ON BAKER'S PATCHWORK CONJECTURE FOR DIAGONAL PADÉ APPROXIMANTS

D. S. LUBINSKY

Abstract. We prove that for entire functions f of finite order, there is a sequence of integers $\mathcal{S}$ such that as $n \to \infty$ through S,

$$\min \left\{ \left| f - [n/n] \right| (z) , \left| f - [n-1/n-1] \right| (z) \right\} \to 0$$

uniformly for z in compact subsets of the plane. More generally this holds for sequences of Newton-Padé approximants and for functions whose errors of approximation by rational functions of type (n,n) decays faster than $\exp\left(-n\sqrt{\log n}\right)$. This establishes George Baker's Patchwork Conjecture for large classes of entire functions.

Padé approximation, Multipoint Padé approximants, spurious poles, Baker Patchwork Conjecture. 41A21, 41A20, 30E10.

1. Introduction[1]

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be a formal power series. Given a non-negative integer n, the (n,n) Padé approximant is a rational function $[n/n] = P_n/Q_n$, where P_n, Q_n are polynomials of degree $\leq n$ with Q_n not identically 0 and

$$\left(fQ_n - P_n \right)(z) = O\left(z^{2n+1} \right).$$

The convergence of Padé approximants is a much studied subject. One of the pitfalls of the method is the phenomenon of spurious poles, namely poles that do not reflect the analytic properties of the function f. For this reason, the most general results, such as the Nuttall-Pommerenke theorem, involve convergence in capacity, rather than uniform convergence. In 1961, Baker, Gammel, and Wills nevertheless conjectured that at least a subsequence of the diagonal Padé sequence converges locally uniformly.

Baker-Gammel-Wills Conjecture (1961)

Date: February 2, 2020.

[1]Research supported by NSF grant DMS1362208

Let f be meromorphic in $B_1 = \{z : |z| < 1\}$ and analytic at 0. Then there is a subsequence $\{[n/n]\}_{n\in\mathcal{S}}$ of $\{[n/n]\}_{n\geq 1}$ that converges uniformly to f in compact subsets of B_1 omitting poles of f.

The author showed in 2001 [26] that the conjecture is false, by considering the Rogers-Ramanujan function with a value of q on the unit circle. A.P. Buslaev quickly followed [6] with an analytic counterexample, formed from an algebraic function, and then showed that even the Rogers-Ramanujan function provides an analytic counterexample [7]. One of the unresolved issues is whether the Baker-Gammel-Wills conjecture is valid for entire functions, or perhaps even functions meromorphic in the whole plane. To date, there is still no counterexample. The author proved [21] that the Baker-Gammel-Wills conjecture is true for most entire functions in the sense of category, and subsequently that a more general form involving multipoint Padé approximants [ ] also holds in the sense of category.

After his original conjecture was disproved, George Baker [3] noted that in the counterexamples, just two subsequences together provide locally uniform convergence in the unit ball. He went on to conjecture that a patchwork of finitely many subsequences can provide locally uniform convergence for functions meromorphic in the ball [4].

Here is a precise statement:

George Baker's Patchwork Conjecture (2005)
Let the function f be analytic in $\overline{B_1} = \{z : |z| \leq 1\}$ except for a finite number of poles in the interior. There there exists a finite number of infinite subsequences $\{\mathcal{S}_k\}_{k=1}^{L}$ of positive integers such that these subsequences can be patched together in such a manner that for any $z \in \overline{B_1}$, for some $1 \leq k \leq L$,

$$\lim_{n\to\infty, n\in\mathcal{S}_k} [n/n](z) = f(z)$$

on the sphere.

Here on the sphere means in the chordal metric - so that at poles of f, the approximants diverge to ∞ in absolute value. In this paper, we shall show Baker's patchwork conjecture is true for entire functions whose errors of rational approximation decay sufficently rapidly, and in particular for all entire functions of finite order. Moreover, we obtain a sequence of integers $\mathcal{S}$ such that either $[n/n]$ or $[n-1/n-1]$ converges for $n \in \mathcal{S}$, so just two subsequences are enough.

We note that one consequence of the Nuttall-Pommerenke theorem, is that for functions f meromorphic in the plane (and more generally with singularities of capacity 0), there is a subsequence $\mathcal{S}$ of integers and a set $\mathcal{E}$ of capacity 0, such that

$$\liminf_{n\to\infty, n\in\mathcal{S}} |f-[n/n]|(z)^{1/n} = 0, z \in \mathbb{C}\backslash\mathcal{E}.$$

Baker's Patchwork Conjecture tries to avoid that unknown set $\mathcal{E}$.

For any compact set $K \subset \mathbb{C}$ and a function f continuous on C, we define

$$E_{nn}(f;K) = \inf\left\{\left\|f-\frac{P}{Q}\right\|_{L_\infty(K)} : \deg(P), \deg(Q) \le n\right\}.$$

A special case of our results is:

Theorem 1
Assume that f is entire and that

$$\lim_{n\to\infty} E_{nn}(f;B_1)^{1/\left(n\sqrt{\log n}\right)} = 0. \tag{1.1}$$

Then there is an infinite sequence of positive integers $\mathcal{S}$ such that uniformly in compact subsets of the plane

$$\lim_{n\to\infty, n\in\mathcal{S}} \min\{|f-[n/n]|(z), |f-[n-1/n-1]|(z)\} = 0.$$

Remarks
(a) The condition (1.1) is satisfied by all entire functions of finite order: indeed for those functions

$$\lim_{n\to\infty} E_{nn}(f;B_1)^{1/(n\log n)} = 0.$$

We believe the result above holds for all entire functions.
(b) Note that this does not imply locally uniform convergence of either $[n/n]$ or $[n-1/n-1]$.
(c) We discuss the density of the the "good" subsequence in Section ?
(d) Given $\rho \in (0,1)$, we can also ensure that

$$\lim_{n\to\infty, n\in\mathcal{S}} \left(\min\left\{\frac{|f-[n/n]|(z)}{E_{nn}(f;B_1)^\rho}, \frac{|f-[n-1/n-1]|(z)}{E_{n-1,n-1}(f;B_1)^\rho}\right\}\right) = 0,$$

so that the convergence rate is close to optimal.

Our method also allows us to treat Newton-Padé approximation. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of not necessarily distinct points in the plane and

$$\omega_n(z) = \prod_{j=1}^{n} (z - z_j).$$

We say $R_n = P_n/Q_n$ where P_n, Q_n have degree at most n and Q_n is not identically 0, is a Newton-Pade approximant to f if

$$\frac{fQ_n - P_n}{\omega_{2n+1}}$$

is analytic at the zeros of ω_{2n+1}. Note that as n increases, we keep earlier interpolation points. Theorem 1 is a special case of :

Theorem 2
Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of not necessarily distinct points lying in a compact set in the plane. Let $\{R_n\}$ be the corresponding Newton-Padé approximants to an entire function satisfying (1.1). Then there is an infinite sequence of positive integers $\mathcal{S}$ such that uniformly in compact subsets of the plane

$$\lim_{n\to\infty, n\in\mathcal{S}} (\min\{|f - R_n|(z), |f - R_{n-1}|(z)\}) = 0.$$

We can also handle functions that are only analytic in an open set containging the interpolation points, but withe errors of approximation satisfying something like ( ).

Theorem 3

The paper is organized as follows:

In the sequel,

$$B_R = \{z : |z| < R\}.$$

cap denotes logarithmic capacity, while m_2 denotes planar measure.

2. Ideas of Proof

Write $R_n = P_n/Q_n$, with some normalization of P_n, Q_n and

$$\Delta_n = fQ_n - P_n. \tag{2.1}$$

Then

$$P_{n+1}Q_n - P_nQ_{n+1} = \Delta_nQ_{n+1} - \Delta_{n+1}Q_n \tag{2.2}$$

vanishes at the zeros of ω_{2n+1}. But then as the left-hand side is a polynomial of degree at most $2n+1$, so for some constant A_n,

$$P_{n+1}Q_n - P_nQ_{n+1} = A_n\omega_{2n+1}. \tag{2.3}$$

Hence also

$$\Delta_nQ_{n+1} - \Delta_{n+1}Q_n = A_n\omega_{2n+1}. \tag{2.4}$$

Now comes the key observation. Suppose that for some ζ_n that is not an interpolation point, and $m = n, n+1$, we have, say,

$$|f - R_m|(\zeta_n) > 1.$$

(If this inequality was initially only known at an interpolation point, then by lower semi-continuity, it would also hold in a neighborhood, so would hold at some ζ_n that is not an interpolation point). Then for $m = n, n+1$,

$$|\Delta_m|(\zeta_n) > |Q_m(\zeta_n)|.$$

Substituing these inequalities into (2.4) gives

$$|A_n||\omega_{2n+1}(\zeta_n)| \leq 2|\Delta_n\Delta_{n+1}|(\zeta_n).$$

If Γ is a simple closed curve enclosing ζ_m, the maximum modulus principle gives

$$|A_n| \leq 2\left\|\frac{\Delta_n\Delta_{n+1}}{\omega_{2n+1}}\right\|_{L_\infty(\Gamma)}.$$

Here ω_{2n+1} may be controlled. So we have a bound on $|A_n|$ decaying roughly like the *square* of $\|\Delta_n\|$, whereas it really ought to decay like $\|\Delta_n\|$. It is this simple fact that makes our proofs work.

Next, we choose $m < n$ and write, using (2.3),

$$R_n - R_m = \sum_{j=m}^{n-1}\frac{A_j\omega_{2j+1}}{Q_jQ_{j+1}},$$

or equivalently

$$P_nQ_m - P_mQ_n = Q_nQ_m\sum_{j=m}^{n-1}\frac{A_j\omega_{2j+1}}{Q_jQ_{j+1}}$$

and

$$\Delta_nQ_m - \Delta_mQ_n = Q_nQ_m\sum_{j=m}^{n-1}\frac{A_j\omega_{2j+1}}{Q_jQ_{j+1}}. \tag{2.5}$$

Then also

$$|f - R_n| \leq |f - R_m| + \sum_{j=m}^{n-1}\left|\frac{A_j\omega_{2j+1}}{Q_jQ_{j+1}}\right|$$

We can use Polya's estimate on the size of a set where monic polynomials are small, to bound $\sum_{j=m}^{n-1} \left| \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}} \right|$ outside a set of not too large measure. (Cartan's lemma could also be used for this). This leads to estimates for $P_m Q_n - P_n Q_m$ on a set of fixed positive area. Potential theory (Bernstein-Walsh's inequality) then provides estimates for $P_m Q_n - P_n Q_m$ on any disk. This in turn allows us to show that

$$|f - R_n| < |f - R_m| + \text{ a small term.}$$

If n is large enough compared to m, and lies in a suitable subsequence if integers, then this contradicts the rate of approximation provided by Newton-Padé approximants. It follows that the ζ_n above does not exist, at least for a subsequence. Of course the rigorous details involve work.

3. Preliminary Lemmas

We start with a simple growth lemma. We use area measure rather than one dimensional Hausdorff measure and Cartan's lemma, and instead of capacity as it leads to smaller estimates for the size of an exceptional set. In the sequel, m_2 denotes planar Lebesgue measure.

Lemma 3.1
Let K be a compact set in $|z| \leq R$ of positive capacity.
(a) Then for $n \geq 1$ and polynomials P of degree $\leq n$,

$$\|P\|_{L_\infty(B_R)} \leq \left(\frac{2R}{cap\,(K)} \right)^n \|P\|_{L_\infty(K)}.$$

(b) Assume now that K is a compact set in $|z| \leq R$ of positive area. Then for $n \geq 1$ and polynomials P of degree $\leq n$,

$$\|P\|_{L_\infty(B_R)} \leq 3^n \left(\frac{\pi R^2}{m_2\,(K)} \right)^{\frac{n}{2}} \|P\|_{L_\infty(K)}.$$

Proof
(a) Let us assume, as we may, that P is monic of degree n. Let μ be the equilibrium measure for K in the sense of potential theory, and g be the Green function for K. Thus

$$g\,(z) = \int \log |z - t|\, d\mu\,(t) - \log cap\,(K)$$

We have for z in the unbounded component of $\mathbb{C} \backslash K$, [Ransford]

$$|P\,(z)| \leq \|P\|_{L_\infty(K)}\, e^{ng(z)}.$$

Here for $|z| \leq R$, we see that

$$g(z) \leq \log(|z| + R) - \log cap(K) \leq \log(2R) - \log cap(K).$$

Then for $|z| = R$,

$$|P(z)| \leq \left(\frac{2R}{cap(K)}\right)^n \|P\|_{L_\infty(K)}.$$

The maximum modulus principle also shows that this holds for all $|z| \leq R$.

(b) Normalize P as follows:

$$P(z) = c \prod_{|v| \leq 2R} (z - v) \prod_{|v| \geq 2R} \left(1 - \frac{z}{v}\right).$$

We may assume that $c \neq 0$. Assume there are k terms in the first product and ℓ in the second. Choose ε such that

$$\|P\|_{L_\infty(K)} = \varepsilon^{\deg(P)} = \varepsilon^{k+\ell}.$$

Then in K,

$$\varepsilon^{k+\ell} \geq |c| \left| \prod_{|v| \leq 2R} (z - v) \right| \left(\frac{1}{2}\right)^\ell$$

so

$$\left| \prod_{|v| \leq 2R} (z - v) \right| \leq |c|^{-1} 2^\ell \varepsilon^{k+\ell}.$$

By Polya's lemma, for any $\delta > 0$

$$m_2\left(\left\{z : \left| \prod_{|v| \leq 2R} (z - v) \right| \leq \delta^k \right\}\right) \leq \pi \delta^2.$$

so

$$m_2(K) \leq \pi \left[|c|^{-1} 2^\ell \varepsilon^{k+\ell}\right]^{2/k}.$$

So

$$|c| \leq 2^\ell \varepsilon^{k+\ell} \left(\frac{\pi}{m_2(K)}\right)^{\frac{k}{2}}.$$

From our normalization, and choice of ε,

$$\begin{aligned}
\|P\|_{L_\infty(B_R)} &\leq |c|\,(2R)^k\left(\frac{3}{2}\right)^{\ell} \\
&\leq 2^{\ell}\varepsilon^{k+\ell}\left(\frac{\pi}{m_2(K)}\right)^{\frac{k}{2}}(3R)^k\left(\frac{3}{2}\right)^{\ell} \\
&\leq 3^{k+\ell}\left(\frac{\pi R^2}{m_2(K)}\right)^{\frac{k}{2}}\|P\|_{L_\infty(K)} \\
&\leq 3^{n}\left(\frac{\pi R^2}{m_2(K)}\right)^{\frac{n}{2}}\|P\|_{L_\infty(K)}
\end{aligned}$$

as $m_2(K) \leq \pi R^2$.■

Next, a well known consequence of Polya's Lemma on the area of lemniscates:

Lemma 3.2
Let $R \geq 1 > \varepsilon > 0$. Let Q be a polynomial of degree $\leq n$, admitting representation

$$Q(z) = \prod_{|v|\leq 2R}(z-v)\prod_{|v|\geq 2R}\left(1-\frac{z}{v}\right)$$

We then say Q is normalized w.r.t. the ball B_{2R}.
(a) Then for $|z| \leq R$,

$$|Q(z)| \leq (3R)^n$$

while if k is the number of zeros of Q in B_{2R},

$$\frac{1}{|Q(z)|} \leq \frac{2^n}{\varepsilon^k}.$$

for $|z| \leq R$, $z \notin \mathcal{E}$, where cap$(\mathcal{E}) \leq \varepsilon$ and $m_2(\mathcal{E}) \leq \pi\varepsilon^2$.
(b) If $S \geq 1$,

$$\|Q\|_{L_\infty(B_S)} \geq 2^{-n}.$$

Proof
(a) Suppose n_1 is the degree of Q. Let k be the number of zeros in $|z| \leq 2R$ and ℓ be the number of zeros outside this disk. We see that for $|z| \leq R$,

$$|Q(z)| = \prod_{|v|\leq 2R}(R+2R)\prod_{|v|>2R}\left(1+\frac{1}{2}\right) \leq (3R)^n.$$

Next for $|z| \le R$,

$$\frac{1}{|Q(z)|} = \frac{1}{\left|\prod_{|v|\le 2R}(z-v)\right|\left|\prod_{|v|>2R}\left(1-\frac{z}{v}\right)\right|} \le \left(\varepsilon^{-k}\right)\left(2^{\ell}\right) \le \left(\frac{2}{\varepsilon}\right)^{n}$$

outside a set $\mathcal{E}$ of planar measure at most $\pi\varepsilon^2$. Also this is outside a set of cap $\le \varepsilon$.

(b) By standard estimates for monic polynomials,

$$\left\|\prod_{|v|\le 2R}(z-v)\right\|_{L_\infty(B_S)} \ge S^k.$$

Also, by the maximum-modulus principle,

$$\left\|\prod_{|v|>2R}\left(1-\frac{z}{v}\right)\right\|_{L_\infty(B_S)} \ge 1.$$

Then by the inequality of Kroo-Pritsker,

$$\begin{aligned}\|Q\|_{L_\infty(B_S)} &\ge 2^{-n+1}\left\|\prod_{|v|\le 2R}(z-v)\right\|_{L_\infty(B_S)}\left\|\prod_{|v|>2R}\left(1-\frac{z}{v}\right)\right\|_{L_\infty(B_S)} \\ &\ge 2^{-n+1}S^k \ge 2^{-n+1}.\end{aligned}$$

■

We shall make substantial use of a result of Goncar and Grigorjan. If f is meromorphic inside a simply connected domain D, then we can form the sum R_f of the principal parts of f in D, so that

$$R_f(z) = \sum_j \sum_{k\ge 1} c_{jk}\left(z-b_j\right)^{-k}$$

where $\{b_j\}$ are the poles of f in D. The analytic part of f in D is then

$$\mathcal{A}f = f - R_f.$$

The following remarkable result is a weaker form of the results of Goncar and Grigorjan, see [13], [15]:

Lemma 3.3

Let D be a bounded simply connected domain with boundary Γ. Let f

be meromorphic in D with poles of total multiplicity at most n, and analytic on Γ. Then

$$\|\mathcal{A}f\|_{L_\infty(\Gamma)} \leq 7n^2 \|f\|_{L_\infty(\Gamma)} .$$

Proof

This follows directly from Theorem 1 in [13, p. 571]. ■

4. Comparing Δ_n and E_{nn} on different sets

In this section, we compare sup norms of Δ_n and E_{nn}, also on different sets. Throughout we assume that

$$\lim_{n\to\infty} E_{nn}\left(f; \overline{B_1}\right)^{1/(n\phi(n))} = 0, \tag{4.1}$$

where $\phi : [1, \infty) \to (0, \infty)$ is an increasing function such that

$$\lim_{x\to\infty} \frac{\phi(2x)}{\phi(x)} = 1.$$

A typical example will be $\phi(x) = \sqrt{\log(1 + x)}$.

We first compare errors of approximation on different sets:

Lemma 4.1

Let T and S be compact sets of positive logarithmic capacity. Let $\eta > 0$. Then for large enough n,

$$E_{nn}(f; T) \leq E_{nn}(f; S)^{1-\frac{\eta}{\phi(n)}} .$$

Remark

In the sequel, we often apply this lemma with ϕ replaced by $\phi/2$ or some other multiple of ϕ.

Proof

Choose $R_n^* = P_n^*/Q_n^*$ such that

$$\|f - R_n^*\|_{L_\infty(T)} = E_{nn}(f; T) .$$

Now choose $R > 1$ so large that B_{R-1} contains both S and T. We initially assume that more generally than (4.1),

$$\lim_{n\to\infty} E_{nn}\left(f; \overline{B_R}\right)^{1/(n\phi(n))} = 0, \tag{4.2}$$

Choose the smallest integer $k \geq n$ such that

$$E_{kk}\left(f; \overline{B_R}\right) \leq E_{nn}(f; T) . \tag{4.3}$$

Then either $k = n$ or

$$E_{k-1,k-1}\left(f; \overline{B_R}\right) > E_{nn}(f; T) . \tag{4.4}$$

Choose $R_k^{\#} = P_k^{\#}/Q_k^{\#}$ such that

$$\left\| f - R_k^{\#} \right\|_{L_\infty(B_R)} = E_{kk}\left(f; \overline{B_R}\right). \tag{4.5}$$

Then from (4.3),

$$\left\| R_k^{\#} - R_n^* \right\|_{L_\infty(T)} \leq 2E_{nn}(f;T).$$

We may normalize the numerators and denominators in R_n^* and $R_k^{\#}$ as we please. It is convenient to normalize them as in Lemma 3.2 so that

$$\begin{aligned} \|Q_n^*\|_{L_\infty(B_R)} &\leq (3R)^n; \\ \left\|Q_k^{\#}\right\|_{L_\infty(B_R)} &\leq (3R)^k. \end{aligned}$$

Then

$$\left\| P_k^{\#} Q_n^* - P_n^* Q_k^{\#} \right\|_{L_\infty(T)} \leq 2(3R)^{k+n} E_{nn}(f;T).$$

By Lemma 3.1, there exists a constant $A > 0$ depending only on T and R such that

$$\left\| P_k^{\#} Q_n^* - P_n^* Q_k^{\#} \right\|_{L_\infty(B_R)} \leq 2(3AR)^{k+n} E_{nn}(f;T).$$

Also by Lemma 3.2,

$$\frac{1}{\left|Q_n^* Q_k^{\#}\right|(z)} \leq 4^{n+k} \tag{4.6}$$

outside a set $\mathcal{E}$ of capacity $\leq \frac{1}{2}$. The radial projection $\{|z| : z \in \mathcal{E}\}$ of this set onto the positive real axis will also have capacity $\leq \frac{1}{2}$. It then follows that we can choose $r \in (R-1, R)$ such that the circle $\Gamma_r = \{z : |z| = r\}$ does not intersect E, and hence the estimate (4.6) holds on Γ_r. Then also

$$\left\| R_k^{\#} - R_n^* \right\|_{L_\infty(\Gamma_r)} \leq 2(12AR)^{k+n} E_{nn}(f;T).$$

Hence using (4.5), (4.3),

$$\|f - R_n^*\|_{L_\infty(\Gamma_r)} \leq E_{nn}(f;T)\left\{1 + 2(12AR)^{k+n}\right\}.$$

From the Gonchar-Grigorjan Lemma, if $\mathcal{A}(f - R_n^*) = f - \mathcal{A}R_n^*$ is the analytic part of $f - R_n^*$ inside Γ_r, we have

$$\|f - \mathcal{A}R_n^*\|_{L_\infty(\Gamma_r)} \leq \left(7n^2\right) E_{nn}(f;T)\left\{1 + 2(12AR)^{k+n}\right\}.$$

As $\mathcal{A}R_n^*$ is also a rational function of type (n,n), we obtain

$$E_{nn}\left(f; \overline{B_r}\right) \leq \left(7n^2\right) E_{nn}(f;T)\left\{1 + 2(12AR)^{k+n}\right\}$$

and since B_r contains B_{R-1} and hence S, so

$$E_{nn}(f;S) \leq \left(7n^2\right) E_{nn}(f;T)\left\{1+2(12AR)^{k+n}\right\}. \tag{4.7}$$

Now if $\eta > 0$,

$$\lim_{n\to\infty}\left(E_{nn}(f;T)^{1/\phi(n)}\right)^{1/n} = 0,$$

so if $k = n$,

$$\left(7n^2\right)\left\{1+2(12AR)^{k+n}\right\} \leq E_{nn}(f;T)^{-\eta/\phi(n)}.$$

If on the other hand $k > n$, then given any constant C, for large enough k,

$$\begin{aligned} C^k &\leq E_{k-1,k-1}(f;T)^{-\eta/\phi(k)} \\ &\leq E_{nn}(f;T)^{-\eta/\phi(k)} \leq E_{nn}(f;T)^{-\eta/\phi(n).} \end{aligned}$$

It follows from (4.7) that

$$E_{nn}(f;S) \leq E_{nn}(f;T)^{1-\frac{1}{\phi(n)}}.$$

We still need to deal with the assumption (4.2), which is more general than our original (4.1) involving only $R = 1$. To do this, we proceed as follows: first note that for any $R > 0$, we have as f is entire,

$$\lim_{n\to\infty} E_{nn}^{\frac{1}{2n}}\left(f;\overline{B_R}\right) = 0.$$

Then choosing $\hat{\phi}(x) = 2$, $x \in [1,\infty)$, we have

$$\lim_{n\to\infty} E_{nn}^{1/\left(n\hat{\phi}(n)\right)}\left(f;\overline{B_R}\right) = 0.$$

Our proof above applied to $S = \overline{B_R}$, $T = \overline{B_1}$ and $\hat{\phi}$ rather than ϕ shows that for large enough n,

$$E_{nn}\left(f;\overline{B_R}\right) \leq E_{nn}\left(f;\overline{B_1}\right)^{1-\frac{1}{\phi(n)}} = E_{nn}\left(f;\overline{B_1}\right)^{1/2}$$

Then the more general (4.2) follows from (4.1), so our extra hypothesis is satisfied. ■

Lemma 4.2
Assume that f is analytic in an open connected set U that contains 0. Let K be a compact subset of U and assume that in K, the Newton-Padé approximant R_n has at least N_n poles counting multiplicity. Assume also that K^o is simply connected. Let Γ be a simple closed contour that contains K in its interior and that lies in U. Assume also that Γ

contains all the zeros of ω_{2n+1} *at a distance* $\geq \delta$ *from* Γ.
(a) Then

$$E_{n-N_n,n-N_n}(f;K) \leq C_1^n E_{nn}(f;\Gamma),$$

where C_1 *depends only on* Γ, δ *and* K *but not on* f, n.
(b) We can find for large enough n *a contour* Γ_n *enclosing* K *in its interior but lying inside* Γ*, such that*

$$\|f - R_n\|_{L_\infty(\Gamma_n)} \leq C_2^n E_{nn}(f;\Gamma),$$

where C_2 *depends only on* Γ, δ *and* K *but not on* f, Γ_n, n.

Proof

(a) Let $R_n^* = P_n^*/Q_n^*$ be a best approximant to f on Γ. Write $R_n = P_n/Q_n$. Then we have for z inside Γ,

$$(Q_n^*(fQ_n - P_n))(z) = \frac{1}{2\pi i}\int_\Gamma \frac{Q_n(t)(fQ_n^* - P_n^*)(t)}{t-z}\frac{\omega_{2n+1}(z)}{\omega_{2n+1}(t)}dt.$$

Recall this follows from the fact that $(Q_n^* P_n - P_n^* Q_n)(t)/((t-z)\omega_{2n+1}(t)) = O(t^{-2})$ at ∞. From this we obtain for z inside Γ

$$|f - R_n|(z) \leq \frac{1}{2\pi dist(\Gamma, z)} E_{nn}(f;\Gamma) \sup_{t\in\Gamma} \frac{|Q_n Q_n^*|(t)}{|Q_n Q_n^*|(z)} \frac{\max_{t\in\Gamma}|\omega_{2n+1}(t)|}{\min_{t\in\Gamma}|\omega_{2n+1}(t)|}.$$

Here as all zeros of ω_{2n+1} lie inside Γ at a distance $\geq \delta$ from Γ, so for some constant $c > 0$ depending on Γ, δ, but not on n,

$$\frac{\max_{t\in\Gamma}|\omega_{2n+1}(t)|}{\min_{t\in\Gamma}|\omega_{2n+1}(t)|} \leq c^{2n+1}.$$

We assume that QQ_n^* is normalized w.r.t. B_R for R such that the circle center 0 radius R encloses Γ, as in Lemma 3.2. Then

$$\sup_{t\in\Gamma} \frac{|Q_n Q_n^*|(t)}{|Q_n Q_n^*|(z)} \leq \left(\frac{3R}{\varepsilon}\right)^{2n}$$

outside a union $B_n^\# = \cup_j B_{n,j}$ of at most $2n$ open balls whose sum of diameters is at most $4e\varepsilon$ by Cartan. We now choose ε to be the distance from Γ to K divided by 100. Notice that ε depends only on K, Γ and not on n. Moreover the sum of the diameters of the balls in $B_n^\#$ is at most $\frac{1}{4}$ of the distance from Γ to K. Then we can choose a contour Γ_n between Γ and K that is simple and closed and does not intersect $B_n^\#$ and is at least ε distance from Γ. To see this, let $\mathcal{A}$ be the annular region between K and Γ. If we can't find such a Γ_n in $\mathcal{A}$, then some of the balls in $B_n^\#$ from a continuum that starts at some point in K and ends at some point in Γ, which is impossible as the sum of diameters of all the balls in $B_n^{\#}$ is much smaller than the distance from Γ to K. Then

$$\sup_{z\in\Gamma_n}|f-R_n|(z)\leq\frac{1}{2\pi\varepsilon}E_{nn}(f;\Gamma)\left(\frac{3R}{\varepsilon}\right)^{2n}c^{2n+1}. \tag{4.8}$$

Next we use the lemma of Gonchar-Grigorjan. Let $\mathcal{A}R_n$ denote the analytic part of $[n/n]$ inside Γ_n, that is $[n/n]$ minus the sume of its principal parts inside Γ_n. Also then as f is analytic inside Γ, we have

$$\mathcal{A}(f-R_n)=f-\mathcal{A}R_n.$$

By Gonchar-Grigorjan as $f-R_n$ has at most n poles inside Γ_n, we have

$$\begin{aligned}\sup_{z\in\Gamma_n}|f-\mathcal{A}R_n|(z) &= \sup_{z\in\Gamma_n}|\mathcal{A}(f-R_n)|(z)\\ &\leq 7n^2\sup_{z\in\Gamma_n}|f-R_n|(z).\end{aligned}$$

By the maximum-modulus principle,

$$\sup_{z\in K}|f-\mathcal{A}R_n|(z)\leq 7n^2\sup_{z\in\Gamma_n}|f-R_n|(z).$$

Here since R_n has $\geq N_n$ poles inside K, which lies inside Γ_n, $\mathcal{A}R_n$ is a rational function of type $(n-N_n,n-N_n)$ so we have

$$\begin{aligned}E_{n-N_n,n-N_n}(f;K) &\leq 7n^2\sup_{z\in\Gamma_n}|f-R_n|(z)\\ &\leq C_1^nE_{nn}(f;\Gamma)\end{aligned}$$

where C_1 depends only on Γ,δ and K, by (4.8).
(b) This was established at (4.8). ■

Lemma 4.3
Assume that f is analytic in an open connected set U that contains 0. Let K,L and T be compact subsets of U such that K,L have non-empty interior while T has positive capacity. Let $R_n=P_n/Q_n$ denote the Newton-Padé approximant R_n to f at the zeros of ω_{2n+1} and $\Delta_n=fQ_n-P_n$, where for some $R>0$, Q_n is normalized as in Lemma 3.2.
(a) For n large enough,

$$E_{nn}(f;T)^{1+\frac{1}{\phi(n)}}\leq\|\Delta_n\|_{L_\infty(K)}\leq E_{nn}(f;T)^{1-\frac{1}{\phi(n)}}. \tag{4.9}$$

(b) For n large enough,

$$\|\Delta_n\|_{L_\infty(L)}^{1+\frac{1}{\phi(n)}}\leq\|\Delta_n\|_{L_\infty(K)}\leq\|\Delta_n\|_{L_\infty(L)}^{1-\frac{1}{\phi(n)}}. \tag{4.10}$$

Proof

(a) First assume that K is as in the previous lemma. Then for some contour Γ_n enclosing K but lying inside Γ, we have from Lemma 4.2(b),

$$\|\Delta_n\|_{L_\infty(\Gamma_n)} \leq \|Q_n\|_{L_\infty(\Gamma_n)} C_2^n E_{nn}(f;\Gamma) \leq C_3^n E_{nn}(f;\Gamma)$$

in view of our normalization of Q_n. Then using the fact that for large enough n,

$$C_3^n \leq E_{nn}(f;\Gamma)^{-\frac{-1}{2\phi(n)}} \tag{4.11}$$

we obtain from the maximum modulus principle

$$\|\Delta_n\|_{L_\infty(K)} \leq E_{nn}(f;\Gamma)^{1-\frac{1}{2\phi(n)}}.$$

In view of Lemma 4.1, we then obtain for large enough n,

$$\|\Delta_n\|_{L_\infty(K)} \leq E_{nn}(f;T)^{\left(1-\frac{1}{2\phi(n)}\right)^2} \leq E_{nn}(f;T)^{1-\frac{1}{\phi(n)}}.$$

Next, if K does not satisfy the requirement of Lemma 4.2 that its interior is simply connected, we may simply increase it in size and apply the maximum-modulus principle. So we have the upper bound in (4.9).

For the lower bound, since K has non-empty interior, we may simply assume that K is a ball of radius $r > 0$. By Lemma 3.2, we can choose a circle Γ_n, concentric with the ball K, and of radius between $r/2$ and r such that

$$\frac{1}{|Q_n(z)|} \leq c^n \text{ on } \Gamma_n$$

with c depending only on r. Then

$$\|f - R_n\|_{L_\infty(\Gamma_n)} \leq \|\Delta_n\|_{L_\infty(K)} c^n.$$

By the Gonchar-Grigorjan Lemma, irrespective of if there are poles of R_n inside K or not,

$$E_{nn}(f;\Gamma_n) \leq 7n^2 \|f - \mathcal{A}R_n\|_{L_\infty(\Gamma_n)} \leq 7n^2 c^n \|\Delta_n\|_{L_\infty(K)}.$$

If K_1 is the ball concentric with K but of radius $r/2$, also then

$$\begin{aligned} E_{nn}(f;K_1) &\leq 7n^2 c^n \|\Delta_n\|_{L_\infty(K)} \\ &\leq E_{nn}(f;K_1)^{-\frac{1}{\phi(n)}} \|\Delta_n\|_{L_\infty(K)} \end{aligned}$$

for n large enough. Here we are using that K_1 does not depend on n. So we have the lower bound in (4.9) when T is replaced by K_1. We can

replace it by T using Lemma 4.1, again using $\phi/2$ instead of ϕ.
(b) From (a), applied twice,

$$\begin{aligned}\|\Delta_n\|_{L_\infty(K)} &\leq E_{nn}\left(f;L\right)^{1-\frac{1}{\phi(n)}}\\ &\leq \|\Delta_n\|_{L_\infty(L)}^{\frac{1-\frac{1}{\phi(n)}}{1+\frac{1}{\phi(n)}}}.\end{aligned}$$

Again applying this with ϕ replaced by $\phi/2$ leads to the right inequality in (4.10) and the left-hand one is similar. ■

5. Proof of Theorems 1 and 2

In this section for a given n, we let

$$M_n = \left[A\frac{n}{\phi\left(n\right)}\right],$$

where A is a large enough positive number and $[x]$ denotes the largest integer $\leq x$. We also let

$$\eta_n = E_{nn}\left(f;\overline{B_1}\right)^{1/(n\phi(n))}$$

and choose an infinite sequence of integers $\mathcal{S}$ such that for $n \in \mathcal{S}$,

$$\eta_n \leq \eta_k,\ 1 \leq k \leq n. \tag{5.1}$$

Of course as $\{\eta_n\}$ has limit 0, such an infinite sequence exists. We also assume that

$$\lim_{x\to\infty}\phi\left(x\right) = \infty;$$

while

$$\lim_{x\to\infty}\frac{\phi\left(x\right)}{x} = 0.$$

We first make a simple observation:

Lemma 5.1
(a) Let $\rho > 0$. *If* $A > \rho + 2$, *then for all large enough* $n \in \mathcal{S}$, *we have for any compact set* T *of positive capacity,*

$$E_{n-M_n,n-M_n}\left(f;T\right) > E_{nn}\left(f;T\right)^{1-\frac{\rho}{\phi(n)}}. \tag{5.2}$$

(b) If $A > 4$, *given any ball* B_R, *for large enough* $n \in \mathcal{S}$, R_n *has* $<$ M_n *poles in* B_R, *counting multiplicity.*
Proof
(a) From (5.1),

$$E_{nn}\left(f;\overline{B_1}\right) \leq E_{n-M_n,n-M_n}\left(f;\overline{B_1}\right)^{\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}}.$$

Using Lemma 4.1, we obtain

$$E_{nn}(f;T) \le E_{n-M_n,n-M_n}(f;T)^{\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}\frac{1-\frac{1}{\phi(n)}}{1+\frac{1}{\phi(n)}}}$$

If (5.2) is false, we then obtain

$$E_{nn}(f;T) \le E_{nn}(f;T)^{\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}\frac{\left(1-\frac{1}{\phi(n)}\right)\left(1-\frac{\rho}{\phi(n)}\right)}{1+\frac{1}{\phi(n)}}}. \tag{5.3}$$

Here the exponent is, using that ϕ is increasing, and that $\frac{M_n}{n} = \frac{A}{\phi(n)} + O\left(\frac{1}{n}\right)$,

$$\begin{aligned} &\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}\frac{\left(1-\frac{1}{\phi(n)}\right)\left(1-\frac{\rho}{\phi(n)}\right)}{1+\frac{1}{\phi(n)}} \\ \ge\ & \frac{1}{1-\left(\frac{A}{\phi(n)}+O\left(\frac{1}{n}\right)\right)}(1-\frac{\rho+2}{\phi(n)}+O\left(\frac{1}{\phi(n)^2}\right)) \\ =\ & 1+\frac{A-\rho-2}{\phi(n)}+O\left(\frac{1}{n}\right)+O\left(\frac{1}{\phi(n)^2}\right) \end{aligned}$$

As $A > \rho + 2$, and ( ) holds, we obtain the the exponent in the right -hand side of (5.3) exceeds 1 for large enough n, leading to a contradiction. So we must have (5.2).

(b) Suppose R_n has at least M_n poles in B_R. We may assume R is so large that B_R contains all the interpolation points. By Lemma 4.2,

$$E_{n-M_n,n-M_n}(f;B_R) \le C_1^k E_{nn}(f;\Gamma),$$

where Γ is a suitable contour enclosing B_R, that is independent of n. For n large enough and corresponding k (note that $k/n \to 1$),

$$E_{n-M_n,n-M_n}(f;B_R) \le E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}},$$

contradicting (a). ■

Remark

This shows that R_n has $O\left(\frac{n}{\phi(n)}\right)$ poles in any compact set when $n \in \mathcal{S}$, which is of independent interest.

Proof of Theorem 2

Suppose that the conclusion of Theorem 2 fails. Then there exists $R > 0$ and $A > 0$ such that

$$\liminf_{n\to\infty}\left(\sup_{z\in B_R}(\min\{|f-R_n(z)|,|f-R_{n-1}|(z)\})\right) > A > 0.$$

Then for large enough n, there exists $\zeta_n \in B_R$ such that for $m = n, n+1$,

$$|f - R_m|(\zeta_n) > A. \tag{5.4}$$

By lower semi-continuity, this also holds in a neighborhood of ζ_n, so we may assume that ζ_n is not an interpolation point. Assume that $R_m = P_m/Q_m$ where Q_m is normalized as in Lemma 3.2 We then have for $n = m, m+1$,

$$|Q_m(\zeta_n)| < \frac{1}{A}|\Delta_m(\zeta_n)|.$$

Then with the notation of Section 2,

$$\begin{aligned} |A_n| &= |\omega_{2n+1}(\zeta_n)|^{-1}|\Delta_n Q_{n+1} - \Delta_{n+1}Q_n|(\zeta_n) \\ &\leq \frac{2}{A}\frac{|\Delta_n\Delta_{n+1}|(\zeta_n)}{|\omega_{2n+1}(\zeta_n)|} \leq \frac{2}{A}\left\|\frac{\Delta_n\Delta_{n+1}}{\omega_{2n+1}}\right\|_{L_\infty(B_R)}, \end{aligned} \tag{5.5}$$

by the maximum-modulus principle. We may assume that R is large enough so that all interpolation points are contained in the disk B_{R-1} so that $|\omega_{2n+1}| \geq 1$ on the circle $|z| = R$. We now consider two subcases. Let

$$M_n = \left[5\frac{n}{\phi(n)}\right]$$

and

$$m = n - M_n.$$

Let $n \in \mathcal{S}$.

Case I: R_k has $< M_n$ poles in B_{2R} for $n \geq k \geq n - M_n$

We already know that this is true for $k = n$ by Lemma 5.1. (Recall there R was arbitrary). We let $m = n - M_n$ and use that as in Section 2,

$$P_nQ_m - P_mQ_n = Q_nQ_m\sum_{j=m}^{n-1}\frac{A_j\omega_{2j+1}}{Q_jQ_{j+1}}. \tag{5.6}$$

Let

$$\varepsilon_n = (n+1)^{-1/2}.$$

Now by Lemma 3.2,

$$\frac{1}{|Q_j(z)|} \leq \frac{2^j}{\varepsilon_n^{M_n}}$$

in $B_R \backslash \mathcal{E}_j$ where $m_2(\mathcal{E}_j) \leq \pi\varepsilon_n^2$. Let

$$\mathcal{E}_{m,n} = \bigcup_{j=m}^{n}\mathcal{E}_j,$$

so that

$$m_2\left(\mathcal{E}_{m,n}\right) \leq (n-m+1)\,\pi\varepsilon_n^2 \leq \pi.$$

It follows that in $B_R \backslash \mathcal{E}_{m,n}$,

$$|P_m Q_n - P_n Q_m| \leq \frac{2}{A}(3R)^{n+m}\sum_{j=m}^{n-1}\varepsilon_n^{-2M_n-1}\frac{\left\|\omega_{2j+1}\right\|_{L_\infty(B_R)}}{\min_{|t|=R}\left|\omega_{2j+1}(t)\right|}\left\|\Delta_j\Delta_{j+1}\right\|_{L_\infty(B_R)}.$$

Here as all zeros of ω_{2j+1} lie in B_{R-1}, so for some C_1 depending only on R,

$$\frac{\left\|\omega_{2j+1}\right\|_{L_\infty(B_R)}}{\min_{|t|=R}\left|\omega_{2j+1}(t)\right|} \leq C_1^j$$

As $B_R \backslash \mathcal{E}_{m,n}$ has area at least $\pi R^2 - \pi$, so by Lemma 3.1, and this last inequality,

$$\left\|P_m Q_n - P_n Q_m\right\|_{L_\infty(B_{2R})} \leq C_2^n \sum_{j=m}^{n-1}\varepsilon_n^{-2M_n}\left\|\Delta_j\Delta_{j+1}\right\|_{L_\infty(B_R)}$$

where C_2 depends only on R and not on m, n. The same estimate then holds for $\Delta_m Q_n - \Delta_n Q_m$ and hence if Γ_n is any simple closed curve inside B_R,

$$\min_{t\in\Gamma_n}\left|Q_n(t)\right|\left\|\Delta_m\right\|_{L_\infty(\Gamma_n)} \leq \left\|\Delta_n\right\|_{L_\infty(B_{2R})}(6R)^m + C_2^n\sum_{j=m}^{n-1}\varepsilon_n^{-2M_n}\left\|\Delta_j\Delta_{j+1}\right\|_{L_\infty(B_R)}.$$

Here by Lemma 3.1, we may choose a curve Γ_n lying in the annulus inside B_{2R} and outisde the ball B_R such that $|Q_n| \geq 1$ on Γ_n. Thus the maximum modulus principle gives

$$\left\|\Delta_m\right\|_{L_\infty(B_R)} \leq \left\|\Delta_n\right\|_{L_\infty(B_{2R})}(6R)^m + C_2^n\sum_{j=m}^{n-1}\varepsilon_n^{-2M_n}\left\|\Delta_j\Delta_{j+1}\right\|_{L_\infty(B_R)}.$$

Using Lemmas 4.1 and 4.3, we then have for large enough n, and corresponding m, as well as using the monotonicity of errors of best approximation,

$$E_{mm}^{1+\frac{1}{\phi(m)}}(f;B_R) \leq E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}} + (n-m)\,\varepsilon_n^{-2M_n}E_{mm}(f;B_R)^{2-\frac{2}{\phi(n)}}.$$

(5.7)

Recall that $m = n - \left[6\frac{n}{\phi(n)}\right]$. By Lemma 5.1,

$$\frac{E_{mm}^{1+\frac{1}{\phi(m)}}(f;B_R)}{E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}}} > E_{nn}(f;B_R)^{\left(1+\frac{1}{\phi(m)}\right)\left(1-\frac{4}{\phi(n)}\right)-\left(1-\frac{2}{\phi(n)}\right)}$$

$$= E_{nn}(f;B_R)^{-\frac{1}{\phi(n)}(1+o(1))} \to \infty \tag{5.8}$$

as $n \to \infty$. Next,

$$\begin{aligned} & \frac{(n-m)\,\varepsilon_n^{-2M_n} E_{mm}(f;B_R)^{2-\frac{2}{\phi(n)}}}{E_{mm}^{1+\frac{1}{\phi(m)}}(f;B_R)} \\ \leq\ & n e^{M_n \log(n+1)} E_{mm}(f;B_R)^{1-\frac{3}{\phi(n)}} \\ \leq\ & n \exp\left(M_n \log(n+1) - \xi_m m \phi(m)\left(1-\frac{3}{\phi(n)}\right)\right), \end{aligned}$$

where $\{\xi_m\}$ has limit ∞ by our hypothesis that

$$\lim_{n\to\infty} E_{nn}^{1/n\phi(n)} = 0.$$

Here as $m/n \to 1$ and $\phi(m)/\phi(n) \to 1$ as $n \to \infty$,

$$\begin{aligned} & M_n \log(n+1) - \xi_m m\phi(m)\left(1-\frac{3}{\phi(n)}\right) \\ =\ & n\phi(n)\left\{5\frac{\log n}{\phi(n)^2}(1+o(1)) - \xi_n(1+o(1))\right\} \\ \to\ & -\infty. \end{aligned}$$

Since

$$\frac{\log n}{\phi(n)^2} = O(1).$$

Thus

$$\frac{(n-m)\,\varepsilon_n^{-2M_n} E_{mm}(f;B_R)^{2-\frac{2}{\phi(n)}}}{E_{mm}^{1+\frac{1}{\phi(m)}}(f;B_R)} \to 0$$

as $m \to \infty$. This and (5.7) show that (?) is impossible for large enough n, and we have a contradiction. Thus if Case I holds, our original hypothesis that (5.4) is true fails.

Case II: R_k has $\geq M_n$ poles in B_{2R} for at least one k with $n \geq k \geq n - M_n$

In this case, we choose ℓ to be the largest such k, so that R_k has $< M_n$ poles for $n \geq k > \ell$ but R_ℓ has $\geq M_n$ poles in B_{2R}. Recall that R_n cannot have that many poles so necessarily $\ell < n$. We then proceed much as above, but using

$$P_n Q_\ell - P_\ell Q_n = Q_n Q_m \sum_{j=\ell}^{n-1} \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}}. \tag{5.9}$$

For $j > \ell$, we let $\mathcal{E}_j$ be as above. For $j = \ell$, we instead let $\mathcal{E}_\ell$ be the set on which

$$\frac{1}{|Q_\ell(z)|} \leq \left(\frac{1}{2}\right)^\ell$$

so that $m_2(\mathcal{E}_\ell) \leq \pi\left(\frac{1}{2}\right)^2$. We now define

$$\mathcal{E}_{\ell,n} = \bigcup_{j=\ell}^{n} \mathcal{E}_j,$$

so that

$$m_2(\mathcal{E}_{\ell,n}) \leq \pi\left(\frac{1}{2}\right)^2 + \pi.$$

This is still much smaller than $m_2(B_R)$, so proceeding as above, we obtain

$$\min_{t\in\Gamma_n} |Q_n|(t) \|\Delta_\ell\|_{L_\infty(B_R)} \leq \|\Delta_n\|_{L_\infty(B_{2R})} (6R)^m + C_2^n \left\{ \begin{array}{c} 2^n \varepsilon_n^{-M_n} \|\Delta_\ell \Delta_{\ell+1}\|_{L_\infty(B_R)} \\ + \sum_{j=\ell+1}^{n-1} \varepsilon_n^{-2M_n} \|\Delta_j \Delta_{j+1}\|_{L_\infty(B_R)} \end{array} \right\}$$

and hence as above,

$$E_{\ell\ell}^{1+\frac{1}{\phi(\ell)}}(f;B_R) \leq E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}} + (n-\ell)\,\varepsilon_n^{-2M_n} E_{\ell\ell}(f;B_R)^{2-\frac{2}{\phi(\ell)}}$$

Exactly as before, for large enough n,

$$E_{\ell\ell}^{1+\frac{1}{\phi(\ell)}}(f;B_R) >> (n-\ell)\,\varepsilon_n^{-2M_n} E_{\ell\ell}(f;B_R)^{2-\frac{2}{\phi(\ell)}}.$$

To deal with the first term, we now use our assumption in Case II that R_ℓ has $\geq M_n$ poles. Using Lemma 4.2, and Lemmas 4.1, 4.3, we obtain

$$E_{\ell-M_n,\ell-M_n}(f;B_R) \leq E_{\ell\ell}(f;B_R)^{1-\frac{1}{\phi(\ell)}},$$

so

$$\begin{aligned} & E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}} / E_{\ell\ell}^{1+\frac{1}{\phi(\ell)}}(f;B_R) \\ \leq\ & E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}} / E_{\ell-M_n,\ell-M_n}^{1+\frac{1}{\phi(\ell)}}(f;B_R) \\ \leq\ & E_{nn}(f;B_R)^{1-\frac{2}{\phi(n)}} / E_{n-M_n,n-M_n}^{1+\frac{1}{\phi(\ell)}}(f;B_R) \to 0 \text{ as } n\to\infty. \end{aligned}$$

Again we have a contradiction. Theorem 2 is proven. ■

References

[1] A. I. Aptekarev, M. L. Yattselev, *Padé approximants for functions with branch points - strong asymptotics of Nuttall-Stahl polynomials*, Acta Mathematica, 215(2015), 217–280.

[2] G.A. Baker, P. Graves-Morris, *Padé Approximants*, 2nd Edition, Cambridge University Press, Cambridge, 1996.

[3] G.A. Baker, *Some structural properties of two counter-examples to the Baker–Gammel–Wills conjecture*, J. Comput. Appl. Math., 161 (2003), 371 – 391.

[4] G.A. Baker, *Counter-examples to the Baker-Gammel-Wills conjecture and patchwork convergence*, J. Comput. Appl. Math. 179 (2005), 1–14.

[5] B. Beckermann, G. Labahn, A. C. Matos, *On rational functions without Froissart doublets,* manuscript.

[6] V. I. Buslaev, *The Baker-Gammel-Wills Conjecture in the Theory of Padé Approximants*, Math. USSR. Sbornik, 193(2002), 811–823.

[7] 13. V. I. Buslaev, *Convergence of the Rogers-Ramanujan Continued Fraction*, Math.USSR. Sbornik, 194(2003), 833–856.

[8] V. I. Buslaev, A. A. Gončar, and S. P. Suetin, *On Convergence of Subsequences of the mth Row of a Padé Table*, Math. USSR Sbornik, 48(1984), 535–540.

[9] F. Cala Rodrigues, G. Lopez Lagomasino, *Exact Rates of Convergence of Multipoint Padé Approximants*, Constr. Approx., 14(1988), 259-272.

[10] T. Claeys and F. Wielonsky, *On Sequences of Rational Interpolants of the Exponential Function with Unbounded Interpolation Points*, J. Approx. Theory, 171(2013), 1-32.

[11] K.A. Driver, *Simultaneous Rational Approximants for a Pair of Functions with Smooth Maclaurin Series Coefficients*, J. Approximation Theory, 83(1995), 308-329.

[12] J. Gilewicz, Y. Kryakin, *Froissart doublets in Padé approximation in the case of polynomial noise,* J. Comp. Appl. Math., 153(2003), 235–242.

[13] A. Gonchar and L. Grigorjan, *On Estimates of the Norm of the Holomorphic Component of a Meromorphic Function*, Math. USSr Sbornik, 28(1976), 571-575.

[14] A. Gonchar, G. Lopez Lagomasino, *Markov's Theorem for Multipoint Padé Approximants*, Math. Sbornik, 105(1978), 512-524.

[15] L. Grigorjan, *Estimates of the Norm of the Holomorphic Components of Functions Meromorphic in Domains with a Smooth Boundary*, Math. USSR Sbornik, 29(1976), 139-146.

[16] D.V. Khristoforov, *On the Phenomenon of Spurious Interpolation of Elliptic Functions by Diagonal Padé Approximant*s, Mathematical Notes, 87(2010), 564-574.

[17] R. Kovacheva, *Zeros of Padé error functions for functions with smooth Maclaurin coefficients,* J. Approximation Theory, 83(1995), 371-391.

[18] Yu. Labych and A. Starovoitov, *Trigonometric Padé Approximants for Functions with regularly decreasing Fourier coefficients*, Mathematics Sbornik, 200(2009), 1051-1074.

[19] P. Lancaster, M. Tismenetsky, *The Theory of Matrices,* Academic Press, San Diego, 1985.

[20] Eli Levin and D.S. Lubinsky, *Rows and Diagonals of the Walsh Array for Entire Functions with Smooth Maclaurin Series Coefficients*, Constructive Approximation, 6(1990), 257-286.

[21] D.S. Lubinsky, *Padé Tables of a Class of Entire Functions*, Proc. Amer. Math. Soc., 94(1985), 399-405.

[22] D.S. Lubinsky, *Padé Tables of Entire Functions of Very Slow and Smooth Growth*, Constr. Approx., 1(1985), 349-358.

[23] D.S. Lubinsky, *Padé Tables of Entire Functions of Very Slow and Smooth Growth II*, Constr. Approx., 4(1988), 321-339.

[24] D.S. Lubinsky, *Distribution of Poles of Diagonal Rational Approximants to Functions of Fast Rational Approximability*, Constructive Approximation, 7(1991), 501-519.

[25] D.S. Lubinsky, *Spurious Poles in Diagonal Rational Approximation*, (in) Progress in Approximation Theory (eds. A.A. Gončar and E.B. Saff), Springer-Verlag, Berlin, 1992, pp. 191-213.

[26] D.S. Lubinsky, *Rogers-Ramanujan and the Baker-Gammel-Wills (Padé) Conjecture*, Annals of Mathematics, 157(2003), 847-889.

[27] D.S. Lubinsky, *Exact Interpolation, Spurious Poles, and Uniform Convergence of Multipoint Padé Approximants*, to appear in Math. Sbornik.

[28] L.M. Milne-Thomson, *The Calculus of Finite Differences*, Chelsea, New York, 1933.

[29] E. A. Rakhmanov, *On the Convergence of Padé Approximants in Classes of Holomorphic Functions*, Math. USSR Sbornik, 40(1981), 149–155.

[30] T. Ransford,

[31] N. Rusak and A. P. Starovoitov, *Padé approximants for entire functions with regularly decreasing Taylor coefficients,* Mathematics Sbornik 193(9), 63–92 (2002) [Russian Acad. Sci. Sb.Math. 193 (9), 1303–1332 (2002)].

[32] H. Stahl, *Spurious Poles in Padé Approximation*, J. Comp. Appl. Math., 9(1998), 511-527.

[33] S.P. Suetin, *Distribution of the Zeros of Padé polynomials and analytic continuation*, Russian Math. Surveys, 70(2015),901-951.

[34] H. Wallin, *The Convergence of Padé Approximants and the Size of the Power Series Coefficients*, Applicable Anal., 4(1974), 235-251.

[35] F. Wielonsky, *Riemann-Hilbert Analysis and Uniform Convergence of Rational Interpolants to the Exponential Function*, J. Approx. Theory, 131(2004), 100-148.

[36] M. Yattselev, *Meromorphic Approximation: Symmetric Contours and Wandering Poles*, manuscript.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu