Linear-Convex Optimal Steady-State Control

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Abstract—We consider the problem of designing a feedback
controller for a multivariable linear time-invariant system which
regulates an arbitrary system output to the solution of an equality-
constrained convex optimization problem despite unknown con-
stant exogenous disturbances; we term this the linear-convex
optimal steady-state (OSS) control problem. We introduce the
notion of an optimality model, and show that the existence of an
optimality model is sufficient to reduce the OSS control problem
to a stabilization problem. This yields a constructive design
framework for optimal steady-state control that unifies and
extends existing design methods in the literature. We illustrate
the approach via an application to optimal frequency control
of power networks, where our methodology recovers centralized
and distributed controllers reported in the recent literature.

Index Terms—Reference tracking and disturbance rejection,
output regulation, convex optimization, online optimization

I. INTRODUCTION

Many engineering systems are required to operate at an
"optimal" steady-state defined by the solution of a constrained
optimization problem that seeks to minimize operational costs
while satisfying operational constraints. Consider, for example,
the problem of optimizing the production setpoints of gener-
ators in an electric power system while maintaining supply-
demand balance and system stability. The current approach
involves a time-scale separation between the optimization and
control objectives: optimal generation setpoints are computed
offline using demand projections and a model of the network,
then the operating points are dispatched as reference commands
to local controllers at each generation site [1]. This process is
repeated with a fixed update rate: a new optimizer is computed,
dispatched, and tracked. If the model is precise and the supply
and demand of power change on a time scale that is slow
compared to the update rate, then this method is adequate.

If however the model is poor, and the optimizer changes
rapidly (as is the case for power networks with a high penetra-
tion of renewable energy sources) the conventional approach
can be inefficient [2]; costs are increased as a result of operating
sub-optimally. It would then be advantageous to (i) enhance
robustness by incorporating feedback, and (ii) eliminate the
time-scale separation, by combining the local controllers with

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a feedback-based online optimization algorithm, so that the
optimal operating condition could be tracked in real time; see
[3]–[12].for power system control approaches in this direction.

The same theme of real-time regulation of system variables
to optimal values emerges in diverse areas. Fields of appli-
cation besides the power network control example mentioned
already include network congestion management [13], [14],
chemical processing [15], wind turbine power capture [16], and
temperature regulation in energy-efficient buildings [17]. The
breadth of applications motivates the need for a *general* theory
and design procedure for controllers that regulate a plant to a
maximally efficient operating point defined by an optimization
problem, even as the optimizer changes over time due to
changing market prices, disturbances to the plant dynamics,
and operating constraints that depend on exogenous variables.
We refer to the problem of designing such a controller as the
optimal steady-state (OSS) control problem.

A number of recent publications have formulated problem
statements and solutions for variants of the OSS control
problem [18]–[25]. Many of the currently-proposed controllers,
however, have limited applicability: some solutions only apply
to systems of a special form [22]; some require asymptotic
stability of the uncontrolled plant [24], [25]; some attempt to
optimize only the steady-state input [20] or output [19], [23],
[26], [27] alone; some apply only to equality-constrained [25]
or unconstrained optimization problems [28].

Broadly speaking, these design methodologies consist of
modifying an off-the-shelf optimization algorithm to accept
system measurements; the algorithm then produces a converg-
ing estimate of the optimal steady-state control input, yielding a
feedback controller. This procedure, while modular, unnecessar-
ily restricts the design space of dynamic controllers. Moreover,
none of the reported approaches adequately considers the
impact of the system model on the achievable optimal operating
points. Our goal in this paper is to present a framework which
widens the design space of optimal steady-state controllers
while fully incorporating the system model into the steady-
state optimization problem.

A. Contributions

We consider the *linear-convex OSS control problem*, in which
the plant is a finite-dimensional linear time-invariant (LTI) state-
space system, the optimization problem has a convex cost
function and affine equality constraints, and the disturbances
are constant in time. This paper has three contributions. First,
we introduce the notion of an optimality model, a dynamic filter
which reduces the OSS control problem to a stabilization prob-
lem, and provide several explicit constructions of optimality
models. Second, we prove that for any of our optimality models,

the existence of a stabilizing controller is guaranteed under mild assumptions when the objective function of the optimization problem is quadratic. Third and finally, we apply our results to a problem from power systems control, and show that our general methodology is flexible enough to recover centralized and distributed controllers from the recent literature.

B. Notation

The symbol $\bullet$ in $\mathbb{R}^{\bullet \times \bullet}$ indicates that the dimension is unspecified. For a class C^1 map $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ denotes its gradient. When the arguments of a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are separated by a semicolon, $\nabla f(x; y)$ refers to the gradient of f with respect to its first argument, evaluated at (x, y). The symbol $\mathbb{0}$ denotes a matrix or vector of zeros whose dimensions can be inferred from context. The symbol $\mathbb{1}_n$ denotes the n-vector of all ones. For scalars or column vectors $\{v_1, v_2, \ldots, v_k\}$, $\mathrm{col}(v_1, v_2, \ldots, v_k)$ is a column vector obtained by vertical concatenation of $v_1, \ldots, v_k$. For vectors α and β, the notation $\alpha \geq \beta$ indicates that every entry of α is greater than or equal to the corresponding entry of β. For symmetric matrices A and B, $A \succ B$ means $A - B$ is positive definite, while $A \succeq B$ means $A - B$ is positive semidefinite.

II. PROBLEM STATEMENT

In the linear-convex optimal steady-state control problem, our objective is to design a feedback controller for a linear time-invariant plant so that a specified output is asymptotically driven to a cost-minimizing steady-state, determined by the solution of a convex optimization problem. In contrast to a standard static optimization problems, we must contend with closed-loop stability in addition to optimizing a set of decision variables. The plant is a linear time-invariant system subject to an unknown constant disturbance $w \in \mathbb{R}^{n_w}$

$$
\begin{aligned}
\dot{x} &= Ax + Bu + B_w w, & x(0) &\in \mathbb{R}^n, \\
y &= Cx + Du + Qw, & & \quad\quad (1) \\
y_{\mathrm{m}} &= h_{\mathrm{m}}(x, u, w).
\end{aligned}
$$

For reasons that will become clear (see Assumption 3.3), the measurements y_{m} are permitted to be general nonlinear functions of state, input, and disturbance. The vector $y \in \mathbb{R}^p$ is the *optimization output*, containing states, tracking errors, and control inputs that should be driven to cost-minimizing values in equilibrium.

We will explicitly enforce that the optimization of y is consistent with steady-state operation of the plant. Let $\overline{Y}(w)$ be the set of optimization outputs achievable from a forced equilibrium of (1):

$$
\begin{aligned}
\overline{Y}(w) := \{ \bar{y} \in \mathbb{R}^p \mid & \text{ there exists an } (\bar{x}, \bar{u}) \text{ such that} \\
& \mathbb{0} = A\bar{x} + B\bar{u} + B_w w \quad\quad (2) \\
& \bar{y} = C\bar{x} + D\bar{u} + Qw \}.
\end{aligned}
$$

We rewrite $\overline{Y}(w)$ in algebraic form so that we may include membership in $\overline{Y}(w)$ as a constraint of the optimization problem in standard equality form. For each w, the set $\overline{Y}(w)$ is an affine subspace of $\mathbb{R}^p$. It may therefore be written as the sum of a (non-unique) "offset vector" and a unique subspace,

which we denote by $\mathbf{sub}(\overline{Y}(w))$. In the following lemma, we construct a matrix G whose columns span this unique subspace.

*Lemma 2.1 (**Construction of G**):* Fix a $\tilde{y}(w) \in \overline{Y}(w)$. If $\mathcal{N} \in \mathbb{R}^{(n+m) \times \bullet}$ is a matrix such that

$$
\mathbf{range}\, \mathcal{N} = \mathbf{null} \begin{bmatrix} A & B \end{bmatrix},
$$

then the columns of the matrix

$$
G := \begin{bmatrix} C & D \end{bmatrix} \mathcal{N} \in \mathbb{R}^{p \times \bullet} \quad\quad (3)
$$

span the subspace $\mathbf{sub}(\overline{Y}(w))$. $\triangle$

The proof is straightforward and is omitted. Note that when A is invertible, one may select

$$
\mathcal{N} := \begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix}
$$

which yields $G = -CA^{-1}B + D$. This is precisely the DC gain matrix of the $u \to y$ channel for the plant (1). One may think of G in (3) as a generalization of this, which one can compute regardless of whether or not A is invertible.

From Lemma 2.1 it follows that

$$
\bar{y} \in \overline{Y}(w) \iff \text{ there exists } v \in \mathbb{R}^\bullet \text{ s.t. } \bar{y} = \tilde{y}(w) + Gv \quad (4)
$$

Now let $G_\perp \in \mathbb{R}^{\bullet \times p}$ be any full-row-rank matrix satisfying $\mathbf{null}\, G_\perp = \mathbf{range}\, G$. Then from (4), one finds that

$$
\overline{Y}(w) = \{ \bar{y} \in \mathbb{R}^p \mid G_\perp \bar{y} = b(w) \}. \quad\quad (5)
$$

where $b(w) := G_\perp \tilde{y}(w)$. We will see shortly that, for our controller design, the matrix $G_\perp$ is important and the vector $b(w)$ is unimportant.

We can now formulate an optimization problem to determine the desired optimal point for $\bar{y}$ as

$$
\begin{aligned}
&\underset{\bar{y} \in \mathbb{R}^p}{\text{minimize}} && f(\bar{y}; w) && \quad (6a) \\
&\text{subject to} && G_\perp \bar{y} = b(w) && \quad (6b) \\
&&& H\bar{y} = Lw. && \quad (6c)
\end{aligned}
$$

The cost f in (6) is our steady-state performance criterion; we assume f is differentiable and convex in $\bar{y}$ for each w. The constraint (6b) is the equilibrium constraint just discussed. The constraints (6c) represent n_{ec} engineering equality constraints determined by the matrices $H \in \mathbb{R}^{n_{\mathrm{ec}} \times p}$ and $L \in \mathbb{R}^{n_{\mathrm{ec}} \times n_w}$. We assume that for every w, the problem (6) has a unique optimizer $\bar{y}^\star$, and a feasible region with non-empty relative interior.

A general nonlinear feedback controller for (1) is given by

$$
\begin{aligned}
\dot{x}_{\mathrm{c}} &= f_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}}), & x_{\mathrm{c}}(0) &\in \mathbb{R}^{n_{\mathrm{c}}}, \\
u &= h_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}}). & & \quad\quad (7)
\end{aligned}
$$

The function f_{c} is assumed to be locally Lipschitz in x_{c} and continuous in y_{m}, while h_{c} is assumed to be continuous. The dynamics of the closed-loop system consist of (1) and (7).

Our objective in linear-convex OSS control (for brevity, we will omit "linear-convex" in the sequel) is to drive the optimization output y of the plant (1) to the solution $\bar{y}^\star(w)$ of the convex optimization problem (6) using a feedback controller while ensuring well-posedness and stability of the closed-loop system. The formal statement is as follows. For

a given w, the closed-loop system is said to be *well-posed* if the control input u is uniquely defined for any choice of $(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$, i.e., the equation $u = h_c(x_c, h_m(x, u, w))$ is uniquely solvable in u.

*Problem 2.2 (*OSS Control*):* For the plant (1), design, if possible, a dynamic feedback controller of the form (7) such that for every w:

  (i) the closed-loop system is well-posed;
 (ii) the closed-loop system possesses a globally asymptotically stable equilibrium point;
(iii) for every initial condition of the closed-loop system, $\lim\limits_{t \to \infty} y(t) = \bar{y}^\star(w)$. $\triangle$

*Remark 2.3 (*Constant Disturbances*):* We assume throughout that the unmeasured disturbances w are constant, which will lead us to incorporate integral action into our controllers; this is by far the most important case in practice. In reality of course, disturbances (and hence, the optimal operating point) will change in a non-stepwise fashion over time, and the quality of tracking will depend on the rate of variation of the disturbance and on the closed-loop bandwidth. For example, if $\dot{w}$ is bounded, the integral-type controllers we develop will track the optimal operating point with bounded error (see, e.g., [25]). This is acceptable in practice, and we defer more detailed disturbance models to future studies. $\triangle$

*Remark 2.4 (*Relation to Optimal Control*):* The OSS control problem appears similar to an infinite-horizon optimal tracking control problem; however, the two are distinct in both their assumptions and demands. In the latter, one minimizes a cost functional over system trajectories leading to a HJB equation; determining the optimal feedback policy is computationally expensive and the policy will require state and disturbance measurements. The OSS control problem is much less demanding; we ask only for optimal behaviour asymptotically, not optimal trajectories. As a result, we encounter no computational bottlenecks, and do not need to assume the full plant state and all disturbances are measurable. $\triangle$

*Remark 2.5 (*Relation to Extremum-Seeking Control*):* The OSS control problem is similar to the online optimization problems considered in the extremum-seeking control literature, e.g., [29], [30]. Extremum seeking is a model-free optimization method, which introduces a sinusoidal probing signal into the system to estimate the static relationship between the control inputs and the desired performance index. In contrast, OSS control explicitly incorporates the equilibrium model of the plant into the optimization problem for this same purpose, and as such does not require the introduction of a probing signal. $\triangle$

Under the assumptions on the optimization problem (6), the *Karush-Kuhn-Tucker (KKT) conditions* are necessary and sufficient for optimality [31, Sections 5.2.3 and 5.5.3]. For each w, the optimal solution $\bar{y}^\star \in \mathbb{R}^p$ is characterized as the unique vector such that $\bar{y}^\star$ is feasible for (6) and there exist $\lambda^\star \in \mathbb{R}^r, \mu^\star \in \mathbb{R}^{n_{ec}}$ such that $(\bar{y}^\star, \lambda^\star, \mu^\star)$ satisfies the *gradient condition*

$$0 = \nabla f(\bar{y}^\star; w) + G_\perp^\mathsf{T} \lambda^\star + H^\mathsf{T} \mu^\star. \tag{8}$$

III. OSS CONTROLLER DESIGN FRAMEWORK

The main difficulty in solving the OSS control problem is that the optimizer $\bar{y}^\star(w)$ is unknown, and thus the optimality error $y - \bar{y}^\star(w)$ cannot be directly computed. In our design framework, we propose using a dynamic filter called an *optimality model* to convert the OSS control problem to a related *output regulation* problem. One then solves this output regulation problem using an integral controller and a stabilizing controller. An optimality model therefore reduces the OSS control problem to a stabilization problem. For background on output regulation and integral controllers, see [32] and [33, Section 12.3].

A. Optimality Models and Reduction to Stabilization Problem

An optimality model is a filter applied to the measured output y_m of the plant that produces a signal ϵ which acts as a *proxy* for the optimality error $y - \bar{y}^\star(w)$. To make this idea precise, consider a filter (φ, h_ϵ) with state $\xi \in \mathbb{R}^{n_\xi}$, input y_m, output $\epsilon \in \mathbb{R}^{n_\epsilon}$, and dynamics

$$\dot{\xi} = \varphi(\xi, y_m), \quad \epsilon = h_\epsilon(\xi, y_m). \tag{9}$$

*Definition 3.1 (*Optimality Model*):* The filter (9) is said to be an *optimality model* (for the OSS control problem, Problem 2.2) if the following implication holds: if the triple $(\bar{x}, \bar{\xi}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^m$ satisfies

$$\begin{aligned} 0 &= A\bar{x} + B\bar{u} + B_w w \\ 0 &= \varphi(\bar{\xi}, h_m(\bar{x}, \bar{u}, w)) \\ 0 &= h_\epsilon(\bar{\xi}, h_m(\bar{x}, \bar{u}, w)) \end{aligned} \tag{10}$$

then the pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfies

$$\bar{y}^\star(w) = C\bar{x} + D\bar{u} + Qw.$$

In the OSS control framework, the optimality model is cascaded with the plant, and we then attempt to solve the (constant disturbance) output regulation problem with ϵ as the (measurable) error signal. This converts the OSS control problem to stabilization of the *augmented plant*

$$\begin{aligned} \dot{x} &= Ax + Bu + B_w w, & \text{(11a)} \\ \dot{\xi} &= \varphi(\xi, h_m(x, u, w)), & \text{(11b)} \\ \dot{\eta} &= \epsilon := h_\epsilon(\xi, h_m(x, u, w)) & \text{(11c)} \end{aligned}$$

using a *stabilizer*

$$\begin{aligned} \dot{x}_s &= f_s(x_s, \eta, \xi, y_m, \epsilon), & \text{(12a)} \\ u &= h_s(x_s, \eta, \xi, y_m, \epsilon). & \text{(12b)} \end{aligned}$$

This design framework (Figure 1) is justified by the following theorem, a proof of which may be found in the appendix.

*Theorem 3.2 (*Reduction of OSS to Stabilization*):* Suppose that (φ, h_ϵ) is an optimality model. If the stabilizer (f_s, h_s) is designed such that the closed-loop system (11)–(12) is well-posed and possesses a globally asymptotically stable equilibrium point for every w, then the controller (11b), (11c), (12a), (12b) solves the OSS control problem. $\triangle$

Solving the OSS control problem therefore amounts to (i) designing an optimality model and (ii) designing (if possible) a stabilizer for the augmented plant.
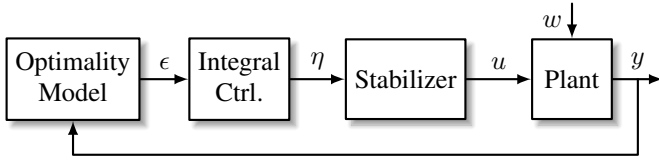
Fig. 1: Block diagram of OSS control architecture.

B. Optimality Model Design

1) The Gradient Condition: According to Definition 3.1, an optimality model encodes sufficient conditions for optimality when it is in equilibrium with the plant and its output ϵ is held at zero. We can incorporate the KKT conditions — which are sufficient for optimality under our assumptions — into an optimality model for this purpose.

Note that the gradient condition (8) involves the dual variable $\lambda^\star$ associated with the equilibrium constraints. Dual variables associated with equality constraints are typically calculated using an integrator on the constraint violation: see [19, Equation (4a)], or [22, Equation (8f)], for example. Unfortunately integrating the equilibrium constraint violation $G_\perp y - b(w)$ is impossible, since w is unknown. Luckily, doing so is also unnecessary, since the constraint $G_\perp y = b(w)$ is satisfied at any forced equilibrium point of the physical plant, by the definition of $G_\perp$ and $b(w)$; recall (2) and (5). We now describe how to incorporate the gradient condition (8) into an optimality model without calculating $\lambda^\star$.

Let G be the matrix of Lemma 2.1. Recall that $\mathbf{null}\, G_\perp = \mathbf{range}\, G$; by taking the orthogonal complement of both sides, it follows that $\mathbf{range}\, G_\perp^\mathsf{T} = \mathbf{null}\, G^\mathsf{T}$. Rearranging (8) to read $G_\perp^\mathsf{T}(-\lambda^\star) = \nabla f(\bar{y}^\star; w) + H^\mathsf{T}\mu^\star$, we see that $\nabla f(\bar{y}^\star; w) + H^\mathsf{T}\mu^\star \in \mathbf{range}\, G_\perp^\mathsf{T} = \mathbf{null}\, G^\mathsf{T}$. That is, the existence of a triple $(\bar{y}^\star, \lambda^\star, \mu^\star)$ satisfying (8) is equivalent to the existence of a pair $(\bar{y}^\star, \mu^\star)$ satisfying

$$G^\mathsf{T}\left(\nabla f(\bar{y}^\star; w) + H^\mathsf{T}\mu^\star\right) = \mathbb{0}. \tag{13}$$

The left-hand side of this equation is a natural choice for inclusion in the proxy error signal ϵ, since driving ϵ to zero will then enforce the gradient KKT condition.

Note however that there are other equivalent ways we could rewrite the gradient condition. Define a matrix T such that

$$\mathbf{range}\, T = \mathbf{null}\begin{bmatrix} G_\perp \\ H \end{bmatrix}. \tag{14}$$

We have $\mathbf{null}\, T^\mathsf{T} = (\mathbf{range}\, T)^\perp$, which means

$$\mathbf{null}\, T^\mathsf{T} = \left(\mathbf{null}\begin{bmatrix} G_\perp \\ H \end{bmatrix}\right)^\perp = \mathbf{range}\begin{bmatrix} G_\perp^\mathsf{T} & H^\mathsf{T} \end{bmatrix}.$$

Rearrange (8) to read $G_\perp^\mathsf{T}(-\lambda^\star) + H^\mathsf{T}(-\mu^\star) = \nabla f(\bar{y}^\star, w)$, and we see that $\nabla f(\bar{y}^\star, w) \in \mathbf{range}\begin{bmatrix} G_\perp^\mathsf{T} & H^\mathsf{T} \end{bmatrix} = \mathbf{null}\, T^\mathsf{T}$. Therefore, the existence of a triple $(\bar{y}^\star, \lambda^\star, \mu^\star)$ satisfying (8) is equivalent to the existence of a $\bar{y}^\star$ satisfying

$$T^\mathsf{T}\nabla f(\bar{y}^\star; w) = \mathbb{0}. \tag{15}$$

This procedure can also be generalized by including only some rows of H in the construction of T, leading to a hybrid between

(13) and (15); the details are omitted. As we did with (13), we can make the expression on the left-hand side of (15) one of the components of an optimality model's error output.

2) Optimality Models: We are now ready to construct optimality models for OSS control. We shall require our measurement vector to contain some key information about the optimization problem (6).

*Assumption 3.3 (***Measurement Assumptions***):* The measurement vector y_{m} contains the gradient $\nabla f(y, w)$ and the engineering equilibrium constraint violations, $Hy - Lw$. △

The following three propositions present the *output subspace*, *feasible subspace*, and *reduced-error feasible subspace* optimality models. Proving that these filters are indeed optimality models is done by examining the closed-loop equilibria and showing that the resulting equations are equivalent to the KKT conditions.

*Proposition 3.4 (***Output Subspace Optimality Model (OS-OM)***):* Let G be the matrix of Lemma 2.1. The dynamic filter

$$\begin{aligned} \dot{\mu} &= Hy - Lw \\ \epsilon &= G^\mathsf{T}\left(\nabla f(y; w) + H^\mathsf{T}\mu\right) \end{aligned} \tag{16}$$

is an optimality model for the OSS control problem. △

Proof: The proof is similar to the proof of Proposition 3.5. □

*Proposition 3.5 (***Feasible Subspace Optimality Model (FS-OM)***):* Let T be a matrix satisfying (14). The static filter

$$\epsilon = \begin{bmatrix} Hy - Lw \\ T^\mathsf{T}\nabla f(y; w) \end{bmatrix} \tag{17}$$

is an optimality model for the OSS control problem. △

Proof: See the appendix. □

In special circumstances, one can modify the FS-OM above to obtain an optimality model with an error signal of reduced dimension; this reduces the number of integrators required.

*Proposition 3.6 (***Reduced-Error FS-OM (REFS-OM)***):* Let G be the matrix of Lemma 2.1 and let T be a matrix satisfying (14). Then the static filter

$$\epsilon = Hy - Lw + T^\mathsf{T}\nabla f(y; w) \tag{18}$$

is an optimality model for the OSS control problem if $\mathbf{range}\, HG \cap \mathbf{range}\, T^\mathsf{T} = \{\mathbb{0}\}$. △

Proof: See the appendix. □

C. Quadratic Program OSS Control

We now consider the specific case when the optimization problem (6) is an equality-constrained convex quadratic program (QP). We term this variant of the problem QP-OSS control. Under this assumption, the closed-loop system becomes LTI, and we can obtain very explicit results on the existence of a stabilizer (Figure 1). Suppose the optimization problem (6) is of the form

$$\begin{aligned} \underset{\bar{y}\in\mathbb{R}^p}{\text{minimize}} \quad & \tfrac{1}{2}\bar{y}^\mathsf{T}\bar{M}\bar{y} - \bar{y}^\mathsf{T}Nw \\ \text{subject to} \quad & G_\perp\bar{y} = b(w) \\ & H\bar{y} = Lw, \end{aligned} \tag{19}$$

where $M \succeq \mathbb{0}$.[1] Assumption 3.3 in the present context implies that we can take the available measurements y_{m} as a linear function of (x, u, w), i.e., $y_{\mathrm{m}} = C_{\mathrm{m}}x + D_{\mathrm{m}}u + Q_{\mathrm{m}}w$.

Under mild assumptions below, we can ensure that the augmented plant (11) arising from the FS-OM, OS-OM, or REFS-OM is both stabilizable and detectable, which in turn guarantees that a solution of the OSS control problem exists and can be found using standard LTI design methods.

*Theorem 3.7 (**Solvability of QP-OSS Control**):* The QP-OSS control problem is solvable when

  (i) (C_{m}, A, B) is stabilizable and detectable,

 (ii) a unique primal solution to (19) exists, and

(iii) at least one of the following holds:

    a) a unique dual solution to (19) exists;

    b) $\mathbf{range}\,HG \cap \mathbf{range}\,T^{\mathsf{T}} = \{\mathbb{0}\}$ and $(\mathbf{range}\,HG)^{\perp} \cap (\mathbf{range}\,T^{\mathsf{T}})^{\perp} = \{\mathbb{0}\}$. $\triangle$

Proof: When (i), (ii), and (iii)(a) hold, one can show that the augmented plant arising from the FS-OM or OS-OM may be made stabilizable and detectable. When (i), (ii), and (iii)(b) hold, one can show that the augmented plant arising from the REFS-OM is stabilizable and detectable. See the appendix for details. $\square$

IV. CASE STUDY: OPTIMAL FREQUENCY REGULATION IN POWER SYSTEMS

This final section illustrates the application of our theory to a power system control problem. Our main objective is to work through the constructions presented in Section III, and to simultaneously illustrate the many sources of design flexibility within our proposed framework. In particular, we will show that centralized and distributed frequency controllers proposed in the literature are recoverable as special cases of our framework.

The dynamics of synchronous generators in a connected AC power network with n buses and n_{t} transmission lines is modelled in a reduced-network framework by the *swing equations*. The vectors of angular frequency (deviations from nominal) $\omega \in \mathbb{R}^n$ and real power flows $p \in \mathbb{R}^{n_{\mathrm{t}}}$ along the transmission lines obey the dynamic equations

$$\begin{aligned} M\dot{\omega} &= P^{\star} - D\omega - \mathcal{A}p + u \\ \dot{p} &= \mathcal{B}\mathcal{A}^{\mathsf{T}}\omega, \end{aligned} \qquad (20)$$

in which $M \succ \mathbb{0}$ is the (diagonal) inertia matrix, $D \succ \mathbb{0}$ is the (diagonal) damping matrix, $\mathcal{A} \in \{0, 1, -1\}^{n \times n_{\mathrm{t}}}$ is the signed node-edge incidence matrix of the network, $\mathcal{B} \succ \mathbb{0}$ is the diagonal matrix of transmission line susceptances, $P^{\star} \in \mathbb{R}^n$ is the vector of uncontrolled power injections (generation minus demand) at the buses, and $u \in \mathbb{R}^n$ is the controllable reserve power produced by the generators. The incidence matrix satisfies $\mathbf{null}\,\mathcal{A}^{\mathsf{T}} = \mathrm{span}(\mathbb{1}_n)$, and strictly for simplicity we assume that the network is acyclic, in which case $n_{\mathrm{t}} = n-1$ and $\mathbf{null}\,\mathcal{A} = \{\mathbb{0}\}$. We refer to [35, Section VII] for a first-principles derivation of this model, and remark that our

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calculations to follow extend without issue to more complex models which include turbine-governor dynamics.

We consider the *optimal frequency regulation problem* (OFRP), wherein we minimize the total cost $\sum_i J_i(\bar{u}_i)$ of steady-state reserve power production in the system subject to system equilibrium and zero steady-state frequency deviations:

$$\begin{aligned} \underset{\bar{u} \in \mathbb{R}^n, \bar{\omega} \in \mathbb{R}^n}{\mathrm{minimize}} \quad & J(\bar{u}) := \sum_{i=1}^{n} J_i(\bar{u}_i) \\ \text{subject to} \quad & G_{\perp}\,\mathrm{col}(\bar{u}, \bar{\omega}) = b(w) \\ & F\bar{\omega} = \mathbb{0}. \end{aligned} \qquad (21)$$

We shall compute the matrix $G_{\perp}$ of the equilibrium constraints shortly; the vector $b(w)$ is unimportant for controller design. The matrix F encodes the steady-state frequency constraint. We will specify the requirements on F later in this section.

With state vector $x := \mathrm{col}(\omega, p)$, the dynamics (20) can be put into the standard LTI form (1) with matrices

$$A := \begin{bmatrix} -M^{-1}D & -M^{-1}\mathcal{A} \\ \mathcal{B}\mathcal{A}^{\mathsf{T}} & \mathbb{0} \end{bmatrix}, \; B = B_w := \begin{bmatrix} M^{-1} \\ \mathbb{0} \end{bmatrix}.$$

We select the optimization output as $y := \mathrm{col}(u, \omega)$, so that

$$C := \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ I_n & \mathbb{0} \end{bmatrix} \quad D := \begin{bmatrix} I_n \\ \mathbb{0} \end{bmatrix}, \qquad (22)$$

and we take the measured output as $y_{\mathrm{m}} = \mathrm{col}(u, F\omega)$.

We will demonstrate the use of the feasible subspace and reduced-error feasible subspace optimality models of Propositions 3.5 and 3.6. We begin by constructing the matrix G of Lemma 2.1 and a matrix T satisfying (14). We first construct a matrix $\mathcal{N}$ satisfying $\mathbf{range}\,\mathcal{N} = \mathbf{null}\begin{bmatrix} A & B \end{bmatrix}$. One may verify that choosing

$$\mathcal{N} := \begin{bmatrix} \mathbb{1}_n & \mathbb{0} \\ \mathbb{0} & I_n \\ D\mathbb{1}_n & \mathcal{A} \end{bmatrix} \qquad (23)$$

yields the required property. Using (23) and (22), we calculate $G = \begin{bmatrix} C & D \end{bmatrix}\mathcal{N}$ to be

$$G = \begin{bmatrix} D\mathbb{1}_n & \mathcal{A} \\ \mathbb{1}_n & \mathbb{0} \end{bmatrix}.$$

Next, we construct a full-row-rank matrix $G_{\perp} \in \mathbb{R}^{n \times 2n}$ satisfying $\mathbf{null}\,G_{\perp} = \mathbf{range}\,G$. We find that selecting

$$G_{\perp} := \begin{bmatrix} \mathbb{1}_n\mathbb{1}_n^{\mathsf{T}} & -(\mathbb{1}_n^{\mathsf{T}}D\mathbb{1}_n)I_n \end{bmatrix}$$

yields the required property. We identify the matrix H of the engineering equality constraints in (6) for the problem (21) as $H := \begin{bmatrix} \mathbb{0} & F \end{bmatrix}$. Following (14), we select a matrix T satisfying

$$\mathbf{range}\,T = \mathbf{null}\begin{bmatrix} \mathbb{1}_n\mathbb{1}_n^{\mathsf{T}} & -(\mathbb{1}_n^{\mathsf{T}}D\mathbb{1}_n)I_n \\ \mathbb{0} & F \end{bmatrix}. \qquad (24)$$

The null space on the right-hand side of (24) is spanned by vectors of the form $\mathrm{col}(v, \mathbb{0})$ where $\mathbb{1}_n^{\mathsf{T}}v = 0$. Inspired by approaches in multi-agent control, we introduce a connected, weighted and directed communication graph $\mathcal{G}_{\mathrm{c}} = (\{1, \ldots, n\}, \mathcal{E}_{\mathrm{c}})$ between the buses, with associated *Laplacian matrix* $L_{\mathrm{c}} \in \mathbb{R}^{n \times n}$. We assume the directed graph $\mathcal{G}_{\mathrm{c}}$ contains a globally reachable node.[2] Under this assumption, we have

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that $\mathbf{rank}(L_\mathrm{c}) = n - 1$ with $\mathbf{null}(L_\mathrm{c})$ spanned by $\mathbb{1}_n$. It follows that (24) holds with $T = \begin{bmatrix} L_\mathrm{c}^\mathsf{T} \\ \mathbb{0} \end{bmatrix}$.

It further holds that L_c has a left null space of dimension one spanned by a nonnegative and non-zero vector $\mathsf{w} \in \mathbb{R}^n$. Assuming that F is selected such that $\mathsf{w}^\mathsf{T} F \mathbb{1}_n \neq \mathbb{0}$, the range condition of Proposition 3.6 is satisfied, and we may apply the REFS-OM (18) to obtain the optimality model

$$\epsilon = F\omega + L_\mathrm{c}\nabla J(u). \tag{25}$$

Therefore, one option for an OSS controller is

$$\dot{\eta} = F\omega + L_\mathrm{c}\nabla J(u) \tag{26a}$$
$$u = -K_\mathrm{p}\omega - K_\mathrm{i}\eta, \tag{26b}$$

where $K_\mathrm{p}, K_\mathrm{i}$ are gain matrices that should be selected for closed-loop stability/performance. With $F := I_n$, $K_\mathrm{p} = \mathbb{0}$ and $K_\mathrm{i} = \frac{1}{k}I_n$ for $k > 0$, this design reduces to the *distributed-averaging proportional-integral* (DAPI) frequency control scheme; see [5], [37], [38].

We can obtain several other control schemes by instead applying the FS-OM as our optimality model. Let $F := \mathsf{c}^\mathsf{T}$, where c is a vector of convex combination coefficients satisfying $\mathsf{c}_i \geq 0$ and $\sum_{i=1}^n \mathsf{c}_i = 1$. Define $\widetilde{L}_\mathrm{c} \in \mathbb{R}^{(n-1)\times n}$ as the matrix obtained by eliminating the first row from L_c and set $T := \begin{bmatrix} \widetilde{L}_\mathrm{c}^\mathsf{T} \\ \mathbb{0} \end{bmatrix}$. This choice of T also satisfies (24). The FS-OM (17) yields the optimality model

$$\epsilon = \begin{bmatrix} \mathsf{c}^\mathsf{T}\omega \\ \widetilde{L}_\mathrm{c}\nabla J(u) \end{bmatrix}. \tag{27}$$

It follows that one option for an OSS controller is

$$\dot{\eta}_1 = \mathsf{c}^\mathsf{T}\omega \tag{28a}$$
$$\dot{\eta}_2 = \widetilde{L}_\mathrm{c}\nabla J(u) \tag{28b}$$
$$u = -K_\mathrm{p}\omega - K_1\eta_1 - K_2\eta_2. \tag{28c}$$

where again K_p, K_1, K_2 are gain matrices. The interpretation of this (novel) controller is that one agent collects frequency measurements and implements the integral control (28a), while the other agents average their marginal costs via (28b).

If the objective function J is a positive definite quadratic, one can use Theorem 3.7 to show that a solution to the present OSS control problem is guaranteed to exist. Specifically, for $F := I_n$, one uses Theorem 3.7 with condition (iii)b, and for $F := \mathsf{c}^\mathsf{T}$, one uses Theorem 3.7 with condition (iii)a. Moreover, the augmented plant defined by the use of either (26a) or (28a)-(28b) can be shown to be stabilizable and detectable using the proof of Theorem 3.7.

As a final example, we can recover the "gather-and-broadcast" scheme of [7] from the optimality model (27) as follows. Assume that each J_i is strictly convex, and retain the integral controller (28a). Next, using the fact that $\mathbf{null}\,\widetilde{L}_\mathrm{c} = \mathrm{span}(\mathbb{1}_n)$, select the input u to zero the second component of ϵ:

$$\widetilde{L}_\mathrm{c}\nabla J(u) = \mathbb{0} \iff \nabla J(u) = \alpha\mathbb{1}_n \text{ for all } \alpha \in \mathbb{R}$$
$$\iff u = (\nabla J)^{-1}(\alpha\mathbb{1}_n), \text{ for all } \alpha \in \mathbb{R}.$$

Selecting $\alpha = \eta$ leads to the gather-and-broadcast controller

$$\dot{\eta} = \sum_{i=1}^n \mathsf{c}_i\omega_i, \quad u_i = (\nabla J_i)^{-1}(\eta). \tag{29}$$

In summary, several recent frequency control schemes, and the novel scheme (28), can be recovered as special cases of our general control framework. The full potential of our methodology for the design of improved power system control will be an area for future study.

V. Conclusions

We have studied in detail the linear-convex OSS control problem, wherein we design a controller to guide an LTI system to the solution of an optimization problem despite unknown, constant exogenous disturbances. We introduced the idea of an optimality model, the existence of which allows us to reduce the OSS control problem to a stabilization problem, and presented several candidate filters which under weak conditions are indeed optimality models. The flexibility of the OSS control framework was illustrated through a case study in power system control, showing that its application recovers several existing schemes in the literature.

Future work will present the analogous discrete-time and sampled-data OSS control problems, along with a more detailed study of applications in power system control. A large number of open problems and directions exist, including but not limited to: OSS control for nonlinear systems subject to time-varying disturbances, flexibility of the framework for distributed/decentralized control, formulations and solutions of hierarchical, competitive, and approximate OSS control problems, and the application of the OSS control framework to the design of new optimization algorithms.

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Appendix

Proof of Theorem 3.2: By assumption, the closed-loop system (11)–(12) is well-posed and possesses a globally asymptotically stable equilibrium point for each w; hence, the first two requirements of Problem 2.2 are satisfied. It remains to show that $\lim_{t\to\infty} y(t) = y^\star(w)$ for each w and every initial condition. Since the closed-loop system possesses a globally asymptotically stable equilibrium point for each w, there exists a unique solution $(\bar{x}, \bar{\xi}, \bar{\eta}, \bar{x}_\mathrm{s})$ to the steady-state equations

$$\mathbb{0} = A\bar{x} + B\bar{u} + B_w w \qquad \mathbb{0} = \varphi(\bar{\xi}, \bar{y}_\mathrm{m})$$
$$\bar{y}_\mathrm{m} = h_\mathrm{m}(\bar{x}, \bar{u}, w) \qquad \mathbb{0} = h_\epsilon(\bar{\xi}, \bar{y}_\mathrm{m})$$
$$\mathbb{0} = f_\mathrm{s}(\bar{x}_\mathrm{s}, \bar{\eta}, \bar{\xi}, \bar{y}_\mathrm{m}, \mathbb{0})$$
$$\bar{u} = h_\mathrm{s}(\bar{x}_\mathrm{s}, \bar{\eta}, \bar{\xi}, \bar{y}_\mathrm{m}, \mathbb{0})$$

for each w. Since (φ, h_ϵ) is an optimality model, the pair $(\bar{x}, \bar{u})$ satisfies $\bar{y}^\star(w) = C\bar{x} + D\bar{u} + Qw$. Because this equilibrium point attracts all trajectories of the closed-loop system and $y(t)$ is continuous, it must be the case that $\lim_{t\to\infty} y(t) = y^\star(w)$ for every w and every initial condition. Therefore, the controller (11b), (11c), (12a), (12b) solves the OSS control problem. $\square$

Proof of Proposition 3.5: For each w, consider the solutions $(\bar{x}, \bar{u})$ to:

$$\mathbb{0} = A\bar{x} + B\bar{u} + B_w w \tag{30a}$$
$$\bar{y} = C\bar{x} + D\bar{u} + Qw \tag{30b}$$
$$\mathbb{0} = H\bar{y} - Lw \tag{30c}$$
$$\mathbb{0} = T^\mathsf{T} \nabla f_0(\bar{y}; w). \tag{30d}$$

The equations (30) correspond to the equations (10) in the definition of an optimality model. We show that (30) imply the KKT conditions. The first two equations (30a) and (30b) imply $\bar{y} \in \overline{Y}(w)$, which is equivalent to the first set of equality constraints, (6b). The equation (30c) is the engineering equality constraint, (6c). Finally, because the feasible subspace property holds, (30d) implies the gradient condition (15). Since the KKT conditions are sufficient for optimality, the following implication holds for all w: if $(\bar{x}, \bar{\mu}, \bar{u})$ satisfy (30), then $(\bar{x}, \bar{u})$ satisfy

$$\bar{y}^\star(w) = C\bar{x} + D\bar{u} + Qw.$$

The filter (17) satisfies the criterion of Definition 3.1, and is therefore an optimality model. $\square$

Proof of Proposition 3.6: For each w, consider the solutions $(\bar{x}, \bar{u})$ to

$$\mathbb{0} = A\bar{x} + B\bar{u} + B_w w \tag{31a}$$
$$\bar{y} = C\bar{x} + D\bar{u} + Qw \tag{31b}$$
$$\mathbb{0} = H\bar{y} - Lw + T^\mathsf{T} \nabla f_0(\bar{y}; w). \tag{31c}$$

The equations (31) correspond to the equations (10) in the definition of an optimality model. By assumption, the feasible

region of the optimization problem (6) is non-empty: hence, there exists a $y(w)$ such that

$$G_\perp y(w) = b(w) \tag{32a}$$

$$H y(w) = Lw. \tag{32b}$$

Equations (31a) and (31b) imply that $G_\perp \bar{y} = b(w)$. Equation (32a) and the fact that $\mathbf{null}\, G_\perp = \mathbf{range}\, G$ imply there exists a v such that $\bar{y} = y(w) + Gv$. Substituting this expression for $\bar{y}$ into (31c) and making use of (32b), we see that

$$\mathbb{0} = HGv + T^\mathsf{T} \nabla f_0(\bar{y}; w). \tag{33}$$

Since

$$\mathbf{range}\, HG \cap \mathbf{range}\, T^\mathsf{T} = \{\mathbb{0}\}, \tag{34}$$

(33) and (34) imply

$$\mathbb{0} = HGv$$
$$\mathbb{0} = T^\mathsf{T} \nabla f_0(\bar{y}; w)$$

for every w. Since $H\bar{y} - Lw = HGv$,

$$\mathbb{0} = H\bar{y} - Lw$$
$$\mathbb{0} = T^\mathsf{T} \nabla f_0(\bar{y}; w)$$

for every w. The remainder of the proof proceeds like the proof of Proposition 3.5. $\square$

Proof of Theorem 3.7: We will apply the classic result [32, Theorem 1], which provides necessary and sufficient conditions for stabilizability and detectability of the augmented plant (11). We examine the application of [32, Theorem 1] to the augmented plant corresponding to each of the three optimality models — the FS-OM, the OS-OM, and the REFS-OM — in sequence. First note that condition (i) is exactly conditions (a) and (b) of [32, Theorem 1]; condition (e) of [32, Theorem 1] is automatically satisfied here. We will show that conditions (c) and (d) of [32, Theorem 1] are equivalent to conditions (ii) and (iii)a in the case of the FS-OM and the OS-OM. In the case of the REFS-OM, the property $\mathbf{range}\, HG \cap \mathbf{range}\, T^\mathsf{T} = \{\mathbb{0}\}$ of condition (iii)b is required for the filter (18) to be an optimality model by Proposition 3.6; we will show that condition (ii) and the second property of (iii)b, $(\mathbf{range}\, HG)^\perp \cap (\mathbf{range}\, T^\mathsf{T})^\perp = \{\mathbb{0}\}$, are equivalent to conditions (c) and (d) of [32, Theorem 1]. We first require the following lemmas.

*Lemma A.1 (**Unique Primal Solution**):* Suppose the optimization problem (19) is feasible, and let $T \in \mathbb{R}^{p \times \bullet}$ be any matrix satisfying $\mathbf{range}\, T = \mathbf{null}\, \begin{bmatrix} G_\perp \\ H \end{bmatrix}$. Then (19) has a unique optimizer if and only if $v^\mathsf{T} M v > 0$ on $\mathbf{range}\, T$.

Proof: Fix a member $\tilde{y}(w)$ of the feasible set of (19). Since $\mathbf{range}\, T = \mathbf{null}\, \begin{bmatrix} G_\perp \\ H \end{bmatrix}$, we can rewrite the optimization problem (19) as $\text{minimize}_{\bar{y} \in \mathbb{R}^p, \, v \in \mathbf{range}\, T}\, \frac{1}{2}\bar{y}^\mathsf{T} M \bar{y} - \bar{y}^\mathsf{T} N w$ subject to the constraint $\bar{y} = \tilde{y}(w) + v$. Eliminating $\bar{y}$ and writing $v = T'r$, where T' is a full-column-rank matrix satisfying $\mathbf{range}\, T' = \mathbf{range}\, T$ and $r \in \mathbb{R}^\bullet$ is a new decision variable, we obtain the equivalent problem $\text{minimize}_{r \in \mathbb{R}^\bullet}\, \frac{1}{2}r^\mathsf{T} T'^\mathsf{T} M T' r + r^\mathsf{T} T'^\mathsf{T} (M\tilde{y}(w) - Nw) + \tilde{y}(w)^\mathsf{T} (M\tilde{y}(w) - Nw)$. This unconstrained QP has a unique

optimizer $r^\star$ if and only if $T'MT' \succ \mathbb{0}$, which is equivalent to M being positive definite on $\mathbf{range}\, T$. $\square$

*Lemma A.2 (**Unique Dual Solution**):* Suppose the optimization problem (19) has a unique primal solution $\bar{y}^\star$. The corresponding dual solution is unique if and only if the matrix $\begin{bmatrix} G_\perp \\ H \end{bmatrix}$ is full row rank.

Proof: Let $\bar{y}^\star$ denote the unique primal solution of (19). Under our assumptions, the pair $(\lambda^\star, \mu^\star)$ is a dual solution if and only if $(\lambda^\star, \mu^\star)$ satisfies the gradient KKT condition (8), which in the present context is given by $\mathbb{0} = M\bar{y}^\star - Nw + G_\perp^\mathsf{T} \lambda^\star + H^\mathsf{T} \mu^\star$. The assumption of a primal solution implies that at least one dual solution $(\lambda^\star, \mu^\star)$ to the preceding exists; this solution is unique if and only if $\begin{bmatrix} G_\perp^\mathsf{T} & H^\mathsf{T} \end{bmatrix}$ is full column rank. $\square$

We move on to the main proof; we will show that when using the FS-OM (17), the OSS control problem is solvable if and only if the stated conditions (i),(ii),(iii)a hold; a similar argument can be made for the OS-OM. A modified version of the same argument can be made for the REFS-OM when (i),(ii),(iii)b hold. Condition (i) is exactly conditions (a) and (b) of [32, Theorem 1]; condition (e) of [32, Theorem 1] is automatically satisfied here. We show conditions (ii) and (iii)a are equivalent to conditions (c) and (d) of [32, Theorem 1]. Define the matrices $\mathcal{N}$, G, and $G_\perp$ as done in Section II, and without loss of generality, assume T in (14) is selected to have full column rank. The augmented plant using the FS-OM is

$$\dot{x} = Ax + Bu + B_w w$$
$$\dot{\eta} = \begin{bmatrix} HC \\ T^\mathsf{T} MC \end{bmatrix} x + \begin{bmatrix} HD \\ T^\mathsf{T} MD \end{bmatrix} u - \begin{bmatrix} L \\ T^\mathsf{T} N \end{bmatrix} w.$$

Following [32, Equation (13)], we check whether

$$\mathcal{R}_{\text{FS}} := \left[\begin{array}{c|c} I_n & \mathbb{0} \\ \hline \mathbb{0} & \begin{bmatrix} H \\ T^\mathsf{T} M \end{bmatrix} \end{array} \right] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{35}$$

has full row rank. Let $\mathrm{col}(\alpha, \beta, \gamma) \in \mathbf{null}\, \mathcal{R}_{\text{FS}}^\mathsf{T}$, so that

$$\begin{bmatrix} \alpha \\ H^\mathsf{T} \beta + MT\gamma \end{bmatrix}^\mathsf{T} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathbb{0}. \tag{36}$$

Multiplying on the right by $\mathcal{N}$ and recalling that $\mathbf{range}\, \mathcal{N} = \mathbf{null}\, \begin{bmatrix} A & B \end{bmatrix}$ and also that $G = \begin{bmatrix} C & D \end{bmatrix} \mathcal{N}$, we find

$$(H^\mathsf{T} \beta + MT\gamma)^\mathsf{T} G = \mathbb{0}. \tag{37}$$

Hence, $H^\mathsf{T} \beta + MT\gamma \in (\mathbf{range}\, G)^\perp$. Because $(\mathbf{range}\, G)^\perp = \mathbf{range}\, G_\perp^\mathsf{T}$ by the definition of $G_\perp$, the above is equivalent to the existence of a vector v such that

$$H^\mathsf{T} \beta + MT\gamma = G_\perp^\mathsf{T} v. \tag{38}$$

Recall that $\mathbf{range}\, T = (\mathbf{null}\, G_\perp) \cap (\mathbf{null}\, H)$, so $G_\perp T = \mathbb{0}$ and $HT = \mathbb{0}$. Multiplying (38) on the left by $\gamma^\mathsf{T} T^\mathsf{T}$ we find

$$\gamma^\mathsf{T} T^\mathsf{T} MT\gamma = 0. \tag{39}$$

For the sufficient direction, we show that if conditions (ii) and (iii)a hold, then $\mathrm{col}(\alpha, \beta, \gamma) = \mathbb{0}$. From condition (ii), it follows by Lemma A.1 that the matrix $T^\mathsf{T} MT$ is positive definite and hence from (39) that $\gamma = \mathbb{0}$. Equation (38) then implies that

$$\begin{bmatrix} v & -\beta \end{bmatrix}^\mathsf{T} \begin{bmatrix} G_\perp \\ H \end{bmatrix} = \mathbb{0}. \tag{40}$$

By condition (iii)a and Lemma A.2, the coefficient matrix (40) in has full row rank, and hence (40) implies that $v = \mathbb{0}$ and $\beta = \mathbb{0}$. Equation (36) then implies that $\alpha^{\mathsf{T}} \begin{bmatrix} A & B \end{bmatrix} = \mathbb{0}$. Since (A, B) is stabilizable, the left null space of $\begin{bmatrix} A & B \end{bmatrix}$ is empty. Therefore $\alpha = \mathbb{0}$, we conclude that $\mathcal{R}_{\mathrm{FS}}$ has full row rank, and thus by [32, Theorem 1] the augmented plant is stabilizable/detectable; it follows by Theorem 3.2 that the OSS control problem is solvable.

For the necessary direction, we show that if either of conditions (ii) or (iii)a fails, then we can construct $\mathrm{col}(\alpha, \beta, \gamma) \neq \mathbb{0}$ satisfying (36), which in turn will violate the transmission zero condition in [32, Theorem 1] and show the augmented plant is not stabilizable. Suppose (iii)a fails. Then by Lemma A.2 there exists a nonzero solution to (40). It cannot be the case that $\beta = \mathbb{0}$, for then v would be zero since $G_\perp$ is full row rank by construction. As a result, if we set $\gamma := \mathbb{0}$, (37) implies that there exists a $\bar{\beta} \neq \mathbb{0}$ such that $\bar{\beta}^{\mathsf{T}} HG = \mathbb{0}$. We observe $\bar{\beta}^{\mathsf{T}} HG = \begin{bmatrix} \bar{\beta}^{\mathsf{T}} HC & \bar{\beta}^{\mathsf{T}} HD \end{bmatrix} \mathcal{N} = \mathbb{0}$. Since $\mathbf{range}\,\mathcal{N} = \mathbf{null}\begin{bmatrix} A & B \end{bmatrix}$, the preceding implies that

$$\begin{bmatrix} C^{\mathsf{T}} H^{\mathsf{T}} \bar{\beta} \\ D^{\mathsf{T}} H^{\mathsf{T}} \bar{\beta} \end{bmatrix} \in \left(\mathbf{null}\begin{bmatrix} A & B \end{bmatrix}\right)^{\perp} = \mathbf{range}\begin{bmatrix} A^{\mathsf{T}} \\ B^{\mathsf{T}} \end{bmatrix}. \quad (41)$$

As a result, a solution $\bar{\alpha}$ exists to $\begin{bmatrix} C^{\mathsf{T}} H^{\mathsf{T}} \bar{\beta} \\ D^{\mathsf{T}} H^{\mathsf{T}} \bar{\beta} \end{bmatrix} = \begin{bmatrix} A^{\mathsf{T}} \\ B^{\mathsf{T}} \end{bmatrix} \bar{\alpha}$. Let $\bar{\alpha}$ satisfy the preceding. Then $\mathrm{col}(\alpha, \beta, \gamma) := \mathrm{col}(-\bar{\alpha}, \bar{\beta}, \mathbb{0})$ satisfies (36). Now suppose instead condition (ii) fails. Then by Lemma A.1 there exists a $\bar{\gamma} \neq \mathbb{0}$ such that $\bar{\gamma}^{\mathsf{T}} T^{\mathsf{T}} MT\bar{\gamma} = 0$, which by positive semidefiniteness of $T^{\mathsf{T}} MT$ implies that $T^{\mathsf{T}} MT\bar{\gamma} = \mathbb{0}$, and hence that $MT\bar{\gamma} = \mathbb{0}$. It follows that the vector $\mathrm{col}(\alpha, \beta, \gamma) := \mathrm{col}(\mathbb{0}, \mathbb{0}, \bar{\gamma})$ satisfies (36).

We now show that when using the OS-OM (16), conditions (ii) and (iii)a are equivalent to conditions (c) and (d) of [32, Theorem 1].

The augmented plant when using the OS-OM is

$$\dot{x} = Ax + \mathbb{0}\mu + Bu + B_w w$$
$$\dot{\mu} = HCx + \mathbb{0}\mu + HDu - Lw$$
$$\dot{\eta} = G^{\mathsf{T}} MCx + G^{\mathsf{T}} H^{\mathsf{T}} \mu + G^{\mathsf{T}} MDu - G^{\mathsf{T}} Nw.$$

Therefore, we examine whether the matrix

$$\mathcal{R}_{\mathrm{OS}} := \begin{bmatrix} A & \mathbb{0} & B \\ HC & \mathbb{0} & HD \\ G^{\mathsf{T}} MC & G^{\mathsf{T}} H^{\mathsf{T}} & G^{\mathsf{T}} MD \end{bmatrix} \quad (42)$$

is full row rank. Let $\mathrm{col}(\alpha, \beta, \gamma) \in \mathbf{null}\,\mathcal{R}_{\mathrm{OS}}^{\mathsf{T}}$, which is equivalent to

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A & \mathbb{0} & B \\ HC & \mathbb{0} & HD \\ G^{\mathsf{T}} MC & G^{\mathsf{T}} H^{\mathsf{T}} & G^{\mathsf{T}} MD \end{bmatrix} = \mathbb{0}. \quad (43)$$

One may rewrite the above equivalently as

$$\begin{bmatrix} \alpha \\ H^{\mathsf{T}} \beta + MG\gamma \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathbb{0} \quad (44a)$$
$$HG\gamma = \mathbb{0}. \quad (44b)$$

Multiplying (44a) on the right by $\mathcal{N}$ and recalling that $\mathbf{range}\,\mathcal{N} = \mathbf{null}\begin{bmatrix} A & B \end{bmatrix}$ and also that $G = \begin{bmatrix} C & D \end{bmatrix} \mathcal{N}$, we find

$$(H^{\mathsf{T}} \beta + MG\gamma)^{\mathsf{T}} G = \mathbb{0}.$$

Expanding the above yields

$$\gamma^{\mathsf{T}} G^{\mathsf{T}} MG + \beta^{\mathsf{T}} HG = \mathbb{0}.$$

We multiply on the right by γ and make use of (44b) to find

$$\gamma^{\mathsf{T}} G^{\mathsf{T}} MG\gamma = 0. \quad (45)$$

By (44b) we have that $G\gamma \in \mathbf{null}\,H$. By definition, $G\gamma \in \mathbf{null}\,G_\perp$ also. Since $\mathbf{range}\,T = (\mathbf{null}\,G_\perp) \cap (\mathbf{null}\,H)$, there exists a vector v such that $G\gamma = Tv$. Using (45), this v satisfies

$$v^{\mathsf{T}} T^{\mathsf{T}} MTv = \mathbb{0}.$$

The remainder of the proof proceeds like the proof in the case of the FS-OM following equation (39).

Finally, we show that when using the REFS-OM (18), condition (ii) and the second property of condition (iii)b, $(\mathbf{range}\,HG)^{\perp} \cap (\mathbf{range}\,T^{\mathsf{T}})^{\perp} = \{\mathbb{0}\}$, are equivalent to conditions (c) and (d) of [32, Theorem 1].

The augmented plant using the REFS-OM is

$$\dot{x} = Ax + Bu + B_w w$$
$$\dot{\eta} = (HC + T^{\mathsf{T}} MC)x + (HD + T^{\mathsf{T}} MD)u$$
$$\quad - (Lw + T^{\mathsf{T}} N)w.$$

Therefore, we examine whether the matrix

$$\mathcal{R}_{\mathrm{RE}} := \begin{bmatrix} I & \mathbb{0} \\ \mathbb{0} & H + T^{\mathsf{T}} M \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (46)$$

is full row rank. Let $\mathrm{col}(\alpha, \beta) \in \mathbf{null}\,\mathcal{R}_{\mathrm{RE}}^{\mathsf{T}}$, which is equivalent to the equations

$$\begin{bmatrix} \alpha^{\mathsf{T}} \\ \beta^{\mathsf{T}}(H + T^{\mathsf{T}} M) \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathbb{0} \quad (47)$$

Multiplying on the right by $\mathcal{N}$ and recalling that $\mathbf{range}\,\mathcal{N} = \mathbf{null}\begin{bmatrix} A & B \end{bmatrix}$ and also that $G = \begin{bmatrix} C & D \end{bmatrix} \mathcal{N}$, we find

$$\beta^{\mathsf{T}}(H + T^{\mathsf{T}} M)G = \mathbb{0}. \quad (48)$$

Hence, $H^{\mathsf{T}} \beta + MT\beta \in (\mathbf{range}\,G)^{\perp}$. Because $(\mathbf{range}\,G)^{\perp} = \mathbf{range}\,G_\perp^{\mathsf{T}}$ by the definition of $G_\perp$, (48) is equivalent to the existence of a vector v such that

$$H^{\mathsf{T}} \beta + MT\beta = G_\perp^{\mathsf{T}} v. \quad (49)$$

Recall that $\mathbf{range}\,T = (\mathbf{null}\,G_\perp) \cap (\mathbf{null}\,H)$, so $G_\perp T = \mathbb{0}$ and $HT = \mathbb{0}$. Multiplying (49) on the left by $\beta^{\mathsf{T}} T^{\mathsf{T}}$, we find that $\beta^{\mathsf{T}} T^{\mathsf{T}} MT\beta = 0$.

For the sufficient direction, we show that if conditions (ii),(iii)b hold then $\mathrm{col}(\alpha, \beta) = \mathbb{0}$. By Lemma A.1, condition (ii) implies M is positive definite on $\mathbf{range}\,T$, so it follows from the above that $T\beta = \mathbb{0}$, or equivalently that $\beta \in (\mathbf{range}\,T^{\mathsf{T}})^{\perp}$. It follows then from (48) that $\beta^{\mathsf{T}} HG = \mathbb{0}$, implying that $\beta \in (\mathbf{range}\,HG)^{\perp}$ also. From condition (iii)b we have $(\mathbf{range}\,HG)^{\perp} \cap (\mathbf{range}\,T^{\mathsf{T}})^{\perp} = \{\mathbb{0}\}$, so we conclude that $\beta = \mathbb{0}$. Equation (47) then reads $\alpha^{\mathsf{T}} \begin{bmatrix} A & B \end{bmatrix} = \mathbb{0}$, from which we conclude $\alpha = \mathbb{0}$ since (A, B) is stabilizable.

For the necessary direction, we show that if any one of the conditions (ii),(iii)b fails, then we can construct a vector $\mathrm{col}(\alpha, \beta) \neq \mathbb{0}$ satisfying (47). Suppose (ii) fails, so that by

Lemma A.1, there exists a $\bar{\beta} \neq \mathbb{0}$ such that $\bar{\beta}^\mathsf{T} T^\mathsf{T} M T \bar{\beta} = 0$ but $T\bar{\beta} \neq \mathbb{0}$. Equation (48) implies that a solution $\bar{\alpha}$ exists to

$$\begin{bmatrix} C^\mathsf{T}(H^\mathsf{T}\bar{\beta} + MT\bar{\beta}) \\ D^\mathsf{T}(H^\mathsf{T}\bar{\beta} + MT\bar{\beta}) \end{bmatrix} = \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix} \bar{\alpha}$$

using the same reasoning as in the proof in the case of the FS-OM starting at (41). With such an $\bar{\alpha}$, $\mathrm{col}(\alpha, \beta) := \mathrm{col}(-\bar{\alpha}, \bar{\beta})$ satisfies (47).

Now suppose (iii)b fails. Then there exists a $\bar{\beta} \neq \mathbb{0}$ such that $T\bar{\beta} = \mathbb{0}$ and $\bar{\beta}^\mathsf{T} HG = \mathbb{0}$. The same reasoning as the proof in the case of the FS-OM starting at (41) shows that a solution $\bar{\alpha}$ exists to

$$\begin{bmatrix} C^\mathsf{T} H^\mathsf{T} \bar{\beta} \\ D^\mathsf{T} H^\mathsf{T} \bar{\beta} \end{bmatrix} = \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix} \bar{\alpha}.$$

With such an $\bar{\alpha}$, $\mathrm{col}(\alpha, \beta) := \mathrm{col}(-\bar{\alpha}, \bar{\beta})$ satisfies (47). $\square$

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