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Anomaly resolution via decomposition

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In this paper, we apply decomposition to orbifolds with quantum symmetries to resolve anomalies. Briefly, it has been argued by, e.g. Wang–Wen–Witten, Tachikawa that an anomalous orbifold can sometimes be resolved by enlarging the orbifold group so that the pullback of the anomaly to the larger orbifold group is trivial. For this procedure to resolve the anomaly, one must specify a set of phases in the larger orbifold, whose form is implicit in the extension construction. There are multiple choices of consistent phases, which give rise to physically distinct resolutions. We apply decomposition, and find that theories with enlarged orbifold groups are equivalent to (disjoint unions of copies of) orbifolds by nonanomalous subgroups of the original orbifold group. In effect, decomposition implies that enlarging the orbifold group is equivalent to making it smaller. We provide a general conjecture for such descriptions, which we check in a number of examples.

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1. Introduction

This paper is devoted to a study of anomalous orbifolds and their resolutions, a subject that has been of renewed interest, see e.g. Refs. 1–9. In particular, this paper utilizes decomposition (which relates e.g. two-dimensional theories with one-form symmetries to disjoint unions) and quantum symmetries to make certain families of anomaly resolutions explicit, following up our previous work.[8–10]

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Briefly, it was argued in Refs. 3 and 4 that given an anomalous orbifold $[X/G]$, with anomaly $\alpha \in H^3(G, U(1))$, one way to resolve the anomaly is to find an extension

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1\,, \tag{1.1}$$

chosen so that the anomaly $\alpha \in H^3(G, U(1))$ is in the image of some $B \in H^1\big(G, H^1(K, U(1))\big)$, under the differential

$$d_2 : H^1(G, H^1(K, U(1))) \to H^3(G, U(1)) \tag{1.2}$$

of the Lyndon–Hochschild–Serre spectral sequence. (For simplicity, throughout this paper, we restrict to central extensions.) For such an extension, the pullback $\pi^*\alpha$ is trivial in $H^3(\Gamma, U(1))$, and the resulting orbifold $[X/\Gamma]$ is expected to be anomaly-free.

Construction of suitable extensions Γ has been discussed elsewhere, see e.g. Refs. 1, 3, 6–8.

However, as also noted in e.g. Sec. 2 of Ref. 3, to physically define the Γ orbifold, we need to specify more than just Γ itself, we must also specify the action of Γ. The Γ action is partially specified by saying that K acts trivially on X; however, as we shall see explicitly in examples, that does not uniquely specify the action, and in any event, does not suffice to make $[X/\Gamma]$ anomaly-free in general.

First, to uniquely specify the Γ orbifold, we interpret $B \in H^1\big(G, H^1(K, U(1))\big)$, the same mathematical quantity that was chosen to be in the preimage of the anomaly α, as a physical quantum symmetry in the sense of Ref. 10. (The physical relevance of that mathematical quantity was also discussed, from a slightly different perspective, in Sec. 2 of 3.)

Second, we apply decomposition to explain the crucial role the quantum symmetry B plays in resolving the anomaly. Briefly, in this context,[8–10] decomposition relates orbifolds with quantum symmetries to disjoint unions of orbifolds by smaller groups. We will see explicitly that for B such that $d_2 B = \alpha$, if we interpret B as defining a quantum symmetry, then the resulting orbifold $[X/\Gamma]_B$ is indeed anomaly-free, and in fact by virtue of decomposition is equivalent to (a disjoint union of) orbifolds by anomaly-free subgroups of G, subgroups such that the restriction of α is trivial. (If Γ is a "minimal" choice of extension, then one gets a single orbifold, but if Γ is larger than needed, in some sense, one gets a disjoint union of several orbifolds.)

In effect, this means that resolving an anomalous G orbifold by an orbifold by an extension Γ is physically equivalent to replacing the G orbifold with (copies of) an orbifold by a nonanomalous subgroup of G. (Related observations concerning module categories have been made in the mathematics literature, see e.g. Example 9.7.2 of Ref. 11.) Put another way, in a certain sense, making the orbifold group larger is equivalent to making it smaller, a duality somewhat reminiscent of T-duality on a circle, with the anomalous orbifold playing a role analogous to the self-dual radius.

Furthermore, which resolution is obtained depends upon the choice of B. For a fixed anomaly α, for a fixed extension Γ, there may be multiple choices of quantum symmetry B such that $d_2 B = \alpha$ (not to mention choices of discrete torsion), each of which can yield a physically distinct, nonanomalous, theory.

We also emphasize merely working with a resolution Γ does not suffice to resolve the anomaly: if the G orbifold is anomalous, then we will see explicitly that taking vanishing quantum symmetry in the Γ orbifold yields anomalous results, regardless of Γ. To resolve the anomaly, one must not only take an extension Γ but in addition one must pick a nonzero quantum symmetry B, such that $d_2 B$ coincides with the anomaly α.

We begin in Sec. 2 by reviewing quantum symmetries and decomposition. Specifically, in this paper we crucially use a generalization of ordinary quantum symmetries discussed in Refs. 8 and 10 to make sense of the anomaly-resolution procedure. We also review decomposition, which relates e.g. orbifolds with trivially-acting subgroups to disjoint unions of copies of orbifolds by effectively-acting groups.

In Sec. 3 we apply decomposition to simplify the anomaly resolution procedure described earlier. Briefly, decomposition makes it clear that the effect of enlarging the orbifold group and turning on a quantum symmetry is equivalent to working with (disjoint unions of) orbifolds by nonanomalous subgroups. We check that prediction explicitly in Sec. 4 in a number of examples.

We close in App. A with a collection of some pertinent results on degree-three group cohomology, as arises in describing anomalies.

We reiterate that throughout this paper, we assume that Γ is a central extension of G by K.

Finally, a remark on nomenclature. Across our several papers on decomposition and quantum symmetries, we have unfortunately mixed additive and multiplicative notations. For example, a trivial quantum symmetry is sometimes written additively, as $B = 0$, and sometimes multiplicatively, as $B(g) = 1$ for all $g \in G$.

2. Review of Quantum Symmetries and Decomposition

Let us quickly review quantum symmetries in orbifolds and properties of the resulting quantum field theories, as described in much greater detail in Ref. 10.

Quantum symmetries (as we use the term in Ref. 10) arise in orbifolds in which a subgroup of the orbifold group acts trivially. Consider an orbifold $[X/\Gamma]$ where Γ is a central extension of G by $K \subset \Gamma$ in which K acts trivially on X:

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1\,. \tag{2.1}$$

Typically in this paper, $[X/G]$ will be an anomalous orbifold, with anomaly $\alpha \in H^3(G, U(1))$, and the extension above will be chosen[a] so as to resolve the anomaly, in part.

[a]Details of how the resolutions are chosen are described in e.g. Subsec. 2.7 of Ref. 3 and Subsec. 5.1 of Ref. 1, and more efficient versions given in Refs. 6–8. Our focus in this paper will be on understanding the physics of the resolutions, not the resolutions per se.

To resolve the orbifold, one must pick a quantum symmetry, which for K central is a bihomomorphism

$$B : G \times K \to U(1) \tag{2.2}$$

defining phases acquired by G-twist fields under the action of K (see also Sec. 2 of Ref. 3). In terms of genus-one partition functions, this means

$$\underset{gz}{\square}_{h} = B(\pi(h), z)\left(\underset{g}{\square}_{h}\right), \quad \underset{g}{\square}_{hz} = B(\pi(g), z)^{-1}\left(\underset{g}{\square}_{h}\right), \tag{2.3}$$

for $z \in K$ and g, $h \in \Gamma$ a commuting pair. The resulting possible values of the quantum symmetry B are classified by elements of

$$H^1\big(G, H^1(K, U(1))\big)\,. \tag{2.4}$$

Sometimes these quantum symmetries can be constructed from discrete torsion. The relationship between quantum symmetries, discrete torsion, and anomalies is encoded in the exact sequence

$$\big(\operatorname{Ker} i^* \subset H^2(\Gamma, U(1))\big) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1))\,, \tag{2.5}$$

where $\iota : K \to \Gamma$ is inclusion and in the sequence above,

$$\iota^* : H^2(\Gamma, U(1)) \to H^2(K, U(1))\,. \tag{2.6}$$

(This is part of a seven-term exact sequence,[12] which was discussed in greater detail in our previous papers.[8,9]) Briefly, for discrete torsion $\omega \in \operatorname{Ker} \iota^* \subset H^2(\Gamma, U(1))$, its image as a quantum symmetry is[b]

$$\frac{1}{\beta(\omega)(\bar{g})} = \frac{\omega(z, s(\bar{g}))}{\omega(s(\bar{g}), z)}\,, \tag{2.7}$$

for $z \in K$, $\bar{g} \in G$, and $s : G \to \Gamma$ any section. (Independence from the choice of section is ultimately due to the fact that $\omega \in \operatorname{Ker} \iota^*$.) Similarly, the map d_2, which coincides with a differential in the Lyndon–Hochschild–Serre spectral sequence, is

$$(d_2 B)(\bar{g}_1, \bar{g}_2, \bar{g}_3) = B\big(\bar{g}_1, s_2 s_3 s_{23}^{-1}\big)\,, \tag{2.8}$$

where $s_i = s(\bar{g}_i)$ and $s : G \to \Gamma$ is any section. (Note that the quantity $s_2 s_3 s_{23}^{-1}$ is the extension class of the extension Γ of G by K.)

In this language, we see that discrete torsion (whose restriction to K is trivial) defines a quantum symmetry, but the quantum symmetries we need, with nontrivial images under d_2, are not produced by discrete torsion. The extension Γ is chosen so that there exists a B for which $d_2 B$ is the anomaly $\alpha \in H^3(G, U(1))$. Furthermore, the reader should note that the anomaly does not uniquely determine B — for example, it can be shifted by (the image of) discrete torsion. The physics of the resulting resolution will depend upon the choice of B, as we shall see, not just the choice of $\alpha = d_2 B$.

[b]The fact that $1/\beta$ appears rather than β itself is a consequence of definitions.

Now, orbifolds in which subgroups of the orbifold group act trivially are equivalent to disjoint unions of orbifolds by smaller groups, known as "universes." This is known as decomposition (see e.g. Refs. 9, 13–15), and has analogs in more general gauge theories with higher-form symmetries and various generalizations, as has been discussed in e.g. Refs. 16–22.

In the current circumstances, for an orbifold $[X/\Gamma]$ with a quantum symmetry B and discrete torsion $\omega \in H^2(\Gamma, U(1))$, the pertinent version of decomposition was described in Ref. 10. We will not do detailed decomposition computations in this paper, instead citing results described elsewhere, but for completeness, we outline the pertinent results here, which are for the case $\iota^*\omega = 0$.

(1) Suppose that $\iota^*\omega = 0$ and $\beta(\omega)$ is nontrivial, then

$$\mathrm{QFT}\big([X/\Gamma]_{B,\omega}\big) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}(B/\beta(\omega))}}{\mathrm{Ker}(B/\beta(\omega))}\right]_{\hat{\omega}_0}\right). \tag{2.9}$$

(2) Suppose that $\iota^*\omega = 0$ and $\omega = \pi^*\bar{\omega}$ for $\bar{\omega} \in H^2(G, U(1))$. Then,

$$\mathrm{QFT}([X/\Gamma]_{B,\omega}) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B}\right]_{\bar{\omega}+\hat{\omega}_0}\right). \tag{2.10}$$

In each case, $\hat{\omega}_0$ denotes discrete torsion on components, and is discussed in Ref. 10.

We will use decomposition to simplify Γ orbifolds with quantum symmetries, and will see in examples that if B resolves the anomaly (meaning that $d_2 B = \alpha$), then the result of decomposition will be manifestly anomaly-free orbifolds, involving anomaly-free subgroups of G.

3. Application of Decomposition to Anomalies

So far we have discussed Γ orbifolds, central extensions of G orbifolds, with quantum symmetries defined by $B \in H^1\big(G, H^1(K, U(1))\big)$, and described a general conjecture relating those orbifolds to simpler orbifolds by subgroups of G, generalizing decomposition.[9,13,14,16]

In this section we describe the application to curing anomalies.

One way to cure an anomaly $\alpha \in H^3(G, U(1))$ in a G orbifold is to extend G to a larger (finite) group Γ with $B \in H^1(G, H^1(K, U(1)))$ such that $d_2(B) = \alpha$. The B appearing in the mathematics defines, physically, a quantum symmetry, as we have discussed.

Then, a good intuition for the general claim of the previous subsection, which we will confirm in examples, is that B reduces G to a subgroup that does not participate in the anomaly, and if K is larger than needed to resolve the anomaly, then one gets multiple copies, in the spirit of decomposition.[9,13,14,16] In particular, from (2.8), we have immediately that the restriction of α to $\mathrm{Ker}\ B$ is trivial,

$$\alpha|_{\mathrm{Ker}\, B} = 0 \in H^3(\mathrm{Ker}\, B, U(1))\,, \tag{3.1}$$

since $\alpha = d_2(B)$ and the restriction of $[B]$ to Ker B vanishes. This guarantees that the orbifold by Ker $B \subset G$ is nonanomalous.

Now, if the Γ orbifold is given discrete torsion ω, and if $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$, then as discussed in the previous subsection and also in Ref. 9, $\beta(\omega)$ is an element of $H^1(G, H^1(K, U(1)))$, and so contributes to the quantum symmetry. However, $\beta(\omega)$ cannot change the anomaly cancellation, as $d_2 \circ \beta = 0$. We can see that using the seven-term exact sequence (App. I.6 of Ref. 23),[12] that can be derived from the Lyndon–Hochschild–Serre spectral sequence

$$0 \to H^1(G, U(1)) \xrightarrow{\pi^*} H^1(\Gamma, U(1)) \xrightarrow{\iota^*} H^1(K, U(1)) \xrightarrow{d_2} H^2(G, U(1)) \xrightarrow{\pi^*} \mathrm{Ker}(\iota^*) \xrightarrow{\beta} H^1(G, H^1(K, U(1))) \xrightarrow{d_2} H^3(G, U(1)) \,. \quad (3.2)$$

If a given element of $H^1\big(G, H^1(K, U(1))\big)$ is in the image of β, the image of discrete torsion (whose restriction to K is trivial), then its image in $H^3(G, U(1))$ under d_2 must vanish. Thus, the B's that arise when trivializing anomalies are in some sense orthogonal to those arising from discrete torsion in the considerations of Ref. 9.

The idea of resolving an anomalous orbifold by instead orbifolding by a subgroup (and possibly turning on discrete torsion) has also appeared in e.g. Subsec. 5.3 of Ref. 2. Here we have arrived at the same result, by gauging an extension of the anomalous group. In some sense, we see that those two approaches to anomalous orbifolds are equivalent to one another. (See also Example 9.7.2 of Ref. 11 for a related discussion of module categories in the mathematics literature.)

4. Examples

4.1. *Anomalous cyclic groups extended to larger cyclic groups*

It was argued in Ref. 6 and Sec. 3 of Ref. 8 that an anomalous $\mathbb{Z}_N$ orbifold with anomaly $\alpha \in H^3(\mathbb{Z}_N, U(1)) = \mathbb{Z}_N$ of order k (meaning $\alpha^k = 1$) can be trivialized by an extension to $\mathbb{Z}_{kN}$, so that $\pi^*\alpha$ is trivial for $\pi : \mathbb{Z}_{kN} \to \mathbb{Z}_N$. Now, to actually implement that anomaly-free resolution, one must pick quantum symmetries, and we shall do exactly that in this section, enumerating quantum symmetries and explaining their effects, explicitly demonstrating that any quantum symmetry B such that $d_2 B = \alpha$ resolves the anomaly in an extension to $\mathbb{Z}_{kN}$.

4.1.1. *Anomalous $\mathbb{Z}_2$ extended to $\mathbb{Z}_{2k}$*

In this subsection, we will add anomalies to examples studied in Subsec. 4.1 of Ref. 10. Consider an anomalous orbifold $[X/\mathbb{Z}_2]$, with anomaly $\alpha \in H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. We resolve the anomaly by extending the orbifold group by $K = \mathbb{Z}_k$ to $\Gamma = \mathbb{Z}_{2k}$:

$$1 \to \mathbb{Z}_k \to \mathbb{Z}_{2k} \to \mathbb{Z}_2 \to 1 \,, \quad (4.1)$$

and by turning on a quantum symmetry B such that $d_2 B = \alpha$. To be nontrivial, we assume that k is even. The case $k = 2$ corresponds to a minimal resolution.

Let us also briefly comment on how this extension trivializes the pullback of anomaly in the case k is even. Let $\pi : \mathbb{Z}_{2k} \to \mathbb{Z}_2$ denote the projection, and note that since α is of order two, $\pi^*\alpha$ must be either 1 or g^k, where g generates $\mathbb{Z}_{2k}$. One can compute a coboundary-invariant phase, as in Eq. (3.2) of Ref. 8, to show that

$$\prod_{i=1}^{2k-1} \pi^*\alpha(-1, i, 1) = \prod_{i=1}^{2k-1} (-)^i = (-)^k \,, \tag{4.2}$$

hence if k is odd, then $\pi^*\alpha$ cannot be trivial. It remains to show that in the case k is even, $\pi^*\alpha = 1$. To do so, note that π acts as reduction mod 2, and factors through $\mathbb{Z}_4$ when k is even:

$$\pi(n) = a(b(n)), \tag{4.3}$$

where $b : \mathbb{Z}_{2k} \to \mathbb{Z}_4$ is reduction mod 4 and $a : \mathbb{Z}_4 \to \mathbb{Z}_2$ is reduction mod 2. Hence, for k even, it suffices to show that $a^*\alpha$ is trivial, and a trivialization was given in Eq. (3.10) of Ref. 8.

Now, to physically construct the orbifold resolving the anomaly, we are instructed to turn on any quantum symmetry $B \in H^1\big(G, H^1(K, U(1))\big)$ in the preimage of α in the orbifold $[X/\Gamma]$. For this Γ, $H^2(\Gamma, U(1)) = 0$, so $H^1\big(G, H^1(K, U(1))\big)$ injects into $H^3(G, U(1))$. Furthermore,

$$H^1\big(G, H^1(K, U(1))\big) = \mathrm{Hom}\big(\mathbb{Z}_2, H^1(\mathbb{Z}_k, U(1))\big) = \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_k) \tag{4.4}$$

$$= \begin{cases} \mathbb{Z}_2 & k \text{ even}\,, \\ 0 & k \text{ odd}\,. \end{cases} \tag{4.5}$$

Thus, so long as k is even, there exists precisely one B that trivializes the anomaly α. However, if k is odd, the anomaly cannot be trivialized by extending $\mathbb{Z}_2$ by $\mathbb{Z}_k$.

Let us assume that k is even, so that the anomaly can be trivialized.

We argued in Subsec. 4.1 of Ref. 10 that for the possible values of B,

$$\mathrm{QFT}\big([X/\mathbb{Z}_{2k}]_B\big) = \begin{cases} \mathrm{QFT}\left(\coprod_k [X/\mathbb{Z}_2]\right) & B = 0\,, \\ \mathrm{QFT}\left(\coprod_{k/2} X\right) & B \neq 0\,. \end{cases} \tag{4.6}$$

In this case, since α is nontrivial, the B that trivializes α is also the nontrivial element. If instead we pick the trivial $B = 0$, then the resulting QFT is anomalous, as it is merely copies of the anomalous orbifold $[X/\mathbb{Z}_2]$. (Furthermore, since all twisted sectors appear, there is no chance of cancelling the anomaly merely by removing certain offending modular orbits from the partition function.)

On the other hand, for the B that trivializes the anomaly, we see that the quantum field theory is well defined: a sum over copies of X, not an anomalous orbifold of X.

Thus, we see explicitly that the combination of working with a larger orbifold group and turning on a quantum symmetry resolves the anomaly. In the case of the minimal resolution (for which $k = 2$), this resolution is equivalent to replacing the original anomalous orbifold ($[X/\mathbb{Z}_2]$) by an orbifold by a nonanomalous subgroup. Since there are no nontrivial subgroups, that means replacing $[X/\mathbb{Z}_2]$ by X itself. For nonminimal resolutions (for which $k > 2$), one simply gets copies.

4.1.2. *Anomalous $\mathbb{Z}_3$ extended to $\mathbb{Z}_9$*

In this subsection we start with an anomalous $G = \mathbb{Z}_3$ orbifold, and remove the anomaly $\alpha \in H^3(G, U(1))$ by extending G by $K = \mathbb{Z}_3$ to $\Gamma = \mathbb{Z}_9$:

$$1 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 1\,, \tag{4.7}$$

and of course turning on a suitable quantum symmetry B.

In this case,

$$B \in \mathrm{Hom}\big(G, H^1(K, U(1)\big) = \mathbb{Z}_3\,. \tag{4.8}$$

In Subsec. 4.1.3 of Ref. 10, we computed that

$$\mathrm{QFT}([X/\Gamma]_B) = \begin{cases} \mathrm{QFT}\left(\coprod_3 [X/\mathbb{Z}_3]\right) & B = 0\,, \\ \mathrm{QFT}(X) & B \text{ nontrivial}\,. \end{cases} \tag{4.9}$$

The anomaly is trivialized by any $B \in H^1\big(G, H^1(K, U(1))\big)$ such that $d_2 B = \alpha$. In this case, $H^2(\Gamma, U(1)) = 0$, so $H^1\big(G, H^1(K, U(1))\big)$ injects into $H^3(G, U(1))$. In this case, since α is nontrivial, B is also nontrivial (but since decomposition gives the same answer for both nontrivial values of B, we do not need to track which nontrivial element B corresponds to).

For $B = 0$, the anomaly should not be trivialized, and indeed the QFT is just copies of the original anomalous orbifold. For B nontrivial, the QFT is well defined, simply X itself.

4.2. *Anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ extended to $\mathbb{Z}_2 \times \mathbb{Z}_4$*

In this subsection, we start with an anomalous $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and remove the anomaly $\alpha \in H^3(G, U(1))$ by extending by $K = \mathbb{Z}_2$ to $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1\,, \tag{4.10}$$

and by turning on a suitable quantum symmetry B. In this case, one can also turn on ordinary discrete torsion $\omega \in H^2(\Gamma, U(1))$, so we have several choices we can make to resolve the orbifold.

Write the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \bar{a}, \bar{b} \rangle$. Using the fact that

$$H^1\big(G, H^1(K, U(1))\big) = \mathbb{Z}_2 \times \mathbb{Z}_2\,, \tag{4.11}$$

Table 1. Summary of decomposition results for $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]_{B,\omega}$ for various values of B, ω, from Subsec. 4.2 of Ref. 10. For non-trivial B, adding discrete torsion has no effect in these cases.

$B(\bar{a})$	$B(\bar{b})$	Without d.t.	With d.t.
$+1$	$+1$	$\coprod_2[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$	$\coprod_2[X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm dt}$
-1	$+1$	$[X/\mathbb{Z}_2 = \langle\bar{b}\rangle]$	$[X/\mathbb{Z}_2 = \langle\bar{b}\rangle]$
$+1$	-1	$[X/\mathbb{Z}_2 = \langle\bar{a}\rangle]$	$[X/\mathbb{Z}_2 = \langle\bar{a}\rangle]$
-1	-1	$[X/\mathbb{Z}_2 = \langle\bar{a}\bar{b}\rangle]$	$[X/\mathbb{Z}_2 = \langle\bar{a}\bar{b}\rangle]$

the possible values of B are characterized by their values on $\bar{a}$, $\bar{b}$. Furthermore, since $H^2(\mathbb{Z}_2 \times \mathbb{Z}_4, U(1)) = \mathbb{Z}_2$ (see e.g. App. D.2 of Ref. 9), we can also turn in discrete torsion in this theory.

This example was computed in Subsec. 4.2 of Ref. 10, where it was shown that the quantum field theory of $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]_{B,\omega}$ takes the values listed in Table 1.

To understand how quantum symmetries can resolve anomalies in this case, we next compute $d_2 B$. We will do so manually. In principle,

$$(d_2 B)(\bar{g}_1, \bar{g}_2, \bar{g}_3) = B(\bar{g}_1, e(\bar{g}_2, \bar{g}_3))\,, \tag{4.12}$$

for $\bar{g}_i \in G$, and where e denotes the extension class of Γ. Recall that for a section $s : G \to \Gamma$, the extension class e is given explicitly as a cocycle[c]

$$e(\bar{g}_1, \bar{g}_2) = s_1 s_2 s_{12}^{-1}\,, \tag{4.13}$$

where $s_i = s(\bar{g}_i)$. For this case, $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$, where $\Gamma = \langle a, b\rangle$, $a^2 = 1 = b^4$. In these conventions, $K = \mathbb{Z}_2 = \langle b^2\rangle$. The projection π maps $a \mapsto \bar{a}$, $b \mapsto \bar{b}$, and we pick the section s given by

$$s(1) = 1\,, \quad s(\bar{a}) = a\,, \quad s(\bar{b}) = b\,, \quad s(\bar{a}\bar{b}) = ab\,. \tag{4.14}$$

Then, we compute explicitly that the values of $e(\bar{g}_1, \bar{g}_2)$ are given as in the following table:

e	1	$\bar{a}$	$\bar{b}$	$\bar{a}\bar{b}$
1	1	1	1	1
$\bar{a}$	1	1	1	1
$\bar{b}$	1	1	b^2	b^2
$\bar{a}\bar{b}$	1	1	b^2	b^2

[c]This cochain is coclosed so long as K is central, which we assume throughout this paper.

Next, we compute $d_2 B$ for various choices of B.

(1) In the trivial case $B(\bar{a}) = +1 = B(\bar{b})$, we find that $d_2 B(\bar{g}_1, \bar{g}_2, \bar{g}_3) = 1$, and so the cohomology class of $d_2 B$ is trivial.

(2) Consider the case $B(\bar{a}) = -1$, $B(\bar{b}) = +1$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{a},\bar{b},\bar{b}) &= -1 = (d_2B)(\bar{a},\bar{b},\bar{a}\bar{b}) = (d_2B)(\bar{a},\bar{a}\bar{b},\bar{b}) \\ &= (d_2B)(\bar{a},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.15)\\ (d_2B)(\bar{a}\bar{b},\bar{b},\bar{b}) &= -1 = (d_2B)(\bar{a}\bar{b},\bar{b},\bar{a}\bar{b}) = (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{b}) \\ &= (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.16)\end{aligned}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2 B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{a}\bar{b}} = -1$, $\epsilon_{\bar{a}} = +1 = \epsilon_{\bar{b}}$. Thus, for this B, $d_2 B$ is nontrivial in cohomology, and can be used to resolve an anomalous $\langle \bar{a}\bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

(3) Next, consider the case $B(\bar{a}) = +1$, $B(\bar{b}) = -1$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{b},\bar{b},\bar{b}) &= -1 = (d_2B)(\bar{b},\bar{b},\bar{a}\bar{b}) = (d_2B)(\bar{b},\bar{a}\bar{b},\bar{b}) \\ &= (d_2B)(\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.17)\\ (d_2B)(\bar{a}\bar{b},\bar{b},\bar{b}) &= -1 = (d_2B)(\bar{a}\bar{b},\bar{b},\bar{a}\bar{b}) = (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{b}) \\ &= (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.18)\end{aligned}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2 B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{b}} = -1 = \epsilon_{\bar{a}\bar{b}}$, $\epsilon_{\bar{a}} = +1$. Thus, for this B, $d_2 B$ is nontrivial in cohomology, and can be used to resolve an anomalous $\langle \bar{b} \rangle, \langle \bar{a}\bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

(4) Next, consider the case $B(\bar{a}) = -1 = B(\bar{b})$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{a},\bar{b},\bar{b}) &= -1 = (d_2B)(\bar{a},\bar{b},\bar{a}\bar{b}) = (d_2B)(\bar{a},\bar{a}\bar{b},\bar{b}) \\ &= (d_2B)(\bar{a},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.19)\\ (d_2B)(\bar{b},\bar{b},\bar{b}) &= -1 = (d_2B)(\bar{b},\bar{b},\bar{a}\bar{b}) = (d_2B)(\bar{b},\bar{a}\bar{b},\bar{b}) \\ &= (d_2B)(\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.20)\end{aligned}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2 B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{b}} = -1$, $\epsilon_{\bar{a}} = +1 = \epsilon_{\bar{a}\bar{b}}$. Thus, for this B, $d_2 B$ is nontrivial in cohomology, and can be used to resolve an anomalous $\langle \bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, let us compare to the results of decomposition listed in Table 1.

(1) For the first choice of B, for which $B(\bar{a}) = +1 = B(\bar{b})$, the cohomology class of $d_2 B$ is trivial, and so no anomaly resolution is guaranteed by the method we outlined. Comparing to the results of decomposition, we see that copies of the anomalous orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ appear, consistent with the fact that in this case, no anomalies are expected to be resolved.
(2) In the next case, for which $B(\bar{a}) = -1$ and $B(\bar{b}) = +1$, an anomaly in the subgroup $\langle \bar{a}\bar{b} \rangle$ can be resolved, but the other subgroups $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are assumed nonanomalous. In this case, the physical theory for this quantum symmetry is the orbifold $[X/\langle \bar{b} \rangle]$, which is anomaly-free.
(3) In the next case, for which $B(\bar{a}) = +1$ and $B(\bar{b}) = -1$, an anomaly in the subgroups $\langle \bar{b} \rangle$, $\langle \bar{a}\bar{b} \rangle$ can be resolved, but the other subgroup $\langle \bar{a} \rangle$ is assumed nonanomalous. In this case, the physical theory for this quantum symmetry is the orbifold $[X/\langle \bar{a} \rangle]$, which is anomaly-free.
(4) In the last case, for which $B(\bar{a}) = -1 = B(\bar{b})$, an anomaly in the subgroup $\langle \bar{b} \rangle$ can be resolved, but the other subgroups $\langle \bar{a} \rangle$, $\langle \bar{a}\bar{b} \rangle$ are assumed nonanomalous. In this case, the physical theory for the quantum symmetry is the orbifold $[X/\langle \bar{a}\bar{b} \rangle]$, which is anomaly-free.

In each case, the effect of turning on a quantum symmetry is to reduce the orbifold group to a nonanomalous subgroup (correlated to the choice of B), precisely as advertised.

4.3. *Anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ extended to D_4*

In this section, we start with an anomalous $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and remove the anomaly $\alpha \in H^3(G, U(1))$ by extending by $K = \mathbb{Z}_2$ to $\Gamma = D_4$:

$$1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1\,, \tag{4.21}$$

and by turning on a suitable quantum symmetry B. In this case, one can also turn on ordinary discrete torsion $\omega \in H^2(\Gamma, U(1))$, so we have several choices we can make to resolve the orbifold.

Write the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \bar{a}, \bar{b} \rangle$. Using the fact that

$$H^1(G, H^1(K, U(1))) = \mathbb{Z}_2 \times \mathbb{Z}_2\,, \tag{4.22}$$

the possible values of B are characterized by their values on $\bar{a}$, $\bar{b}$.

This example was computed in Subsec. 4.3 of Ref. 10, where from both decomposition and explicit computation it was demonstrated that the quantum field theory of $[X/\Gamma]_{B,\omega}$ took the values listed in Table 2.

To understand how quantum symmetries can resolve anomalies in this case, we next compute $d_2 B$. We will do so manually, as we did in the last example. In principle,

$$(d_2 B)(\bar{g}_1, \bar{g}_2, \bar{g}_3) = B(\bar{g}_1, e(\bar{g}_2, \bar{g}_3))\,, \tag{4.23}$$

Table 2. Summary of decomposition results for $[X/D_4]_{B,\omega}$ for various values of B, ω, from Subsec. 4.3 of Ref. 10.

$B(\bar{a})$	$B(\bar{b})$	Without d.t.	With d.t.
$+1$	$+1$	$[X/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm dt}$	$[X/\mathbb{Z}_2 = \langle \bar{b} \rangle]$
-1	$+1$	$[X/\mathbb{Z}_2 = \langle \bar{b} \rangle]$	$[X/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm dt}$
$+1$	-1	$[X/\mathbb{Z}_2 = \langle \bar{a} \rangle]$	$[X/\mathbb{Z}_2 = \langle \bar{a}\bar{b} \rangle]$
-1	-1	$[X/\mathbb{Z}_2 = \langle \bar{a}\bar{b} \rangle]$	$[X/\mathbb{Z}_2 = \langle \bar{a} \rangle]$

for $\bar{g}_i \in G$, and where e denotes the extension class of Γ,

$$e(\bar{g}_1, \bar{g}_2) = s_1 s_2 s_{12}^{-1} \,, \tag{4.24}$$

where $s_i = s(\bar{g}_i)$ for s a section. For this case, $\Gamma = D_4$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, write

$$D_4 = \{1, a, b, z, az, bz, ab, ba = abz\} \,, \tag{4.25}$$

where $z^2 = 1 = a^2$ and $b^2 = z$, z generates the center, $\pi(a) = \bar{a}$, and $\pi(b) = \bar{b}$. Then, take the section to be given by

$$s(\bar{a}) = a \,, \quad s(\bar{b}) = b \,, \quad s(\bar{a}\bar{b}) = ab \,. \tag{4.26}$$

Then, we compute explicitly that the values of $e(\bar{g}_1, \bar{g}_2)$ are given as in the following table:

e	1	$\bar{a}$	$\bar{b}$	$\bar{a}\bar{b}$
1	1	1	1	1
$\bar{a}$	1	1	1	1
$\bar{b}$	1	z	z	1
$\bar{a}\bar{b}$	1	z	z	1

Next, we compute $d_2 B$ for various choices of B.

(1) In the trivial case $B(\bar{a}) = +1 = B(\bar{b})$, we find that $d_2 B(\bar{g}_1, \bar{g}_2, \bar{g}_3) = 1$, and so the cohomology class of $d_2 B$ is trivial.

(2) Consider the case $B(\bar{a}) = -1$, $B(\bar{b}) = +1$. In this case it is straightforward to compute

$$\begin{aligned} (d_2 B)(\bar{a}, \bar{b}, \bar{a}) &= -1 = (d_2 B)(\bar{a}, \bar{b}, \bar{b}) = (d_2 B)(\bar{a}, \bar{a}\bar{b}, \bar{a}) \\ &= (d_2 B)(\bar{a}, \bar{a}\bar{b}, \bar{b}) \,, \end{aligned} \tag{4.27}$$

$$\begin{aligned} (d_2 B)(\bar{a}\bar{b}, \bar{b}, \bar{a}) &= -1 = (d_2 B)(\bar{a}\bar{b}, \bar{b}, \bar{b}) = (d_2 B)(\bar{a}\bar{b}, \bar{a}\bar{b}, \bar{a}) \\ &= (d_2 B)(\bar{a}\bar{b}, \bar{a}\bar{b}, \bar{b}) \,, \end{aligned} \tag{4.28}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{a}} = +1 = \epsilon_{\bar{b}} = \epsilon_{\bar{a}\bar{b}}$, and so for this B, d_2B is trivial in cohomology. (In fact, since this choice of B is equivalent to turning on discrete torsion, it should not be a surprise that it is in the kernel of d_2.)

(3) Next, consider the case $B(\bar{a}) = +1$, $B(\bar{b}) = -1$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{b},\bar{b},\bar{a}) &= -1 = (d_2B)(\bar{b},\bar{b},\bar{b}) = (d_2B)(\bar{b},\bar{a}\bar{b},\bar{a}) \\ &= (d_2B)(\bar{b},\bar{a}\bar{b},\bar{b})\,, \end{aligned} \tag{4.29}$$

$$\begin{aligned}(d_2B)(\bar{a}\bar{b},\bar{b},\bar{a}) &= -1 = (d_2B)(\bar{a}\bar{b},\bar{b},\bar{b}) = (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{a}) \\ &= (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{b})\,, \end{aligned} \tag{4.30}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{b}} = -1$, $\epsilon_{\bar{a}} = \epsilon_{\bar{a}\bar{b}} = +1$, and so for this B, d_2B is nontrivial in cohomology, and can be used to resolve anomalies in the $\langle \bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup.

(4) Finally, consider the case $B(\bar{a}) = -1$, $B(\bar{(b)}) = -1$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{a},\bar{b},\bar{a}) &= -1 = (d_2B)(\bar{a},\bar{b},\bar{b}) = (d_2B)(\bar{a},\bar{a}\bar{b},\bar{a}) \\ &= (d_2B)(\bar{a},\bar{a}\bar{b},\bar{b})\,, \end{aligned} \tag{4.31}$$

$$\begin{aligned}(d_2B)(\bar{b},\bar{b},\bar{a}) &= -1 = (d_2B)(\bar{b},\bar{b},\bar{b}) = (d_2B)(\bar{b},\bar{a}\bar{b},\bar{a}) \\ &= (d_2B)(\bar{b},\bar{a}\bar{b},\bar{b})\,, \end{aligned} \tag{4.32}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{b}} = -1$, $\epsilon_{\bar{a}} = \epsilon_{\bar{a}\bar{b}} = +1$, the same as for the previous choice of B, and so for this B, d_2B is nontrivial in cohomology, and again can be used to resolve anomalies in the $\langle \bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup.

Now, let us compare to the results of decomposition listed in Table 2. For the first two choices of B, for which $B(\bar{a}) = +1$, the cohomology class of d_2B is trivial, and so no anomaly resolution is guaranteed by the method we outlined. Comparing to the results of decomposition, for these two choices of B, we see that for some choices of discrete torsion, copies of the anomalous orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ appear, consistent with the fact that the method described is not anticipated to resolve any anomalies. (Curiously, for other values of discrete torsion, only the orbifold $[X/\mathbb{Z}_2 = \langle \bar{b} \rangle]$ appears, which would resolve an anomaly that is in the $\langle \bar{a} \rangle$ or $\langle \bar{a}\bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.)

For the last two choices of B, for which $B(\bar{b}) = -1$, the cohomology class of d_2B is nontrivial, characterized by $\epsilon_{\bar{b}} = -1$, and so we expect that these choices of B can resolve an anomaly in $\langle \bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$, so long as $\langle \bar{a} \rangle$ and $\langle \bar{a}\bar{b} \rangle$ are anomaly-free. This is indeed consistent with the results of decomposition in Table 2: for both

pertinent choices of B, for all choices of discrete torsion, the QFTs are consistent, in that they only involve orbifolds by anomaly-free subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Thus, to summarize, we see explicitly that our prescription works in this example: for choices of B such that $d_2 B$ matches the anomaly, the orbifold $[X/\Gamma]_{B,\omega}$ is indeed anomaly-free, and its precise physics is determined by choices of B (with fixed image under d_2) and discrete torsion ω. The reader should further note that this particular extension (D_4) can only be used to resolve anomalies in the subgroup $\langle \bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$; for other anomalous subgroups, different resolutions are required.

This example was also studied, from a different perspective, in Subsec. 5.2.1 of Ref. 8.

4.4. *Anomalous* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *extended to* $\mathbb{H}$

Next, we consider an example that is closely related to the previous one. Here, we again start with an anomalous $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and this time remove the anomaly $\alpha \in H^3(G, U(1))$ by extending by $K = \mathbb{Z}_2$ to $\Gamma = \mathbb{H}$, the eight-element finite group of quaternions,

$$1 \to \mathbb{Z}_2 \to \mathbb{H} \xrightarrow{\pi} \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1\,, \tag{4.33}$$

and by turning on a suitable quantum symmetry B. This extension is different from that discussed in the previous section, and so we will see a different pattern of anomaly resolution.

As before, write the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \bar{a}, \bar{b} \rangle$, so that using

$$H^1\big(G, H^1(K, U(1))\big) = \mathbb{Z}_2 \times \mathbb{Z}_2\,, \tag{4.34}$$

the possible values of B are characterized by their values on $\bar{a}$, $\bar{b}$. In our conventions,

$$\pi(\pm i) = \bar{a}\,, \quad \pi(\pm j) = \bar{b}\,, \quad \pi(\pm k) = \bar{a}\bar{b}\,. \tag{4.35}$$

Unlike the case of D_4, the group $\mathbb{H}$ does not admit discrete torsion: $H^2(\mathbb{H}, U(1)) = 0$. Applying decomposition,[9] it is straightforward to show that $\mathrm{QFT}([X/\mathbb{H}]_B)$ for the various choices of B are as given in Table 3.

Table 3. Summary of decomposition results for $[X/\mathbb{H}]_B$ for various values of B. The trivial case, $B(\bar{a}) = +1 = B(\bar{b})$, is discussed in Subsec. 5.3 of Ref. 13.

$B(\bar{a})$	$B(\bar{b})$	Theory
$+1$	$+1$	$[X/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{dt}}$
-1	$+1$	$[X/\mathbb{Z}_2 = \langle \bar{b} \rangle]$
$+1$	-1	$[X/\mathbb{Z}_2 = \langle \bar{a} \rangle]$
-1	-1	$[X/\mathbb{Z}_2 = \langle \bar{a}\bar{b} \rangle]$

To understand how quantum symmetries can resolve anomalies in this case, we next compute d_2B. As before, in principle,

$$(d_2B)(\bar{g}_1, \bar{g}_2, \bar{g}_3) = B(\bar{g}_1, e(\bar{g}_2, \bar{g}_3))\,, \tag{4.36}$$

for $\bar{g}_i \in G$, and where e denotes the extension class of Γ. What distinguishes this example from the previous one is that the extension class e is different.

Let us compute the extension class e explicitly. For a section $s : G \to \Gamma$, it can be written

$$e(\bar{g}_1, \bar{g}_2) = s_1 s_2 s_{12}^{-1}\,, \tag{4.37}$$

where $s_i = s(\bar{g}_i)$. We pick the section given by

$$s(1) = 1\,, \quad s(\bar{a}) = i\,, \quad s(\bar{b}) = j\,, \quad s(\bar{k}) = k\,. \tag{4.38}$$

Then, we compute explicitly that the values of $e(\bar{g}_1, \bar{g}_2)$ are given as in the following table:

e	1	$\bar{a}$	$\bar{b}$	$\bar{a}\bar{b}$
1	1	1	1	1
$\bar{a}$	1	-1	1	-1
$\bar{b}$	1	-1	-1	1
$\bar{a}\bar{b}$	1	1	-1	-1

One sees immediately that this extension class differs from that appearing in the D_4 extension we considered in the previous section.

Next, we compute d_2B for various choices of B.

(1) In the trivial case $B(\bar{a}) = +1 = B(\bar{b})$, we find that $d_2B(\bar{g}_1, \bar{g}_2, \bar{g}_3) = 1$, and so the cohomology class of d_2B is trivial.

(2) Consider the case $B(\bar{a}) = -1$, $B(\bar{b}) = +1$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{a}, \bar{a}, \bar{a}) = -1 &= (d_2B)(\bar{a}, \bar{a}, \bar{a}\bar{b}) = (d_2B)(\bar{a}, \bar{b}, \bar{a}) = (d_2B)(\bar{a}, \bar{b}, \bar{b}) \\ &= (d_2B)(\bar{a}, \bar{a}\bar{b}, \bar{b}) = (d_2B)(\bar{a}, \bar{a}\bar{b}, \bar{a}\bar{b})\,, \end{aligned} \tag{4.39}$$

$$\begin{aligned}(d_2B)(\bar{a}\bar{b}, \bar{a}, \bar{a}) = -1 &= (d_2B)(\bar{a}\bar{b}, \bar{a}, \bar{a}\bar{b}) = (d_2B)(\bar{a}\bar{b}, \bar{b}, \bar{a}) = (d_2B)(\bar{a}\bar{b}, \bar{b}, \bar{b}) \\ &= (d_2B)(\bar{a}\bar{b}, \bar{a}\bar{b}, \bar{b}) = (d_2B)(\bar{a}\bar{b}, \bar{a}\bar{b}, \bar{a}\bar{b})\,, \end{aligned} \tag{4.40}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2B \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{a}} = -1 = \epsilon_{\bar{a}\bar{b}}$, $\epsilon_{\bar{b}} = +1$, hence for this choice of B, d_2B is nontrivial in cohomology, and can be used to resolve anomalies in the subgroups $\langle \bar{a} \rangle, \langle \bar{a}\bar{b} \rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

(3) Next, consider the case $B(\bar{a}) = +1$, $B(\bar{b}) = -1$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{b},\bar{a},\bar{a}) = -1 &= (d_2B)(\bar{b},\bar{a},\bar{a}\bar{b}) = (d_2B)(\bar{b},\bar{b},\bar{a}) = (d_2B)(\bar{b},\bar{b},\bar{b}) \\ &= (d_2B)(\bar{b},\bar{a}\bar{b},\bar{b}) = (d_2B)(\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.41)\\ (d_2B)(\bar{a}\bar{b},\bar{a},\bar{a}) = -1 &= (d_2B)(\bar{a}\bar{b},\bar{a},\bar{a}\bar{b}) = (d_2B)(\bar{a}\bar{b},\bar{b},\bar{a}) = (d_2B)(\bar{a}\bar{b},\bar{b},\bar{b}) \\ &= (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{b}) = (d_2B)(\bar{a}\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.42)\end{aligned}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2B \in H^3(\mathbb{Z}_2\times\mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{b}} = -1 = \epsilon_{\bar{a}\bar{b}}$, $\epsilon_{\bar{a}} = +1$, hence for this choice of B, d_2B is nontrivial in cohomology, and can be used to resolve anomalies in the subgroups $\langle\bar{b}\rangle$, $\langle\bar{a}\bar{b}\rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

(4) Finally, consider the case $B(\bar{a}) = -1 = B(\bar{b})$. In this case it is straightforward to compute

$$\begin{aligned}(d_2B)(\bar{a},\bar{a},\bar{a}) = -1 &= (d_2B)(\bar{a},\bar{a},\bar{a}\bar{b}) = (d_2B)(\bar{a},\bar{b},\bar{a}) = (d_2B)(\bar{a},\bar{b},\bar{b}) \\ &= (d_2B)(\bar{a},\bar{a}\bar{b},\bar{b}) = (d_2B)(\bar{a},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.43)\\ (d_2B)(\bar{b},\bar{a},\bar{a}) = -1 &= (d_2B)(\bar{b},\bar{a},\bar{a}\bar{b}) = (d_2B)(\bar{b},\bar{b},\bar{a}) = (d_2B)(\bar{b},\bar{b},\bar{b}) \\ &= (d_2B)(\bar{b},\bar{a}\bar{b},\bar{b}) = (d_2B)(\bar{b},\bar{a}\bar{b},\bar{a}\bar{b})\,, \qquad (4.44)\end{aligned}$$

with all other entries $+1$. From the invariants (A.2)–(A.7), we find that the cohomology class of $d_2B \in H^3(\mathbb{Z}_2\times\mathbb{Z}_2, U(1))$ is characterized by $\epsilon_{\bar{a}} = -1 = \epsilon_{\bar{b}}$, $\epsilon_{\bar{a}\bar{b}} = +1$, hence for this choice of B, d_2B is nontrivial in cohomology, and can be used to resolve anomalies in the subgroups $\langle\bar{a}\rangle$, $\langle\bar{b}\rangle \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, let us compare to the results of decomposition listed in Table 3.

(1) For the first choice of B, for which $B(\bar{a}) = +1 = B(\bar{b})$, the cohomology class of d_2B is trivial, and so no anomaly resolution is guaranteed by the method we outlined. Comparing to the results of decomposition, we see that copies of the anomalous orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ appear, consistent with the fact that in this case, no anomalies are expected to be resolved.

(2) In the next case, for which $B(\bar{a}) = -1$ and $B(\bar{b}) = +1$, the subgroups $\langle\bar{a}\rangle$, $\langle\bar{a}\bar{b}\rangle$ are (potentially) anomalous, but $\langle\bar{b}\rangle$ is nonanomalous, and indeed the physical theory for this quantum symmetry involves only an orbifold by the nonanomalous subgroup $\langle\bar{b}\rangle$.

(3) In the next case, for which $B(\bar{a}) = +1$ and $B(\bar{b}) = -1$, the subgroups $\langle\bar{b}\rangle$, $\langle\bar{a}\bar{b}\rangle$ are (potentially) anomalous, but $\langle\bar{a}\rangle$ is nonanomalous, and indeed the physical theory for this quantum symmetry involves only an orbifold by the nonanomalous subgroup $\langle\bar{a}\rangle$.

(4) In the last case, for which $B(\bar{a}) = -1 = B(\bar{b})$, the subgroups $\langle\bar{a}\rangle$, $\langle\bar{b}\rangle$ are (potentially) anomalous, but $\langle\bar{a}\bar{b}\rangle$ is nonanomalous, and indeed the physical theory for this quantum symmetry involves only an orbifold by the nonanomalous subgroup $\langle\bar{a}\bar{b}\rangle$.

In each case, the effect of turning on a quantum symmetry is to reduce the orbifold group to a nonanomalous subgroup (correlated to the choice of B), precisely as advertised.

This example was also studied, from a different perspective, in App. A of Ref. 8.

4.5. *Anomalous* $\mathbb{Z}_2$ *extended to* $\mathbb{Z}_2 \times \mathbb{Z}_2$

This example will be of a different form than previous examples we have considered: in this case, the extension itself will not trivialize the anomaly. Nevertheless, we will see that the anomaly can be removed via a choice of discrete torsion.

Consider an anomalous orbifold $[X/\mathbb{Z}_2]$, with anomaly $\alpha \in H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, and extend the orbifold group to $K \times G = \mathbb{Z}_2 \times \mathbb{Z}_2$, with $K = \mathbb{Z}_2$ acting trivially on X:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2 \to 1\,. \tag{4.45}$$

Since the extension splits, $d_2 B$ is trivial for any choice of quantum symmetry B, and so there is no choice of B such that $\alpha = d_2 B$ for any nontrivial α.

Nevertheless, let us apply decomposition to various choice of quantum symmetry B and discrete torsion ω. The possible quantum symmetries are

$$H^1\big(G, H^1(K, U(1))\big) = \mathbb{Z}_2\,, \tag{4.46}$$

which we will enumerate via their action on the generator of G, which we will denote a. If we let ω denote the nontrivial choice of discrete torsion in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ then from e.g. Subsec. 5.1 of Ref. 9, we know that $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$. Applying the decomposition conjecture of Ref. 10, we find the quantum field theory of $[X/\Gamma]_{B,\omega}$, as listed in Table 4.

First, Table 4 confirms that if there is no quantum symmetry and no discrete torsion, then the anomaly is not resolved, one gets instead two copies of the anomalous $\mathbb{Z}_2$ orbifold. However, if either there is a quantum symmetry or there is discrete torsion, the anomaly is resolved, despite the fact that in no case in this example does $d_2 B = \alpha$. (Intuitively, discrete torsion or nonzero B in the Γ orbifold has the effect of canceling out some of the sectors appearing in the G orbifold.) We therefore emphasize that the anomaly resolution method we have primarily focused on elsewhere in this paper is not the only way to resolve anomalies.

Table 4. Results for $\mathrm{QFT}([X/\Gamma]_{B,\omega})$ for various choices of B, ω. As $\beta(\omega) \neq 0$, the result of turning on discrete torsion is essentially to exchange the results for the two values of B.

$B(a)$	Without d.t.	With d.t.
$+1$	$\coprod_2 [X/\mathbb{Z}_2]$	X
-1	X	$\coprod_2 [X/\mathbb{Z}_2]$

This example was also considered, from a different perspective, in Subsec. 5.1.2 of Ref. 8.

5. Conclusions

In this paper, we have applied decomposition to simplify the anomaly-resolution program described in e.g. Refs. 1–3, by showing that enlarging the orbifold group and turning on a quantum symmetry phase is equivalent to (copies of) orbifolding by a nonanomalous subgroup of the original orbifold group.

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Appendix A. Notes on Degree-Three Group Cohomology

In this Appendix, to make this paper self-contained, we collect pertinent results on $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = (\mathbb{Z}_2)^3$.

Explicit cocycles representing this group cohomology were given in Eq. (5.26) of Ref. 8, which we repeat below. Writing $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, c\}$, we can represent $\alpha \in H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ as

$$\begin{array}{lll}
\alpha(a,a,a) = \epsilon_a\,, & \alpha(a,a,b) = 1\,, & \alpha(a,a,c) = \epsilon_a\,, \\
& \alpha(a,b,a) = 1\,, & \alpha(a,b,b) = 1\,, \\
& \alpha(a,b,c) = 1\,, & \alpha(a,c,a) = \epsilon_a\,, \\
& \alpha(a,c,b) = 1\,, & \alpha(a,c,c) = \epsilon_a\,, \\
\alpha(b,a,a) = 1\,, & \alpha(b,a,b) = 1\,, & \alpha(b,a,c) = 1\,, \\
& \alpha(b,b,a) = \epsilon_a\epsilon_b\epsilon_c\,, & \alpha(b,b,b) = \epsilon_b\,, \\
& \alpha(b,b,c) = \epsilon_a\epsilon_c\,, & \alpha(b,c,a) = \epsilon_a\epsilon_b\epsilon_c\,, \\
& \alpha(b,c,b) = \epsilon_b\,, & \alpha(b,c,c) = \epsilon_a\epsilon_c\,, \\
\alpha(c,a,a) = \epsilon_a\,, & \alpha(c,a,b) = 1\,, & \alpha(c,a,c) = \epsilon_a\,, \\
& \alpha(c,b,a) = \epsilon_a\epsilon_b\epsilon_c\,, & \alpha(c,b,b) = \epsilon_b\,, \\
& \alpha(c,b,c) = \epsilon_a\epsilon_c\,, & \alpha(c,c,a) = \epsilon_b\epsilon_c\,, \\
& \alpha(c,c,b) = \epsilon_b\,, & \alpha(c,c,c) = \epsilon_c\,,
\end{array} \tag{A.1}$$

where $\epsilon_{a,b,c}$ represent the cohomology class, each $\epsilon^2 = 1$, and where if any index is 1, $\alpha = 1$.

Now, of course, a given cocycle need not have the precise form above to represent a given cohomology class, so below we list some invariants that can be computed

from a cocycle to determine the cohomology class (we assume that α is normalized, so that $\alpha(1,g,h) = \alpha(g,1,h) = \alpha(g,h,1) = 1$ for all $g,\, h \in \mathbb{Z}_2 \times \mathbb{Z}_2$):

$$\alpha(a,a,a) = \epsilon_a\,, \tag{A.2}$$

$$\alpha(b,b,b) = \epsilon_b\,, \tag{A.3}$$

$$\alpha(c,c,c) = \epsilon_c\,, \tag{A.4}$$

$$\alpha(a,b,a)\alpha(a,ab,a) = \epsilon_a\,, \tag{A.5}$$

$$\alpha(b,c,b)\alpha(b,bc,b) = \epsilon_b\,, \tag{A.6}$$

$$\alpha(c,a,c)\alpha(c,ac,c) = \epsilon_c\,. \tag{A.7}$$

(These were extracted from Eq. (3.2) of Ref. 8 and Eqs. (5.36)–(5.41) of Ref. 8.) It is straightforward to check that the products of cocycles on the left side of each equation are invariant under coboundaries.

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