THE EFFECT OF ADDING ENDPOINT MASSPOINTS ON BOUNDS FOR ORTHOGONAL POLYNOMIALS

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Abstract. Let ν be a positive measure supported on $[-1,1]$, with infinitely many points in its support. Let $\{p_n(\nu,x)\}_{n\geq 0}$ be its sequence of orthonormal polynomials. Suppose we add masspoints at ± 1, giving a new measure $\mu = \nu + M\delta_1 + N\delta_{-1}$. How much larger can $|p_n(\mu,0)|$ be than $|p_n(\nu,0)|$? We study this question for symmetric measures, and give more precise results for ultraspherical weights. Under quite general conditions, such as ν lying in the Nevai class, it turns out that the growth is no more than $1+o(1)$ as $n\to\infty$.

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1. Results

Let μ be a finite positive Borel measure on the real line with infinitely many points in its support, and all finite moments

$$\int t^j d\mu(t), \; j = 0,1,2,\ldots .$$

Then we may define orthonormal polynomials

$$p_n(\mu,x) = \gamma_n(\mu)x^n + \ldots, \gamma_n(\mu) > 0,$$

$n = 0,1,2,\ldots$ satisfying the orthonormality conditions

$$\int p_n(\mu,x)\,p_m(\mu,x)\,d\mu(x) = \delta_{mn}.$$

The zeros of $p_n(\mu,x)$ are denoted by

$$x_{nn}(\mu) < x_{n-1,n}(\mu) < \ldots < x_{2n}(\mu) < x_{1n}(\mu).$$

The nth reproducing kernel for μ is

$$K_n(\mu,x,t) = \sum_{j=0}^{n-1} p_j(\mu,x)\,p_j(\mu,t) = \frac{\gamma_{n-1}}{\gamma_n}(\mu)\,\frac{p_n(\mu,x)\,p_{n-1}(\mu,t) - p_{n-1}(\mu,x)\,p_{n-1}(\mu,t)}{x-t}.$$

The three term recurrence relation has the form

$$(x - b_n(\mu))\,p_n(\mu,x) = a_{n+1}(\mu)\,p_{n+1}(\mu,x) + a_n(\mu)\,p_{n-1}(\mu,x),$$

where

$$a_n(\mu) = \frac{\gamma_{n-1}}{\gamma_n}(\mu).$$

A central problem in the theory of orthonormal polynomials is to establish bounds on $p_n(\mu,x)$, and there is an extensive literature. See for example [1], [3], [5], [8], [12], [14]. In this paper, our goal is to assess how adding masspoints at ± 1 can increase the size of the orthonormal polynomial at the origin. We take advantage of

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the fact that a lot is known about the orthogonal polynomials for measures formed by adding such masspoints. Differential equations and other identities have been obtained, asymptotics as $n \to \infty$ have been established, and Sobolev analogues have been investigated. See [2], [4], [7], [10], [11] for some references.

Consider a fixed positive measure ν supported on $[-1,1]$ with infinitely many points in its support, and that is symmetric about 0, so that $\nu([-b,-a]) = \nu([a,b])$ for all $[a,b] \subset [-1,1]$. Fix $S > 0$. We let $\mathcal{M}(\nu, S)$ denote the class of all measures

$$\mu = \nu + M\delta_1 + N\delta_{-1} \tag{1}$$

where $M, N \geq 0$ and $M + N \leq S$. We let $\mathcal{M}(\nu)$ denote the class of all measures of this form with $M, N \geq 0$ and no restriction on $M + N$.

We shall need some auxiliary parameters that depend only on n and ν. For even integers n, we set

$$r_n = \frac{\gamma_{n-1}}{\gamma_n}(\nu)\frac{p_{n-1}(\nu,1)}{p_n(\nu,1)} = -\frac{K_n(\nu,-1,1)}{p_n^2(\nu,1)}. \tag{2}$$

The second formula for r_n follows from the Christoffel-Darboux formula, and symmetry of ν. Also let

$$\begin{aligned} U_n &= K_n(\nu,1,1) - K_n(\nu,-1,1); \\ V_n &= K_n(\nu,1,1) + K_n(\nu,-1,1). \end{aligned} \tag{3}$$

We note that it follows from the recurrence relation that $0 < r_n < 1$, while the symmetry of ν and Cauchy-Schwarz show that $U_n, V_n > 0$ (see (27) below).

We prove:

Theorem 1.1
Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support. Assume also that ν is symmetric, so that $\nu([-b,-a]) = \nu([a,b])$ for all subintervals $[a,b]$ of $[-1,1]$. Let $n \geq 2$ be even. Then

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \max\left\{1, \frac{U_n^2}{V_n V_{n+1}}\right\}. \tag{4}$$

Moreover,

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \frac{U_n^2}{V_n V_{n+1}} > 1 \tag{5}$$

iff

$$\frac{2p_n^2(\nu,1)}{V_n} > \frac{1-2r_n}{r_n^2}. \tag{6}$$

Remarks
(a) We have been unable to find a measure for which (6) fails, but nor have we been able to prove that it is always true. It is true for all even Jacobi weights and large enough n, as we shall see below.
(b) Interestingly enough, the supremum in (4) is not attained. It occurs as $M = N \to \infty$. However, we note that for a large class of measures, it decays to 1 as $n \to \infty$:

Corollary 1.2
Assume in addition to the hypotheses of Theorem 1.1, that ν lies in the Nevai class, so that the recurrence coefficients satisfy

$$\lim_{n\to\infty} a_n(\nu) = \frac{1}{2}. \tag{7}$$

Then

$$\lim_{n\to\infty} \left(\sup_{\mu\in\mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)} \right)^2 \right) = 1. \tag{8}$$

Remarks
(i) Note that since ν is symmetric about 0, $b_n(\nu) = 0$ for all n.
(ii) The only property that we use of the Nevai class is *subexponential growth* at 1 :

$$\lim_{n\to\infty} p_n(\nu,1)^2 / K_n(\nu,1,1) = 0.$$

Next, we consider the case where we maximize over the class $\mathcal{M}(\nu,S)$. For a given $S>0$, and given n, let

$$X_S = p_n^2(\nu,1) \frac{S + S^2 U_n/2}{S^2 U_n V_n/4 + S K_n(\nu,1,1) + 1}. \tag{9}$$

In the course of our proofs, we shall show that X_S is an increasing function of $S>0$, and its limit as $S\to\infty$ coincides with the left-hand side of (6). We prove:

Theorem 1.3
Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support. Assume also that ν is symmetric, so that $\nu([-b,-a]) = \nu([a,b])$ for all subintervals $[a,b]$ of $[-1,1]$. Let $n\geq 2$ be even and $S>0$ and let $\mathcal{M}(\nu,S)$ denote the class of measures defined above.
(a) There exists $\mu^* = \nu + M^*\delta_1 + N^*\delta_{-1} \in \mathcal{M}(\nu,S)$ satisfying

$$|p_n(\mu^*,0)| = \max\{|p_n(\mu,0)| : \mu\in\mathcal{M}(\nu,S)\}. \tag{10}$$

(b) If $X_S < \frac{1-2r_n}{r_n^2}$, then $M^* = N^* = 0$, $\mu^* = \nu$, and

$$|p_n(\mu^*,0)| = |p_n(\nu,0)|. \tag{11}$$

(c) If $X_S > \frac{1-2r_n}{r_n^2}$, then $M^* = N^* = \frac{S}{2}$, $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$, and

$$\begin{aligned} &\left(\frac{p_n(\mu^*,0)}{p_n(\nu,0)}\right)^2 \\ = \quad & \frac{\left(S^2U_n^2/4 + SU_n + 1\right)^2}{\left(S^2U_nV_n/4 + SK_n(\nu,1,1)+1\right)\left(S^2U_nV_{n+1}/4 + SK_{n+1}(\nu,1,1)+1\right)} > 1. \end{aligned}$$

(12)

(d) If $X_S = \frac{1-2r_n}{r_n^2}$, then there are two extremal measures, namely $\mu^* = \nu$, and $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$, and (11) holds.
(e) In all cases,

$$\max_{\mu\in\mathcal{M}(\nu,S)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \max\left\{1, \frac{(1+r_nX_S)^2}{1+X_S}\right\}.$$

Thus the extremal measure is always symmetric. It is also unique, except when $X_S = \frac{1-2r_n}{r_n^2}$. For even Jacobi weights (or equivalently ultraspherical weights), we obtain more explicit results:

Theorem 1.4
Let $\alpha > -1$ *and*

$$\nu'(t) = \left(1-t^2\right)^{\alpha}, t \in (-1,1). \tag{13}$$

For even $n \geq 2$, *the inequality (5) holds, and*

$$\begin{aligned} &\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 \\ = \;& 1 + \frac{\left(\frac{1}{n+\alpha}\right)^2 2(\alpha+1)\{1+\frac{2\alpha+1}{n}\}}{1+2\frac{\alpha+1}{n+\alpha}+\frac{\alpha+1}{(n+\alpha)^2}\left\{\alpha-1-\frac{2(2\alpha+1)}{n}\right\}} \\ = \;& 1+\frac{2(\alpha+1)}{(n+\alpha)^2}+O\left(n^{-3}\right). \end{aligned} \tag{14}$$

Thus for all $\alpha > -1$, the supremum exceeds 1 for large enough n, but decays to 1 with rate $O\left(n^{-2}\right)$ as $n \to \infty$. For fixed S, we prove:

Theorem 1.5
Let ν, n *be as in Theorem 1.4 and let* $S > 0$. *Let* $\mu^* = \nu + M^*\delta_1 + N^*\delta_{-1} \in \mathcal{M}(\nu, S)$ *be an extremal measure satisfying (10).*
(a) Suppose $-1 < \alpha < -\frac{1}{2}$. *Then there exists* $n_0(\alpha)$ *such that for* $n \geq n_0(\alpha)$, $r_n > \frac{1}{2}$. *Moreover, for* $n \geq n_0(\alpha)$ *and for all* $S > 0$, $M^* = N^* = \frac{S}{2}$ *and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$.
(b) Suppose $\alpha > -\frac{1}{2}$. *Then there exists* $n_0(\alpha)$ *such that for* $n \geq n_0(\alpha)$, $r_n < \frac{1}{2}$. *Then for* $n \geq n_0(\alpha)$ *and* $S > 0$ *so small that* $X_S < \frac{1-2r_n}{r_n^2}$, $M^* = N^* = 0$ *and* $\mu^* = \nu$. *For* $n \geq n_0(\alpha)$ *and* $X_S = \frac{1-2r_n}{r_n^2}$, *we may take* $\mu^* = \nu$, *or* $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$. *For* $n \geq n_0(\alpha)$ *and* $X_S > \frac{1-2r_n}{r_n^2}$, $M^* = N^* = \frac{S}{2}$ *and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$.
(c) Suppose $\alpha = -\frac{1}{2}$. *Then* $r_n = \frac{1}{2}$. *For* $n \geq 2$, $M^* = N^* = \frac{S}{2}$ *and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$.

Observe that if $\alpha > -\frac{1}{2}$, the extremal measure is $\mu^* = \nu$ for small enough S, but once S increases beyond a certain threshold, $\mu^* = \nu + \frac{S}{2}(\delta_{-1}+\delta_1)$. It is possible to give a more explicit form to the expression for the sup in (10) for ultraspherical weights, but it is messy and so omitted.

This paper is organized as follows: In Section 2, we present a basic identity. In Section 3, we first prove Theorem 1.3 and then Theorem 1.1 and Corollary 1.2. In Section 4, we first prove Theorem 1.4 and then Theorem 1.5.

In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurences.

2. The Basic Identity

Throughout this section, ν satisfies the hypotheses of Theorem 1.1. Recall that r_n, U_n, V_n and X_S are defined by (2), (3) and (9). Our analysis is based on the identity in Lemma 2.2 below. We do not claim that it is new, as identities of this type are commonly used in analyzing measures with added masspoints, but derive it in a form that we can apply it:

Theorem 2.1
Let $n \geq 2$ be even. Let $M, N \geq 0$ and

$$\mu = \nu + M\delta_1 + N\delta_{-1}.$$

Let

$$x = x(M,N) = p_n^2(\nu,1)\frac{2MNU_n + M + N}{MNU_nV_n + (M+N)K_n(\nu,1,1)+1}. \tag{15}$$

(a) Then

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = g(x) := \frac{(1+r_nx)^2}{1+x}.$$

(b) If $r_n < \frac{1}{2}$, the function g is a strictly decreasing function of $x \in (0, \frac{1-2r_n}{r_n})$ and is a strictly increasing function of $x \in (\frac{1-2r_n}{r_n}, \infty)$.
(c) If $r_n \geq \frac{1}{2}$, the function g is a strictly increasing function of $x \in (0,\infty)$.
(d) $g(x) > 1$ iff

$$x > \frac{1-2r_n}{r_n^2}. \tag{16}$$

while $g(x) = 1$ iff $x = \frac{1-2r_n}{r_n}$ or $x = 0$.

We begin the proof with

Lemma 2.2
(a) Let

$$\pi_{n-1}(y) = p_n(\mu,y) - \frac{\gamma_n(\mu)}{\gamma_n(\nu)}p_n(\nu,y); \tag{17}$$

$$A = \begin{bmatrix} 1+MK_n(\nu,1,1) & -MK_n(\nu,1,-1) \\ -NK_n(\nu,1,-1) & 1+NK_n(\nu,1,1) \end{bmatrix}; \tag{18}$$

and

$$d = MNU_nV_n + (M+N)K_n(\nu,1,1)+1. \tag{19}$$

(a) Then

$$p_n(\mu,y) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)}\left\{p_n(\nu,y) + \frac{p_n(\nu,1)}{d}\begin{bmatrix} -NK_n(\nu,y,-1) \\ -MK_n(\nu,y,1)\end{bmatrix}^T A \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}. \tag{20}$$

(b)

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\left\{1 + \frac{p_n^2(\nu,1)}{d}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T A^T \begin{bmatrix} N \\ M \end{bmatrix}\right\} = 1. \tag{21}$$

Proof

(a) Using orthogonality, we see that

$$\begin{aligned}\pi_{n-1}(y) &= \int_{-1}^{1} K_n(\nu, y, t)\, \pi_{n-1}(t)\, d\nu(t) \\ &= \int_{-1}^{1} K_n(\nu, y, t)\, p_n(\mu, t)\, d\nu(t) \\ &= -MK_n(\nu, y, 1)\, p_n(\mu, 1) - NK_n(\nu, y, -1)\, p_n(\mu, -1).\end{aligned}$$

(22)

Taking $y = -1$ and $y = 1$, and gathering the terms involving $p_n(\mu, \pm 1)$, gives the matrix equation

$$\begin{bmatrix} 1 + NK_n(\nu, -1, -1) & MK_n(\nu, -1, 1) \\ NK_n(\nu, 1, -1) & 1 + MK_n(\nu, 1, 1) \end{bmatrix} \begin{bmatrix} p_n(\mu, -1) \\ p_n(\mu, 1) \end{bmatrix} = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \begin{bmatrix} p_n(\nu, -1) \\ p_n(\nu, 1) \end{bmatrix}.$$

The determinant d of the matrix can be put into the form in (19), if we take account of the definition (3) of U_n, V_n. Solving the matrix equation and using the symmetry of ν gives

$$\begin{aligned}\begin{bmatrix} p_n(\mu, -1) \\ p_n(\mu, 1) \end{bmatrix} &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \frac{1}{d} \begin{bmatrix} 1 + MK_n(\nu, 1, 1) & -MK_n(\nu, 1, -1) \\ -NK_n(\nu, 1, -1) & 1 + NK_n(\nu, 1, 1) \end{bmatrix} \begin{bmatrix} p_n(\nu, 1) \\ p_n(\nu, 1) \end{bmatrix} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, 1)\, \frac{A}{d} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.\end{aligned}$$

(23)

From (22) and this last identity,

$$\pi_{n-1}(y) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \frac{p_n(\nu, 1)}{d} \begin{bmatrix} -NK_n(\nu, y, -1) \\ -MK_n(\nu, y, 1) \end{bmatrix}^T A \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then (20) follows from the definition of π_{n-1}.

(b) We obtain equations for $\frac{\gamma_n(\mu)}{\gamma_n(\nu)}$ in two ways:

$$\begin{aligned}&\int_{-1}^{1} \pi_{n-1}^2(y)\, d\nu(y) \\ &= \int_{-1}^{1} p_n^2(\mu, y)^2\, d\nu(y) - 2\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 + \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\ &= 1 - Mp_n(\mu, 1)^2 - Np_n(\mu, -1)^2 - \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2.\end{aligned}$$

Also, from (22),

$$\begin{aligned}&\int_{-1}^{1} \pi_{n-1}^2(y)\, d\nu(y) \\ &= \int_{-1}^{1} \left(-NK_n(\nu, y, -1)\, p_n(\mu, -1) - MK_n(\nu, y, 1)\, p_n(\mu, 1)\right)^2 d\nu(y) \\ &= N^2 p_n^2(\mu, -1)\, K_n(\nu, -1, -1) + M^2 p_n^2(\mu, 1)\, K_n(\nu, 1, 1) + 2MN p_n(\mu, -1)\, p_n(\mu, 1)\, K_n(\nu, -1, 1).\end{aligned}$$

Then using the last two equations and solving for $1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2$,

$$\begin{aligned}
&1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\
=\;& p_n^2(\mu,-1)\left\{N+N^2K_n(\nu,-1,-1)\right\}+p_n^2(\mu,1)\left\{M+M^2K_n(\nu,1,1)\right\} \\
&+2MNp_n(\mu,-1)\,p_n(\mu,1)\,K_n(\nu,-1,1) \\
=\;& \begin{bmatrix} p_n(\mu,-1) \\ p_n(\mu,1) \end{bmatrix}^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 1+NK_n(\nu,1,1) & MK_n(\nu,-1,1) \\ NK_n(\nu,-1,1) & 1+MK_n(\nu,1,1) \end{bmatrix} \begin{bmatrix} p_n(\mu,-1) \\ p_n(\mu,1) \end{bmatrix} \\
=\;& d\begin{bmatrix} p_n(\mu,-1) \\ p_n(\mu,1) \end{bmatrix}^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} A^{-1} \begin{bmatrix} p_n(\mu,-1) \\ p_n(\mu,1) \end{bmatrix}
\end{aligned}$$

Using (23) gives

$$\begin{aligned}
&1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\
=\;& \frac{p_n^2(\nu,1)}{d}\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T A^T \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

and (21) follows.■

Proof of Theorem 2.1(a)
Setting $y=0$ in (20), squaring and multiplying by the factor {} in (21) gives

$$\begin{aligned}
& p_n^2(\mu,0)\left\{1+\frac{p_n^2(\nu,1)}{d}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T A^T \begin{bmatrix} N \\ M \end{bmatrix}\right\} \\
=\;& \left\{p_n(\nu,0)+\frac{p_n(\nu,1)}{d}\begin{bmatrix} -NK_n(\nu,0,-1) \\ -MK_n(\nu,0,1) \end{bmatrix}^T A \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}^2.
\end{aligned}$$

(24)

Here from the Christoffel-Darboux formula and as $p_{n-1}(\nu,0)=0$, while $p_{n-1}(\nu,-1)=-p_{n-1}(\nu,1)$,

$$K_n(\nu,0,\pm 1)=-\frac{\gamma_{n-1}}{\gamma_n}(\nu)\,p_n(\nu,0)\,p_{n-1}(\nu,1)$$

so using Christoffel-Darboux again,

$$p_n(\nu,1)\,K_n(\nu,0,\pm 1)=p_n(\nu,0)\,K_n(\nu,-1,1)\,.$$

Thus (24) becomes

$$\begin{aligned}
& \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2\left\{1+\frac{p_n^2(\nu,1)}{d}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T A^T \begin{bmatrix} N \\ M \end{bmatrix}\right\} \\
=\;& \left\{1-\frac{K_n(\nu,-1,1)}{d}\begin{bmatrix} N \\ M \end{bmatrix}^T A \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}^2.
\end{aligned}$$

(25)

Here from (18) and (19), followed by (15),

$$\begin{aligned}
&\frac{p_n^2(\nu,1)}{d}\begin{bmatrix}1\\1\end{bmatrix}^T A^T\begin{bmatrix}N\\M\end{bmatrix}\\
&= p_n^2(\nu,1)\frac{N+M+2MNU_n}{MNU_nV_n+(M+N)K_n(\nu,1,1)+1}=x.
\end{aligned}$$

Also, from (2),

$$K_n(\nu,-1,1)=-\frac{\gamma_{n-1}}{\gamma_n}(\nu)\,p_n(\nu,1)\,p_{n-1}(\nu,1)=-r_np_n^2(\nu,1)$$

(26)

so (25) becomes

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2\{1+x\}=\{1+r_nx\}^2.$$

■

Proof of Theorem 2.1 (b), (c), (d)
A calculation shows that

$$g(x)=r_n^2x+\left(2r_n-r_n^2\right)+\frac{(r_n-1)^2}{1+x}$$

so

$$g'(x)=r_n^2\left\{1-\frac{\left(1-\frac{1}{r_n}\right)^2}{(1+x)^2}\right\}.$$

Thus $g'(x)$ is an increasing function of $x\in[0,\infty)$, with limit $r_n^2>0$ as $x\to\infty$. Also

$$g'(x)=0\Leftrightarrow 1+x=\pm\left(1-\frac{1}{r_n}\right)$$

so as $x>0$, and $r_n>0$,

$$g'(x)=0\Leftrightarrow x=\frac{1-2r_n}{r_n}.$$

Then if $r_n<\frac{1}{2}$, it follows that $g(x)$ decreases in $\left(0,\frac{1-2r_n}{r_n}\right)$ and increases in $\left(\frac{1-2r_n}{r_n},\infty\right)$. If $r_n\geq\frac{1}{2}$, it follows that $g(x)$ increases in $[0,\infty)$. Finally

$$\begin{aligned}
g(x) &> 1\Leftrightarrow 1+2r_nx+r_n^2x^2>1+x\\
&\Leftrightarrow x>\frac{1-2r_n}{r_n^2},
\end{aligned}$$

as $x>0$. Also $g(x)=1$ iff $x=0$ or $x=\frac{1-2r_n}{r_n^2}$.■

3. Proof of Theorems 1.1 and 1.3

Recall that $x = x(M,N)$ is given by (15). We begin with

Lemma 3.1
(a) For $M, N \geq 0$,

$$\frac{\partial x}{\partial M} > 0; \frac{\partial x}{\partial N} > 0.$$

(b) The maximum of $x = x(M,N)$ in the triangular region $T = \{(M,N) : 0 \leq M, N \text{ and } M + N \leq S\}$ occurs when and only when

$$M = N = \frac{S}{2}.$$

(c) Moreover, the maximum is

$$x = X_S = p_n^2(\nu,1) \frac{S^2 U_n/2 + S}{S^2 U_n V_n/4 + S K_n(\nu,1,1) + 1}.$$

(d)

$$X_\infty := \lim_{S\to\infty} X_S = \frac{2p_n^2(\nu,1)}{V_n}.$$

Proof
(a) Note that from Cauchy-Schwarz, and as $p_j(\nu,-1) = (-1)^j p_j(\nu,1)$,

$$\begin{aligned}
|K_n(\nu,-1,1)| &= \left|\sum_{j=0}^{n-1} p_j(\nu,1)\, p_j(\nu,-1)\right| \\
&< \sum_{j=0}^{n-1} |p_j(\nu,1)\, p_j(\nu,-1)| \\
&\leq \sqrt{K_n(\nu,1,1)\, K_n(\nu,-1,-1)} = K_n(\nu,1,1)
\end{aligned}$$

so that

$$U_n, V_n > 0. \tag{27}$$

Next, using $V_n - 2K_n(\nu,1,1) = -U_n$, and from (15),

$$\begin{aligned}
&\frac{1}{p_n^2(\nu,1)} (MNU_nV_n + (M+N)K_n(\nu,1,1) + 1)^2 \left(\frac{\partial x}{\partial M}\right) \\
=\;& (2NU_n + 1)(MNU_nV_n + (M+N)K_n(\nu,1,1) + 1) - (2MNU_n + M + N)(NU_nV_n + K_n(\nu,1,1)) \\
=\;& MNU_n\{(2NU_n+1)V_n - 2(NU_nV_n + K_n(\nu,1,1))\} \\
&+ (M+N)\{(1+2NU_n)K_n(\nu,1,1) - (NU_nV_n + K_n(\nu,1,1))\} + 2NU_n + 1 \\
=\;& MNU_n\{V_n - 2K_n(\nu,1,1)\} + (M+N)\{NU_n(2K_n(\nu,1,1) - V_n)\} + 2NU_n + 1 \\
=\;& MNU_n\{-U_n\} + (M+N)\{NU_n^2\} + 2NU_n + 1 \\
=\;& (NU_n + 1)^2 > 0.
\end{aligned}$$

Thus

$$\frac{\partial x}{\partial M} = p_n^2(\nu,1)\frac{(NU_n+1)^2}{d^2}.$$

Then as $U_n > 0$, $\frac{\partial x}{\partial M} > 0$ and similarly $\frac{\partial x}{\partial N} > 0$.
(b) Since $\frac{\partial x}{\partial M} > 0, \frac{\partial x}{\partial N} > 0$ for all $M, N \geq 0$, so there are no critical points within

the interior of the triangle. Moreover, it then follows that the maximum cannot occur on the axes $M=0$ or $N=0$, so occurs when $M+N=S$. Then on this line segment,

$$\begin{aligned} x &= p_n^2(\nu,1)\frac{2M(S-M)U_n+S}{M(S-M)U_nV_n+SK_n(\nu,1,1)+1} \\ &= \frac{p_n^2(\nu,1)}{V_n}\left\{2+\frac{SV_n-2SK_n(\nu,1,1)-2}{M(S-M)U_nV_n+SK_n(\nu,1,1)+1}\right\} \\ &= \frac{p_n^2(\nu,1)}{V_n}\left\{2-\frac{SU_n+2}{M(S-M)U_nV_n+SK_n(\nu,1,1)+1}\right\}. \end{aligned}$$

(28)

Here we have used the definition of U_n, V_n. Since $S\geq 0$ is fixed and $U_n, V_n>0$, this last expression is an increasing function of $M(S-M)$ and in turn that is maximized over $M\in[0,S]$ when and only when $M=\frac{S}{2}$.
(c) This follows by substituting $M=N=\frac{S}{2}$ into the first line in (28).
(d) This is immediate from (c). ■

Proof of Theorem 1.3(a)
We can choose sequences $\{M_m\}$ and $\{N_m\}$ of nonnegative numbers with $0\leq M_m+N_m\leq S$ and if

$$\mu_m=\nu+M_m\delta_1+N_m\delta_{-1},$$

then

$$\lim_{m\to\infty}|p_n(\mu_m,0)|=\sup\{|p_n(\mu,0)|:\mu\in\mathcal{M}(\nu,S)\}.$$

By passing to a subsequence, and relabeling, we can assume that $\{\mu_m\}$ converges weakly to μ^* while $M_m\to M^*$ and $N_m\to N^*$ so that $\mu^*=\nu+M^*\delta_1+N^*\delta_{-1}$. Then for each fixed $j\geq 0$,

$$\lim_{m\to\infty}\int t^j d\mu_m(t)=\int t^j d\mu^*(t).$$

It follows from the determinantal representation of orthonormal polynomials [9, p. 57], [16, p. 23] that

$$|p_n(\mu^*,0)|=\lim_{m\to\infty}|p_n(\mu_m,0)|=\sup\{|p_n(\mu,0)|:\mu\in\mathcal{M}(\nu,S)\}.$$

■

Proof of Theorem 1.3(b)
We're assuming that $X_S<\frac{1-2r_n}{r_n^2}$. Of course this is possible only if $r_n<\frac{1}{2}$, since $X_S>0$. Let $0\leq M,N$ and $M+N\leq S$ and $\mu=\nu+M\delta_1+N\delta_{-1}$. By Theorem 2.1, if $x=x(M,N)$, we have

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2=\frac{(1+r_nx)^2}{1+x}=g(x).$$

Here by Lemma 3.1, $0\leq x\leq X_S<\frac{1-2r_n}{r_n^2}$, so Theorem 2.1(d) shows that

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2<1,$$

unless $x = 0$. It follows that the maximum possible value of $\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2$ for $\mu \in \mathcal{M}(\nu, S)$ occurs iff $M = N = 0$. ■

Proof of Theorem 1.3(c)
We're assuming that $X_S > \frac{1-2r_n}{r_n^2}$. By Theorem 2.1, if $x = x(M, N)$, we have

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \frac{(1+r_n x)^2}{1+x} = g(x)$$

is maximal when x is large as possible under the restrictions $0 \le M, N$ and $M+N \le S$. By Lemma 3.1, this occurs iff $M = N = \frac{S}{2}$, and then $x = X_S$. Here from (2) and (9),

$$r_n X_S = -SK_n(\nu,-1,1)\frac{1+SU_n/2}{S^2U_nV_n/4 + SK_n(\nu,1,1)+1}$$

so

$$\begin{aligned} & 1+r_nX_S \\ = & \frac{S^2(U_nV_n - 2U_nK_n(\nu,-1,1))/4 + S(K_n(\nu,1,1) - K_n(\nu,-1,1)) + 1}{S^2U_nV_n/4 + SK_n(\nu,1,1)+1} \\ = & \frac{S^2U_n^2/4 + SU_n + 1}{S^2U_nV_n/4 + SK_n(\nu,1,1)+1} \end{aligned}$$

while

$$\begin{aligned} & 1+X_S \\ = & \frac{S^2U_n\left[V_n + 2p_n^2(\nu,1)\right]/4 + S\left[K_n(\nu,1,1) + p_n^2(\nu,1)\right] + 1}{S^2U_nV_n/4 + SK_n(\nu,1,1)+1} \\ = & \frac{S^2U_nV_{n+1}/4 + SK_{n+1}(\nu,1,1)+1}{S^2U_nV_n/4 + SK_n(\nu,1,1)+1} \end{aligned}$$

Then

$$\begin{aligned} & \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \frac{(1+r_nX_S)^2}{1+r_nX_S} \\ = & \frac{\left(S^2U_n^2/4 + SU_n + 1\right)^2}{(S^2U_nV_n/4 + SK_n(\nu,1,1)+1)(S^2U_nV_{n+1}/4 + SK_{n+1}(\nu,1,1)+1)}. \end{aligned}$$

By Theorem 2.1(d), and as $X_S > \frac{1-2r_n}{r_n^2}$, this exceeds 1. ■

Proof of Theorem 1.3(d)
Here as $X_S = \frac{1-2r_n}{r_n^2}$, we have $g(X_s) = 1 = g(0)$, and for any other value of $x = x(M, N)$ we have $g(x) < 1$. ■

Proof of Theorem 1.3(e)
It follows from Theorem 2.1 and Lemma 3.1, that for a given $S > 0$,

$$\sup_{\mu\in\mathcal{M}(\nu,S)}\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \max\left\{1, \frac{(1+r_nX_S)^2}{1+X_S}\right\}$$

and moreover the sup is attained. Indeed if $X_S \le \frac{1-2r_n}{r_n^2}$, the maximum is 1, while if $X_S > \frac{1-2r_n}{r_n^2}$, the maximum is achieved when $M = N = \frac{S}{2}$. If $X_S = \frac{1-2r_n}{r_n^2}$, the maximum is achieved when $M = N = \frac{S}{2}$ and $M = N = 0$. ■

Proof of Theorem 1.1
From Lemma 3.1, Theorem 1.3(e) and (12),

$$\begin{aligned}
&\sup_{\mu\in\mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 \\
&= \lim_{S\to\infty} \sup_{\mu\in\mathcal{M}(\nu,S)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 \\
&= \max\left\{1, \frac{\left(U_n^2/4\right)^2}{\left(U_nV_n/4\right)U_nV_{n+1}/4}\right\} \\
&= \max\{1, \frac{U_n^2}{V_nV_{n+1}}\}.
\end{aligned}$$

Finally, the above considerations show that we can drop the 1 in the max, that is

$$\sup_{\mu\in\mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \frac{U_n^2}{V_nV_{n+1}} > 1$$

iff for large enough S, $X_S > \frac{1-2r_n}{r_n^2}$, which is true iff (recall Lemma 3.1(d))

$$\frac{2p_n^2(\nu,1)}{V_n} = X_\infty > \frac{1-2r_n}{r_n^2}. \tag{29}$$

■

We have been unable to resolve if (29) is always true. Here is an equivalent form:

Lemma 3.2
The inequality (29) is equivalent for even n to

$$\frac{-K_n(\nu,-1,1)}{K_{n+1}(\nu,-1,1)} > \frac{K_n(\nu,1,1)}{K_{n+1}(\nu,1,1)}.$$

Proof
From the second identity in (2),

$$\frac{1-2r_n}{r_n^2} = \frac{p_n^2(\nu,1)}{K_n(\nu,1,-1)^2}\left[2K_n(\nu,1,-1) + p_n^2(\nu,1)\right],$$

so (29) is equivalent to

$$\begin{aligned}
\frac{2p_n^2(\nu,1)}{V_n} &> \frac{p_n^2(\nu,1)}{K_n(\nu,-1,1)^2}\left(p_n^2(\nu,1) + 2K_n(\nu,-1,1)\right) \\
&\Leftrightarrow\ 2K_n(\nu,-1,1)^2 > \left(K_n(\nu,1,1) + K_n(\nu,-1,1)\right)\left(p_n^2(\nu,1) + 2K_n(\nu,-1,1)\right) \\
&\Leftrightarrow\ 0 > \left(K_n(\nu,1,1) + K_n(\nu,-1,1)\right)p_n^2(\nu,1) + 2K_n(\nu,1,1)K_n(\nu,-1,1) \\
&\Leftrightarrow\ 0 > \left(K_n(\nu,1,1) + p_n^2(\nu,1)\right)K_n(\nu,-1,1) + \left(K_n(\nu,-1,1) + p_n^2(\nu,1)\right)K_n(\nu,1,1) \\
&\Leftrightarrow\ 0 > K_{n+1}(\nu,1,1)K_n(\nu,-1,1) + K_{n+1}(\nu,-1,1)K_n(\nu,1,1) \\
&\Leftrightarrow\ \frac{-K_n(\nu,-1,1)}{K_{n+1}(\nu,-1,1)} > \frac{K_n(\nu,1,1)}{K_{n+1}(\nu,1,1)}.
\end{aligned}$$

Here we are using $K_n(\nu,-1,1) < 0 < K_{n+1}(\nu,-1,1)$. ■

Proof of Corollary 1.2
By the Christoffel-Darboux formula, and symmetry of ν, ,

$$|K_n(\nu,-1,1)| / K_n(\nu,1,1) = \frac{\gamma_{n-1}}{\gamma_n} \frac{|p_n(\nu,1)\, p_{n-1}(\nu,1)|}{K_n(1,1)}.$$

Here as the support of ν is $[-1,1]$, $\frac{\gamma_{n-1}}{\gamma_n} \leq 2$ [9, p. 41, Lemma 7.2] while as ν lies in the Nevai class, we have subexponential growth [13, Thm. 2.1, p. 218]:

$$\lim_{n\to\infty} p_n(\nu,1)^2 / K_n(\nu,1,1) = 0.$$

See also [6], [15]. It follows that

$$\lim_{n\to\infty} \frac{U_n}{K_n(\nu,1,1)} = 1 = \lim_{n\to\infty} \frac{V_n}{K_n(\nu,1,1)}$$

and also

$$\lim_{n\to\infty} \frac{V_n}{V_{n+1}} = 1.$$

Thus

$$\lim_{n\to\infty} \frac{U_n^2}{V_n V_{n+1}} = 1$$

and Theorem 1.1 gives the result. ■

4. Proof of Theorems 1.4 and 1.5

Let us first recall the values of some orthogonal polynomial quantities for the ultraspherical weight (or even Jacobi weight)

$$\nu'(t) = \left(1-t^2\right)^{\alpha},\ t \in (-1,1).$$

Here $\alpha > -1$ is fixed. Throughout this section, we drop the parameter ν in $p_n(\nu,x)$ etc. The classical Jacobi polynomials $P_n^{(\alpha,\alpha)}$ are normalized by [16, p. 58]

$$P_n^{(\alpha,\alpha)}(1) = \binom{n+\alpha}{n}. \tag{30}$$

The leading coefficient of $P_n^{(\alpha,\alpha)}$ is [16, p. 63]

$$2^{-n}\binom{2n+2\alpha}{n}.$$

Also, the orthonormal polynomial is given by [16, p. 68]

$$p_n(x) = c_n P_n^{(\alpha,\alpha)}(x), \tag{31}$$

where

$$c_n = \left\{ \frac{2n+2\alpha+1}{2^{2\alpha+1}} \frac{\Gamma(n+1)\,\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)^2} \right\}^{1/2}, \tag{32}$$

so that

$$p_n(1) = c_n \binom{n+\alpha}{n} \tag{33}$$

and

$$\gamma_n = c_n 2^{-n} \binom{2n+2\alpha}{n}. \tag{34}$$

Furthermore, taking account that our reproducing kernel sums to $n-1$ while that in [16] adds to n, [16, p. 71, eqn. (4.5.3)]

$$K_n(x,1) = 2^{-2\alpha-1} \frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)} P_{n-1}^{(\alpha+1,\alpha)}(x) \tag{35}$$

so that

$$K_n(1,1) = 2^{-2\alpha-1} \frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)} \binom{n+\alpha}{n-1} \tag{36}$$

while using that $P_{n-1}^{(\alpha+1,\alpha)}(-x) = (-1)^{n-1} P_{n-1}^{(\alpha,\alpha+1)}(x)$,

$$K_n(-1,1) = (-1)^{n-1} 2^{-2\alpha-1} \frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)} \binom{n-1+\alpha}{n-1}. \tag{37}$$

The proofs of this section involve several straightforward calculations. We shall exclude some of the line by line computations.

Lemma 4.1
Let $n \geq 2$ be even.
(a)

$$\frac{p_{n-1}(1)}{p_n(1)} = \left(1 - \frac{1+2\alpha}{n} + \eta_n\right)^{1/2}, \tag{38}$$

where

$$\eta_n = (2\alpha+1) \frac{n(4\alpha+1) + 2\alpha(2\alpha+1)}{(2n+2\alpha+1)(n+2\alpha)n}. \tag{39}$$

(b)

$$\begin{aligned} r_n &= \frac{1}{2}\left(1 + \frac{1-4\alpha^2}{4(n+\alpha)^2 - 1}\right)^{1/2} \left(1 - \frac{1+2\alpha}{n} + \eta_n\right)^{1/2} \\ &= \frac{1}{2}\left(1 - \frac{1+2\alpha}{2n} + O\left(n^{-2}\right)\right). \end{aligned}$$

(40)

(c)

$$\frac{1-2r_n}{r_n^2} = \frac{2(1+2\alpha)}{n}\left(1 + O\left(n^{-1}\right)\right). \tag{41}$$

(d)

$$X_\infty = \frac{2p_n^2(1)}{V_n} > \frac{1-2r_n}{r_n^2}. \tag{42}$$

(e)

$$\frac{2p_n^2(1)}{K_n(1,1)} = 4\left(\frac{\alpha+1}{n}\right)\left(1 + \frac{1}{2(n+\alpha)}\right). \tag{43}$$

(f)

$$\frac{-K_n(-1,1)}{K_n(1,1)} = \frac{\alpha+1}{n+\alpha}. \tag{44}$$

Proof
(a) Firstly using (32),

$$\begin{aligned}\frac{c_{n-1}}{c_n} &= \left(\frac{2n+2\alpha-1}{2n+2\alpha+1}\frac{\Gamma(n)\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(n+2\alpha+1)}\frac{\Gamma(n+\alpha+1)^2}{\Gamma(n+\alpha)^2}\right)^{1/2} \\ &= \left(\frac{2n+2\alpha-1}{2n+2\alpha+1}\frac{(n+\alpha)^2}{n(n+2\alpha)}\right)^{1/2}\end{aligned}$$

so by (34), and a straightforward calculation,

$$\begin{aligned}\frac{\gamma_{n-1}}{\gamma_n} &= 2\frac{c_{n-1}}{c_n}\binom{2n-2+2\alpha}{n-1}\Big/\binom{2n+2\alpha}{n} \\ &= \frac{1}{2}\left(1+\frac{1-4\alpha^2}{4(n+\alpha)^2-1}\right)^{1/2}.\end{aligned}$$

(45)

Next, from (33),

$$\begin{aligned}&\frac{p_{n-1}(1)}{p_n(1)} \\ &= \frac{c_{n-1}\binom{n-1+\alpha}{n-1}}{c_n\binom{n+\alpha}{n}} \\ &= \left(\frac{2n+2\alpha-1}{2n+2\alpha+1}\frac{n}{n+2\alpha}\right)^{1/2} \\ &= \left(1-\frac{1+2\alpha}{n}+\eta_n\right)^{1/2},\end{aligned}$$

where

$$\begin{aligned}\eta_n &= -2\left[\frac{1}{2n+2\alpha+1}-\frac{1}{2n}+\frac{\alpha}{n+2\alpha}-\frac{\alpha}{n}\right]+\frac{4\alpha}{(2n+2\alpha+1)(n+2\alpha)} \\ &= (2\alpha+1)\frac{n(4\alpha+1)+2\alpha(2\alpha+1)}{(2n+2\alpha+1)(n+2\alpha)n},\end{aligned}$$

again, by a straightforward calculation.
(b) From (45) and (38),

$$\begin{aligned}r_n &= \frac{\gamma_{n-1}}{\gamma_n}\frac{p_{n-1}(1)}{p_n(1)} \\ &= \frac{1}{2}\left(1+\frac{1-4\alpha^2}{4(n+\alpha)^2-1}\right)^{1/2}\left(1-\frac{1+2\alpha}{n}+\eta_n\right)^{1/2} \\ &= \frac{1}{2}\left(1-\frac{1+2\alpha}{2n}+O\left(n^{-2}\right)\right).\end{aligned}$$

(c) This follows immediately from (b).
(d) Recall from Lemma 3.2 that

$$\frac{2p_n^2(\nu,1)}{V_n} = X_\infty > \frac{1-2r_n}{r_n^2}$$

is equivalent to

$$\frac{-K_n(-1,1)}{K_{n+1}(-1,1)} > \frac{K_n(1,1)}{K_{n+1}(1,1)}. \tag{46}$$

Now substitute in our values from (36) and (37):

$$\begin{aligned}\frac{-K_n(-1,1)}{K_{n+1}(-1,1)} &= \frac{\left(\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha)}\right)\binom{n-1+\alpha}{n-1}}{\left(\frac{\Gamma(n+2\alpha+2)}{\Gamma(n+1+\alpha)}\right)\binom{n+\alpha}{n}} \\ &= 1-\frac{2\alpha+1}{n+2\alpha+1}.\end{aligned}$$

(47)

Also

$$\begin{aligned}\frac{K_n(1,1)}{K_{n+1}(1,1)} &= \frac{\left(\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha)}\right)\binom{n+\alpha}{n-1}}{\left(\frac{\Gamma(n+2\alpha+2)}{\Gamma(n+1+\alpha)}\right)\binom{n+1+\alpha}{n}} \\ &= \frac{n+\alpha}{n+2\alpha+1}\frac{n}{n+\alpha+1} \\ &= \left(1-\frac{\alpha+1}{n+2\alpha+1}\right)\left(1-\frac{\alpha+1}{n+\alpha+1}\right) \\ &= 1-(\alpha+1)\left[\frac{1}{n+2\alpha+1}+\frac{1}{n+\alpha+1}\right]+\frac{(\alpha+1)^2}{(n+2\alpha+1)(n+\alpha+1)} \\ &= 1-\frac{2(\alpha+1)}{n+2\alpha+1}-\frac{(\alpha+1)\alpha}{(n+\alpha+1)(n+2\alpha+1)}+\frac{(\alpha+1)^2}{(n+2\alpha+1)(n+\alpha+1)}\end{aligned}$$

so recalling (46) and (47), we want to check when

$$\frac{2\alpha+1}{n+2\alpha+1} < \frac{2(\alpha+1)}{n+2\alpha+1}+\frac{(\alpha+1)\alpha}{(n+\alpha+1)(n+2\alpha+1)}-\frac{(\alpha+1)^2}{(n+2\alpha+1)(n+\alpha+1)}$$

which is equivalent to

$$\begin{aligned}0 &< 1+\frac{(\alpha+1)\alpha}{(n+\alpha+1)}-\frac{(\alpha+1)^2}{(n+\alpha+1)} \\ &= 1-\frac{\alpha+1}{n+\alpha+1}.\end{aligned}$$

which is true for all even $n \geq 2$.
(e) From (33), (36), and then (32),

$$\begin{aligned}\frac{2p_n^2(1)}{K_n(1,1)} &= \frac{2\left\{c_n\binom{n+\alpha}{n}\right\}^2}{2^{-2\alpha-1}\frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)}\binom{n+\alpha}{n-1}} \\ &= 4(\alpha+1)\left(1+\frac{1}{2(n+\alpha)}\right)\frac{1}{n}.\end{aligned}$$

(f) From (36), (37),

$$\frac{-K_n(-1,1)}{K_n(1,1)} = \frac{\binom{n-1+\alpha}{n-1}}{\binom{n+\alpha}{n-1}} = \frac{\alpha+1}{n+\alpha}.$$

■

Proof of Theorem 1.4

As shown in the previous lemma, we have the inequality (42) for $n \geq 2$. For such n, we have from Theorem 1.1 that

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \frac{U_n^2}{V_n V_{n+1}}.$$

Here from (44),

$$\begin{aligned} U_n &= K_n(1,1)\left\{1 - \frac{K_n(-1,1)}{K_n(1,1)}\right\} = K_n(1,1)\left\{1 + \frac{\alpha+1}{n+\alpha}\right\}; \\ V_n &= K_n(1,1)\left\{1 + \frac{K_n(-1,1)}{K_n(1,1)}\right\} = K_n(1,1)\left\{1 - \frac{\alpha+1}{n+\alpha}\right\}; \end{aligned}$$

and from (43) and (44), and as $p_n(-1) = p_n(1)$,

$$\begin{aligned} V_{n+1} &= K_n(1,1)\left\{1 + \frac{K_n(-1,1)}{K_n(1,1)} + \frac{2p_n^2(1)}{K_n(1,1)}\right\} \\ &= K_n(1,1)\left\{1 - \frac{\alpha+1}{n+\alpha} + 4\left(\frac{\alpha+1}{n}\right)\left(1 + \frac{1}{2(n+\alpha)}\right)\right\} \\ &= K_n(1,1)\{1 + 3\frac{\alpha+1}{n+\alpha} + \frac{2(\alpha+1)(2\alpha+1)}{n(n+\alpha)}\} \end{aligned}$$

so

$$\begin{aligned} V_n V_{n+1} &= K_n^2(1,1)\left\{\begin{array}{c} 1 + 2\frac{\alpha+1}{n+\alpha} + \frac{2(\alpha+1)(2\alpha+1)}{n(n+\alpha)} \\ -3\left(\frac{\alpha+1}{n+\alpha}\right)^2 - \frac{2(\alpha+1)^2(2\alpha+1)}{n(n+\alpha)^2} \end{array}\right\} \\ &= K_n^2(1,1)\left\{\begin{array}{c} 1 + 2\frac{\alpha+1}{n+\alpha} \\ +\frac{\alpha+1}{n+\alpha}\left\{\frac{\alpha-1}{n+\alpha} - \frac{2(2\alpha+1)}{n(n+\alpha)}\right\} \end{array}\right\}. \end{aligned}$$

Then by yet another calculation,

$$\begin{aligned} &\frac{U_n^2}{V_n V_{n+1}} \\ &= \frac{1 + 2\frac{\alpha+1}{n+\alpha} + \left(\frac{\alpha+1}{n+\alpha}\right)^2}{1 + 2\frac{\alpha+1}{n+\alpha} + \frac{\alpha+1}{n+\alpha}\left\{\frac{\alpha-1}{n+\alpha} - \frac{2(2\alpha+1)}{n(n+\alpha)}\right\}} \\ &= 1 + \frac{\left(\frac{1}{n+\alpha}\right)^2 2(\alpha+1)\{1 + \frac{2\alpha+1}{n}\}}{1 + 2\frac{\alpha+1}{n+\alpha} + \frac{\alpha+1}{(n+\alpha)^2}\left\{\alpha - 1 - \frac{2(2\alpha+1)}{n}\right\}}. \end{aligned}$$

■

Proof of Theorem 1.5

(a) From Lemma 4.1(b), as $\alpha < -\frac{1}{2}$, so $r_n > \frac{1}{2}$ for $n \geq n_0(\alpha)$. Then $\frac{1-2r_n}{r_n^2} < 0$ for

$n \geq n_0(\alpha)$, so for all $S \geq 0$, $X_S > 0 > \frac{1-2r_n}{r_n^2}$. By Theorem 1.3(c), the extremal measure has the form $\nu + \frac{S}{2}(\delta_1 + \delta_{-1})$.
(b) From Lemma 4.1(b), as $\alpha > -\frac{1}{2}$, so $r_n < \frac{1}{2}$ for $n \geq n_0(\alpha)$. From Lemma 4.1(d), $X_\infty > \frac{1-2r_n}{r_n^2}$ while from (9), $X_0 = 0 < \frac{1-2r_n}{r_n^2}$. Also we know X_S is an increasing function of S. By Theorem 1.3(b), (c), there is a threshold S^* such that for $0 \leq S < S^*$, the extremal measure is ν, while for $S > S^*$, the extremal measure is $\nu + \frac{S}{2}(\delta_1 + \delta_{-1})$. For $S = S^*$, where $X_{S^*} = \frac{1-2r_n}{r_n^2}$, there are two extremal measures, namely ν and $\nu + \frac{S}{2}(\delta_1 + \delta_{-1})$.
(c) For $\alpha = -\frac{1}{2}$, (39) and (40) show that $r_n = \frac{1}{2}$, so $X_S > 0 = \frac{1-2r_n}{r_n^2}$ for all S and the extremal measure is $\nu + \frac{S}{2}(\delta_1 + \delta_{-1})$ for all $S \geq 0$. ■

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