^\bullet) \xrightarrow{\delta} H^\ell(X_\sigma, \mathcal{K}^\bullet), \quad (4.10)$$

in which both maps are the identity on $W_\sigma^{\ell-1}(\mathcal{K}^\bullet)$ (and zero on the other summand). If $\mathcal{X}_{\Delta_\sigma} = f^{-1}(\Delta_\sigma)$ and $p \in \Delta_\sigma \setminus \{\sigma\}$, then (4.10) is really just the map $H_{X_\sigma}^\ell(\mathcal{X}_{\Delta_\sigma}) \rightarrow H^\ell(\mathcal{X}_{\Delta_\sigma}, X_p) \rightarrow H^\ell(X_\sigma)$ with $\mathcal{K}^\bullet[-1]$ -coefficients; the composite is the same if the middle term is replaced by $H^\ell(\mathcal{X})$. Defining the *phantom cohomology at σ* by

$$H_{\text{ph}, \sigma}^\ell(\mathcal{K}^\bullet) := \ker \left\{ \text{sp} : H^\ell(X_\sigma, \mathcal{K}^\bullet) \rightarrow H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet) \right\}, \quad (4.11)$$

we therefore have

$$\begin{aligned} H_{\text{ph}, \sigma}^\ell(\mathcal{K}^\bullet) &= W_\sigma^{\ell-1}(\mathcal{K}^\bullet) = \text{im}(\delta \circ \delta^\vee) \\ &= \text{im} \left(H_{X_\sigma}^\ell(\mathcal{X}, \mathcal{K}^\bullet) \xrightarrow{\mathcal{I}_*^\sigma} \mathbb{H}^\ell(\mathcal{X}, \mathcal{K}^\bullet[-1]) \xrightarrow{\mathcal{I}_*^\sigma} H^\ell(X_\sigma, \mathcal{K}^\bullet) \right). \end{aligned}$$

Denote $\text{gy} := \mathcal{I}_*^\sigma \circ \mathcal{I}_*^\sigma$ in the sequel.

5 Decomposition theorem over a curve (2): consequences

Continuing for the moment with $\mathcal{K}^\bullet \in \text{MHM}(\mathcal{X})$ semisimple (but otherwise arbitrary), there are a couple of different ways to relate the special fiber cohomology and the limiting cohomology. The immediate consequence of the first triangle of (4.3) is the *vanishing cycle sequence* (of MHS)

$$\cdots \rightarrow H^\ell(X_\sigma, \mathcal{K}^\bullet) \xrightarrow{\text{sp}} H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet) \xrightarrow{\text{can}} H_{\text{van}, \sigma}^\ell(\mathcal{K}^\bullet) \xrightarrow{\delta} H^{\ell+1}(X_\sigma, \mathcal{K}^\bullet) \rightarrow \cdots, \quad (5.1)$$

which is useful whenever one has methods to compute $\phi_t \mathcal{K}^\bullet$, a subject taken up in Part II.

Also evident from the identifications in §4 is the *Clemens–Schmid sequence*

$$0 \rightarrow H_{\text{lim}, \sigma}^{\ell-2}(\mathcal{K}^\bullet)_{T_\sigma}(-1) \xrightarrow{\text{sp}^\vee} H_{X_\sigma}^\ell(\mathcal{X}, \mathcal{K}^\bullet) \xrightarrow{\text{gy}} H^\ell(X_\sigma, \mathcal{K}^\bullet) \xrightarrow{\text{sp}} H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)^{T_\sigma} \rightarrow 0, \quad (5.2)$$

which does away with the vanishing cohomology. The *local invariant cycle theorem* expressed by surjectivity of sp can be seen more briefly by just taking stalks on both sides of (4.2).

- Remark 5.1** (i) A more elegant approach to (5.2) can be formulated in terms of the octahedral axiom (cf. [66, Rm. 5.2.2] and also the “Appendix” below); though one must still invoke the DT to get (4.2) (equiv. (5.2.2.3) in *loc. cit.*).
- (ii) In general, if X is a complex analytic space, $F : X \rightarrow \Delta$ is proper, and $K \in D^b\text{MHM}(X)$ is self-(Verdier-)dual, then (a) ${}^pR^jF_*K$ decomposes ($\forall j$) and (b) C–S (5.2) holds ($\forall \ell$) are equivalent. (This follows at once from duality of can and $\widehat{\text{var}}$.) Also see Theorem A.4 in the “Appendix” below.

There are two amplifications that make (5.2) more useful: first, one can extend it to a longer sequence of MHS by using the unipotent parts:

$$H^\ell(X_\sigma, \mathcal{K}^\bullet) \xrightarrow{\text{sp}} H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)^u \xrightarrow{N_\sigma} H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)^u(-1) \xrightarrow{\text{sp}^\vee} H_{X_\sigma}^{\ell+2}(\mathcal{X}, \mathcal{K}^\bullet)(-1).$$

(Notice that $\{\ker(N_\sigma) \subset H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)^u\} = \{\ker(T_\sigma - I) \subset H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)\} = \{\ker(\text{can}) \subset H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)\}$.) Second, the hard Lefschetz part of DT implies isomorphisms $\mathbb{V}^{-j}(\mathcal{K}^\bullet) \xrightarrow{h^j} \mathbb{V}^j(\mathcal{K}^\bullet)(j)$ and $W^{-j}(\mathcal{K}^\bullet) \xrightarrow{h^j} W^j(\mathcal{K}^\bullet)(j)$, hence

$$H_{\text{lim}, \sigma}^{-\ell}(\mathcal{K}^\bullet) \xrightarrow[\hbar^\ell]{\cong} H_{\text{lim}, \sigma}^\ell(\mathcal{K}^\bullet)(\ell) \text{ and } H_{\text{ph}, \sigma}^{-\ell+1}(\mathcal{K}^\bullet) \xrightarrow[\hbar^\ell]{\cong} H_{\text{ph}, \sigma}^{\ell+1}(\mathcal{K}^\bullet)(\ell). \quad (5.3)$$

In addition to these local results, we mention one consequence of a global flavor: the *generalized Shioda formula*, which for $\mathcal{S}$ a complete curve reads

$$\begin{aligned} \mathbb{H}^k(\mathcal{X}, \mathcal{K}^\bullet) &\cong \bigoplus_{i=-1}^1 \text{Gr}_{p\mathcal{L}}^i \mathbb{H}^k(\mathcal{X}, \mathcal{K}^\bullet) \cong \bigoplus_{i=-1}^1 \mathbb{H}^i(\mathcal{S}, {}^pR^{k-i}f_*\mathcal{K}^\bullet) \\ &\cong \bigoplus_{i=-1}^1 \left\{ \mathbb{H}^{i+1}(\mathcal{S}, J_*\mathbb{V}^{k-i}(\mathcal{K}^\bullet)) \oplus \mathbb{H}^i(\mathcal{S}, \iota_*W^{k-i}(\mathcal{K}^\bullet)) \right\} \\ &= H^0(U, \mathbb{V}^{k+1}(\mathcal{K}^\bullet)) \oplus \left\{ \text{IH}^1(\mathcal{S}, \mathbb{V}^k(\mathcal{K}^\bullet)) \oplus \bigoplus_{\sigma} W_{\sigma}(\mathcal{K}^\bullet) \right\} \\ &\quad \oplus H_c^2(U, \mathbb{V}^{k-1}(\mathcal{K}^\bullet)), \end{aligned} \quad (5.4)$$

where we remark that the last term $\cong H^0(U, \mathbb{V}^{k-1}(\mathcal{K}^\bullet))^\vee(-1)$. On the other hand, if $\mathcal{S}$ is a quasi-projective curve, the last term is simply omitted. In either case, $\mathbb{H}^k(\mathcal{X}, \mathcal{K}^\bullet)$ surjects onto the first ($i = -1$) term, i.e. the *global invariant cycle theorem* holds.

Now we specialize to the case $\mathcal{K}^\bullet = \text{IC}_{\mathcal{X}}^\bullet$, noting (in light of Remark 3.2(ii)) that we can relax the hypotheses on $f, \mathcal{X}, \mathcal{S}$ somewhat if we ignore the hard Lefschetz statements. By considering that $\text{IC}_{\mathcal{X}}^\bullet|_{\mathcal{X}^{\text{sm}}} = \mathbb{Q}_{\mathcal{X}^{\text{sm}}}[n+1]$ on the smooth part of $\mathcal{X}$ (to get the degrees right), one arrives at identifications $\mathbb{H}^k(\mathcal{X}, \text{IC}_{\mathcal{X}}^\bullet) = \text{IH}^{k+n+1}(\mathcal{X})$, $H^k(X_\sigma, \text{IC}_{\mathcal{X}}^\bullet) = H^{k-1}(\iota_\sigma^*Rf_*\text{IC}_{\mathcal{X}}^\bullet) = \text{IH}^{k+n}(\mathcal{X}_\Delta)$, and $H_{X_\sigma}^k(\mathcal{X}, \text{IC}_{\mathcal{X}}^\bullet) = H^{k-1}(\iota_\sigma^!Rf_*\text{IC}_{\mathcal{X}}^\bullet) = \text{IH}_c^{k+n}(\mathcal{X}_\Delta)$. Accordingly we write

$$\text{IH}_{\text{ph}, \sigma}^{k+n} := \text{im} \left\{ \text{IH}_c^{k+n}(\mathcal{X}_\Delta) \rightarrow \text{IH}^{k+n}(\mathcal{X}_\Delta) \right\} = H_{\text{ph}, \sigma}^k(\text{IC}_{\mathcal{X}}^\bullet), \quad (5.5)$$

and note that there are morphisms (of MHS) from $H^{k+n}(\mathcal{X}_\Delta) \cong H^{k+n}(X_\sigma)$ [resp. $H_c^{k+n}(\mathcal{X}_\Delta) \cong H_{n-k+2}(X_\sigma)(-n-1)$], $H_{\text{ph}, \sigma}^{k+n} := \text{im}\{H_{n-k+2}(X_\sigma)(-n-1) \xrightarrow{\text{gy}}$

$H^{n+k}(X_\sigma)\}$ to $\mathrm{IH}^{k+n}(\mathcal{X}_\Delta)$ [resp. $\mathrm{IH}_c^{k+n}(\mathcal{X}_\Delta)$, $\mathrm{IH}_{\mathrm{ph},\sigma}^{k+n}$], which in general are neither injective nor surjective.⁷ As the restriction of $\mathrm{IC}_\bullet^\bullet$ to a fiber X_p over $p \in U$ is $\mathrm{IC}_{X_p}^\bullet[1]$, we also write $\mathbb{V}^k(\mathrm{IC}_\bullet^\bullet) =: \mathcal{IH}_f^{k+n}$, and

$$\begin{cases} \mathrm{IH}_{\mathrm{lim},\sigma}^{k+n} := \psi_{tJ*} \mathcal{IH}_f^{k+n} = \mathbb{H}^k(X_\sigma, {}^p\psi_t \mathrm{IC}_\bullet^\bullet) = H_{\mathrm{lim},\sigma}^k(\mathrm{IC}_\bullet^\bullet) \\ \mathrm{IH}_{\mathrm{van},\sigma}^{k+n} := \phi_{tJ*} \mathcal{IH}_f^{k+n} = \mathbb{H}^k(X_\sigma, {}^p\phi_t \mathrm{IC}_\bullet^\bullet) = H_{\mathrm{van},\sigma}^k(\mathrm{IC}_\bullet^\bullet). \end{cases}$$

With this notation (and $\ell = k - n$), (5.1)–(5.4) become

$$\cdots \rightarrow \mathrm{IH}^k(\mathcal{X}_\Delta) \xrightarrow{\mathrm{sp}} \mathrm{IH}_{\mathrm{lim},\sigma}^k(X_t) \xrightarrow{\mathrm{can}} \mathrm{IH}_{\mathrm{van},\sigma}^k(X_t) \xrightarrow{\delta} \mathrm{IH}^{k+1}(\mathcal{X}_\Delta) \rightarrow \cdots, \quad (5.6)$$

$$0 \rightarrow \mathrm{IH}_{\mathrm{lim},\sigma}^{k-2}(X_t)_{T_\sigma}(-1) \xrightarrow{\mathrm{sp}^\vee} \mathrm{IH}_c^k(\mathcal{X}_\Delta) \rightarrow \mathrm{IH}^k(\mathcal{X}_\Delta) \xrightarrow{\mathrm{sp}} \mathrm{IH}_{\mathrm{lim},\sigma}^k(X_t)^{T_\sigma} \rightarrow 0, \quad (5.7)$$

$$\mathrm{IH}_{\mathrm{lim},\sigma}^{n-k} \xrightarrow[h^k]{\cong} \mathrm{IH}_{\mathrm{lim},\sigma}^{n+k}(k), \quad \mathrm{IH}_{\mathrm{ph},\sigma}^{n-k+1} \xrightarrow[h^k]{\cong} \mathrm{IH}_{\mathrm{ph},\sigma}^{n+k+1}(k), \quad (5.8)$$

$$\mathrm{IH}^k(\mathcal{X}) \cong H^0(U, \mathcal{IH}_f^k) \oplus \{\mathrm{IH}^1(\mathcal{S}, \mathcal{IH}_f^{k-1}) \oplus \bigoplus_{\sigma} \mathrm{IH}_{\mathrm{ph},\sigma}^k\} \oplus H_c^2(U, \mathcal{IH}_f^{k-2}). \quad (5.9)$$

Note that $\ker(\mathrm{can}) = \mathrm{IH}_{\mathrm{lim},\sigma}^k(X_t)^{T_\sigma}$, and (5.6) splits at all but the $\mathrm{IH}_{\mathrm{lim}}$ terms.

Remark 5.2 To compute $\mathrm{IH}^k(\mathcal{X}_\Delta)$ (say, for use in (5.7)) one needs to know $\mathcal{I}_\sigma^* \mathrm{IC}_\bullet^\bullet$, which was studied by Dimca and Saito in [26]. In general, $\mathcal{I}_\sigma^* \mathrm{IC}_\bullet^\bullet[-1]$ is perverse, and there is a map $c_\sigma: \mathbb{Q}_{X_\sigma}[n] \rightarrow \mathcal{I}_\sigma^* \mathrm{IC}_\bullet^\bullet[-1]$ with kernel ${}^p\mathcal{H}^{-1}\mathbb{Q}_\mathcal{X}[n+1]$ and cokernel $W_n {}^p\mathcal{H}^0\mathbb{Q}_\mathcal{X}[n+1]$. If $\mathbb{Q}_\mathcal{X}[n+1]$ (hence $\mathbb{Q}_{X_\sigma}[n]$) is perverse (e.g. if $\mathcal{X}$ has local complete intersection singularities) then the kernel vanishes and $[c_\sigma]: H^k(X_\sigma) \rightarrow \mathrm{IH}^k(\mathcal{X}_\Delta)$ is injective on the top Gr_k^W . (See also Example 3.4 and Remark 8.1.) If $\mathcal{I}_\sigma^* \mathrm{IC}_\bullet^\bullet[-1] = \mathbb{Q}_{X_\sigma}[n]$, then $\mathrm{IC}_\bullet^\bullet = \mathbb{Q}_\mathcal{X}[n+1]$ and one just uses (5.11) instead of (5.7).

If $\mathcal{X}$ is smooth, then $\mathrm{IC}_\bullet^\bullet = \mathbb{Q}_\mathcal{X}[n+1]$ and we simply replace IH resp. $\mathcal{IH}_f$ everywhere by H resp. $\mathcal{H}_f$, except for the parabolic cohomology group $\mathrm{IH}^1(\mathcal{S}, \mathcal{H}_f^{k-1})$ in (5.9). The rank of the latter may be computed by the *Euler-Poincaré formula*, which reads

$$\mathrm{rk}(\mathrm{IH}^1(\mathcal{S}, \mathbb{V})) = \sum_{\sigma \in \Sigma} \mathrm{rk}(\mathbb{V}_v / \mathbb{V}_v^{T_\sigma}) - \chi(\mathcal{S}) \cdot \mathrm{rk}(\mathbb{V}_v) + h^1(\mathcal{S}) \cdot \mathrm{rk}(\mathbb{V}_c), \quad (5.10)$$

if $\mathbb{V} = \mathbb{V}_c \oplus \mathbb{V}_v$ is the splitting into fixed (constant) and variable parts for the local system underlying a PVHS. Next, for the two exact sequences of MHS we have:

Theorem 5.3 *For $\mathcal{X}$ smooth, the Clemens–Schmid sequence reads*

$$0 \rightarrow H_{\mathrm{lim},\sigma}^{k-2}(X_t)_{T_\sigma}(-1) \xrightarrow{\mathrm{sp}^\vee} H_{2n-k+2}(X_\sigma)(-n-1) \xrightarrow{\mathrm{gy}} H^k(X_\sigma) \xrightarrow{\mathrm{sp}} H_{\mathrm{lim},\sigma}^k(X_t)^{T_\sigma} \rightarrow 0. \quad (5.11)$$

⁷ One also has maps from $H^*(X_\sigma) \rightarrow \mathrm{IH}^*(X_\sigma)$, but $\mathrm{IH}^*(X_\sigma)$ is different from $\mathrm{IH}^*(\mathcal{X}_\Delta)$ (e.g., when $\mathcal{X}_\Delta$ is smooth and X_σ is not).

while the vanishing cycle sequence is

$$0 \rightarrow H_{\text{ph},\sigma}^k \rightarrow H^k(X_\sigma) \xrightarrow{\text{sp}} H_{\text{lim},\sigma}^k(X_t) \xrightarrow{\text{can}} H_{\text{van},\sigma}^k(X_t) \rightarrow H_{\text{ph},\sigma}^{k+1} \rightarrow 0. \quad (5.12)$$

Proof These follow directly from (5.6)–(5.7) since X_σ is a deformation retract of $\mathcal{X}_\Delta$. $\square$

Remark 5.4 One can actually prove (5.11) without invoking Theorem 3.1 using the equivalence of (a) and (b) in Remark 5.1 (with $K = \mathbb{Q}_{\mathcal{X}}[n+1]$), together with C–S for semistable degenerations. If

$$\begin{array}{ccc} \mathcal{Y}_\Delta & \xrightarrow{\quad \tilde{\pi} \quad} & \mathcal{X}_\Delta \\ g \downarrow & & \downarrow f \\ \Delta & \xrightarrow{\quad \pi \quad} & \Delta \end{array}$$

is a semistable reduction ($\pi(t) = t^\kappa$), then (5.11) for $g \implies Rg_*\mathbb{Q}_{\mathcal{Y}}$ decomposes $\pi \xrightarrow{\text{finite}} Rf_*R\tilde{\pi}_*\mathbb{Q}_{\mathcal{Y}} = R\pi_*Rg_*\mathbb{Q}_{\mathcal{Y}}$ decomposes. Since $\mathbb{Q}_{\mathcal{X}}$ is a direct factor of $R\tilde{\pi}_*\mathbb{Q}_{\mathcal{Y}}$, this also decomposes, and so (5.11) holds for f . (In fact, this amounts to a direct proof of the DT in this case.) We thank Saito for this remark; see also Remarks A.4(iii) in his “Appendix”.

Finally, we record two important facts about the terms in the sequences (5.11)–(5.12). For our purposes here $\text{sing}(X_\sigma)$ contains any “nonreduced” components of X_σ (where $\text{ord}(t) > 1$).

Proposition 5.5 For $\mathcal{X}$ smooth and $d_{\text{sing}} := \dim(\text{sing}(X_\sigma))$:

- (i) $H_{\text{ph},\sigma}^k := \ker(\text{sp}) = \delta(H_{\text{van},\sigma}^{k-1}(X_t)) = \text{im}(\text{gy})$ is pure of weight k (and level $\leq k-2$) and a direct summand of $H^k(X_\sigma)$; and
- (ii) $H_{\text{van},\sigma}^k(X_t)$ (hence $H_{\text{ph},\sigma}^{k+1}$) is zero outside the range $n - d_{\text{sing}} \leq k \leq n + d_{\text{sing}}$.

Proof (i) The MHS $H_1 := H_{2n-k+2}(X_\sigma)(-n-1)$ has weights $\geq k$, while $H_2 := H^k(X_\sigma)$ has weights $\leq k$. Therefore $\text{gy}: H_1 \rightarrow H_2$ is split and factors through $\text{Gr}_k^W H_1$, which (with complexification dual to $\text{Gr}_{2n-k+2}^W H^{2n-k+2}(X_\sigma)_{\mathbb{C}} = H^{n,n-k+2}(X_\sigma) \oplus \dots \oplus H^{n-k+2,n}(X_\sigma)$) has level $\leq k-2$.

- (ii) As ${}^p\phi_t\mathbb{Q}_{\mathcal{X}}[n+1]$ is supported on $\text{sing}(X_\sigma)$, its perversity implies the existence of a stratification $\hat{S}_\bullet$ ($\dim \hat{S}_q = q$) such that the cohomology sheaves $\mathcal{H}^j({}^p\phi_t\mathbb{Q}_{\mathcal{X}}[n+1]|_{\hat{S}_q \setminus \hat{S}_{q-1}})$ vanish unless $-d_{\text{sing}} \leq j \leq -q$. Hence in the hypercohomology spectral sequence

$$E_2^{i,j} = \mathbb{H}^i(\text{sing}(X_\sigma), \mathcal{H}^j({}^p\phi_t\mathbb{Q}_{\mathcal{X}}[n+1])) \implies H_{\text{van},\sigma}^{i+j+n}(X_t),$$

all nonzero terms lie in $\{i \geq 0, i+2j \leq 0, j \geq -d_{\text{sing}}\}$, so $E_\infty^{i,j} = 0$ outside $-d_{\text{sing}} \leq i+j \leq d_{\text{sing}}$. $\square$

Remark 5.6 More generally, part (ii) and (5.12) hold (for H_{van}^k only) if $\mathcal{X}$ has local complete intersection singularities since then $\mathbb{Q}_{\mathcal{X}}[n+1]$ is still perverse. (Note that $\dim(\text{sing}(\mathcal{X})) \leq d_{\text{sing}}$.) This is because the derivation of (5.1) made no use of the Decomposition Theorem.

Corollary 5.7 *If $\mathcal{X}$ is smooth, then $\text{Gr}_F^0 H^k(X_\sigma) \cong \text{Gr}_F^0 H_{\text{lim}}^k(X_t)^{T_\sigma}$ and $W_{k-1} H^k(X_\sigma) \cong W_{k-1} H_{\text{lim}}^k(X_\sigma)^{T_\sigma}$.*

6 Decomposition theorem over a curve (3): examples

If $\mathcal{X}$ is smooth and X_σ has only isolated singularities ($d_{\text{sing}} = 0$), then by Proposition 5.5(ii) the vanishing cycle sequence becomes

$$0 \rightarrow H^n(X_\sigma) \xrightarrow{\text{sp}} H_{\text{lim},\sigma}^n(X_t) \xrightarrow{\text{can}} H_{\text{van},\sigma}^n(X_t) \xrightarrow{\delta} H^{n+1}(X_\sigma) \xrightarrow{\text{sp}} H_{\text{lim},\sigma}^{n+1}(X_t) \rightarrow 0 \quad (6.1)$$

with $H^k(X_\sigma) \xrightarrow{\cong} H_{\text{lim},\sigma}^k(X_t)$ in all other degrees ($k \neq n, n+1$). In particular, $T_\sigma = I$ on $H_{\text{lim},\sigma}^k(X_t)$ for $k \neq n$, while Clemens–Schmid reduces to

$$\begin{cases} 0 \rightarrow H_{\text{lim},\sigma}^{n-1}(X_t) \rightarrow H_{n+1}(X_\sigma)(-n-1) \rightarrow H^{n+1}(X_\sigma) \rightarrow H_{\text{lim},\sigma}^{n+1}(X_t) \rightarrow 0 \\ \text{and } H^n(X_\sigma) \cong H_{\text{lim},\sigma}^n(X_t)^{T_\sigma}. \end{cases} \quad (6.2)$$

These sequences are also valid when $n = 1$ and the curve X_σ has nonreduced components: certainly $H_{-1}(X_\sigma)(-2) \xrightarrow{\text{gy}} H^1(X_\sigma)$ is zero and (assuming f connected) $T_\sigma = I$ on H_{lim}^2 and H_{lim}^0 , which gives (6.2) hence (6.1).

We illustrate (6.1)–(6.2) for two simple examples, then relate H_{van} to “tails” appearing in the semistable reduction process.

Example 6.1 Let $\mathcal{X} \xrightarrow{f} \mathbb{P}^1$ be a smooth minimal elliptic surface with section, and singular fibers of types $2I_1$, I_6^* , II , and IV^* (e.g., obtained from base-change and quadratic twist of the elliptic modular surface for $\Gamma_1(3)$). These have $m_\sigma = 1, 11, 1$, resp. 7 components, with $H_{\text{ph},\sigma}^2 \cong \mathbb{Q}(-1)^{\oplus(m_\sigma-1)}$; and $\deg(\mathcal{H}_{f,e}^{1,0}) = \frac{1}{12}(2 \cdot 1 + 12 + 2 + 8) = 2 \implies X \text{ K3}$. (Here $\mathcal{H}_{f,e}^{1,0}$ is Deligne’s canonical extension of $\mathcal{H}_f^{1,0} := R^1 f_*^{\text{sm}} \Omega_{\mathcal{X}}^1$ to $\mathbb{P}^1$; see [30, §III] for the contributions of the Kodaira singular fiber types to its degree.) In (5.9), the end terms are generated by the class of the (zero-) section and a fiber, while $\text{IH}^1(\mathbb{P}^1, \mathcal{H}_f^1)$ has rank 4 hence Hodge numbers $(1, 2, 1)$. (This rank comes either from Euler–Poincaré or from subtracting the Picard rank $\rho = 2 + \sum(m_\sigma - 1) = 18$ from 22.)

The Hodge–Deligne diagrams for the first three terms of (6.1) ($n = 1$) are well-known for each of these four degenerations. We display them in Fig. 1, writing numbers for $h^{p,q} \neq 1$, eigenvalues $\neq 1$ of T_σ^{ss} in braces, and $N := \log(T_\sigma^{\text{un}})$.

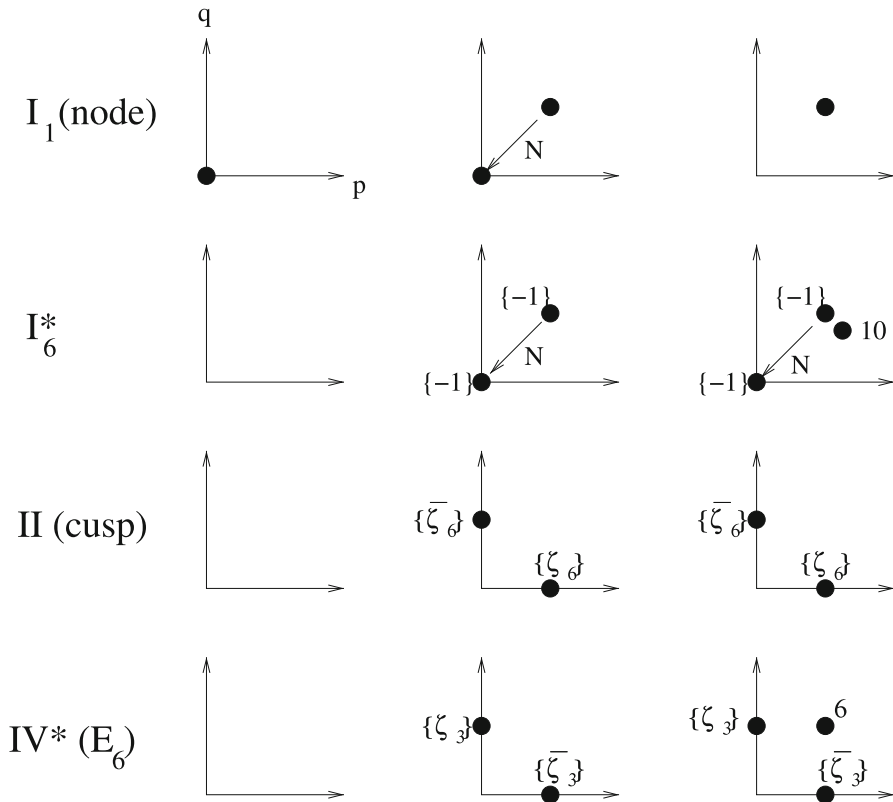


Fig. 1 Hodge–Deligne diagrams for Example 6.1

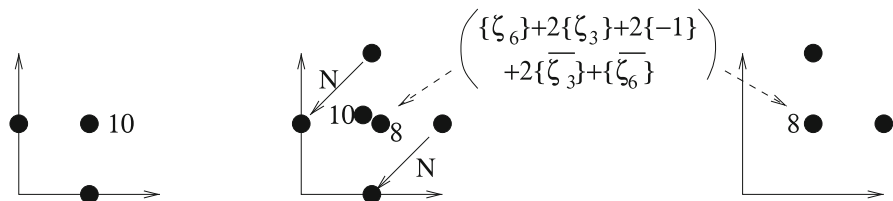


Fig. 2 Hodge–Deligne diagrams for Example 6.2

Example 6.2 Let $\mathcal{X}_\Delta \xrightarrow{f} \Delta$ be a family of $K3$ surfaces acquiring a single $\tilde{E}_8$ singularity: locally, $f \sim x^2 + y^3 + z^6 + \lambda xyz$. Then all $H_{\text{ph},0}^k$ are zero, and the first three terms of (6.1) ($n = 2$) are displayed in Fig. 2.

If $\mathfrak{X}_\Delta$ is the (singular) base-change by $t \mapsto t^6$, then these terms are unchanged except that the action of T^{ss} trivializes – which means that (5.11) now fails. (As we shall see explicitly in Example 7.1, $\mathbb{Q}_{\mathfrak{X}_\Delta}[3]$ is not semisimple.) Of course (5.6)–(5.7) still apply: in particular, $\text{IH}^2(\mathfrak{X}_\Delta) \cong H^2(X_0) \oplus \mathbb{Q}(-1)^{\oplus 8}$ and $\text{IH}_{\text{ph},0}^2 = \{0\}$.

On the other hand, performing a weighted blow-up of $f + t^6 = 0$ at the origin yields the semistable/slc model $\mathcal{Y}_\Delta \xrightarrow{t} \Delta$, with $Y_0 = \tilde{X}_0 \cup \mathcal{E}$ and $\tilde{X}_0 \cap \mathcal{E} =: E$. Here $E = \{f = 0\} \subset \mathbb{P}[1 : 2 : 3]$ is an elliptic curve and the “tail” $\mathcal{E} = \{f + t^6 = 0\} \subset \mathbb{P}[1 : 1 : 2 : 3]$ a del Pezzo surface of degree 1 ($\rho = 9$). As we have seen in Example 2.2, the extension class of $H^2(\tilde{X}_0)$ by $H^1(E)$ in

$$0 \rightarrow H^1(E) \rightarrow H^2(Y_0) \rightarrow H^2(\tilde{X}_0) \oplus H^2(\mathcal{E}) \rightarrow H^2(E) \rightarrow 0$$

(i.e. in $H^2(X_0)$) can be nontorsion. However, that of $\ker(H^2(\mathcal{E}) \rightarrow H^2(E))$ by $H^1(E)$ is torsion due to the eigenspace decomposition under the *original* T^{ss} and the fact that we have not altered H_{lim}^1 in (6.2). This will change if we take a more general pullback of the form

$$t^6 + a_5 t^5 z + t^4 (a_4 z^2 + b_4 y) + \cdots + x^2 + y^3 + z^6 + \lambda xyz = 0,$$

as then $t \mapsto \zeta_6 t$ no longer induces an automorphism of $H^2(\mathcal{E}, E)$.

We briefly explain how the relation between the “tail” $(\mathcal{E}, E)$ and the vanishing cohomology generalizes for isolated singularities. Consider the scenario

$$\begin{array}{ccccc}
 \Delta & \xlongequal{\quad} & \Delta & \xrightarrow{(\cdot)^K} & \Delta \\
 \uparrow F & & \uparrow & & \uparrow f \\
 \mathcal{Y} & \xrightarrow{\pi} & \mathfrak{X} & \xrightarrow{\rho} & \mathcal{X} \\
 \uparrow \tilde{\mathcal{I}} & & \uparrow \mathcal{I} & & \uparrow \\
 \tilde{X}_0 \cup \mathcal{E} & \xlongequal{\quad} & Y_0 & \xrightarrow{\pi_0} & X_0 & \xlongequal{\quad} & X_0 \\
 \uparrow & & \uparrow \iota_p & & \uparrow \\
 \tilde{X}_0 \cap \mathcal{E} & \xlongequal{\quad} & E & \longrightarrow & \{p\}
 \end{array} \tag{6.3}$$

where $\mathcal{X}, \mathcal{Y}, \tilde{X}_0$ are smooth, $X_0 = (f)$ is reduced and irreducible, $p \in \text{sing}(X_0) = \text{sing}(\mathfrak{X})$ is isolated, ρ is cyclic base-change and F is semistable ($\implies Y_0$ SNCD). First we look at the case of $\mathcal{E}, E$ irreducible and smooth:

Proposition 6.3 *As a mixed Hodge structure, $H_{\text{van}}^k(X_t)$ is the reduced cohomology $\tilde{H}^k(\mathcal{E} \setminus E)$, and this vanishes for $k \neq n$.*

Proof Applying ${}^p\phi_t$ to the distinguished triangle in $D_c^b(\mathfrak{X})$

$$\text{Cone}(\alpha)[-1] \rightarrow \mathbb{Q}_{\mathfrak{X}}[n+1] \xrightarrow{\alpha} R\pi_* \mathbb{Q}_{\mathcal{Y}}[n+1] \xrightarrow{+1}$$

yields a triangle in $D_c^b(X_0)$ with terms:

$$\begin{aligned} \bullet \quad {}^p\phi_t \text{Cone}(\alpha)[-1] &= \mathcal{I}^* \text{Cone}(\alpha)[-1] = \text{Cone}(\mathcal{I}^* \mathbb{Q}_{\mathcal{X}}[n] \rightarrow R\pi_{0*} \tilde{\mathcal{I}}^* \mathbb{Q}_{\mathcal{Y}}[n]) \\ &= \text{Cone}(\mathbb{Q}_{X_0}[n] \rightarrow R\pi_{0*} \mathbb{Q}_{Y_0}[n]) \\ &= (\iota_p)_* \text{Cone}(\mathbb{Q}[n] \rightarrow R\Gamma \mathbb{Q}_{\mathcal{E}}[n]) \end{aligned}$$

using the fact that ${}^p\psi_t(\alpha)$ is an isomorphism;

$$\bullet \quad {}^p\phi_t \mathbb{Q}_{\mathcal{X}}[n+1] = {}^p\phi_t \mathbb{Q}_{\mathcal{X}}[n+1]$$

since the base-change doesn't affect the first vanishing cycle triangle; and

$$\begin{aligned} \bullet \quad {}^p\phi_t R\pi_* \mathbb{Q}_{\mathcal{Y}}[n+1] &= R\pi_{0*} {}^p\phi_t \mathbb{Q}_{\mathcal{Y}}[n+1] = R\pi_{0*} \mathbb{Q}_E(-1)[n-1] \\ &= \iota_{p*} R\Gamma \mathbb{Q}_E(-1)[n-1], \end{aligned}$$

where we used the fact that ${}^p\phi_t$ of the constant sheaf is $\mathbb{Q}(-1)[-2]$ at a node (Example 6.1). It is immediate that ${}^p\phi_t \mathbb{Q}_{\mathcal{X}}[n+1]$ is $(\iota_p)_*$ of

$$\begin{aligned} \text{Cone}(R\Gamma \mathbb{Q}_E(-1)[-2] \oplus \mathbb{Q} \rightarrow R\Gamma \mathbb{Q}_{\mathcal{E}}[n]) &\simeq \text{Cone}(\mathbb{Q} \rightarrow R\Gamma \mathbb{Q}_{\mathcal{E} \setminus E}[n]) \\ &\simeq \oplus_k \tilde{H}^k(\mathcal{E} \setminus E)[n-k]; \end{aligned}$$

which being perverse must vanish outside degree 0. $\square$

In the more general case where $\mathcal{E}$ is a union of smooth $\{\mathcal{E}_i\}_{i=1}^e$, $H_{\text{van}}^k(X_t)$ is still 0 for $k \neq n$ by Proposition 5.5(ii), but $H_{\text{van}}^n(X_t)$ is not as straightforward as in Proposition 6.3. To see what one can say, write $\mathcal{E}_0 := \tilde{X}_0$, $\mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i$, $\mathcal{E}_i^* := \mathcal{E}_i \setminus (\cup_{j \in \{0, \dots, e\} \setminus \{i\}} \mathcal{E}_j)$,

$$\mathcal{E}^{[k]} := \coprod_{\substack{I \subseteq \{1, \dots, e\} \\ |I| = k+1}} \mathcal{E}_I \subset \coprod_{\substack{I \subseteq \{0, \dots, e\} \\ |I| = k+1}} \mathcal{E}_I =: Y_0^{[k]} \xrightarrow{\mathcal{I}^{[k]}} Y_0,$$

and $\mu := \dim(H_{\text{van}}^n(X_t))$ for the Milnor number.

Theorem 6.4 (i) *The associated graded $\text{Gr}_{\bullet}^W H_{\text{van}}^n(X_t)$ is a subquotient of*

$$\bigoplus_{k=0}^n \left\{ H^{n-k}(\mathcal{E}^{[k]}) \oplus \left(H^{n-k}(Y_0^{[k]}) \otimes \tilde{H}^*(\mathbb{P}^k) \right) \right\}.$$

$$(ii) \quad \mu = (-1)^n \left\{ -1 + \sum_{i=1}^e \chi(\mathcal{E}_i^*) \right\}.$$

Proof We shall see in Part II that for a semistable fiber at $t = 0$, ${}^p\phi_t\mathbb{Q}_{\mathcal{Y}}[n+1] \in \text{Perv}(Y_0)$ has terms in degrees $(k-n)1-n$ thru 0 with

$$({}^p\phi_t\mathbb{Q}_{\mathcal{Y}})^{k+1} = \mathcal{I}_*^{[k]}(\oplus_{\ell=1}^k \mathbb{Q}_{Y_0^{[k]}(-\ell)}) = \mathcal{I}_*^{[k]}\mathbb{Q}_{Y_0^{[k]}} \otimes \tilde{H}^*(\mathbb{P}^k).$$

The rest of the proof of Proposition 6.3 is unchanged, and so ${}^p\phi_t\mathbb{Q}_{\mathcal{X}}[n+1]$ is $(\iota_p)_*$ of the MHS $(H_{\text{van}}^n(X_t) =)$

$$\begin{aligned} & \mathbb{H}^n\{\text{Cone}(\mathbb{Q} \oplus R\Gamma {}^p\phi_t\mathbb{Q}_{\mathcal{Y}} \rightarrow R\Gamma\mathbb{Q}_{\mathcal{E}})\} \\ &= \mathbb{H}^n\{\widetilde{R\Gamma}\mathbb{Q}_{\mathcal{E}} \rightarrow R\Gamma\mathbb{Q}_{Y_0^{[1]}} \otimes \tilde{H}^*(\mathbb{P}^1) \rightarrow \cdots \rightarrow R\Gamma\mathbb{Q}_{Y_0^{[n]}} \otimes \tilde{H}^*(\mathbb{P}^n)\} \\ &= \mathbb{H}^n\{\widetilde{R\Gamma}\mathbb{Q}_{\mathcal{E}^{[0]}} \rightarrow R\Gamma\mathbb{Q}_{\mathcal{E}^{[1]}} \oplus (R\Gamma\mathbb{Q}_{Y_0^{[1]}} \otimes \tilde{H}^*(\mathbb{P}^1)) \rightarrow \\ & \quad \cdots \rightarrow R\Gamma\mathbb{Q}_{\mathcal{E}^{[n]}} \oplus (R\Gamma\mathbb{Q}_{Y_0^{[n]}} \otimes \tilde{H}^*(\mathbb{P}^n))\}, \end{aligned}$$

giving (i). This also shows that $\chi(H_{\text{van}}^*(X_t)) + 1 = (-1)^n\mu + 1$ is given by

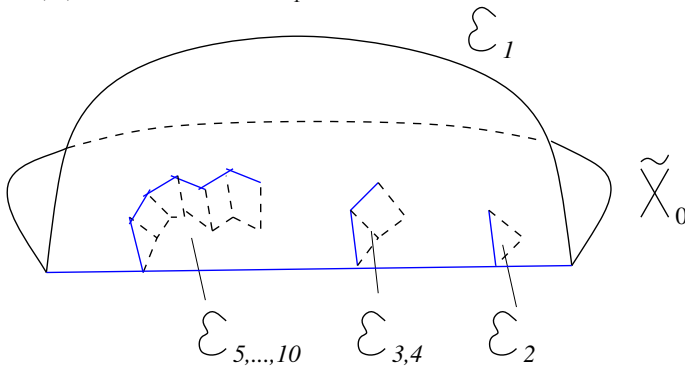
$$\begin{aligned} & \sum_{k=0}^n (-1)^k \{(\chi(\mathbb{P}^k) - 1)\chi(Y_0^{[k]}) + \chi(\mathcal{E}^{[k]})\} \\ &= \sum_{i=1}^e \chi(\mathcal{E}_i) + \sum_{k=1}^n (-1)^k (k\chi(Y_0^{[k]}) + \chi(\mathcal{E}^{[k]})) \\ &= \sum_{i=1}^e \chi(\mathcal{E}_i^*), \end{aligned}$$

which yields (ii). □

While $H_{\text{van}}^n(X_t)$ in our setting (6.3) can in general have weights from 0 to $2n$, Theorem 6.4(i) makes it clear that the graded pieces are directly related to strata of the tail, while (ii) is a close cousin of the theorem of A’Campo [1]. The proof of (i) actually yields a more precise computation of $\text{Gr}_{\bullet}^W H_{\text{van}}^*$ related to the “motivic Milnor fiber” of [25], and which we shall use systematically in Part II.

Example 6.5 (see also [54, Sect. 6]) Suppose X_0 has a Dolgachev singularity of type E_{12} , viz. $f \sim x^2 + y^3 + z^7$ locally. Taking $\kappa = 42$ yields (for $\mathfrak{X}$) $t^{42} + x^2 + y^3 + z^7 = 0$, whose weighted blow-up produces a singular fiber $X'_0 \cup \mathcal{E}'$ with $X'_0 \cap \mathcal{E}' \cong \mathbb{P}^1 \ni p_1, p_2, p_6$ and $X'_0, \mathcal{E}'$ having A_k singularities at p_k . After a toric resolution, we arrive at the SSR $\mathcal{Y}$, with $\mathcal{E}_1 = \tilde{\mathcal{E}}'$ a K3 surface and $\mathcal{E}_2, \dots, \mathcal{E}_{10}$ toric Fanos; $\mathcal{E}_1$ meets $\tilde{X}_0$

and each $\mathcal{E}_{i=2,\dots,10}$ in a $\mathbb{P}^1$, and $H^2(\mathcal{E}_1^*)$ has Hodge numbers $(1, 10, 1)$.



The other $\mathcal{E}_i^*$ are all $\mathbb{G}_m^2$ or $\mathbb{G}_m \times \mathbb{A}^1$, so have $\chi = 0$, which yields $\mu = \chi(\mathcal{E}_i^*) - 1 = 12$; indeed, $H_{\text{van}}^2(X_t)$ is just $H^2(\mathcal{E}_1^*) = H_{\text{tr}}^2(\mathcal{E}_1)$ in this case. The moral is that toric components of $\mathcal{E}$ arising from resolving canonical singularities (here the threefold $A_1 + A_2 + A_6$ singularities) won't complicate the result much beyond the case of $\mathcal{E}$ smooth. A general reason for this is given by Proposition 8.3 below.

We conclude by sketching a geometric application of the more general form (5.2) of Clemens–Schmid, where $\mathcal{K}^\bullet$ is not $\text{IC}_{\mathcal{X}}^\bullet$. (Full details will appear elsewhere.)

Example 6.6 Let $\mathcal{K}^\bullet := \text{IC}_{\mathcal{X}}^\bullet(\tilde{\mathbb{L}})$, where:

- $\mathcal{X} \xrightarrow{f} \mathbb{P}_t^1$ is the minimal smooth compactification of the elliptic curve family $w^2 = tz(z-1)(z+1) + t^2$ with $\Sigma = \{0, \pm \frac{2}{3\sqrt{3}}, \infty\}$;
- $\mathbb{L}$ is the rank two local system on $\mathbb{P}_z^1 \setminus \{0, \pm 1, \infty\}$ arising from relative H^1 of the rational elliptic surface $\mathcal{E} \rightarrow \mathbb{P}_z^1$ with fibers I_2 at ± 1 and I_4 at $0, \infty$; and
- $\tilde{\mathbb{L}} := \pi^* \mathbb{L}$ is the pullback local system on $\mathcal{Y} := \mathcal{X} \setminus \overline{\pi^{-1}(0, \pm 1, \infty)} \hookrightarrow \mathcal{X}$, where $\mathcal{X} \xrightarrow{\pi} \mathbb{P}_z^1$ is given by $\pi(t, w, z) := z$.

Taking $\ell = 0$ in (5.2), one checks that for $\sigma \in \Sigma$, $\text{gr} \mathcal{Y} = 0$ and so

$$H^0(X_\sigma, \text{IC}_{\mathcal{X}}^\bullet(\tilde{\mathbb{L}})) \xrightarrow[\cong]{\text{sp}} H_{\text{lim}, \sigma}^0(\text{IC}_{\mathcal{X}}^\bullet(\tilde{\mathbb{L}}))^{T_\sigma} \cong H_{\text{tr}}^2(S_t)_{\text{lim}, \sigma}^{T_\sigma} \quad (6.4)$$

where $S_t := \widetilde{X_t \times_{\mathbb{P}_z^1} \mathcal{E}}$ is a family of (smooth) surfaces over $\mathbb{P}_t^1 \setminus \Sigma$ introduced by Katz [37].

For $t \notin \Sigma$, $H^2(S_t) = \text{IH}^1(X_t, \tilde{\mathbb{L}})$ has rank 7 by Euler–Poincaré (5.10), with Hodge numbers $(2, 3, 2)$. Viewed as a weight-2 VHS on $\mathbb{P}^1 \setminus \Sigma$, it has geometric monodromy group G_2 , as shown by an arithmetic argument in *op. cit.* and by a direct calculation of the monodromies in [28], both quite painstaking. However, we can use (6.4) to quickly deduce the Hodge–Deligne diagrams for $H_{\text{tr}}^2(S_t)_{\text{lim}, \sigma}$ (Fig. 3); in particular, at $\sigma = 0$ and ∞ this is much easier than using a smooth compactification of the family of surfaces.

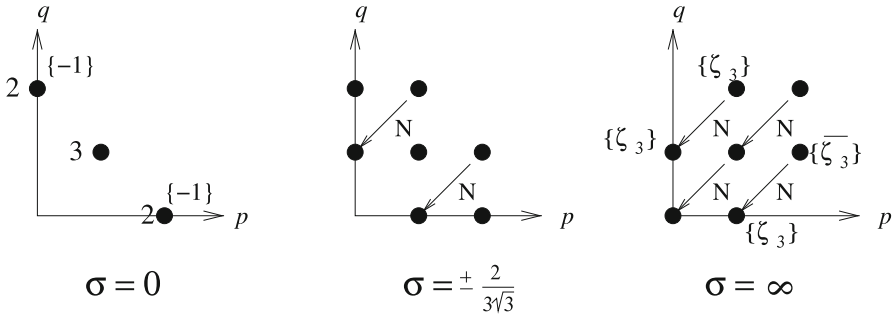


Fig. 3 Hodge–Deligne diagrams for Example 6.6

$\sigma = \pm \frac{2}{3\sqrt{3}}$: X_σ has Kodaira type I_1 , with node $p_\sigma \in Y_\sigma \xrightarrow{J^\sigma} X_\sigma$, and $\mathcal{I}_\sigma^* \mathcal{K}^\bullet = \overline{J_*^\sigma(\tilde{\mathbb{L}}|_{Y_\sigma})}[1]$. The LHS of (6.4) is thus $H^1(X_\sigma, J_*^\sigma \tilde{\mathbb{L}}|_{Y_\sigma})$, which is an extension of $H^1(\tilde{X}_\sigma, \rho^* J_*^\sigma \tilde{\mathbb{L}}|_{Y_\sigma})$ (weight 2 and rank 3 by (5.10)) by $H^1(E_{p_\sigma})$, where $\rho: \tilde{X}_\sigma \rightarrow X_\sigma$ is the normalization and E_{p_σ} is the fiber of $\mathcal{E}$ over $\pi(p_\sigma)$.

$\sigma = 0$: $X_0 = \cup_{i=0}^4 P_i$ has type I_0^* , 5 $\mathbb{P}^1$'s with P_0 meeting each P_i in a single node p_i . The divisor $Z = \mathcal{X} \setminus \mathcal{Y}$ contains $\{P_i\}_{i=1}^4$, and has additional components meeting $\{P_i\}_{i=2}^4$ at $\{q_i\}_{i=2}^4$. So $\mathcal{H}^{-1} \mathcal{I}_0^* \mathcal{K}^\bullet$ is $J_*^0 \tilde{\mathbb{L}}|_{P_0 \setminus \{p_i\}_{i=1}^4}$ on P_0 and $\mathbb{Q}(0)$ on the $\{P_i\}_{i=1}^4$, while $\mathcal{H}^0 \mathcal{I}_0^* \mathcal{K}^\bullet$ is $\iota_*^q \mathbb{Q}(-1)$ on the $\{P_i\}_{i=2}^4$. Conclude that $H^1(X_0, \mathcal{H}^{-1} \mathcal{I}_0^* \mathcal{K}^\bullet) = 0$ by Mayer-Vietoris and (5.10), so that $\text{LHS}(6.4) \cong H^0(X_0, \mathcal{H}^0 \mathcal{I}_0^* \mathcal{K}^\bullet) \cong \mathbb{Q}(-1)^{\oplus 3}$. (The key observation here was that the pullback of a unipotent degeneration of weight 1 rank 2 HS by $(x, y) \mapsto xy$ has local $\text{IH}^1 \cong \mathbb{Q}(-1)$, cf. (10.3)–(10.4).)

$\sigma = \infty$: $X_\infty = \cup_{i=1}^3 Q_i$ has type IV, 5 concurrent $\mathbb{P}^1$'s meeting at p_∞ . One finds $Q_i \cap Z = \{p_0^i, p_1^i, p_{-1}^i, p_\infty\}$ for $i = 1, 2$, while Z contains Q_3 and another component meeting it in a node q . Therefore $\mathcal{H}^{-1} \mathcal{I}_\infty^* \mathcal{K}^\bullet$ is $J_*^\infty \tilde{\mathbb{L}}|_{Q_i \cap \mathcal{Y}}$ on Q_1, Q_2 , and $\mathbb{Q}(0)$ on Q_3 ; whereas $\mathcal{H}^0 \mathcal{I}_\infty^* \mathcal{K}^\bullet$ is just $\iota_*^q \mathbb{Q}(-1)$ on Q_3 . One gets a sequence

$$0 \rightarrow H^1(X_\infty, \mathcal{H}^{-1} \mathcal{I}_\infty^* \mathcal{K}^\bullet) \rightarrow \text{LHS}(6.4) \rightarrow H^0(X_\infty, \mathcal{H}^0 \mathcal{I}_\infty^* \mathcal{K}^\bullet) \xrightarrow{d_2} H^2(X_\infty, \mathcal{H}^{-1} \mathcal{I}_\infty^* \mathcal{K}^\bullet) \rightarrow 0 \quad (6.5)$$

with d_2 an isomorphism and first term $\mathbb{Q}(0)$.

To get $\ker(N)$ (instead of $\ker(T_\sigma - I)$) at 0 and ∞ , one performs a base-change by $t \mapsto t^2$ [resp. t^3] followed by a proper modification to replace X_0 [resp. X_∞] by a smooth elliptic curve. The computations then proceed as above.

7 Decomposition theorem for an isolated singularity

Let $\tilde{\mathfrak{X}} \xrightarrow{\pi} \mathfrak{X}$ be the resolution of an isolated singularity $p \xhookrightarrow{!} \mathfrak{X}$ with exceptional divisor $\mathcal{E}$ (not assumed smooth or normal-crossings). With $d := \dim(\mathfrak{X})$, (3.1) specializes to

$$R\pi_* \mathbb{Q}_{\tilde{\mathfrak{X}}}[d] \simeq \text{IC}_{\mathfrak{X}}^\bullet \oplus \iota_* \left(\bigoplus_i V^i[-i] \right), \quad (7.1)$$

where the stalk cohomologies of the intersection complex $H^j(\iota^* \mathrm{IC}_{\mathfrak{X}}^\bullet)$ [resp. $H^{-j}(\iota^! \mathrm{IC}_{\mathfrak{X}}^\bullet)$] vanish for $j > 0$. Writing $\mathcal{N}_{\mathcal{E}}$ for the preimage of a ball about p , we apply $H^j \circ \iota^*$ [resp. $H^{-j} \circ \iota^!$] to (7.1) to find

$$V^i \cong \begin{cases} H^{d+i}(\mathcal{N}_{\mathcal{E}}), & i > 0 \\ H_c^{d+i}(\mathcal{N}_{\mathcal{E}}), & i \leq 0 \end{cases} \cong \begin{cases} H^{d+i}(\mathcal{E}), & i > 0 \\ H_{d-i}(\mathcal{E})(-d), & i \leq 0. \end{cases} \quad (7.2)$$

The hard Lefschetz property $V^{-j}(-j) \cong V^j$ (as $\mathbb{Q}$ -MHS) therefore reads $H_{d+j}(\mathcal{E})(-d-j) \cong H^{d+j}(\mathcal{E})$, which implies that the V^i are all pure.

Example 7.1 Specialize the scenario (6.3) to f a family of elliptic curves with cuspidal (type II) fiber X_0 , and $\kappa = 6$. The local equation of $\mathfrak{X}$ at p is then $x^2 + y^3 + t^6 + \cdots$, i.e. an $\tilde{E}_8$ (simple elliptic) singularity, with $\tilde{X}_0 \cong \mathbb{P}^1$ and $\mathcal{E}$ a CM elliptic curve. We have $R\pi_* \mathbb{Q}_{\mathcal{Y}}[2] \cong \mathrm{IC}_{\mathfrak{X}}^\bullet \oplus \iota_* \mathbb{Q}(-1)$ and a short-exact sequence

$$0 \rightarrow \iota_* H^1(\mathcal{E}) \rightarrow \mathbb{Q}_{\mathfrak{X}}[2] \rightarrow \mathrm{IC}_{\mathfrak{X}}^\bullet \rightarrow 0 \quad (7.3)$$

in $\mathrm{Perv}(\mathfrak{X})$, which we claim is *not split*. (Hence, as remarked in Example 3.4, the DT for f does not apply to $\mathcal{K}^\bullet = \mathbb{Q}_{\mathfrak{X}}[2]$.)

Indeed, were (7.3) split, $\mathbb{Q}_{\mathfrak{X}}[2]$ would be semisimple hence (as in §4)

$${}^p\phi_t \mathbb{Q}_{\mathfrak{X}}[2] = \mathrm{im}(\mathrm{can}) \oplus \ker(\widetilde{\mathrm{var}}) = \mathrm{im}({}^p\psi_t \mathbb{Q}_{\mathfrak{X}}[2]) \oplus \ker(T^6 - I),$$

where everything is supported on p . But T^6 acts trivially (since the eigenvalues of T are $\zeta_6^{\pm 1}$, cf. Example 6.1), while can is onto, a contradiction. (Alternatively, one could take the long-exact hypercohomology sequence of (7.3) and observe that the connecting homomorphism $\delta : \mathrm{IH}^1(\mathfrak{X}) = \mathbb{H}^{-1}(\mathrm{IC}_{\mathfrak{X}}^\bullet) \rightarrow H^1(\mathcal{E})$ is an isomorphism in view of (5.7).)

More generally, the argument shows that sequences like (7.3) are non-split if the order of a nontrivial eigenvalue of T_{ss} divides the base-change exponent κ . In particular, this applies to the sequence $0 \rightarrow \iota_* H^2(\mathcal{E} \setminus E) \rightarrow \mathbb{Q}_{\mathfrak{X}}[3] \rightarrow \mathrm{IC}_{\mathfrak{X}}^\bullet \rightarrow 0$ implicit in Example 6.2. Since the DT then applies only to $\mathrm{IC}_{\mathfrak{X}}^\bullet$ (not $\mathbb{Q}_{\mathfrak{X}}[n+1]$), we have only (5.7) (and not (5.11)) for $\mathfrak{X}$.

As an immediate consequence of (7.1)–(7.2), we find that

$$H^k(\tilde{\mathfrak{X}}) \twoheadrightarrow H^k(\mathcal{E}) \quad \text{for } k \geq d. \quad (7.4)$$

Now suppose that $\mathfrak{X} = X_0$ appears as the singular fiber in a family $\mathcal{X} \rightarrow \Delta$ with $n = d$ and $\mathcal{X}$ smooth (and write $\mathcal{E} \subset \tilde{X}_0 \xrightarrow{\pi} X_0$ for the exceptional divisor). For $k < n$, the Clemens–Schmid and vanishing cycle sequences give $H^k(X_0) \cong H_{\mathrm{lim}}^k(X_t)^T = H_{\mathrm{lim}}^k(X_t)$, which is pure since T acts by the identity. So for $1 < k < n$, in the exact sequence

$$\rightarrow H^{k-1}(X_0) \xrightarrow{\pi^*} H^{k-1}(\tilde{X}_0) \xrightarrow{\iota_{\mathcal{E}}^*} H^{k-1}(\mathcal{E}) \xrightarrow{\tau} H^k(X_0) \rightarrow,$$

purity of $H^k(X_0) \implies \tau = 0 \implies \iota_{\mathcal{E}}^*$ surjective; therefore

$$H^k(\tilde{X}_0) \twoheadrightarrow H^k(\mathcal{E}) \quad \text{for} \quad k \leq n-2 \quad \text{and} \quad k \geq n. \quad (7.5)$$

Of course, our assumption implies that p is a hypersurface singularity, so that $\mathbb{Q}_{X_0}[d]$ is perverse; (7.5) can then also be derived from the resulting exact sequence $0 \rightarrow \iota_* W \rightarrow \mathbb{Q}_{X_0}[n] \rightarrow \mathrm{IC}_{X_0}^\bullet \rightarrow 0$ (in fact, we only need an isolated l.c.i. singularity here). The Clemens–Schmid and vanishing cycle sequences’ real strength is in using the smooth fibers’ cohomology to further constrain those of X_0 and $\mathcal{E}$.

8 Cyclic base-change and quotients

We return to a scenario analogous to (6.3), but where the singularities need not be isolated. Begin with a flat projective family $f : \mathcal{X}' \rightarrow \Delta$ with $\Xi' := \mathrm{sing}(X'_0) \supseteq \mathrm{sing}(\mathcal{X}')$ ($\implies f^{-1}(\Delta^*) = \mathcal{X}' \setminus X'_0$ smooth), and recall that for us $X'_0 = \cup D_i$ is the *reduction* of the divisor $(f) = \sum \kappa_i D_i, \kappa_i \in \mathbb{Z}_{\geq 0}$. For lack of a less self-contradictory terminology, we shall say that $\mathcal{X}'$ has *reduced special fiber* if all $\kappa_i = 1$; this implies in particular that X'_0 is Cartier.

Let $g : \mathcal{X}'' \rightarrow \Delta$ be a second family with a finite surjective morphism $\rho : \mathcal{X}'' \rightarrow \mathcal{X}'$ over a cyclic quotient $t \mapsto t^\kappa$; and fix log resolutions $\mathcal{Y}', \mathcal{Y}''$ of $(\mathcal{X}', X'_0), (\mathcal{X}'', X''_0)$ to have a diagram

$$\begin{array}{ccccccc} Y'_0 \subset \mathcal{Y}' & \xrightarrow{\pi'} & \mathcal{X}' & \xleftarrow{\rho} & \mathcal{X}'' & \xleftarrow{\pi''} & \mathcal{Y}'' \supset Y''_0 \\ & \searrow & \downarrow f & & \downarrow g & \swarrow & \\ & & \Delta & \xleftarrow{(\cdot)^\kappa} & \Delta & & \end{array} \quad (8.1)$$

Writing $\mathcal{X}, \mathcal{Y}$, etc. when we want to make a statement independent of the decoration, we assume that the log resolutions π are isomorphisms off X_0 and write $Y_0 = \tilde{X}_0 \cup \mathcal{E}$. Denote the monodromies by $T' := T_0 = T_0^{\mathrm{ss}} e^{N_0}$ and $T'' = T_0^\kappa$.

8.1 Cyclic base-change

An important special case of (8.1) is where:

- ρ is the base-change, so that $X'_0 = X''_0 =: X_0$;
- $\mathcal{Y}''$ is the semi-stable reduction of $\mathcal{X}'$ (so κ must satisfy $(T_0^{\mathrm{ss}})^\kappa = I$); and
- $\mathcal{X}'$ (hence $\mathcal{X}''$) has reduced special divisor.

In this case the SNCD $Y_0 = \tilde{X}_0 \cup \mathcal{E}$, with $\tilde{X}_0 \twoheadrightarrow X_0$ birational. When $\mathcal{X}$ ($= \mathcal{X}'$ or $\mathcal{X}''$) is not smooth (though we continue to assume $\mathcal{X} \setminus X_0$ smooth), we would first like to “quantify” the failure of the local invariant cycle theorem for $\mathcal{X}$.

Begin with the diagram of split short-exact sequences

$$\begin{array}{ccccc}
 0 & & & & 0 \\
 & \searrow & & \swarrow & \\
 & \mathrm{IH}_{\mathrm{ph}}^k(\mathcal{X}) & & & H_{\mathrm{ph}}^k(Y_0) \\
 & \searrow & \xrightarrow{\gamma^*} & \swarrow & \\
 & \mathrm{IH}^k(\mathcal{X}) & & H^k(Y_0) & \\
 & \searrow & & \swarrow & \\
 & & H_{\mathrm{lim}}^k(Y_t)^T & & \\
 & \searrow & & \swarrow & \\
 0 & & & & 0
 \end{array}$$

where $\Xi \xleftarrow{\alpha} \mathcal{E} \xrightarrow{\beta} Y_0 \xrightarrow{\gamma} X_0$ and $\widetilde{\gamma}^*$ is induced by $\mathbb{Q}_{\mathcal{X}}[n+1] \rightarrow \mathrm{IC}_{\mathcal{X}}^{\bullet}$. By the decomposition theorem for $\mathcal{Y} \rightarrow \mathcal{X}$, the curved arrows are split injections as well. Clearly then

$$H_{\mathrm{ph}}^k(X_0) := (\widetilde{\gamma}^*)^{-1} \left(\mathrm{IH}_{\mathrm{ph}}^k(\mathcal{X}) \right) = (\gamma_*)^{-1} \left(H_{\mathrm{ph}}^k(Y_0) \right)$$

and

$$\overline{H}^k(X_0) := H^k(X_0)/H_{\mathrm{ph}}^k(X_0) \cong \mathrm{im} \left\{ H^k(X_0) \xrightarrow{\mathrm{sp}} H_{\mathrm{lim}}^k(X_t)^T \right\} \quad (8.2)$$

are independent of choices (of whether $\mathcal{X} = \mathcal{X}'$ or $\mathcal{X}''$, and of $\mathcal{Y}$).

Remark 8.1 Since $\mathrm{IH}_{\mathrm{ph}}^k(\mathcal{X}) = \mathrm{IH}^k(\mathcal{X}) \cap H_{\mathrm{ph}}^k(Y_0) \subset H^k(Y_0)$,

$$\begin{array}{ccc}
 \mathrm{IH}^k(\mathcal{X}) & \cong & \mathrm{IH}_{\mathrm{ph}}^k(\mathcal{X}) \oplus H_{\mathrm{lim}}^k(X_t)^T \\
 \downarrow & & \downarrow \parallel \\
 H^k(Y_0) & \cong & H_{\mathrm{ph}}^k(Y_0) \oplus H_{\mathrm{lim}}^k(X_t)^T
 \end{array}$$

exhibits the MHS $\mathrm{IH}^k(\mathcal{X})$ as a “lower bound” on the cohomology of any resolution.

On the other hand, the monodromy invariants $H_{\mathrm{lim}}^k(X_t)^T$ are certainly not independent of the choice of $\mathcal{X}$, and so the cokernel of sp in (8.2) cannot be. In view of the exact sequence

$$\rightarrow H^k(X_0) \rightarrow H^k(Y_0) \oplus H^k(\Xi) \xrightarrow{\beta^* - \alpha^*} H^k(\mathcal{E}) \xrightarrow{\delta} H^{k+1}(X_0) \rightarrow$$

we compute

$$\begin{aligned}
 \mathrm{coker} \left(H^k(X_0) \rightarrow H_{\mathrm{lim}}^k(X_t)^T \right) &= \mathrm{coker} \left(\overline{H}^k(X_0) \xrightarrow{\gamma^*} \overline{H}^k(Y_0) \right) \\
 &= \ker \left\{ \frac{H^k(\mathcal{E})}{\alpha^* H^k(\Xi) + \beta^* H_{\mathrm{ph}}^k(Y_0)} \xrightarrow{\delta} H^{k+1}(X_0) \right\} =: \overline{H}^k(\mathcal{E}). \quad (8.3)
 \end{aligned}$$

Accordingly, the replacement for Clemens–Schmid in this general context becomes

$$0 \rightarrow \overline{H}^k(X_0) \rightarrow H_{\lim}^k(X_t)^T \rightarrow \overline{H}^k(\mathcal{E}) \rightarrow 0. \quad (8.4)$$

Specializing a bit more, suppose the pre-base-change family $\mathcal{X}'$ is smooth: then $\mathrm{IH}^k(\mathcal{X}') = H^k(X_0)$ itself is the “lower bound” for $H^k(Y'_0)$, and $\overline{H}^k(\mathcal{E}') = \{0\}$. Since $\mathcal{Y}'' \rightarrow \mathcal{X}'$ is a proper morphism between equidimensional smooth manifolds (cf. [81]), we have $(H^k(\mathcal{X}'') = H^k(X_0) =) H^k(\mathcal{X}') \hookrightarrow H^k(\mathcal{Y}'')$ which yields

$$\begin{array}{ccccc} H^k(X_0) & \hookrightarrow & \mathrm{IH}^k(\mathcal{X}'') & \hookrightarrow & H^k(Y''_0) \\ \parallel & & \parallel & & \parallel \\ H_{\mathrm{ph}}^k(X_0) & \hookrightarrow & \mathrm{IH}_{\mathrm{ph}}^k(\mathcal{X}'') & \hookrightarrow & H_{\mathrm{ph}}^k(Y''_0) \\ \oplus & & \oplus & & \oplus \\ H_{\lim}^k(X_t)^{T_0} & \hookrightarrow & H_{\lim}^k(X_t)^{T_0^\kappa} & = & H_{\lim}^k(X_t)^{T_0^\kappa} \end{array} \quad (8.5)$$

(where $(\cdot)^{T_0^\kappa} = \ker(N_0)$) and

$$\overline{H}^k(\mathcal{E}'') = H_{\lim}^k(X_t)^{T_0^\kappa} / H_{\lim}^k(X_t)^{T_0}. \quad (8.6)$$

So while (by Corollary 5.7) Gr_F^0 and W_{k-1} of $H^k(X_0)$ and $H_{\lim}^k(X_t)^{T_0}$ agree, this is false in general if we replace T_0 by T_0^κ .

8.2 Cyclic quotient singularities

Rather than obtaining $\mathcal{X}''$ from $\mathcal{X}'$, we may wish to define $\mathcal{X}' := \mathcal{X}''/G$, where $G = \langle g \rangle \cong \mathbb{Z}/\kappa\mathbb{Z}$ acts nontrivially on X_0 . In this case, $\mathcal{X}'$ will essentially never be smooth or have reduced special fiber. Nevertheless, it is *always* true (no need to assume $\mathcal{X}''$ smooth; cf. [11, Th. III.7.2]) that

$$H^*(X'_0) = H^*(\mathcal{X}') \xrightarrow[\rho^*]{\cong} H^*(\mathcal{X}'')^G = H^*(X''_0)^G. \quad (8.7)$$

We have $g^* \in \mathrm{Aut}(H^*(X''_0))$, $T' \in \mathrm{Aut}(H_{\lim}^*(X'_t))$, $T'' \in \mathrm{Aut}(H_{\lim}^*(X''_t))$, and $H_{\lim}^*(X'_t) = H_{\lim}^*(X''_t) =: H_{\lim}^*$. One may perhaps know g^* and T'' , and wish to determine T' : for instance, if the quotient has been used to form a singularity of higher index (on X'_0) from one of index 1 (on X''_0). To that end we have the following

Proposition 8.2 *T' extends the action of g^* on $\mathrm{im}\{H^*(X''_0) \rightarrow H_{\lim}^*\} =: \bar{H}^*(X''_0)$ to all of $H_{\lim}^*$. In particular, we have $\bar{H}^*(X'_0) = (H_{\lim}^*)^{T'} \cap \bar{H}^*(X''_0)$, and so the local invariant cycle theorem holds for $\mathcal{X}'$ if it holds for $\mathcal{X}''$.*

Proof If we analytically continue a basis $\{C_i\} \subset H_*(X''_t)$ to $H_*(X''_{\zeta_{\kappa t}})$, then writing $g(C_i)$ (call this $\tilde{g}_*$) in terms of these translates is just g_* on the “global” cycles and corresponds to clockwise monodromy “downstairs” (in t^κ). The local-system monodromy T' on cohomology is the *transpose* of the latter (cf. [27]): so $T' = {}^t\tilde{g}_* = \tilde{g}^*$, and $\tilde{g}^*|_{\bar{H}^*(X''_0)} = g^*$. $\square$

Now suppose $\mathcal{X}''$ is smooth. Since Clemens–Schmid (5.11) holds for $\mathcal{X}''$, taking G -invariant parts exactly gives

$$0 \rightarrow H_{\lim}^{k-2}(X'_t)_{T_0}(-1) \xrightarrow{\mathbb{S}\mathbb{P}^\vee} H_{2n-k+2}(X'_0)(-n-1) \xrightarrow{\rho_* \circ \text{gy} \circ \rho^*} H^k(X'_0) \xrightarrow{\mathbb{S}\mathbb{P}} H_{\lim}^k(X'_t)^{T_0} \rightarrow 0, \quad (8.8)$$

since $((H_{\lim}^*)^{T_0^\kappa})^G = (H_{\lim}^*)^{T_0}$. (As quotient singularities, those of $\mathcal{X}'$ are rational [44], but the results of Sect. 9 for rational singularities are weaker than this.) Further, it is often possible to deduce T' from g^* and T'' in this case. The action of $T'' = (T_0^{\text{ss}})^\kappa e^{\kappa N_0}$ on $H_{\lim}^*$ extends to one of $\mathfrak{sl}_2 \times \langle (T_0^{\text{ss}})^\kappa \rangle$, compatibly with the Deligne bigrading $H_{\lim, \mathbb{C}}^* = \bigoplus_{p,q} (H_{\lim}^*)^{p,q}$. Accordingly, it suffices to determine the choice of κ^{th} root T' of T'' on $\ker(N_0) \subset H_{\lim}^*$. That of T'' (resp. T') decomposes $\ker(N_0) \cong \bigoplus_{k \in \mathbb{N}} W_k^{\oplus m_k}$ (resp. $\bigoplus_{\ell \in \mathbb{N}} V_\ell^{\oplus n_\ell}$) over $\mathbb{Q}$, where $\det\{(\lambda I - (T_0^{\text{ss}})^\kappa)|_{W_k}\}$ (resp. $\det\{(\lambda I - T_0^{\text{ss}})|_{V_\ell}\}$) is the k^{th} (resp. ℓ^{th}) cyclotomic polynomial, and so the issue is to compute the $\{n_\ell\}$ given $\{m_k\}$. The point here is that since $\bar{H}^*(X'_0) = (H_{\lim}^*)^{T''} = W_1^{\oplus m_1}$, g^* determines the n_ℓ for all $\ell|\kappa$, and one can sometimes deduce the others from the formula $V_\ell = W_{\ell/\langle \phi(\ell)/\phi(\ell/(\ell, \kappa)) \rangle}^{\oplus \{ \phi(\ell)/\phi(\ell/(\ell, \kappa)) \}}$. For instance, if $\kappa|k$ then the only possibility is $W_k^{\oplus m_k} = V_{\kappa k}^{\oplus m_k/\kappa}$. Conversely, this puts constraints on the set of $\mathbb{Z}/\kappa\mathbb{Z}$ by which one can even consider taking cyclic quotients.

8.3 Relative quotients

A more general quotient scenario is where $G \xrightarrow{\theta} \mathbb{Z}/\kappa\mathbb{Z}$; in §1.7.2, θ was an isomorphism. Now we consider the opposite extreme, where $\kappa = 1$. More precisely, let $\mathcal{X}' \xrightarrow{f} \Delta$ be flat, projective, and smooth over Δ^* , with $X'_0 = D_a \cup D_b$ generically a reduced NCD along $D_{ab} = D_a \cup D_b$. Suppose that each $x \in X'_0$ has a neighborhood $\mathcal{V}$ arising as a finite group quotient

$$\begin{array}{ccccc} \tilde{V}_a \cup \tilde{V}_b = s^{-1}(0) & \hookrightarrow & \tilde{\mathcal{V}} & \xrightarrow[\rho]{/G} & \mathcal{V} \hookleftarrow t^{-1}(0) = V_a \cup V_b \\ & & \searrow s & & \swarrow t=f|_{\mathcal{V}} \\ & & \Delta & & \end{array}$$

of a semistable degeneration $\tilde{\mathcal{V}}/\Delta$. Then the vanishing cycles of f behave exactly as in a SSD:

Proposition 8.3 *In this situation, ${}^p\phi_f \mathbb{Q}_{\mathcal{X}'} \cong \mathbb{Q}_{D_{ab}}(-1)[-2]$.*

Proof Working locally, since t is G -invariant

$$\begin{aligned} {}^p\phi_t \mathbb{Q}_{\mathcal{V}} &= {}^p\phi_t(R\rho_* \mathbb{Q}_{\tilde{\mathcal{V}}})^G = ({}^p\phi_t R\rho_* \mathbb{Q}_{\tilde{\mathcal{V}}})^G = (R\rho_* {}^p\phi_t \mathbb{Q}_{\tilde{\mathcal{V}}})^G \\ &= (R\rho_* \mathbb{Q}_{\tilde{V}_{ab}}(-1)[-2])^G = \mathbb{Q}_{V_{ab}}(-1)[-2], \end{aligned}$$

where V_{ab} denotes $V_a \cap V_b$ etc. □

Remark 8.4 The above scenario arises frequently via weighted blow-ups: typically one has

$$\begin{array}{ccc}
 \mathcal{X}' & \hookrightarrow & \text{Bl}_{\underline{w}}(\mathbb{A}^{n+2}) = \mathbb{P}_{\tilde{\Sigma}_{\underline{w}}} \\
 \downarrow b & & \downarrow \beta \\
 \mathfrak{X} & \hookrightarrow & \mathbb{A}_{x_0, \dots, x_{n+1}}^{n+2} \\
 \downarrow f & \nearrow p & \downarrow x_0 \mapsto t \\
 \mathbb{A}_t^1 & &
 \end{array}$$

where $\mathfrak{X}$ has a singularity at $\underline{0}$ arising from cyclic base-change, $\underline{w} = (1, w_1, \dots, w_{n+1})$ and $\tilde{\Sigma}_{\underline{w}} = \text{fan}\{e_0, e_1, \dots, e_{n+1}, e_0 + \sum_{i=1}^{n+1} w_i e_i\}$. (In particular, Proposition 8.3 explains Example 6.5, which will be generalized in Part II.) The exceptional divisor of β is $\mathbb{WP}(\underline{w}) \cong \mathbb{P}_{\Sigma_{\underline{w}}}$, with $\Sigma_{\underline{w}} = \text{fan}\{e_1, \dots, e_{n+1}, -\sum_{i=1}^{n+1} w_i e_i\}$ the image of $\tilde{\Sigma}_{\underline{w}}$ under the projection $e_0 \mapsto -\sum w_i e_i$; the proper transform of $t^{-1}(0)$ is $\mathbb{P}_{\Sigma_0} (\cong \mathbb{A}^{n+1})$ where $\Sigma_0 = \text{fan}\{e_1, \dots, e_{n+1}\}$. Assuming that $\mathcal{X}'$ meets $\mathbb{P}_{\Sigma_{\underline{w}}}$ with multiplicity 1 (so $D_a = \mathcal{X}' \cap \mathbb{P}_{\Sigma_{\underline{w}}}$ and $D_b = \mathcal{X}' \cap \mathbb{P}_{\Sigma_0}$), it will suffice to exhibit $\mathbb{P}_{\tilde{\Sigma}_{\underline{w}}}$ locally as a finite quotient of $\mathbb{A}^{n+2}$ not branched along $\mathbb{P}_{\Sigma_{\underline{w}}}$ or $\mathbb{P}_{\Sigma_0}$.

Writing $\sigma_i := \mathbb{R}_{>0}\langle e_0, \dots, \hat{e}_i, \dots, e_{n+1}, e_0 + \sum_{i=1}^{n+1} w_i e_i \rangle$ and $N_i := \mathbb{Z}\langle e_0, \dots, \hat{e}_i, \dots, e_{n+1}, e_0 + \sum_{i=1}^{n+1} w_i e_i \rangle$, set $U_i := \text{Spec}(\mathbb{C}[\underline{x}^{\sigma_i \cap \mathbb{Z}^{n+2}}])$ and $\tilde{U}_i := \text{Spec}(\mathbb{C}[\underline{x}^{\sigma_i \cap N_i}]) \cong \mathbb{A}_{u_0, \dots, u_{n+1}}^{n+2}$ (where $u_i = x_i^{\frac{1}{w_i}}$, and $u_j = x_j u_i^{-w_j}$ for $j \neq i$). The natural, generically $w_i : 1$ morphism $\tilde{U}_i \xrightarrow{\tau_i} U_i$ is the quotient by $\mathbb{Z}/w_i \mathbb{Z} \ni 1 \mapsto \{u_i \mapsto \zeta_{w_i} u_i, u_j \mapsto \zeta_{w_i}^{-w_j} u_j \ (j \neq i)\}$. Since $w_0 = 1$ we get $t = x_0 = \frac{x_0}{u_i} u_i = u_0 u_i$ (as a function) so that τ_i is not branched along $\mathbb{P}_{\Sigma_{\underline{w}}}$ or $\mathbb{P}_{\Sigma_0}$, and $s|_{\tilde{U}_i} = t|_{U_i} \circ \tau_i = p \circ \beta \circ \tau_i$ (as a mapping) is a SSD as desired. Note that this exhibits the singularity type in U_i (at the origin) as $\frac{1}{w_i}(1, -1, -w_1, \dots, \hat{w}_i, \dots, -w_{n+1})$; while if the proper transform of $\mathcal{X}'$ passes through a point with $u_k = 0$ for $k \neq j$, the quotient yields a point of type $\frac{1}{\gcd(w_i, w_j)}(1, -1, -w_1, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, -w_{n+1})$ on $\mathcal{X}'$.

Example 8.5 So in Example 6.5, $\underline{w} = (1, 6, 14, 21)$ yields points of type $\frac{1}{2}(1, -1, 1)$, $\frac{1}{3}(1, -1, 1)$, $\frac{1}{7}(1, -1, 1)$ on $\mathcal{X}'$, which become $\frac{1}{2}(1, -1)$, $\frac{1}{3}(1, -1)$, $\frac{1}{7}(1, -1)$ (i.e. A_1, A_2, A_6) on D_a and D_b .

9 Singularities of the minimal model program

Let X be a projective variety with resolution $\tilde{X} \xrightarrow{\epsilon} X$, and extend this⁸ to a cubical hyperresolution $\epsilon^\bullet : X^\bullet \rightarrow X$ [64].

Definition 9.1 (i) X has *rational singularities* $\iff \mathcal{O}_X \xrightarrow{\sim} R\epsilon_* \mathcal{O}_{\tilde{X}}$.

⁸ Note that typically $\tilde{X}$ is only a connected component of X^0 .

(ii) X has *du Bois* singularities $\iff \mathcal{O}_X \xrightarrow{\sim} R\epsilon_*^\bullet \mathcal{O}_{X^\bullet}$.

In general we have $\mathrm{Gr}_F^0 H^k(X) \cong \mathbb{H}^k(X, R\epsilon_*^\bullet \mathcal{O}_{X^\bullet})$, so that if X is du Bois then $\mathrm{Gr}_F^0 H^k(X) \cong H^k(\mathcal{O}_X)$. This last isomorphism clearly also holds if X has rational singularities: since $\mathcal{O}_X \rightarrow R\epsilon_* \mathcal{O}_{\tilde{X}}$ always factors through $R\epsilon_*^\bullet \mathcal{O}_{X^\bullet}$, we get a diagram

$$\begin{array}{ccccc}
 & & H^k(\mathcal{O}_{\tilde{X}}) & & \\
 & \nearrow & \uparrow \cong & \nwarrow & \\
 H^k(X, \mathbb{C}) & \longrightarrow & H^k(\mathcal{O}_X) & \xrightarrow{(*)} & \mathbb{H}^k(X, R\epsilon_*^\bullet \mathcal{O}_{X^\bullet}) \\
 & \searrow & \downarrow (*) & \swarrow \cong & \\
 & & \mathrm{Gr}_F^0 H^k(X, \mathbb{C}) & &
 \end{array}$$

forcing $(*)$ to be injective and surjective. With more work [47], one can show that rational singularities are in fact du Bois.

Now consider a flat projective family $f : \mathcal{X} \rightarrow \Delta$, with $\mathcal{X} \setminus X_0$ smooth and $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ a log-resolution of $(\mathcal{X}, X_0)$ (i.e. $\mathcal{Y}$ smooth, $Y_0 = \pi^{-1}(X_0)$ SNCD) restricting to an isomorphism $\mathcal{Y} \setminus Y_0 \cong \mathcal{X} \setminus X_0$. We shall assume that π extends to a morphism $\bar{\pi} : \bar{\mathcal{Y}} \rightarrow \bar{\mathcal{X}}$ of projective varieties ($\bar{\mathcal{Y}}$ smooth, $\cong$ off X_0), and that f is also extendable to an algebraic morphism (from $\bar{\mathcal{X}}$ to a curve).

Proposition 9.2 *If $\mathcal{X}$ has rational singularities, then sp induces isomorphisms*

$$\mathrm{Gr}_F^0 H^k(X_0) \cong \mathrm{Gr}_F^0 H_{\lim}^k(X_t)^{T_0^{\mathrm{ss}}}$$

for all k .

Proof We have $\mathrm{Gr}_F^0 H_{\lim}^k(X_t)^{T_0^{\mathrm{ss}}} \cong \mathrm{Gr}_F^0 H_{\lim}^k(X_t)^{T_0} \xrightarrow[\mathrm{sp}]{\cong} \mathrm{Gr}_F^0 H^k(Y_0)$, since $\mathcal{Y}$ is smooth and $\mathrm{level}(H_{\mathrm{ph}}^k(Y_0)) \leq k - 2$. Taking Gr_F^0 of the Mayer-Vietoris sequence (cf. [64, Theorem 5.35])

$$\rightarrow H^k(\bar{\mathcal{X}}) \rightarrow H^k(\bar{\mathcal{Y}}) \oplus H^k(X_0) \rightarrow H^k(Y_0) \rightarrow$$

and using that $\bar{\mathcal{X}}$ is (rational $\implies$) du Bois yields

$$\rightarrow H^k(\mathcal{O}_{\bar{\mathcal{X}}}) \rightarrow H^k(\mathcal{O}_{\bar{\mathcal{Y}}}) \oplus \mathrm{Gr}_F^0 H^k(X_0) \rightarrow \mathrm{Gr}_F^0 H^k(Y_0) \rightarrow .$$

Now $\mathcal{O}_{\bar{\mathcal{X}}} \xrightarrow{\sim} R\bar{\pi}_* \mathcal{O}_{\bar{\mathcal{Y}}} \implies H^k(\mathcal{O}_{\bar{\mathcal{X}}}) \xrightarrow{\cong} H^k(\mathcal{O}_{\bar{\mathcal{Y}}}) \implies \mathrm{Gr}_F^0 H^k(X_0) \cong \mathrm{Gr}_F^0 H^k(Y_0)$ gives the result. $\square$

We next make use of an “inversion of adjunction” result of Schwede [71], that when a Cartier divisor (with smooth complement) is du Bois, the ambient variety has only rational singularities. However, this requires us to place an additional constraint on $\mathcal{X}$ to ensure that X_0 is Cartier and remains so after base-change.

Theorem 9.3 *If X_0 has du Bois singularities, and $\mathcal{X}$ has reduced special fiber (i.e. $(f) = X_0$), then*

$$\mathrm{Gr}_F^0 H^k(X_0) \cong \mathrm{Gr}_F^0 H_{\lim}^k(X_t)^{T_0^{\mathrm{ss}}} \cong \mathrm{Gr}_F^0 H_{\lim}^k(X_t) \quad (\forall k). \quad (9.1)$$

Proof Obviously $\mathcal{X}$ has rational singularities by [op. cit.], but this also applies to any finite base-change. So taking $\mathcal{X}' = \mathcal{X}$ in the setting of Sect. 8.1, Proposition 9.2 applies in addition to $\mathcal{X}''$, whose $T'' = T_0^K$ is unipotent. $\square$

Remark 9.4 If $\mathcal{X}$ is smooth, then (regardless of whether $(f) = X_0$) one can show that X_0 is du Bois iff $\mathcal{O}_{X_0} \xrightarrow{\sim} R\pi_*^0 \mathcal{O}_{Y_0}$ [71].

Example 9.5 To see the necessity of the $(f) = X_0$ requirement in Theorem 9.3, consider a smooth $\mathcal{X}$ with elliptic fibers over Δ^* and X_0 a Kodaira type IV* (“ E_6 ”) fiber. Then $(f) = 3D_1 + 2(D_2 + D_3) + (D_4 + D_5 + D_6) \neq \sum_i D_i = X_0$; and sure enough, the conclusion of Theorem 9.3 fails (cf. Example 6.1).

Example 9.6 Assume $\Sigma = \{p\}$ is an isolated quasi-homogeneous singularity of type $(f \sim) F = x^2 + y^3 + z^6 + \lambda xyz$ ($\tilde{E}_8$) resp. $G = x^5 + y^5 + z^2$ (N_{16}). In the first case, X_0 is du Bois. As we can see from Example 6.2, the discrepancy between $H^2(X_0) \cong H_{\lim}^2(X_t)^{T_0}$ and $H_{\lim}^2(X_t)^{T_0^{\mathrm{ss}}} = \ker(N_0)$ consists of 8 (1, 1) classes, and neither differ from $H_{\lim}^2(X_t)$ on Gr_F^0 . Any base-change $F + t^M$ defines a rational 3-fold singularity, since (as one deduces from the absence of integral interior points in the convex hull of $\{(2, 0, 0), (0, 3, 0), (0, 0, 6), (0, 0, 0)\}$) the exceptional divisor $\mathcal{E}$ of the weighted blow-up has $\Omega^2(\mathcal{E}) = \{0\}$.

In the second case (where $T_0 = T_0^{\mathrm{ss}}$), $H^2(\mathcal{E})$ has Hodge numbers (1, 14, 1), so that Gr_F^0 of $H^2(X_0) \cong H_{\lim}^2(X_t)^{T_0}$ and $H_{\lim}^2(X_t)$ differ by 1. The point is that (while $\mathcal{X}$ is smooth) X_0 is not du Bois and neither is (say) $G + t^{10}$; so in particular, $\mathcal{X}''$ will not have rational singularities.

Returning to our resolution $\tilde{X} \xrightarrow{\varepsilon} X$, assume now⁹

- X is normal (smooth in codim. 1, and satisfies S_2)
- X is $\mathbb{Q}$ -Gorenstein (K_X is $\mathbb{Q}$ -Cartier)

and write $K_{\tilde{X}} = \varepsilon^* K_X + \sum_i m_i E_i$ (E_i exceptional prime divisors).

Definition 9.7 X has *terminal* (resp. *canonical*, *log-terminal*, *log-canonical*) singularities $\iff$ all m_i are > 0 (resp. ≥ 0 , > -1 , ≥ -1).

A larger class of singularities is obtained by dropping the “smooth in codimension 1” part of normality:

Definition 9.8 Assume X satisfies S_2 and is $\mathbb{Q}$ -Gorenstein, and has only normal-crossing singularities in codimension 1. Let $\hat{X} \rightarrow X$ be the normalization and $\hat{D}$ the conductor (inverse image of the normal-crossing locus); let $\hat{Y} \xrightarrow{\hat{\pi}} \hat{X}$ be a log-resolution of $(\hat{X}, \hat{D})$. Then X has *semi-log-canonical* (slc) singularities $\iff$ the $m_i \geq -1$ in $K_{\hat{Y}} + \hat{\pi}_*^{-1} \hat{D} = \hat{\pi}^*(K_{\hat{X}} + \hat{D}) + \sum_i m_i \hat{E}_i$ ($\hat{E}_i$ exceptional).

⁹ Serre’s condition S_2 is “algebraic Hartogs”: given any $Z \subset X$ of $\mathrm{codim.} \geq 2$, $j_* \mathcal{O}_{X \setminus Z} = \mathcal{O}_X$; so it easily follows that normality is equivalent to $\varepsilon_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.

We have two related “inversion of adjunction” results here [36,44]: if a Cartier divisor with smooth complement in a normal, $\mathbb{Q}$ -Gorenstein variety is log-terminal (resp. slc), then the ambient variety has only terminal (resp. canonical) singularities. In addition, we know that log-terminal (resp. slc) singularities are rational (resp. du Bois) [40,46,47]. Thus we arrive at the following

Corollary 9.9 *Assume our family $f : \mathcal{X} \rightarrow \Delta$ has normal, $\mathbb{Q}$ -Gorenstein total space, and reduced special fiber X_0 .*

- (i) *If X_0 is slc, then (9.1) holds.*
- (ii) *If X_0 is log-terminal, (9.1) holds and $W_{k-1} \text{Gr}_F^0 H_{\lim}^k(X_t) = \{0\}$.*

Proof For (ii), $\text{Gr}_F^0 H^k(X_0) \cong H^k(\mathcal{O}_{X_0})$ (since X_0 is du Bois) and $\text{Gr}_F^0 H^k(\tilde{X}_0) \cong H^k(\mathcal{O}_{\tilde{X}_0})$, where $H^k(\tilde{X}_0)$ is pure of weight k . Since X_0 has rational singularities, $H^k(\mathcal{O}_{X_0}) \cong H^k(\mathcal{O}_{\tilde{X}_0})$. $\square$

We can think of (i) in terms of Hodge-Deligne numbers as saying that

$$h^k(X_0)^{p,q} = h_{\lim}^k(X_t)^{p,q} \text{ for } p \cdot q = 0,$$

and (ii) as saying that moreover both are zero for $(p, q) = (r, 0)$ or $(0, r)$ with $r \neq k$.

In the log-terminal case, we have

$$\text{Gr}_F^1 H_{\lim}^k(X_t) = \text{Gr}_F^1 H_{\lim}^k(X_t)^{T^{\text{un}}} \quad (9.2)$$

by (ii), which might kindle hopes that perhaps this equals $\text{Gr}_F^1 H^k(X_0)$. Unfortunately, nothing quite this strong is true at any level of generality one can specify in terms of the singularity types described above: for $n = 3$, the nicest such scenario would be where $\mathcal{X}$ is smooth and X_0 has Gorenstein terminal ($\iff$ isolated compound du Val) singularities.

Example 9.10 Such a singularity is given locally by $f \sim x^2 + y^2 + zw^4 + z^2w^2 + z^4w$, whose contribution to $H_{\lim}^3(X_0)$ has nontrivial $(1, 1)$ and $(1, 2)$ parts, with neither part T^{ss} -invariant hence neither appearing in $H^3(X_0)$. This assertion will be justified in Part II.

In any case, here is something one *can* say:

Theorem 9.11 *If X_0 is log-terminal (or more generally, has rational singularities),¹⁰ and $\mathcal{X}$ is smooth, then $\text{Gr}_F^1 H^k(X_0) \cong \text{Gr}_F^1 (H_{\lim}^k(X_t))^{T^{\text{ss}}}$.*

Proof Begin by observing that under the duality functor $\mathbb{D}$ on MHM we have $\mathbb{D}\mathbb{Q}_{\mathcal{X}}[n+1] = \mathbb{Q}_{\mathcal{X}}n+1$ hence (on X_0)

$$\mathbb{D}^p \phi_f^u \mathbb{Q}_{\mathcal{X}}[n+1] = {}^p \phi_f^u \mathbb{Q}_{\mathcal{X}}n+1 \quad (9.3)$$

¹⁰ We emphasize that we do *not* assume isolated singularities here.

by [67, (2.6.2)]. Since $\mathbb{H}^{j-n}(X_0, {}^p\phi_f^u \mathbb{Q}_{\mathcal{X}}[n+1]) = H_{\text{van}}^j(X_t)^{T^{\text{ss}}}$ by definition, and

$$\begin{aligned} \mathbb{H}^{j-n}(X_0, \mathbb{D} {}^p\phi_f^u \mathbb{Q}_{\mathcal{X}}[n+1]) &\cong \mathbb{H}^{n-j}(X_0, {}^p\phi_f^u \mathbb{Q}_{\mathcal{X}}[n+1])^{\vee} \\ &= (H_{\text{van}}^{2n-j}(X_t)^{T^{\text{ss}}})^{\vee}, \end{aligned}$$

taking the direct image of (9.3) by $X_0 \rightarrow \text{pt}$ induces $(\forall j \in \mathbb{Z})$ a perfect pairing

$$H_{\text{van}}^j(X_t)^{T^{\text{ss}}} \times H_{\text{van}}^{2n-j}(X_t)^{T^{\text{ss}}} \rightarrow \mathbb{Q}(-n-1). \quad (9.4)$$

In particular, $F^n H_{\text{van}}^{2n-j}(X_t)^{T^{\text{ss}}}$ is dual to $H_{\text{van}}^j(X_t)^{T^{\text{ss}}} / F^2$ in (9.4).

Now recall that X_0 has (not necessarily isolated) rational singularities, and $\mathcal{X}$ is smooth. By a result of Saito [69, Theorems 0.4–0.6], we therefore have

$$\left(F^n ({}^p\phi_f^u \mathbb{Q}_{\mathcal{X}}[n+1]) \subseteq \right) F^n ({}^p\phi_f \mathbb{Q}_{\mathcal{X}}[n+1]) = \{0\}, \quad (9.5)$$

where $\mathbb{Q}_{\mathcal{X}}[n+1]$ is interpreted as a MHM and F^n is F_{-n} in *op. cit.* Again taking H^{n-j} of the direct image, we conclude $(\forall j \in \mathbb{Z})$ that $F^n H_{\text{van}}^{2n-j}(X_t)^{T^{\text{ss}}} = \{0\}$ hence $H_{\text{van}}^j(X_t)^{T^{\text{ss}}} = F^2 H_{\text{van}}^j(X_t)^{T^{\text{ss}}}$. Taking Gr_F^1 of the T^{ss} -invariant part of the vanishing cycle sequence (1.3), the result follows. $\square$

Remark 9.12 (i) The main issue dealt with in Theorem 9.11 is the vanishing of the $(1, k-1)$ part of $H_{\text{ph}}^k(X_0)$. Indeed, one reduces to this statement as follows: taking T^{ss} -invariants of (9.2) gives $\text{Gr}_F^1 H_{\text{lim}}^k(X_t)^{T^{\text{ss}}} \cong \text{Gr}_F^1 H_{\text{lim}}^k(X_t)^T$; while applying C–S for $\mathcal{X}$ smooth (Theorem 5.3) yields $\text{Gr}_F^1 H_{\text{lim}}^k(X_t)^T \cong \text{sp}(\text{Gr}_F^1 H^k(X_0))$, and $\ker\{\text{sp} : H^k(X_0) \rightarrow H_{\text{lim}}^k(X_t)\} = \delta(H_{\text{van}}^{k-1}(X_t))$ has pure weight k by Proposition 5.5.

- (ii) According to Saito [70], for X_0 du Bois [resp. rational], the conclusions of Theorem 9.3 and Corollary 9.9(i) [resp. Corollary 9.9(ii) and Theorem 9.11] hold if we assume $\mathcal{X}$ is smooth Kähler, f is proper, and X_0 is reduced — in particular, one need not assume that f extends to an algebraic morphism (or that $\mathcal{X}$ extends to a projective variety).
- (iii) The result of Theorem 9.3 also holds for $\mathcal{X}$ a complex analytic space (neither smooth nor extendable-to-algebraic) provided we assume that $\mathcal{X} \setminus X_0$ is smooth, f is *projective* and X_0 is a reduced and irreducible divisor with *rational* singularities [42].

One can say quite a bit more with the aid of spectra, especially in the case of isolated singularities. For example, in Part II we will show that when $\mathcal{X}$ is smooth and X_0 has isolated k -log-canonical singularities in the sense of [63], one has $\text{Gr}_F^j H^k(X_0) \cong \text{Gr}_F^j H_{\text{lim}}^k(X_t)$ for $j = 0, \dots, k$.

10 Decomposition theorem over a polydisk

We conclude by elaborating the consequences of Theorem 3.1 for the simplest multi-parameter setting of all. Let $f : \mathcal{X} \rightarrow \Delta^r$ be a projective map of relative dimension n , equisingular¹¹ over $(\Delta^*)^r$ and each “coordinate $(\Delta^*)^k$ ”; and take $\mathcal{K}^\bullet = \mathrm{IC}_{\mathcal{X}}^\bullet$. For notation we shall use:

- $s_1, \dots, s_r$ for the disk coordinates;
- $U_{\{I\}}$ [resp. $\mathcal{S}_{\{I\}}$] for the coordinate $(\Delta^*)^{r-|I|}$ [resp. $\Delta^{r-|I|}$] where $s_i = 0$ ($\forall i \in I$);
- $\Delta^r \xleftrightarrow{\iota_c} \mathcal{S}_c := \bigcup_{|I|=c} \mathcal{S}_{\{I\}} \xleftrightarrow{j_c} \bigcup_{|I|=c} U_{\{I\}} =: U_c$; and
- $\bigcup_{|I|=c} \{f_{\{I\}}^\circ\} := f_c^\circ : \mathcal{X}_c^\circ \rightarrow U_c$ resp. $\mathcal{X}_c \rightarrow \mathcal{S}_c$ for restrictions of f .

In (3.1), $Z_{d(=r-c)}$ is replaced by $\mathcal{S}_c$, and the pure weight- $(j+n+c)$ VHS $\mathbb{V}_{r-c}^j$ over U_c is rewritten H_c^{j+n+c} (restricting to $\mathrm{H}_{\{I\}}^{j+n+c}$ on each $U_{\{I\}}$ with $|I|=c$). For $c=0$, we write $\mathcal{H}^{j+n} := \mathrm{H}_0^{j+n}$, with fibers $\mathrm{IH}^{j+n}(X_s)$ ($s \in (\Delta^*)^r$) and monodromies $T_1, \dots, T_r$. For $c=1$, the $\mathrm{H}_{\{i\}}^{j+n+1}$ are the phantom IH’s of fibers of $f_{\{i\}}^\circ$.

Remark 10.1 When $\mathcal{X}$ is smooth, $\mathcal{K}^\bullet = \mathbb{Q}_{\mathcal{X}}[n+r]$, and the H_c^{j+n+c} are (pure) sub-VMHS of $R^{j+n+c}(f_c^\circ)_* \mathbb{Q}_{\mathcal{X}_c^\circ}$, with $\mathcal{H}^{j+n} \cong R^{j+n}(f_0^\circ)_* \mathbb{Q}_{\mathcal{X}_0^\circ}$.

With this indexing by codimension, the terms of (3.1) become

$${}^p R^j f_* \mathrm{IC}_{\mathcal{X}}^\bullet \simeq \bigoplus_c \iota_{*}^c j_{!*}^c \mathrm{H}_c^{j+n+c}[r-c], \quad (10.1)$$

so that

$$\begin{aligned} \mathrm{IH}^m(\mathcal{X}) &\cong \mathbb{H}^{m-n-r}(\Delta^r, Rf_* \mathrm{IC}_{\mathcal{X}}^\bullet) \\ &\cong \bigoplus_{j,c} \mathrm{IH}^{m-n-c-j}(\mathcal{S}_c, \mathrm{H}_c^{j+n+c}) \\ &\cong \bigoplus_{c,\ell} \mathrm{IH}^\ell(\mathcal{S}_c, \mathrm{H}_c^{m-\ell}) \end{aligned} \quad (10.2)$$

where $\ell := m - (j+n+c)$. There are two things to note here: first, that¹² $\mathrm{IH}^\ell(\mathcal{S}_c, \mathrm{H}_c^{m-\ell}) = \bigoplus_{|I|=c} \mathrm{IH}^\ell(\mathcal{S}_{\{I\}}, \mathrm{H}_{\{I\}}^{m-\ell}) = \bigoplus_{|I|=c} \mathrm{IH}^\ell(\mathrm{H}_{\{I\}}^{m-\ell})_{\underline{0}}$ are really just sums of local IH groups at $\underline{0}$. These are naturally endowed with *mixed* Hodge structures by setting $\mathrm{H}_{\{I\},\mathrm{lim}}^* := (\prod_{j \notin I} \psi_{s_j}) \mathrm{H}_{\{I\}}^*$ (or just $\mathrm{H}_{\mathrm{lim}}^* := \psi_{s_1} \cdots \psi_{s_r} \mathcal{H}^*$ for $c=0$) and defining Koszul complexes $\mathcal{K}^\bullet(\mathrm{H}_{\{I\}}^*)$ by

$$\mathrm{H}_{\{I\},\mathrm{lim}}^* \rightarrow \bigoplus_{j \notin I} N_j \mathrm{H}_{\{I\},\mathrm{lim}}^*(-1) \rightarrow \bigoplus_{j_1 < j_2 \notin I} N_{j_1} N_{j_2} \mathrm{H}_{\{I\},\mathrm{lim}}^*(-2) \rightarrow \cdots; \quad (10.3)$$

then (as MHSs)

$$\mathrm{IH}^\ell(\mathrm{H}_{\{I\}}^*)_{\underline{0}} \cong H^\ell(\mathcal{K}^\bullet(\mathrm{H}_{\{I\}}^*))^{G_I} \quad (10.4)$$

¹¹ More precisely, we assume that the restrictions of $R^i f_* \mathrm{IC}_{\mathcal{X}}^\bullet \mathbb{Q}$ to $U_{\{I\}}$ are local systems ($\forall i, I$).

¹² For simplicity, we write this as $\mathrm{IH}^\ell(\mathrm{H}_c^{m-\ell})_{\underline{0}}$ below; this notation means the *stalk* cohomology $H^{\ell-r+c} \iota_{\underline{0}}^* \iota_{*}^c j_{!*}^c (\mathrm{H}_c^{m-\ell}[r-c])$, *not* the (costalk) cohomology with *support* at $\underline{0}$.

where G_I is the (finite) group generated by the $\{T_j^{\text{ss}}\}_{j \notin I}$ [13,38]. We shall write $H_{\{I\}, \text{inv}}^* := \text{IH}^0(H_{\{I\}}^*)_{\underline{0}}$ for the $\{T_j\}_{j \notin I}$ -invariants in $H_{\{I\}, \text{lim}}^*$, and $H_{c, \text{inv}}^* = \bigoplus_{|I|=c} H_{\{I\}, \text{inv}}^*$. Obviously, (10.4) vanishes for $\ell > \max\{r - |I| - 1, 0\}$.

Second, the hard Lefschetz isomorphisms (3.2) take the form

$$H_c^{-j+n+c}(-j) \xrightarrow{\cong} H_c^{j+n+c},$$

so that H_c^* is centered about $* = n + c$. Since it is zero for $* > 2n$, it is also zero for $* < 2c$. Taking stock of these vanishings, (10.2) becomes

$$\text{IH}^m(\mathcal{X}) \cong \bigoplus_{c=0}^{\min(r, \lfloor \frac{m}{2} \rfloor)} \bigoplus_{\ell=0}^{\max(0, r-c-1)} \text{IH}^\ell(\mathcal{S}_c, H_c^{m-\ell}). \quad (10.5)$$

Now we introduce two filtrations: the *coniveau filtration* (by codimension of support) is just

$$\mathcal{N}^\alpha \text{IH}^m(\mathcal{X}) := \bigoplus_{c=\alpha}^{\min(r, \lfloor \frac{m}{2} \rfloor)} \bigoplus_{\ell=0}^{\max(0, r-c-1)} \text{IH}^\ell(\mathcal{S}_c, H_c^{m-\ell}); \quad (10.6)$$

while the *shifted perverse Leray filtration* $\mathcal{L}^\alpha := {}^p \mathcal{L}^{\alpha-r}$ is given (cf. (3.4)–(3.5)) by

$$\mathcal{L}^\alpha \text{IH}^m(\mathcal{X}) := \bigoplus_{c=0}^{\min(r, \lfloor \frac{m}{2} \rfloor)} \bigoplus_{\ell=\max(\alpha-c, 0)}^{\max(r-c-1, 0)} \text{IH}^\ell(\mathcal{S}_c, H_c^{m-\ell}). \quad (10.7)$$

The following is essentially a special case of [20]:

Proposition 10.2 $\mathcal{L}^\alpha$ is the kernel of restriction to $f^{-1}(\mathcal{A})$, where $\mathcal{A} \subset \Delta^r$ is a general affine¹³ slice of codimension $\alpha - 1$; and $\mathcal{N}^\bullet \subseteq \mathcal{L}^\bullet$.

Proof The restrictions

$$\text{IH}^\ell(\mathcal{S}_c, H_c^{m-\ell}) \rightarrow \text{IH}^\ell(\mathcal{S}_c \cap \mathcal{A}^{\alpha-1}, H_c^{m-\ell}) \quad (10.8)$$

are either injective or zero. Here the target is computed by a Čech-Koszul double complex, which has the $N_{j_1} \cdots N_{j_\ell} H_{c, \text{lim}}^{m-\ell}(-\ell)$ terms required if and only if $\mathcal{S}_{c+\ell} \cap \mathcal{A}^{\alpha-1}$ is nonempty. So (10.8) is zero $\iff \ell \geq \alpha - c$ (as required).

The inclusion $\mathcal{N}^\alpha \subseteq \mathcal{L}^\alpha$ is now geometrically obvious, though it also follows directly from (10.6)–(10.7) by $c \geq \alpha \implies \alpha - c \leq 0 \implies \max(\alpha - c, 0) = 0$. $\square$

Taken together, these filtrations endow every term in the double sum (10.5) with geometric meaning: from (10.6) and (10.7) we have $\text{Gr}_{\mathcal{L}}^\alpha = \bigoplus_{\ell=0}^\alpha \text{IH}^\ell(H_{\alpha-\ell}^{m-\ell})_{\underline{0}}$ and $\text{Gr}_{\mathcal{N}}^\beta \text{IH}^m(\mathcal{X}) = \bigoplus_{\ell=0}^{\max(r-\beta-1, 0)} \text{IH}^\ell(H_\beta^{m-\ell})_{\underline{0}}$, whereupon

$$\text{IH}^\ell(H_c^{m-\ell})_{\underline{0}} = \text{Gr}_{\mathcal{L}}^{\ell+c} \text{Gr}_{\mathcal{N}}^c \text{IH}^m(\mathcal{X}). \quad (10.9)$$

Recalling that $\mathcal{X}_1 = f^{-1}(\{s_1 \cdots s_r = 0\})$, we also obtain a generalization of (part of) the Clemens–Schmid sequence:

¹³ More precisely, we mean the intersection of $\alpha - 1$ hypersurfaces of the form $L(\underline{s}) = K$, where L is a linear form and K a sufficiently small nonzero constant.

Theorem 10.3 *The sequence of MHS*

$$\mathrm{IH}_{\mathcal{X}_1}^m(\mathcal{X}) \longrightarrow \frac{\mathrm{IH}^m(\mathcal{X})}{\bigoplus_{\ell=1}^{r-1} \mathrm{IH}^\ell(\mathcal{H}^{m-\ell})_{\underline{0}}} \xrightarrow{\mathrm{sp}} \mathrm{IH}_{\mathrm{lim}}^m(X_{\underline{s}})^{T_1, \dots, T_r} \longrightarrow 0 \quad (10.10)$$

is exact.

Proof Actually more is true: $\mathrm{IH}^m(\mathcal{X})$ is the *direct sum* of $\mathcal{N}^1 \mathrm{IH}^m(\mathcal{X}) = \text{image } \{\mathrm{IH}_{\mathcal{X}_1}^m(\mathcal{X}) \rightarrow \mathrm{IH}^m(\mathcal{X})\}$ and $\mathrm{Gr}_{\mathcal{N}}^0 \mathrm{IH}^m(\mathcal{X}) = \bigoplus_{\ell=0}^{r-1} \mathrm{IH}^{r-1}(\mathcal{H}^{m-\ell})_{\underline{0}}$, with $\mathrm{IH}^0(\mathcal{H}^m)_{\underline{0}} = H_{\mathrm{inv}}^m = (H_{\mathrm{lim}}^m)^{T_1, \dots, T_r}$. $\square$

For the remainder of the section, we *assume that $\mathcal{X}$ is smooth*, so that (10.10) becomes¹⁴

$$H_{\mathcal{X}_1}^m(\mathcal{X}) \xrightarrow{i_{X_0}^* i_{\mathcal{X}_1}^*} \frac{H^m(X_{\underline{0}})}{\bigoplus_{\ell=1}^{r-1} \mathrm{IH}^\ell(\mathcal{H}^{m-\ell})_{\underline{0}}} \xrightarrow{\mathrm{sp}} H_{\mathrm{inv}}^m(X_{\underline{s}}) \longrightarrow 0. \quad (10.11)$$

(Note that we are *not* assuming unipotent monodromies.) It is instructive to write out the decomposition (10.5) in detail for small r :

- ($r = 1$) $H^m(X_{\underline{0}}) \cong H_{\mathrm{inv}}^m \oplus \underbrace{H_1^m}_{\mathcal{L}^1}$
- ($r = 2$) $H^m(X_{\underline{0}}) \cong H_{\mathrm{inv}}^m \oplus \underbrace{\mathrm{IH}^1(\mathcal{H}^{m-1})_{\underline{0}} \oplus H_{1,\mathrm{inv}}^m}_{\mathcal{L}^1} \oplus \overbrace{H_2^m}^{\mathcal{L}^2}$
- ($r = 3$) $H^m(X_{\underline{0}}) \cong H_{\mathrm{inv}}^m \oplus \underbrace{\left\{ \begin{array}{c} \mathrm{IH}^1(\mathcal{H}^{m-1})_{\underline{0}} \\ \oplus H_{1,\mathrm{inv}}^m \end{array} \right\}}_{\mathcal{L}^1} \oplus \overbrace{\left\{ \begin{array}{c} \mathrm{IH}^2(\mathcal{H}^{m-2})_{\underline{0}} \oplus \\ \mathrm{IH}^1(H_1^{m-1})_{\underline{0}} \oplus H_{2,\mathrm{inv}}^m \end{array} \right\}}^{\mathcal{L}^2} \oplus \underbrace{H_3^m}_{\mathcal{L}^3}$

in which $H_c^* = 0$ for $* \leq 2c - 1$ (and H_1^* is the phantom cohomology in codimension 1). By Proposition 10.2, $\mathcal{L}^1$ is the kernel of the restriction to a nearby fiber $X_{\underline{s}}$ (i.e. of sp), $\mathcal{L}^2$ of the restriction to a nearby affine line (meeting all coordinate hyperplanes), and so on.

Finally, here are a few examples which illustrate the $r = 2$ scenario (and which all happen to have unipotent monodromy):

Example 10.4 Let $\mathcal{C} \rightarrow \Delta^2$ be a family of curves with smooth total space. Then $H^1(C_{\underline{0}}) = H_{\mathrm{inv}}^1$ and $H^2(C_{\underline{0}}) = H_{\mathrm{inv}}^2 (\cong \mathbb{Q}(-1)) \oplus \mathrm{IH}_0^1(\mathcal{H}^1) \oplus H_{1,\mathrm{inv}}^2$. The simplest example with IH^1 -term nonzero is when $\mathcal{C}$ is a family of elliptic curves with I_1 -fibers on $\{0\} \times \Delta^* \cup \Delta^* \times \{0\}$ (with equal monodromies $N_1 = N_2$) and I_2 -fiber at $\underline{0}$ (cf. [48]); then $H_1^2 = 0$ and $H^2(C_{\underline{0}}) \cong \mathbb{Q}(-1)^{\oplus 2}$. For instance, if we base-change a 1-variable I_1

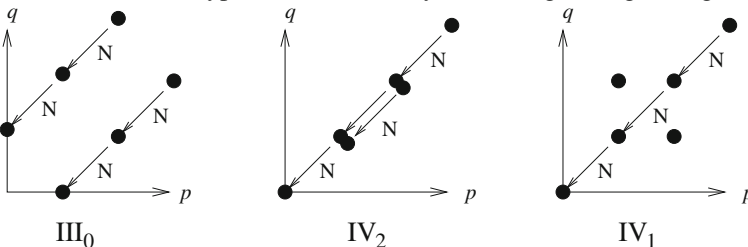
¹⁴ Alternatively one can move the denominator of the middle term to the right-hand term as a direct summand.

degeneration by $(s_1, s_2) \mapsto s_1 s_2$, the nonvanishing of $\mathrm{IH}^1(\mathcal{H}^1)$ simply indicates that without blowing up, we have a singular total space.

Example 10.5 Abramovich and Karu [3] defined a notion of semistable degenerations in more than one parameter; these are characterized by having (i) smooth total space (so that (10.11) applies) and (ii) local structure of a fiber product of SSDs along the coordinate hyperplanes. (In particular, they have unipotent monodromies.) An easy case is that of an “exterior product” of 1-variable SSDs: for instance, let $\mathcal{E} \xrightarrow{g} \Delta$ be a semistable degeneration of elliptic curves with I_k singular fiber, so that $\mathcal{X} := \mathcal{E} \times \mathcal{E} \xrightarrow{g \times g} \Delta \times \Delta$ has fibers $E_{s_1} \times E_{s_2}$. (Locally this takes the form $s_1 = xy$, $s_2 = zw$.)

Regardless of k , we have $\mathrm{IH}^1(\mathcal{H}^*) = 0$. For $k = 1$, the $H_{\{i\}}$ and $H_{\{12\}}^*$ all vanish, so that $H^m(X_0) = H_{\mathrm{inv}}^m$ in all degrees. However, when $k = 2$, we have $H_{\{i\}, \mathrm{inv}}^* = H_{\mathrm{lim}}^{*-2}(E_s)(-1)$, and $H_{\{12\}}^* = H^{*-4}(\mathrm{pt.})(-2)$; the reader may check that $H_{\mathrm{inv}}^m \oplus H_{\{1\}, \mathrm{inv}}^m \oplus H_{\{2\}, \mathrm{inv}}^m \oplus H_{\{12\}}^m$ correctly computes $H^m(E_0 \times E_0)$.

Example 10.6 Mirror symmetry allows for the computation of (unipotent) monodromies $T_i = e^{N_i}$ of families of CY toric hypersurfaces $X_{\underline{s}} \subset \mathbb{P}$ in the “large complex structure limit”. In particular, [49, §8.3] and [31] study two distinct 2-parameter families of $h^{2,1} = 2$ CY 3-folds over $(\Delta^*)^2$ with Hodge-Tate LMHS H_{lim}^3 at the origin. The notation $\langle \mathrm{IV}_1 \mid \mathrm{IV}_2 \mid \mathrm{III}_0 \rangle$ for the first family and $\langle \mathrm{III}_0 \mid \mathrm{IV}_2 \mid \mathrm{III}_0 \rangle$ for the second indicates the LMHS types corresponding to N_1 (on $\{0\} \times \Delta^*$), $N_1 + N_2$ (at $\{0\}$), and N_2 (on $\Delta^* \times \{0\}$). These types are described by their Hodge-Deligne diagrams:



Both variations have $H_{\mathrm{inv}}^3 \cong \mathbb{Q}(0)$ and $H_{\mathrm{inv}}^4 \cong H^2(\mathbb{P})(-1)$ for their respective toric varieties, and of course $\mathrm{IH}^1(\mathcal{H}^*)_0 = 0$ for $* \neq 3$. Let us assume we have smooth compactifications of both families with all $H_{1, \mathrm{inv}}^*$ and H_2^* zero. Then $H^3(X_0) = \mathbb{Q}(0)$ and $H^4(X_0) = H^2(\mathbb{P})(-1) \oplus \mathrm{IH}^1(\mathcal{H}^3)_0$ in both cases; so the key to the topology of X_0 in each case (provided we want a smooth total space) lies in the cohomology of the complex $H_{\mathrm{lim}}^3 \rightarrow N_1 H_{\mathrm{lim}}^3(-1) \oplus N_2 H_{\mathrm{lim}}^3(-1) \rightarrow N_1 N_2 H_{\mathrm{lim}}^3(-2)$. From the LMHS types one immediately deduces that (writing ranks of maps over the arrows) this complex takes the form $\mathbb{C}^6 \xrightarrow{5} \mathbb{C}^3 \oplus \mathbb{C}^4 \xrightarrow{2} \mathbb{C}^2$ in the first case (so that $\mathrm{IH}^1(\mathcal{H}^3) = 0$), and $\mathbb{C}^6 \xrightarrow{5} \mathbb{C}^4 \oplus \mathbb{C}^4 \xrightarrow{2} \mathbb{C}^2$ in the second (so that $\mathrm{IH}^1(\mathcal{H}^3) = \mathbb{Q}(-2)$).

Acknowledgements The authors wish to thank P. Brosnan, P. Gallardo, and L. Migliorini for correspondence closely related to this work, and the IAS for providing the environment in which, some years ago, this series of papers was first conceived. We also thank Green, Griffiths, Pearlstein, and Robles for our fruitful discussions and collaborations on period maps and moduli during the course of the NSF FRG project “Hodge theory, moduli, and representation theory”. Finally, we are grateful to Saito for his careful reading and numerous remarks which led to improvements in the exposition. We also thank Saito for writing an “Appendix” to our paper that explains the connection between C–S and DT in full generality.

Appendix: Decomposition theorem over Stein curves by Morihiko Saito

In this Appendix we prove the following (see Theorem A.4 below):

Theorem A *Let $f : X \rightarrow C$ be a proper surjective morphism of a connected complex manifold X to a connected non-compact curve C . The decomposition theorem for $\mathbf{R}f_*\mathbb{Q}_X$ is equivalent to the Clemens-Schmid exact sequence (or the local invariant cycle theorem) for every singular fiber of f .*

We also show that the weak decomposition always holds if C is a connected non-compact smooth curve, see Corollary A.3 below. Note that the last hypothesis implies that C is *Stein* by Behnke–Stein, see for instance [29, Corollary 26.8]. We have the following.

Corollary A *For $f : X \rightarrow C$ as above, the decomposition theorem holds for $\mathbf{R}f_*\mathbb{Q}_X$, if there is an embedded resolution $\rho : \tilde{X} \rightarrow X$ such that the inverse image $\tilde{X}_c := \rho^{-1}(X_c)$ of any singular fiber $X_c := f^{-1}(c)$ is a divisor with simple normal crossings (not necessarily reduced) and there is a cohomology class $\eta_c \in H^2(\tilde{X}_c, \mathbb{C})$ whose restriction to any irreducible component of $\tilde{X}_c$ is a Kähler class.*

Note that the local invariant cycle theorem holds under the above hypotheses as is well-known, see also Remark A.4(ii) below.

This work is partially supported by JSPS Kakenhi 15K04816.

A.1. t -structure on complex manifolds (see [8]). Let X be a complex manifold, and A be any subfield of $\mathbb{C}$. Let $D_c^b(X, A)$ be the bounded derived category of A -complexes with constructible cohomology sheaves. For $k \in \mathbb{Z}$, we have the full subcategories

$$D_c^b(X, A)^{\leq k} \subset D_c^b(X, A),$$

defined by the following condition for $K^\bullet \in D_c^b(X, A)$:

$$\dim \operatorname{Supp} \mathcal{H}^j K^\bullet \leq k - j \quad (\forall j \in \mathbb{Z}). \quad (\text{A.1.1})$$

Put

$$\begin{aligned} D_c^b(X, A)^{\geq k} &:= \mathbb{D}^{-1}(D_c^b(X, A)^{\leq -k}) \\ &= \{K^\bullet \in D_c^b(X, A) \mid \mathbb{D}K^\bullet \in D_c^b(X, A)^{\leq -k}\}, \\ D_c^b(X, A)^{[k]} &:= D_c^b(X, A)^{\leq k} \cap D_c^b(X, A)^{\geq k}. \end{aligned} \quad (\text{A.1.2})$$

Here, taking an injective resolution $A_X \xrightarrow{\sim} \mathcal{I}^\bullet$, we can define $\mathbb{D}K^\bullet$ by

$$\mathbb{D}K^\bullet = \tau_{\leq k} \mathcal{H}om_A(K^\bullet, \mathcal{I}^\bullet(\dim X)[2 \dim X]) \quad (k \gg 0), \quad (\text{A.1.3})$$

where $\tau_{\leq k}$ is a classical truncation.

By [8], the $D_c^b(X, A)^{[k]}$ are *abelian* full subcategories of $D_c^b(X, A)$, and there are truncation functors

$${}^p\tau_{\leq k} : D_c^b(X, A) \rightarrow D_c^b(X, A)^{\leq k},$$

(similarly for ${}^p\tau_{\geq k}$) together with the cohomological functors

$${}^p\mathcal{H}^k : D_c^b(X, A) \rightarrow D_c^b(X, A)^{[0]},$$

and also the distinguished triangles for $K^\bullet \in D_c^b(X, A)$:

$${}^p\tau_{\leq k-1} K^\bullet \rightarrow {}^p\tau_{\leq k} K^\bullet \rightarrow ({}^p\mathcal{H}^k K^\bullet)[-k] \xrightarrow{+1}.$$

A.2. Curve case. Assume $\dim X = 1$, that is, X is a smooth curve C . It is well-known that $D_c^b(C, A)^{[k]} \subset D_c^b(C, A)$ is defined by the following conditions:

$$\begin{aligned} \dim \operatorname{Supp} \mathcal{H}^k K^\bullet &= 0, \\ \mathcal{H}_{\{c\}}^0 \mathcal{H}^{k-1} K^\bullet &= 0 \quad (\forall c \in C), \\ \mathcal{H}^j K^\bullet &= 0 \quad (\forall j \notin \{k, k-1\}). \end{aligned} \tag{A.2.1}$$

This can be shown using the functor $i_c^!$ for $i_c : \{c\} \hookrightarrow C$ together with duality.

The following proposition and lemma are also well-known:

Proposition A.2 *For any $K^\bullet \in D_c^b(C, A)^{[k]}$, there is a unique finite increasing filtration G on $K^\bullet$ satisfying*

$$\begin{aligned} \operatorname{Gr}_0^G K^\bullet &= (j_* L)[1-k], \\ \dim \operatorname{Supp} \operatorname{Gr}_i^G K^\bullet &= 0 \quad \text{if } |i| = 1, \\ \operatorname{Gr}_i^G K^\bullet &= 0 \quad \text{if } |i| > 1. \end{aligned} \tag{A.2.2}$$

where L is an A -local system on a Zariski-open subset $C' \xrightarrow{j'} C$ which is obtained by restricting $\mathcal{H}^{k-1} K^\bullet$ to C' (note that $|C \setminus C'|$ may be infinite). Moreover $\operatorname{Gr}_{-1}^G K^\bullet$ (resp. $\operatorname{Gr}_1^G K^\bullet$) is the maximal subobject (resp. quotient object) of $K^\bullet$ supported on a discrete subset of C .

Proof Set

$$G_1 K^\bullet := K^\bullet, \quad G_0 K^\bullet := \tau_{\leq k-1} K^\bullet,$$

where $\tau_{\leq k-1}$ is the truncation in the classical sense. By (A.2.1) we have the canonical isomorphisms

$$G_0 K^\bullet = (\mathcal{H}^{k-1} K^\bullet)[1-k], \quad \operatorname{Gr}_1^G K^\bullet = (\mathcal{H}^k K^\bullet)[-k], \tag{A.2.3}$$

together with the short exact sequence of sheaves

$$0 \rightarrow \mathcal{H}^{k-1} K^\bullet \xrightarrow{\iota} j_* L \rightarrow \text{Coker } \iota \rightarrow 0, \quad (\text{A.2.4})$$

inducing a short exact sequence in $D_c^b(C, A)^{[k]}$:

$$0 \rightarrow (\text{Coker } \iota)[-k] \rightarrow (\mathcal{H}^{k-1} K^\bullet)[1-k] \rightarrow (j_* L)[1-k] \rightarrow 0. \quad (\text{A.2.5})$$

So we get (A.2.2), setting $G_{-1} K^\bullet := (\text{Coker } \iota)[-k]$, $G_{-2} K^\bullet := 0$. The last assertion follows from Lemma A.2 below. This finishes the proof of Proposition A.2.

Lemma A.2 *In the notation of Proposition A.2, the shifted direct image sheaf $(j_* L)[1]$ is canonically isomorphic to the intermediate direct image $j_{!*}(L[1])$ in $D_c^b(C, A)^{[0]}$ (see [8]).*

Proof We have the following short exact sequences in $D_c^b(C, A)^{[0]}$:

$$\begin{aligned} 0 &\rightarrow (\text{Coker } \iota') \rightarrow (j_! L)[1] \rightarrow (j_* L)[1] \rightarrow 0, \\ 0 &\rightarrow (j_* L)[1] \rightarrow (\mathbf{R} j_* L)[1] \rightarrow R^1 j_* L \rightarrow 0, \end{aligned} \quad (\text{A.2.6})$$

where $\iota' : j_! L \hookrightarrow j_* L$ is a canonical inclusion. (These two short exact sequences are dual of each other if L in the second sequence is replaced by its dual.) Lemma A.2 then follows.

Lemma A.2 and Proposition A.2 imply the following.

Corollary A.2 *Any simple object of $D_c^b(C, A)^{[k]}$ is either $A_c[-k]$ with A_c a sheaf supported at a point $c \in C$ or $(j_* L)[1-k]$ with L a simple A -local system on a Zariski-open subset $C' \xrightarrow{j} C$.*

(Note, however, that the intermediate direct image $j_{!*}$ and the direct image j_* are not exact functors.)

Remark A.2 The intermediated direct image $j_{!*}(L[1])$ is also written as $\text{IC}_C L$, and is called the *intersection complex* (with local system coefficients).

A.3. Vanishing of higher extension groups. In the case of non-compact curves, we have the following.

Proposition A.3 *If C is a connected non-compact smooth curve, and $K^\bullet, K'^\bullet \in D_c^b(C, A)^{[k]}$, we have*

$$\text{Ext}_{D^b(C, A)}^i(K^\bullet, K'^\bullet) = 0 \quad (i \geq 2). \quad (\text{A.3.1})$$

Proof Since the assertion is independent of $k \in \mathbb{Z}$, we may assume $k = 1$. Set

$$E^\bullet := \mathbf{R}\mathcal{H}om_A(K^\bullet, K'^\bullet) \in D_c^b(C, A).$$

We first reduce the assertion (A.3.1) to

$$E^\bullet \in D_C^b(C, A)^{\leq 1}, \quad (\text{A.3.2})$$

using the well-known isomorphism

$$\begin{aligned} \text{Ext}_{D^b(C, A)}^i(K^\bullet, K'^\bullet) &= H^i(C, \mathbf{R}\mathcal{H}om_A(K^\bullet, K'^\bullet)) \\ &= H^i(C, E^\bullet), \end{aligned} \quad (\text{A.3.3})$$

together with the Riemann-Hilbert correspondence and also Cartan's Theorem B. (Here some finiteness condition would be needed if we use the duality for the direct images of objects of $D_C^b(C, A)$ by the morphism $C \rightarrow \text{pt.}$)

By scalar extension $A \hookrightarrow \mathbb{C}$ we may assume $A = \mathbb{C}$. Let M be the regular holonomic left $\mathcal{D}_C$ -module M corresponding to ${}^p\mathcal{H}^j E^\bullet$ ($j \leq 1$). It has a global good filtration F , and we have the following quasi-isomorphism for $k \gg 0$:

$$C(d : F_k M \rightarrow F_{k+1} M \otimes_{\mathcal{O}_C} \Omega_C^1) \xrightarrow{\sim} \text{DR}(M) = {}^p\mathcal{H}^j E^\bullet.$$

Recall that any connected *non-compact* smooth curve is *Stein* as a consequence of the theory of Behnke-Stein, see for instance [29, Corollary 26.8]. We then get by Cartan's Theorem B

$$H^i(C, {}^p\mathcal{H}^j E^\bullet) = 0 \quad (i > 0).$$

Here one problem is that it is not quite clear whether k exists *globally*, since C is non-compact. For this we can use Proposition A.2 so that the assertion is reduced to the intersection complex case. Then the Deligne extension [22] gives the filtration F with the above k rather explicitly. (Note that Cartan's Theorem B does not necessarily hold for quasi-coherent sheaves, see for instance [66, Remark 2.3.8 (2)].) The assertion (A.3.1) is thus reduced to (A.3.2).

Let $C' \subset C$ be a Zariski-open subset such that $K^\bullet|_{C'}$, $K'^\bullet|_{C'}$ are local systems. Since the assertion (A.3.2) is local, we may assume that $(C, C') = (\Delta, \Delta^*)$ so that $E^\bullet|_{\Delta^*}$ is a local system. The assertion is then reduced to that

$$(\mathcal{H}^j E^\bullet)_0 = \text{Ext}_{D^b(\Delta, \mathbb{C})}^j(K^\bullet, K'^\bullet) = 0 \quad (j \geq 2). \quad (\text{A.3.4})$$

Using the Riemann-Hilbert correspondence, the latter assertion is equivalent to that

$$\text{Ext}_{\mathcal{D}_{\Delta, 0}}^j(M, M') = 0 \quad (j \geq 2), \quad (\text{A.3.5})$$

for any regular holonomic $\mathcal{D}_{\Delta, 0}$ -modules M, M' . This is further reduced to the case where M, M' are simple regular holonomic $\mathcal{D}_{\Delta, 0}$ -modules (using the standard long exact sequences of extension groups). So we may assume that M, M' are of the form

$\mathcal{D}_{\Delta,0}/\mathcal{D}_{\Delta,0}P$ with

$$P = \partial_t, \quad t, \quad t\partial_t - \alpha \quad (\alpha \in \mathbb{C} \setminus \mathbb{Z}), \quad (\text{A.3.6})$$

where t is a coordinate of Δ . This implies a free resolution

$$0 \rightarrow \mathcal{D}_{\Delta,0} \xrightarrow{\cdot P} \mathcal{D}_{\Delta,0} \rightarrow M \rightarrow 0,$$

and shows (A.3.5). (Here it is also possible to use the isomorphism $\mathbf{R}Hom_A(K^\bullet, K'^\bullet) = \delta^!(\mathbb{D}K^\bullet \boxtimes K'^\bullet)$ with $\delta : X \hookrightarrow X \times X$ the diagonal, although the argument is more complicated.) This finishes the proof of Proposition A.3.

From Proposition A.3 we can deduce the following.

Theorem A.3 (Weak Decomposition theorem). *Let C be a connected non-compact smooth curve. For any $K^\bullet \in D_c^b(C, A)$, we have a non-canonical isomorphism*

$$K^\bullet \cong \bigoplus_{j \in \mathbb{Z}} ({}^p\mathcal{H}^j K^\bullet)[-j] \quad \text{in } D_c^b(C, A), \quad (\text{A.3.7})$$

Proof This follows from Proposition A.3 by using the distinguished triangles in (A.1.4) by induction on k . $\square$

Corollary A.3 *Let $f : X \rightarrow C$ be a proper morphism of complex manifolds with C a connected non-compact smooth curve. Let $X' \subset X$ be a Zariski-open subset. Set $f' := f|_{X'}$, ${}^pR^j f'_* := {}^p\mathcal{H}^j \mathbf{R}f'_*$, and $d_X := \dim X$. Then we have a non-canonical isomorphism*

$$\mathbf{R}f'_*(A_{X'}[d_X]) \cong \bigoplus_{j \in \mathbb{Z}} ({}^pR^j f'_*(A_{X'}[d_X]))[-j] \quad \text{in } D_c^b(C, A). \quad (\text{A.3.8})$$

Remark A.3 (i) It is quite unclear whether $\mathbf{R}f'_* A_{X'}$ belongs to $D_c^b(C, A)$ unless we assume that $f' : X' \rightarrow C$ can be extended to a proper morphism of complex manifolds $f : X \rightarrow C$ with $X \setminus X'$ a closed analytic subset of X .

(ii) If $f : X \rightarrow Y$ is a *smooth* projective morphism of complex manifolds, we have the *weak decomposition* (see [21]):

$$\mathbf{R}f_*(A_X[d_X]) \cong \bigoplus_{j \in \mathbb{Z}} ({}^pR^j f_*(A_X[d_X]))[-j] \quad \text{in } D_c^b(Y, A), \quad (\text{A.3.9})$$

using the Leray spectral sequence together with the hard Lefschetz property

$$\ell^j : {}^pR^{-j} f_*(A_X[d_X]) \xrightarrow{\sim} {}^pR^j f_*(A_X[d_X])(j) \quad (j > 0), \quad (\text{A.3.10})$$

since ${}^pR^j f_*(A_X[d_X]) = (R^{d_X-d_Y} f_* A_X)[d_Y]$ in the f smooth case.

(iii) In the f non-smooth case, we need a “spectral object” in the sense of Verdier [79] in order to extend the above argument, see also [66, Lemma 5.2.8]. Note that the

proof of the decomposition theorem in [8, Theorem 6.2.5] is *completely different* from this. It uses mod p reduction, and the coefficients are $\mathbb{C}$, not $\mathbb{Q}$.

(iv) The decomposition theorem in the derived category of mixed Hodge modules was *not* proved in [66] (it follows from [67, (4.5.4)]). We can deduce from [66] only the decompositions of the underlying $\mathbb{Q}$ -complex and the underlying complex of filtered $\mathcal{D}$ -modules together with some compatibility between the decomposition isomorphisms, using Deligne’s argument on the “uniqueness” in [24]. We have to use [66, Lemma 5.2.8 and Proposition 2.1.12] to apply Deligne’s argument respectively to the $\mathbb{Q}$ -complex and the complex of filtered $\mathcal{D}$ -modules (by passing from the derived category of filtered $\mathcal{D}$ -modules to that of the abelian category of graded $\mathcal{B}$ -modules), see also [68, 2.4–5].

A.4. Clemens–Schmid sequence. For $K^\bullet \in D_c^b(\Delta, A)$, we have the following diagram for the *octahedral axiom* of derived categories (see also [66, Remark 5.2.2]):

$$\begin{array}{ccccc}
 i_0^* K^\bullet & \longleftarrow & i_0^! K^\bullet & & i_0^* K^\bullet & \longleftarrow & i_0^! K^\bullet \\
 & \swarrow c & \nwarrow d & & \swarrow d & \nwarrow c & \\
 \downarrow +1d & \xrightarrow{p\varphi_1} & K^\bullet & \xrightarrow{+1} & \downarrow +1c & \xrightarrow{C(N)} & K^\bullet & \xrightarrow{+1} \\
 \nearrow \text{can} & & \searrow \text{var} & & \nearrow +1 & & \searrow d & \\
 {}^p\psi_1 K^\bullet & \xrightarrow{N} & {}^p\psi_1 K^\bullet(-1) & \xrightarrow{N} & {}^p\psi_1 K^\bullet(-1) & & {}^p\psi_1 K^\bullet(-1) &
 \end{array}$$

Here c and d mean respectively commutative and distinguished, and $i_0 : \{0\} \hookrightarrow \Delta$ is the inclusion. We denote respectively by ${}^p\psi_1$, ${}^p\varphi_1$ the *unipotent* monodromy part of the shifted nearby and vanishing cycle functors ${}^p\psi := \psi[-1]$, ${}^p\varphi := \varphi[-1]$ for the coordinate t of Δ .

In the above diagram, the following two distinguished triangles are respectively called the *vanishing cycle triangle* (see [23]) and the *dual vanishing cycle triangle*:

$$\begin{aligned}
 {}^p\psi_1 K^\bullet &\xrightarrow{\text{can}} {}^p\varphi_1 K^\bullet \rightarrow i_0^* K^\bullet \xrightarrow{+1}, \\
 i_0^! K^\bullet &\rightarrow {}^p\varphi_1 K^\bullet \xrightarrow{\text{Var}} {}^p\psi_1 K^\bullet(-1) \xrightarrow{+1}.
 \end{aligned} \tag{A.4.1}$$

These are dual of each other if $K^\bullet$ in the second triangle is replaced by $\mathbb{D}K^\bullet$, see for instance [66, Lemma 5.2.4].

The *Clemens–Schmid sequence* is associated to the outermost part of the above diagram as follows:

$$\begin{aligned}
 &\rightarrow H^{j-1} i_0^! K^\bullet \rightarrow H^{j-1} i_0^* K^\bullet \rightarrow H^{jp} {}^p\psi_1 K^\bullet \xrightarrow{N} H^{jp} {}^p\psi_1 K^\bullet(-1) \\
 &\rightarrow H^{j+1} i_0^! K^\bullet \rightarrow H^{j+1} i_0^* K^\bullet \rightarrow \dots
 \end{aligned} \tag{A.4.2}$$

There are two sequences depending on the parity of $j \in \mathbb{Z}$, see also [14]. Note that this sequence is essentially *self-dual*, more precisely, its dual sequence is isomorphic to the Clemens–Schmid sequence for $\mathbb{D}K^\bullet$. This follows from the duality between the two distinguished triangles in (A.4.1).

For $K^\bullet \in D_c^b(\Delta, A)$, we say that the *Clemens–Schmid exact sequence holds* if the above two sequences are *exact* at every term.

We say that the *local invariant cycle property holds* if the above sequence is exact at the third term, that is, if we have the exactness of

$$H^j i_0^* K^\bullet \rightarrow H^j \psi_1 K^\bullet \xrightarrow{N} H^j \psi_1 K^\bullet(-1) \quad (\forall j \in \mathbb{Z}). \quad (\text{A.4.3})$$

We say that the *strong decomposition holds* for $K^\bullet \in D_c^b(C, A)$ with C a smooth curve if there is a non-canonical isomorphism

$$K^\bullet \cong \bigoplus_{j \in \mathbb{Z}} \text{IC}_C L^j[-j] \oplus \bigoplus_{c \in C, j \in \mathbb{Z}} E_c^j[-j] \quad \text{in } D_c^b(C, A), \quad (\text{A.4.4})$$

where the L^j are local systems defined on a Zariski-open subset of C , and the E_c^j are sheaves (in the classical sense) supported at $c \in C$.

If the weak decomposition holds (that is, if the isomorphism (A.3.7) holds), then the strong decomposition is equivalent to the following *canonical* isomorphisms called the *cohomological decompositions*:

$${}^p \mathcal{H}^j K^\bullet = \text{IC}_C L^j \oplus \bigoplus_{c \in C} E_c^j \quad \text{in } D_c^b(C, A)^{[0]} \quad (\forall j \in \mathbb{Z}).$$

These isomorphisms are *canonical* by *strict support decomposition* (see [66, 5.1.3]), and (A.4.5) is equivalent to the following direct sum decompositions at every $c \in C$:

$$\text{Im can} \oplus \text{Ker Var} = {}^p \varphi_{t,1} {}^p \mathcal{H}^j K^\bullet \quad (\forall j \in \mathbb{Z}), \quad (\text{A.4.5})$$

where t is a local coordinate of (C, c) .

By Theorem A.3, the weak decomposition holds for any complex $K^\bullet \in D_c^b(C, A)$ if C is connected and non-compact.

We have the following.

Theorem A.4 *Let C be a connected non-compact smooth curve. Let $K^\bullet \in D_c^b(C, A)$ with a self-duality isomorphism $\mathbb{D}K^\bullet \cong K^\bullet[m]$ for some $m \in \mathbb{Z}$. Then the following three conditions are equivalent to each other:*

- (a) *The strong decomposition holds.*
- (b) *The Clemens–Schmid exact sequence holds at any $c \in C$.*
- (c) *The local invariant cycle property holds at any $c \in C$.*

Proof We first prove (a) $\Rightarrow$ (b). Restricting to each direct factor of $K^\bullet$, it is enough to consider the case $K^\bullet = j_* L$ with L a local system, where the self-duality assumption $\mathbb{D}K^\bullet \cong K^\bullet[m]$ is forgotten for the moment. (Indeed, in the case $K^\bullet = E_c$, we can use the functorial isomorphisms $i_c^! \circ (i_c)_* = i_c^* \circ (i_c)_* = \text{id}$ with $i_c : \{c\} \hookrightarrow C$ the inclusion.) We have to show the exact sequence

$$0 \rightarrow i_0^* j_* L \rightarrow \psi_1 L \xrightarrow{N} \psi_1 L(-1) \rightarrow H^2 i_0^! j_* L \rightarrow 0, \quad (\text{A.4.6})$$

where $\psi_1 j_* L$ is denoted by $\psi_1 L$. We have the exactness at the second term by definition. This implies the exactness at the third term, if we remember the self-duality

condition $\mathbb{D}K^\bullet \cong K^\bullet[m]$ (which implies that $\mathbb{D}(j_*L) \cong (j_*L')[2]$ for some direct factor $(j_*L')[m'] \subset K^\bullet$) together with the self-duality of the Clemens–Schmid sequence explained after (A.4.2). So the implication (a) $\Rightarrow$ (b) follows.

Assume now condition (c) (since the implication (b) $\Rightarrow$ (c) is trivial). Let $\Delta \subset C$ be a sufficiently small neighborhood of c with coordinate t . Condition (c) implies that

$$\text{Ker can} = \text{Ker } N \quad \text{in } {}^p\psi_{t,1} {}^p\mathcal{H}^j K^\bullet \quad (j \in \mathbb{Z}),$$

using the long exact sequence associated to the vanishing cycle triangle in (A.4.1), since

$$\text{Var} \circ \text{can} = N \quad \text{on } {}^p\psi_{t,1} {}^p\mathcal{H}^j K^\bullet.$$

Considering the coimages and images of can and N , it induces the isomorphism

$$\begin{aligned} \text{Var} : \text{Im can} &\xrightarrow{\sim} \text{Im } N, \quad \text{hence} \\ \text{Im can} \cap \text{Ker Var} &= 0. \end{aligned} \tag{A.4.7}$$

We then get the direct sum decomposition (A.4.6) using the self-duality isomorphism

$$\mathbb{D} {}^p\mathcal{H}^j K^\bullet = {}^p\mathcal{H}^{-j} \mathbb{D} K^\bullet \cong {}^p\mathcal{H}^{m-j} K^\bullet \quad (j \in \mathbb{Z}),$$

since can and Var are dual of each other (up to a sign) as is explained after (A.4.1), see also Remark A.4(i) below. So condition (a) follows. This finishes the proof of Theorem A.4. $\square$

Remark A.4 (i) It is well-known that any indecomposable regular holonomic $\mathcal{D}_0$ -module M (with $\mathcal{D}_0 := \mathcal{D}_{\Delta,0}$) is isomorphic to one of the following:

$$\begin{aligned} \text{(A)} \quad & \mathcal{D}_0/\mathcal{D}_0(\partial_t t)^i \quad (K = \mathbf{R}j_* j^* K) & i \geq 1, \\ \text{(B)} \quad & \mathcal{D}_0/\mathcal{D}_0(t \partial_t)^i \quad (K = j_! j^* K) & i \geq 1, \\ \text{(C)} \quad & \mathcal{D}_0/\mathcal{D}_0(\partial_t t)^{i-1} \partial_t \quad (K = j_* j^* K) & i \geq 1, \\ \text{(D)} \quad & \mathcal{D}_0/\mathcal{D}_0 t (\partial_t t)^i & i \geq 0, \\ \text{(E)} \quad & \mathcal{D}_0/\mathcal{D}_0(\partial_t t - \alpha)^i \quad (K = j_! j^* K = \mathbf{R}j_* j^* K) & i \geq 1, \end{aligned}$$

where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, $K := \text{DR}(M)[-1] \in D_c^b(C, A)^{[1]}$, and i is the rank of the local system $K|_{\Delta^*}$. Note that their duals are respectively (B), (A), (C), (D), (E) (with α changed). We can prove the assertion, for instance, calculating the extension groups of simple regular holonomic $\mathcal{D}_0$ -modules in (A.3.6). The local invariant cycle property implies that indecomposable $\mathcal{D}_0$ -modules of type (A), (C), (D) (with $i = 0$), (E) are allowed, and the self-duality excludes the type (A). Here we use the short exact sequences

$$0 \rightarrow \mathcal{D}_0/\mathcal{D}_0 P \xrightarrow{\cdot Q} \mathcal{D}_0/\mathcal{D}_0 P Q \rightarrow \mathcal{D}_0/\mathcal{D}_0 Q \rightarrow 0.$$

This implies that $\mathcal{O}_{\Delta,0}$ is a $\mathcal{D}_0$ -submodule of an indecomposable regular holonomic $\mathcal{D}_0$ -module M only for type (A), (C). (This classification argument is used in a detailed version of [65].)

(ii) The local invariant cycle theorem holds for a proper morphism of complex manifolds $f : X \rightarrow \Delta$ if there is an embedded resolution $\rho : (\tilde{X}, \tilde{X}_0) \rightarrow (X, X_0)$ such that $\tilde{X}_0$ is a divisor with simple normal crossings (not necessarily reduced) and there is a cohomology class $\eta_c \in H^2(\tilde{X}_0, \mathbb{C})$ whose restriction to any irreducible component of $\tilde{X}_0$ is represented by a Kähler form. This follows for instance from the arguments in [66, 4.2.2 and 4.2.4] (see also arXiv:math/0006162). It is known that the argument in [76] is insufficient, see for instance [32] where the singular fiber is assumed reduced. (It does not seem very clear whether one can prove the semi-stable reduction theorem in the analytic case using the same argument as in the algebraic case.)

(iii) The reduction of the decomposition theorem using a base change is trivial if X is *smooth*. Indeed, assume there is a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi} & Y \\ \downarrow f & & \downarrow g \\ C & \xleftarrow{\pi'} & D \end{array}$$

where X, Y are connected complex manifolds, C, D are curves, f, g are proper surjective morphisms, π' is a finite morphism, and π is a proper and generically finite étale morphism. Then the canonical morphism $A_X \rightarrow \mathbf{R}\pi_* A_Y$ splits by composing it with its dual, using the self-duality of $A_X[d_X]$, $A_Y[d_Y]$ together with $\mathrm{Hom}_{D^b(X,A)}(A_X, A_X) = A$. Moreover, it is known that intersection complexes with local system coefficients are *stable* under the direct images by *finite* morphisms, see [8]. So the decomposition theorem for g implies that for f . (Here C can be singular.)

References

1. A'Campo, N.: La fonction zêta d'une monodromie. *Comment. Math. Helv.* **50**, 233–248 (1975)
2. Allcock, D., Carlson, J.A. and Toledo, D.: The moduli space of cubic threefolds as a ball quotient. *Mem. Am. Math. Soc.* **209**(985), xii+70 (2011)
3. Abramovich, D., Karu, K.: Weak semistable reduction in characteristic 0. *Invent. Math.* **139**(2), 241–273 (2000)
4. Alexeev, V.: Log Canonical Singularities and Complete Moduli of Stable Pairs (1996). [arXiv:alg-geom/9608013](https://arxiv.org/abs/alg-geom/9608013)
5. Adiprasito, K., Liu, G. and Temkin, M.: Semistable Reduction in Characteristic 0. *Sém. Lothar. Combin.* **82B**, Art. 25, 10 (2020)
6. Ash, A., Mumford, D., Rapoport, M., and Tai, Y.S.: *Smooth Compactifications of Locally Symmetric Varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge (2010) (With the collaboration of Peter Scholze)
7. Arapura, D.: The Leray spectral sequence is motivic. *Invent. Math.* **160**(3), 567–589 (2005)
8. Beilinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. *Astérisque* **100**(2), 5–171 (1982)
9. Brosnan, P., Chow, T.Y.: Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Adv. Math.* **329**, 955–1001 (2018)
10. Brosnan, P., Fang, H., Nie, Z., and Pearlstein, G.: Singularities of admissible normal functions, *Invent. Math.* **177**(3), 599–629 (2009). With an appendix by Najmuddin Fakhruddin

11. Bredon, G.E.: Introduction to Compact Transformation Groups, Pure and Applied Mathematics, Vol. 46. Academic Press, New York (1972)
12. Brosnan, P.: Perverse obstructions to flat regular compactifications. *Math. Z.* **290**(1–2), 103–110 (2018)
13. Cattani, E., Kaplan, A., Schmid, W.: L^2 and intersection cohomologies for a polarizable variation of Hodge structure. *Invent. Math.* **87**(2), 217–252 (1987)
14. Clemens, C.H.: Degeneration of Kähler manifolds. *Duke Math. J.* **44**(2), 215–290 (1977)
15. Casalaina-Martin, S., Grushevsky, S., Hulek, K. and Laza, R.: Complete moduli of cubic threefolds and their intermediate Jacobians, *Proc. Lond. Math. Soc.* (3) **122**(2), 259–316 (2021)
16. del Ángel R.P., Doran, C., Kerr, M., Lewis, J., Iyer, J., Müller-Stach, S., and Patel, D.: Specialization of cycles and the K -theory elevator. *Commun. Number Theory Phys.* **13**(2), 299–349 (2019)
17. de Cataldo, M.A.: Decomposition theorem for semisimples. *J. Singul.* **14**, 194–197 (2016)
18. de Cataldo, M.A., and Migliorini, L.: The Hodge theory of algebraic maps. *Ann. Sci. École Norm. Sup.* (4), **38**(5), 693–750 (2005)
19. de Cataldo, M.A. and Migliorini, L.: Hodge-theoretic aspects of the decomposition theorem, *Algebraic geometry—Seattle 2005. Part 2, Proceedings of the Symposium on Pure Math.*, vol. 80, Amer. Math. Soc., Providence, RI, pp. 489–504 (2009). MR 2483945
20. de Cataldo, M.A. and Migliorini, L.: The perverse filtration and the Lefschetz hyperplane theorem. *Ann. of Math.* (2) **171**(3), 2089–2113 (2010)
21. Deligne, P.: Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Inst. Hautes Études Sci. Publ. Math.* No. 35, 259–278 (1968)
22. Deligne, P.: Equations différentielles à points singuliers réguliers, *Lect. Notes in Math.* 163, Springer, Berlin (1970)
23. Deligne, P.: Le formalisme des cycles évanescents, *Lect. Notes in Math.*, vol. 340 (SGA7 II), pp. 82–115. Springer, Berlin (1973)
24. Deligne, P.: Décompositions dans la catégorie dérivée, *Motives. Proc. Sympos. Pure Math.*, 55, Part 1, Amer. Math. Soc. Providence, RI, Seattle, WA, vol. 1994, 115–128 (1991)
25. Denef, J., Loeser, F.: Motivic Igusa zeta functions. *J. Algebraic Geom.* **7**(3), 505–537 (1998)
26. Dimca, A., Saito, M.: Vanishing cycle sheaves of one-parameter smoothings and quasi-semistable degenerations. *J. Alg. Geom.* **21**, 247–271 (2012)
27. Dimca, A., and Saito, M.: Some remarks on limit mixed Hodge structures and spectrum. *An. Ştiinţ. Univ. “Ovidius” Constanţa Ser. Mat.* **22**(2), 69–78 (2014)
28. da Silva Jr., G.: On the Limiting Behavior of Variations of Hodge Structures. Ph.D. thesis, Washington University in St. Louis (2016)
29. Forster, O.: *Lectures on Riemann Surfaces.* Springer, Heidelberg (1977)
30. Green, M., Griffiths, P., Kerr, M.: Some enumerative global properties of variations of Hodge structure. *Moscow Math. J.* **9**, 469–530 (2009)
31. Grimm, T., and Li, C.: The 2-cube $(III_0|IV_2|III_0)$, private communication (2018)
32. Guillén, F., and Navarro Aznar, V.: Sur le théorème local des cycles invariants. *Duke Math. J.* **61**(1), 133–155 (1990)
33. Hassett, B.: Local stable reduction of plane curve singularities. *J. Reine Angew. Math.* **520**, 169–194 (2000)
34. Huybrechts, D.: A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky], *Astérisque* (2012), no. 348, Exp. No. 1040, x, 375–403, *Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042*
35. Jung, S.-J., Kim, I.-K., Yoon, Y., Saito, M.: Hodge Ideals and Spectrum of Isolated Hypersurface Singularities, to appear in *Ann. Inst. Fourier* (2019). [arXiv:1904.02453](https://arxiv.org/abs/1904.02453)
36. Karu, K.: Minimal models and boundedness of stable varieties. *J. Algebraic Geom.* **9**(1), 93–109 (2000)
37. Katz, N.: Elliptic convolution, G_2 , and elliptic surfaces. *Astérisque* **370**, 197–205 (2015)
38. Kashiwara, M., Kawai, T.: The Poincaré lemma for variations of polarized Hodge structure. *Publ. Res. Inst. Math. Sci.* **23**(2), 345–407 (1987)
39. Kulikov, V.S., Kuranov, P.F.: Complex Algebraic Varieties: Periods of Integrals and Hodge Structures, *Algebraic Geometry, III, Encyclopaedia Math. Sci.*, vol. 36, pp. 1–217, 263–270. Springer, Berlin (1998)
40. Kollár, J., Kovács, S.J.: Log canonical singularities are Du Bois. *J. Am. Math. Soc.* **23**(3), 791–813 (2010)
41. Kerr, M., Laza, R.: Hodge Theory of Degenerations, (II): Vanishing Cohomology and Geometric Applications (2020). [arXiv:2006.03953](https://arxiv.org/abs/2006.03953)

42. Kerr, M., Laza, R., Saito, M.: Deformation of Rational Singularities and Hodge Structure (2019). [arXiv:1906.03917](https://arxiv.org/abs/1906.03917)
43. Kollár, J., Laza, R., Saccà, G., Voisin, C.: Remarks on degenerations of hyper-Kähler manifolds. *Ann. Inst. Fourier (Grenoble)* **68**(7), 2837–2882 (2018)
44. Kollár, J., Mori, S.: *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge: With the collaboration of C. H. Clemens and A. Corti (1998)
45. Kollár, J.: *Singularities of the Minimal Model Program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge (2013). With a collaboration of Sándor Kovács
46. Kollár, J.: *Families of Varieties of General Type*, book in preparation (2018)
47. Kovács, S.J.: Rational, log canonical. Du Bois singularities: on the conjectures of Kollár and Steenbrink. *Compositio Math.* **118**(2), 123–133 (1999)
48. Kerr, M., Pearlstein, G.: An Exponential History of Functions with Logarithmic Growth, *Topology of Stratified Spaces*, Math. Sci. Res. Inst. Publ., vol. 58, pp. 281–374. Cambridge Univ. Press, Cambridge (2011)
49. Kerr, M., Pearlstein, G., Robles, C.: Polarized relations on horizontal $SL(2)$ ’s. *Doc. Math.* **24**, 1295–1360 (2019)
50. Kollár, J., Shepherd-Barron, N.I.: Threefolds and deformations of surface singularities. *Invent. Math.* **91**(2), 299–338 (1988)
51. Kulikov, V.S.: *Mixed Hodge Structures and Singularities*, Cambridge Tracts in Mathematics, vol. 132, Cambridge University Press, Cambridge (1998)
52. Laza, R.: The moduli space of cubic fourfolds via the period map, *Ann. Math. (2)* **172**(1), 673–711 (2010)
53. Laza, R.: The KSBA compactification for the moduli space of degree two $K3$ pairs. *J. Eur. Math. Soc. (JEMS)* **18**(2), 225–279 (2016)
54. Laza, R., O’Grady, K.: GIT versus Baily-Borel compactification for quartic $K3$ surfaces, *Geometry of Moduli*, Abel Symp., vol. 14, pp. 217–283. Springer, Cham (2018)
55. Laza, R., O’Grady, K.: Birational geometry of the moduli space of quartic $K3$ surfaces. *Compos. Math.* **155**(9), 1655–1710 (2019)
56. Laza, R., O’Grady, K.: GIT versus Baily-Borel compactification for $K3$ ’s which are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$. *Adv. Math.* **383**, 107680 (2021)
57. Looijenga, E.: Compactifications defined by arrangements. II. Locally symmetric varieties of type IV, *Duke Math. J.* **119**(3), 527–588 (2003)
58. Looijenga, E., Swierstra, R.: The period map for cubic threefolds. *Compos. Math.* **143**(4), 1037–1049 (2007)
59. Laza, R., Saccà, G., Voisin, C.: A hyper-Kähler compactification of the intermediate Jacobian fibration associated with a cubic 4-fold. *Acta Math.* **218**(1), 55–135 (2017)
60. Massey, D.: Notes on Perverse Sheaves and Vanishing Cycles (2016). [arXiv:math/9908107v7](https://arxiv.org/abs/math/9908107v7)
61. Mumford, D., Fogarty, J., Kirwan, F.: *Geometric Invariant Theory*, third ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*, vol. 34, Springer, Berlin (1994)
62. Milnor, J.: *Singular Points of Complex Hypersurfaces*, *Annals of Mathematics Studies*, No. 61, Princeton University Press, Princeton; University of Tokyo Press, Tokyo (1968)
63. Mircea, M., Mihnea, P.: Hodge ideals. *Mem. Am. Math. Soc.* **262**(1268), v+80 (2019)
64. Peters, C.A.M., Steenbrink, J.H.M.: *Mixed Hodge Structures*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.*, vol. 52, Springer, Berlin (2008)
65. Saito, M.: Hodge filtrations on Gauss-Manin systems II. *Proc. Jpn. Acad. Ser. A Math. Sci.* **59**, 37–40 (1983)
66. Saito, M.: Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.* **24**(6), 849–995 (1988)
67. Saito, M.: Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.* **26**, 221–333 (1990)
68. Saito, M.: Decomposition Theorem for Proper Kähler Morphisms, *Tohoku Math. J. (2)* **42**(2), 127–147 (1990)
69. Saito, M.: On b -function, spectrum and rational singularity. *Math. Ann.* **295**(1), 51–74 (1993)
70. Saito, M.: Letter to the Authors (2019)
71. Schwede, K.: A simple characterization of Du Bois singularities. *Compos. Math.* **143**(4), 813–828 (2007)
72. Schnell, C.: An Overview of Miroshiko Saito’s Theory of Mixed Hodge Modules (2014). [arXiv:1405.3096](https://arxiv.org/abs/1405.3096)

73. Shah, J.: Insignificant limit singularities of surfaces and their mixed Hodge structure, *Ann. Math.* (2) **109**(3), 497–536 (1979)
74. Shah, J.: A complete moduli space for $K3$ surfaces of degree 2, *Ann. Math.* (2) **112**(3), 485–510 (1980)
75. Shah, J.: Degenerations of $K3$ surfaces of degree 4. *Trans. Am. Math. Soc.* **263**(2), 271–308 (1981)
76. Steenbrink, J.H.M.: Limits of Hodge structures. *Inv. Math.* **31**, 229–257 (1976)
77. Steenbrink, J.H.M.: Hodge, Mixed, structure on the vanishing cohomology, Real and complex singularities (Proc. Ninth Nordic Summer School, NAVF Sympos. Math., Oslo, : Sijthoff and Noordhoff. Alphen aan den Rijn, vol. **1977**, pp. 525–563 (1976)
78. Steenbrink, J.H.M.: Cohomologically Insignificant Degenerations, *Compositio Math.* **42**(3), 315–320 (1980/81)
79. Verdier, J.-L.: Des catégories dérivées des catégories abéliennes. *Astérisque* **239** (1996)
80. Voisin, C.: Théorème de Torelli pour les cubiques de P^5 . *Invent. Math.* **86**(3), 577–601 (1986)
81. Wells Jr., R.O.: Comparison of de Rham and Dolbeault cohomology for proper surjective mappings. *Pacific J. Math.* **53**, 281–300 (1974)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.