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Bounds for discrepancies in the Hamming space[☆]

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ABSTRACT

We derive bounds for the ball L_p -discrepancies in the Hamming space for $0 < p < \infty$ and $p = \infty$. Sharp estimates of discrepancies have been obtained for many spaces such as the Euclidean spheres and more general compact Riemannian manifolds. In the present paper, we show that the behavior of discrepancies in the Hamming space differs fundamentally because the volume of the ball in this space depends on its radius exponentially while such a dependence for the Riemannian manifolds is polynomial.

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1. Introduction

1.1. Basic definitions

Let $\mathcal{X}_n = \{0, 1\}^n$ be the binary Hamming space which can be also thought of as a linear space $\mathbb{F}_2^n$ over the finite field $\mathbb{F}_2$. The cardinality $|\mathcal{X}_n| = 2^n$. Denote by $B(x, t)$ the ball with center at $x \in \mathcal{X}_n$ and radius $t \geq 0$, i.e., the set of all points $y \in \mathcal{X}_n$ with $d(x, y) \leq t$, where $d(x, y)$ is the Hamming distance. The volume of the ball $v(t) := |B(x, t)| = \sum_{i=0}^t \binom{n}{i}$ is independent of $x \in \mathcal{X}_n$. It is convenient to assume that $B(x, t) = \emptyset$ and $v(t) = 0$ for $t < 0$, and $B(x, t) = \mathcal{X}_n$ and $v(t) = 2^n$ for $t > n$.

For an N -point subset $Z_N \subset \mathcal{X}_n$ and a ball $B(y, t)$ define the local discrepancy as follows:

$$D(Z_N, y, t) = |B(y, t) \cap Z_N| - N 2^{-n} v(t). \quad (1)$$

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We note that $D(Z_N, y, n) = 0$ for any Z_N, y , and thus below we limit ourselves to the values $0 \leq t \leq n - 1$. Define the weighted L_p -discrepancy by

$$D_p(G, Z_N) = \left(\sum_{t=0}^{n-1} g_t \sum_{y \in \mathcal{X}_n} 2^{-n} |D(Z_N, y, t)|^p \right)^{1/p}, \quad 0 < p < \infty, \quad (2)$$

where $G = (g_0, \dots, g_{n-1})$ is a vector of nonnegative weights normalized by

$$\sum_{t=0}^{n-1} g_t = 1. \quad (3)$$

With such a normalization, we have

$$D_p(G, Z_N) \leq D_q(G, Z_N) \quad 0 < p < q < \infty. \quad (4)$$

The L_∞ -discrepancy is defined by

$$D_\infty(I, Z_N) = \max_{t \in I} \max_{y \in \mathcal{X}_n} |D(Z_N, y, t)|, \quad (5)$$

where $I \subseteq \{0, \dots, n - 1\}$ is a subset of the set of the radii.

We also introduce the following extremal discrepancies

$$D_p(G, n, N) = \min_{Z_N \subset \mathcal{X}_n} D_p(G, Z_N), \quad 0 < p < \infty,$$

and

$$D_\infty(I, n, N) = \min_{Z_N \subset \mathcal{X}_n} D_\infty(I, Z_N).$$

These quantities can be thought of as geometric characteristics of the Hamming space.

It is useful to keep in mind the following simple observations:

(i) If $Z_N^c = \mathcal{X}_n \setminus Z_N$ is the complement of $Z_N \subseteq \mathcal{X}_n$, then $D(Z_N, y, t) = -D(Z_N^c, y, t)$, and we have

$$D_p(G, Z_N) = D_p(G, Z_N^c) \quad \text{and} \quad D_p(G, n, N) = D_p(G, n, 2^n - N),$$

for all $0 < p \leq \infty$. Hence, generally it suffices to consider only subsets Z_N with $N \leq 2^{n-1}$. Together with results of [1] on quadratic discrepancies, this remark enables us to identify some new examples of subsets that attain the minimum value of $D_2(G_1, n, 2^n - N)$ with uniform weights

$$G_1 = (1/n, 1/n, \dots, 1/n).$$

Let us give one example. A code $Z_N \in \mathcal{X}_n$ with minimum pairwise distance $2t + 1$ is called *perfect* if spheres of radius t around the codewords fill the entire space $\mathcal{X}_n$ without overlapping. Perfect codes were shown in [1] to have the smallest quadratic discrepancy among all codes of the same cardinality. For instance, for $n = 2^m - 1$, $t = 1$ the well-known *Hamming code* is perfect [13, p.23ff]. Therefore, for $N = 2^n(1 - 2^{-m})$, $m \geq 2$ the code Z_N formed of spheres of radius one around the codewords of the Hamming code (i.e., the union of the n cosets of the Hamming code) is a minimizer of quadratic discrepancy. Another family of minimizers is given by $\mathcal{X}_n \setminus \{y, \bar{y}\}$ for any $y \in \mathcal{X}_n$, where $\bar{y} := 1^n + y$ is a point antipodal to y and $1^n \in \mathcal{X}_n$ denotes the all-ones vector. Some other examples can be also given; see [1]. For the reader's convenience, we emphasize that the quadratic discrepancy $D^{l_2}(Z_N)$ in [1] is related to our definition (2) by $D^{l_2}(Z_N) = 2^n n N^{-2} (D(G_1, Z_N))^2$.

(ii)

Indeed, notice that $B(y, t) = \mathcal{X}_n \setminus B(\bar{y}, n - 1 - t)$, and therefore $D(Z_N, y, t) = D(Z_N, \bar{y}, n - 1 - t)$. Also, obviously,

$$\sum_{y \in \mathcal{X}_n} |D(Z_N, \bar{y}, t)|^p = \sum_{y \in \mathcal{X}_n} |D(Z_N, y, t)|^p,$$

and thus

$$\begin{aligned} D_p(G, Z_N) &= \left(\sum_{t=0}^v \left(g_t 2^{-n} \sum_{y \in \mathcal{X}_n} |D(Z_N, y, t)|^p + g_{n-1-t} 2^{-n} \sum_{y \in \mathcal{X}_n} |D(Z_N, \bar{y}, t)|^p \right) \right)^{1/p} \\ &= \left(2^{-n} \sum_{t=0}^v (g_t + g_{n-1-t}) \sum_{y \in \mathcal{X}_n} |D(Z_N, y, t)|^p \right)^{1/p}. \end{aligned}$$

We conclude that limiting the summation range of t amounts to changing the weights in definition (2). Similar arguments hold true for the L_∞ -discrepancy (5).

1.2. Earlier results

Discrepancies in compact metric measure spaces have been studied for a long time, starting with basic results in the theory of uniform distributions [2,3,14]. In particular, quadratic discrepancy of finite subsets of the Euclidean sphere is related to the structure of the distances in the subset through a well-known identity called Stolarsky's invariance principle [19]. Stolarsky's identity expresses the L_2 -discrepancy of a spherical set as a difference between the average distance on the sphere and the average distance in the set. Recently it has been a subject of renewed attention in the literature. In particular, papers [4,7,15] gave new, simplified proofs of Stolarsky's invariance, while [18] extended Stolarsky's principle to projective spaces and derived asymptotically tight estimates of discrepancy. Sharp bounds on quadratic discrepancy were obtained in [6,8,15,16]. Finally, paper [17] introduced new asymptotic upper bounds on L_p -discrepancies of finite sets in compact metric measure spaces.

A recent paper [1] initiated the study of Stolarsky's invariance in finite metric spaces, deriving an explicit form of the invariance principle in the Hamming space $\mathcal{X}_n$ as well as bounds on the quadratic discrepancy of subsets (codes) in $\mathcal{X}_n$. Explicit formulas were obtained for the uniform weights G_1 . Namely, let $x, y \in \mathcal{X}_n$ be two points with $d(x, y) = w$. Define

$$\lambda(x, y) = \lambda(w) := 2^{n-w} w \binom{w-1}{\lceil \frac{w}{2} \rceil - 1}, \quad w = 0, \dots, n.$$

As shown in [1, Eq. (23)], Stolarsky's identity for $Z_N \subset \mathcal{X}_n$ can be written in the following form:

$$2^n n D_2(G_1, Z_N)^2 = \frac{n N^2}{2^{n+1}} \binom{2n}{n} - \sum_{i,j=1}^N \lambda(d(z_i, z_j)). \quad (6)$$

Using this representation, [1, Cor.5.3, Thm.5.5] further showed that

$$c n^{-3/4} N^{1/2} \left(1 - \frac{N}{2^n}\right)^{1/2} \leq D_2(G_1, n, N) \leq C n^{-1/4} N^{1/2},$$

where c, C are some universal constants. Here the upper bound is proved by random choice and the lower bound by linear programming. The method of linear programming, well known in coding theory [11,12], is applicable to the problem of bounding the quadratic discrepancy because it can be expressed as an energy functional on the code with potential given by λ . Moreover, there exist sequences of subsets (codes) $Z_N \subset \mathcal{X}_n$, $n = 2^m - 1$ whose quadratic discrepancy meets the lower bound. Observe also that if $N = o(2^n)$, then the bounds differ only by a factor of $\sqrt{n}$: for example, if $N \simeq 2^{\alpha n}$, $0 < \alpha < 1$, then

$$N^{1/2} (\log N)^{-3/4} \lesssim D_2(G_1, n, N) \lesssim N^{1/2} (\log N)^{-1/4}, \quad (7)$$

In this short paper we develop the results of [1], proving bounds on $D_p(G, n, N)$, $p \in (0, \infty]$. We also consider a restricted version of the discrepancy $D_p(G, Z_N)$, limiting ourselves to the case of hemispheres in $\mathcal{X}_n$. In other words, we take local discrepancy for $t = (n - 1)/2$ in (1) (n odd) and average its value over the centers of the balls. For the case of the Euclidean sphere, quadratic discrepancy for hemispheres was previously studied in [4,16], which established a version of Stolarsky's invariance for this case.

2. Bounds on $D_p(G, n, N)$

We are interested in universal bounds for discrepancies (2)–(5) for given n, N and $p \in (0, \infty]$ without accounting for the structure of the subset. For the case of finite subsets in compact Riemannian manifolds this problem was recently studied in [17], and we draw on the approach of this paper in the derivations below.

2.1. The case $0 < p < \infty$

We shall consider random subsets $Z_N \subset \mathcal{X}_n$, using the following standard result to handle discrepancies of such subsets.

Lemma 2.1 (Marcinkiewicz–Zygmund Inequality; [10], Sec.10.3). *Let ζ_j , $j \in J$, $|J| < \infty$, be a finite collection of real-valued independent random variables with expectations $\mathbb{E} \zeta_j = 0$ and $\mathbb{E} \zeta_j^2 < \infty$, $j \in J$. Then, we have*

$$\mathbb{E} \left| \sum_{j \in J} \zeta_j \right|^p \leq 2^p (p + 1)^{p/2} \mathbb{E} \left(\sum_{j \in J} \zeta_j^2 \right)^{p/2}, \quad 1 \leq p < \infty.$$

In our first result we construct a random subset Z_N by uniform random choice. Later we will refine this procedure, obtaining a more precise bound on D_p .

Theorem 2.2. *Let G be a vector of weights. For all $N \leq 2^{n-1}$, we have*

$$D_p(G, n, N) \leq \begin{cases} 2(p + 1)^{1/2} N^{1/2} & \text{for } 1 \leq p < \infty, \\ 2^{3/2} N^{1/2} & \text{for } 0 < p < 1. \end{cases} \quad (8)$$

Remark 2.1. Bounds of the type (8) hold true for arbitrary compact metric measure spaces. Theorem 2.2 is given here to compare it with Theorem 2.3. Notice also that the upper bound (7) is better than (8) with $p = 2$ and $G = G_1$ by a logarithmic factor. Such an improvement is obtained in [1] because of the explicit formula (6) for the quadratic discrepancy with the uniform weights G_1 .

Proof. Choose a subset Z_N by selecting the points $\{z_i\}_1^N$ independently and uniformly in $\mathcal{X}_n$. The probability that such a point falls into a subset $\mathcal{E} \in \mathcal{X}_n$ equals $|\mathcal{E}|/|\mathcal{X}_n|$. Therefore, for the local discrepancy (1) of this random subset Z_N we have

$$D(Z_N, y, t) = \sum_{i=1}^N \zeta_i(y, t), \quad (9)$$

where

$$\zeta_i(y, t) = \mathbb{1}_{B(y, t)}(z_i) - \frac{v(t)}{|\mathcal{X}_n|},$$

where $\mathbb{1}_{\mathcal{E}}$ is the indicator function of a subset $\mathcal{E} \subseteq \mathcal{X}_n$. The quantities $\zeta_i(y, t)$ are independent random variables that satisfy $|\zeta_i(y, t)| \leq 1$ and $\mathbb{E} \zeta_i(y, t) = 0$.

Applying the Marcinkiewicz–Zygmund inequality to the sum (9), we obtain

$$\mathbb{E} |D(Z_N, y, t)|^p \leq 2^p (p + 1)^{p/2} N^{p/2}, \quad 1 \leq p < \infty,$$

and, therefore, in view of (3),

$$\mathbb{E} D(G, Z_N)^p \leq 2^p (p+1)^{p/2} N^{p/2}, \quad 1 \leq p < \infty.$$

Thus, there exists a subset $Z_N = Z_N(p) \subset \mathcal{X}_n$, $1 \leq p < \infty$, whose discrepancy is bounded above as in this inequality. For $0 < p < 1$, in view of (4), we can put $Z_N(p) = Z_N(1)$ to complete the proof. $\square$

In some situations the bound of this theorem can be improved relying on the method of *jittered* (or *stratified*) sampling, which uses a partition of the metric space into subsets of small diameter and equal volume. This idea goes back to classical works on discrepancy theory [2], [3, pp. 237–240] and it was used more recently in [5,6,9] for the case of the Euclidean sphere and in [17] for general metric spaces. Below we follow the approach of [17]. In the case of the Hamming space the natural way to proceed is to partition $\mathcal{X}_n$ into sub-hypercubes of a fixed dimension.

In our analysis bounds on the volume of the ball $v(t)$ are crucial. For large n and $t = \lambda n$, $0 \leq \lambda \leq 1$, the well-known bound on $v(t)$ (cf. [13, p. 310]) can be written in the form

$$v(\lambda n) \leq 2^{nH(\lambda)}, \quad (10)$$

where

$$H(\lambda) = \begin{cases} h(\lambda), & \text{if } 0 \leq \lambda \leq 1/2, \\ 1, & \text{if } 1/2 < \lambda \leq 1, \end{cases} \quad (11)$$

and $h(\lambda) = -\lambda \log_2 \lambda - (1-\lambda) \log_2 (1-\lambda)$ is the standard binary entropy, and in general, the bound (10) cannot be improved. Formally speaking, the statement (10) requires λn to be integer, but this does not matter for the asymptotic arguments that we employ.

Theorem 2.3. Let $N = 2^{\alpha n}$, $0 < \alpha < 1$, be a power of 2. Suppose that the weights $g_i = 0$ for $t > \beta n$, $0 < \beta < 1/2$. Then

$$D_p(G, n, N) \leq \begin{cases} 2(p+1)^{1/2} N^{(1-\kappa)/2}, & \text{for } 1 \leq p < \infty, \\ 2^{3/2} N^{(1-\kappa)/2}, & \text{for } 0 < p < 1, \end{cases} \quad (12)$$

where

$$\kappa = \kappa(\alpha, \beta) = \frac{1 - H(1 + \beta - \alpha)}{\alpha} \geq 0. \quad (13)$$

If $\alpha > \frac{1}{2} + \beta$, then the exponent $\kappa(\alpha, \beta) > 0$, and the bound (12) is better than (8).

Proof. Let $V \subset \mathcal{X}_n$ be the k -dimensional subspace, $k = \gamma n$, $0 < \gamma < 1$, consisting of all vectors $(x_1, \dots, x_n)$ with $x_i = 0$ if $i > k$. Let $N = 2^{n-k} = 2^{\alpha n}$, $\alpha = 1 - \gamma$. The affine subspaces

$$V_i = V + s_i, \quad s_i \in \mathcal{X}_n/V$$

form a partition of the Hamming space

$$\mathcal{X}_n = \bigcup_{i=1}^N V_i, \quad V_i \cap V_j = \emptyset,$$

where $|V_i| = 2^{\gamma n}$, $\text{diam } V_i = \gamma n$, where $\text{diam } \mathcal{E} = \max\{d(x_1, x_2) : x_1, x_2 \in \mathcal{E}\}$ denotes the diameter of a subset $\mathcal{E} \subseteq \mathcal{X}_n$.

We consider a subset $Z_N = \{z_i\}_{i=1}^N$ with $z_i \in V_i$, $i = 1, \dots, N$. For such a subset, the local discrepancy (1) can be written as follows

$$D(Z_N, y, t) = \sum_{i=1}^N \zeta_i(y, t), \quad (14)$$

where

$$\zeta_i(y, t) = \mathbb{1}_{\{B(y, t) \cap V_i\}}(z_i) - N \frac{|B(y, t) \cap V_i|}{|\mathcal{X}_n|}.$$

Notice that if $V_i \subset B(y, t)$, then $\zeta_i(y, t) \equiv 0$ (recall that $z_i \in V_i$ and $NV_i = 2^n$). Therefore, the sum (14) takes the form

$$D(Z_N, y, t) = \sum_{i \in J} \zeta_i(y, t),$$

where J is a subset of indices i such that $V_i \cap B(y, t) \neq \emptyset$ but $V_i \not\subset B(y, t)$ (V_i is neither completely inside nor completely outside $B(y, t)$). Since $\text{diam } V_i = k$, we conclude that all V_i , $i \in J$, are contained in the ball $B(y, t + k)$ and do not intersect the ball $B(y, t - k - 1)$. Therefore,

$$|J| |V_i| \leq v(t + k) - v(t - k - 1) \leq v(t + k).$$

Here we estimate $|J|$ from above by the number of sets V_i such that $V_i \subset B(y, t + k)$. We note that discarding the term $v(t - k - 1)$ entails no significant loss in the asymptotics because this term is exponentially small compared to $v(t + k)$. For $t \leq \beta n$, using the bound (11) and $\alpha + \gamma = 1$, we obtain

$$|J| \leq 2^{nH(\beta+\gamma)-\gamma n} = 2^{\alpha n(1-\kappa)} = N^{1-\kappa},$$

where κ is defined in (13).

Now consider a random subset $Z_N = \{z_i\}_1^N$ in which each point z_i is selected independently and uniformly in V_i . For a subset $\mathcal{E} \in V_i$ we have $\Pr(z_i \in \mathcal{E}) = |\mathcal{E}|/|V_i| = N|\mathcal{E}|/|\mathcal{X}_n|$. The quantities $\zeta_i(y, t)$ are bounded independent random variables that satisfy $|\zeta_i(y, t)| \leq 1$ and $\mathbb{E} \zeta_i(y, t) = 0$. Applying the Marcinkiewicz–Zygmund inequality to the sum (14), we obtain

$$\mathbb{E} |D(Z_N, y, t)|^p \leq 2^p (p + 1)^{p/2} N^{p(1-\kappa)/2}$$

and, therefore, in view of (3),

$$\mathbb{E} |D(G, Z_N)|^p \leq 2^p (p + 1)^{p/2} N^{p(1-\kappa)/2}. \quad (15)$$

Thus, there exists a subset $Z_N = Z_N(p) \subset \mathcal{X}_n$, $1 \leq p < \infty$, whose discrepancy is bounded above as in this inequality. For $0 < p < 1$, in view of (4), we can put $Z(p) = Z(1)$ to complete the proof. $\square$

Remark 2.2. We conjecture that the improvement of the discrepancy estimate for weights equal to zero in the neighborhood of $t = n/2$ takes place also for d -dimensional Euclidean spheres $S^d \subset \mathbb{R}^{d+1}$ in the case that the dimension d grows in proportion to the cardinality N . Indeed, the sphere S^d and the Hamming space $\mathcal{X}_n$ share the property that for large dimensions the invariant measure concentrates around the “equator”. This interesting problem deserves a separate study.

2.2. The case $p = \infty$

The following statement is analogous to [17, Prop.2.2]. For $1 \leq p < \infty$, any subset $Z_N \subseteq \mathcal{X}_n$, and $I \subset \{0, 1, \dots, n - 1\}$, we have

$$D_\infty(I, Z_N) \leq |I|^{1/p} 2^{n/p} D_p(G_I, Z_N), \quad (16)$$

where

$$D_p(G_I, Z_N) = \left(\sum_{t=0}^v |I|^{-1} \sum_{y \in \mathcal{X}_n} 2^{-n} |D(Z_N, y, t)|^p \right)^{1/p},$$

is a special L_p -discrepancy with $G_I = (g_1, \dots, g_v)$, where $g_t = |I|^{-1}$ for $t \in I$ and $g_t = 0$ otherwise. Indeed, for $y \in \mathcal{X}_n$ and $t \in I$ we have

$$\begin{aligned} |D(Z_N, y, t)| &\leq \left(\sum_{t \in I} \sum_{y \in \mathcal{X}_n} |D(Z_N, y, t)|^p \right)^{1/p} \\ &= |I|^{1/p} 2^{n/p} \left(\sum_{t \in I} |I|^{-1} \sum_{y \in \mathcal{X}_n} 2^{-n} |D(Z_N, y, t)|^p \right)^{1/p}. \end{aligned}$$

Theorem 2.4. (i) Let $I \subseteq \{0, 1, \dots, n\}$ be an arbitrary subset of the set of radii, and $N \leq 2^{n-1}$. Then

$$D_\infty(I, Z_N) \leq 8(1+n)^{1/2} N^{1/2}. \quad (17)$$

If N increases exponentially, $N \cong 2^{\alpha n}$, then $D_\infty(I, n, N) = O((\log_2 N)^{1/2} N^{1/2})$.

(ii) Let $I \subseteq \{0, 1, \dots, \beta n\}$ be an arbitrary subset of the set of radii $t \leq \beta n$, $0 < \beta < 1/2$, and let $N = 2^{\alpha n} \leq 2^{n-1}$ be a power of 2. Then

$$D_\infty(I, n, N) \leq 8 \left(2 + \frac{\log_2 N}{\alpha} \right)^{1/2} N^{(1-\kappa)/2}, \quad (18)$$

where the exponent $\kappa = \kappa(\alpha, \beta)$ is given in (13). If $\alpha > \frac{1}{2} + \beta$, then the exponent $\kappa(\alpha, \beta) > 0$, and the bound (18) is better than (17).

Proof. Substituting the bounds (8) and (12) into inequality (16), we obtain

$$D_\infty(I, Z_N) \leq n^{1/p} 2^{n/p} 2(p+1)^{1/2} N^{1/2} \quad (19)$$

and

$$D_\infty(I, Z_N) \leq n^{1/p} 2^{n/p} 2(p+1)^{1/2} N^{(1-\kappa)/2}. \quad (20)$$

Now, we put $p = n$ in (19) and (20) to obtain, respectively, (17) and (18). $\square$

3. Discrepancy for hemispheres

Let $\mathcal{X}_n$, $n = 2m + 1$ be the Hamming space. In this section we consider a restricted version of discrepancy where instead of all the ball radii in (2) we consider discrepancy only with respect to the balls of radius m , calling them hemispheres. For any pair of antipodal points $y, \bar{y}$

$$\mathcal{X}_n = B(y, m) \cup B(\bar{y}, m), \quad B(y, m) \cap B(\bar{y}, m) = \emptyset,$$

hence $2^{-n}v(m) = 2^{-n}|B(y, m)| = 1/2$.

For a subset $Z_N \subset \mathcal{X}_n$ define

$$D_p^{(m)}(Z_N) = \left(2^{-n} \sum_{y \in \mathcal{X}_n} |D(Z_N, y, m)|^p \right)^{1/p}, \quad 0 < p < \infty, \quad (21)$$

where

$$D(Z_N, y, m) = |B(y, m) \cap Z_N| - \frac{N}{2}$$

is the local discrepancy defined in (1). In the previous notation $D_p^{(m)}(Z_N) = D_p(G(m), Z_N)$, with weights $G(m) = (g_1, \dots, g_{n-1})$, where $g_m = 1$ and $g_t = 0$ if $t \neq m$. Further, let

$$D_\infty^{(m)}(Z_N) = \max_{y \in \mathcal{X}_n} |D(Z_N, y, m)|.$$

As before, define

$$D_p^{(m)}(n, N) = \min_{Z_N \subset \mathcal{X}_n} D_p^{(m)}(Z_N), \quad p \in (0, \infty].$$

First we address the question of global minimizers of discrepancy.

Theorem 3.1. For the Hamming space $\mathcal{X}_n$ with odd $n = 2m + 1$, we have the following.

(i) Let $N = 2K$ be even, then for all subsets $Z_N \subseteq \mathcal{X}_n$ and $p \in (0, \infty]$

$$D_p^{(m)}(Z_N) \geq 0 \quad (22)$$

with equality for subsets Z_N consisting of K pairs of antipodal points.

(ii) Let $N = 2K + 1$ be odd, then for all subsets $Z_N \subseteq \mathcal{X}_n$ and $p \in (0, \infty]$

$$D_p^{(m)}(Z_N) \geq 1/2 \quad (23)$$

with equality for subsets Z_N consisting of K pairs of antipodal points supplemented with a single point.

In other words, for all $p \in (0, \infty]$ the extremal discrepancy $D_p^{(m)}(n, N) = 0$ if N is even and $D_p^{(m)}(n, N) = 1/2$ if N is odd.

Remark 3.1. The phenomenon of such small discrepancies for hemispheres is also known for Euclidean spheres $S^d \subset \mathbb{R}^{d+1}$, see [4, 15, 16]. The sphere S^d can be represented as a disjoint union of two antipodal hemispheres and the equator. But the equator in this partition is of zero invariant measure and has no effect on the discrepancy. A similar situation holds for the Hamming space $\mathcal{X}_n$ with odd n , because in this case the “equator” with $t = n/2$ is simply an empty set.

Proof. From (21) we conclude that

$$N = |B(y, m) \cap Z_N| + |B(\bar{y}, m) \cap Z_N|,$$

and for any $y \in \mathcal{X}_n$ the local discrepancy can be written as

$$2|D(Z_N, y, m)| = \left| 2|B(y, m) \cap Z_N| - N \right| = \left| |B(y, m) \cap Z_N| - |B(\bar{y}, m) \cap Z_N| \right|. \quad (24)$$

Let $N = 2K$. Inequality (22) holds for all subsets Z_N . If Z_N is formed of K pairs of antipodal points, then $|D(Z_N, y, m)| = 0$ for all $y \in \mathcal{X}_n$. This proves part (i).

Let $N = 2K + 1$. It follows from (24) that $2|D(Z_N, y, m)| \geq 1$, since N is odd and $2|B(y, m) \cap Z_N|$ is even. This implies inequality (23). Furthermore, it also follows from (24) that $2|D(Z_N, y, m)| = 1$ for all $y \in \mathcal{X}_n$ if Z_N consists of K pairs of antipodal points supplemented with a single point. This proves part (ii). $\square$

Thus in particular, any linear code $Z_N \subset \mathcal{X}_n$ that contains the all-ones vector has discrepancy zero (such codes are called self-complementary). Many well-known families of binary linear codes such as the Hamming codes, BCH codes [13], etc. possess this property.

A minor generalization of the above proof implies the following useful relation. Let $Z_N = Z'_N \cup Z''_N$ be a union of two subsets, where Z'_N contains all pairs of antipodal points in Z_N then

$$D_p^{(m)}(Z_N) = D_p^{(m)}(Z'_N), \quad p \in (0, \infty].$$

3.1. Quadratic discrepancy for hemispheres

In this section we consider the discrepancy $D_p^{(m)}(Z_N)$ defined in (21) for the special case $p = 2$. Let $Z_N \subset \mathcal{X}_n$ be a code, where $n = 2m + 1$. For a pair of points $x, y \in \mathcal{X}_n$ such that $d(x, y) = w$ let $\mu_m(x, y) = \mu_m(w) = |B(x, m) \cap B(y, m)|$ be the size of the intersection of the balls of radius t with centers at x and y . By abuse of notation we write μ_m both as a kernel on $\mathcal{X}_n \times \mathcal{X}_n$ and as a function on $\{0, 1, \dots, n\}$. This is possible because $\mu_m(x, y)$ depends only on the distance between x and y . Note that $\mu_m(0) = v(m) = 2^{n-1}$ and $\mu_m(n) = 0$.

In this section we use some more specific facts of coding theory. We refer to [13] for details. For a code $Z_N \subset \mathcal{X}_n$ let

$$A_w = A_w(Z_N) = \frac{1}{N} |\{(z_i, z_j) \in Z_N^2 \mid d(z_i, z_j) = w\}|, \quad w = 0, 1, \dots, n$$

be the normalized number of ordered pairs of points at distance w (the numbers A_w , $w = 0, 1, \dots, n$ form the *distance distribution* of Z_N). Recall that the *dual distance distribution* of the code Z_N is given by

$$A_i^\perp = \frac{1}{N} \sum_{w=0}^n A_w K_i^{(n)}(w), \quad i = 0, 1, \dots, n, \quad (25)$$

where $K_i^{(n)}(x)$ be the binary Krawtchouk polynomial of degree $k = 0, \dots, n$, defined as follows:

$$K_i^{(n)}(x) = \sum_{j=0}^i (-1)^j \binom{x}{j} \binom{n-x}{i-j}. \quad (26)$$

The vector $(A_i^\perp)$ forms the MacWilliams transform of the distance distribution of the code Z_N , and if Z_N is a linear code, it coincides with the weight distribution of the dual code $Z_N^\perp$ [13, pp. 129,138]. The MacWilliams transform is an involution [11, Thm. 3], which enables us to invert relations (25):

$$A_i = \frac{2^n}{N} \sum_{w=0}^n A_w^\perp K_i^{(n)}(w), \quad i = 0, 1, \dots, n. \quad (27)$$

The following result is implied by [1], Lemma 4.1.

Lemma 3.2. *The Krawtchouk expansion of the function $\mu_m(w)$, $w = 0, 1, \dots, n$ has the form*

$$\mu_m(w) = \hat{\mu}_0 + \sum_{\substack{k=1 \\ k \text{ odd}}}^n \hat{\mu}_k K_k^{(n)}(w)$$

where $\hat{\mu}_0 = 2^{n-2}$ and for all $k = 1, 3, \dots, n$

$$\hat{\mu}_k = 2^{-n} \binom{2m}{m}^2 \frac{\binom{m}{(k-1)/2}^2}{\binom{2m}{k-1}^2}.$$

In the next proposition we establish a version of Stolarsky's invariance principle for the quadratic discrepancy $D_2^{(m)}(Z_N)$ defined above in (21).

Proposition 3.3. *We have*

$$2^n N^{-2} D_2^{(m)}(Z_N)^2 = \frac{1}{N} \sum_{w=0}^n A_w \mu_m(w) - 2^{n-2} \quad (28)$$

$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^n \hat{\mu}_k A_k^\perp. \quad (29)$$

Proof. Starting with (21), we compute

$$\begin{aligned} 2^n D_2^{(m)}(Z_N)^2 &= \sum_{y \in \mathcal{X}_n} \left(\sum_{j=1}^N \mathbb{1}_{B(y,m)}(z_j) - \frac{N}{2} \right)^2 = \sum_{y \in \mathcal{X}_n} \left(\sum_{j=1}^N \mathbb{1}_{B(z_j,m)}(y) - \frac{N}{2} \right)^2 \\ &= \sum_{y \in \mathcal{X}_n} \left(\sum_{i,j=1}^N \mathbb{1}_{B(z_i,m)}(y) \mathbb{1}_{B(z_j,m)}(y) - N \sum_{j=1}^N \mathbb{1}_{B(z_j,m)}(y) + \frac{N^2}{4} \right) \\ &= \sum_{i,j=1}^N \sum_{y \in \mathcal{X}_n} \mathbb{1}_{B(z_i,m)}(y) \mathbb{1}_{B(z_j,m)}(y) - 2^{n-2} N^2 \\ &= \sum_{i,j=1}^N \mu_m(z_i, z_j) - 2^{n-2} N^2 = N \sum_{w=0}^n A_w \mu_m(w) - 2^{n-2} N^2, \end{aligned}$$

where the last equality uses the definition of A_w . This proves (28).

To obtain (29), substitute the result of Lemma 3.2 into (28) and then use (25). $\square$

The size of the intersection of the balls can be written in a more explicit form:

$$\mu_m(w) = \sum_{i,j} \binom{w}{i} \binom{n-w}{j}, \quad w = 0, 1, \dots, n,$$

where $i+j \leq m$, $0 \leq w-i+j \leq m$; in particular, $\mu_m(0) = 2^{n-1}$. It is not difficult to show that for any $l = 1, 2, \dots, \lfloor n/2 \rfloor$ we have $\mu_m(2l-1) = \mu_m(2l)$ and otherwise $\mu_m(w)$ is a decreasing function of w .

Let $\langle \mu_m \rangle_{\mathcal{E}}$ be the average value of the kernel $\mu_m(x, y)$ over the subset $\mathcal{E} \subset \mathcal{X}_n$. Since $\langle \mu_m \rangle_{\mathcal{X}_n} = \widehat{\mu}_0$, we can write (28) in the following form:

$$2^n N^{-2} D_2^{(m)}(Z_N)^2 = \langle \mu_m \rangle_{Z_N} - \langle \mu_m \rangle_{\mathcal{X}_n}. \quad (30)$$

Relations (30), (28) are similar to the invariance principle for hemispheres in the case of the Euclidean sphere, [4, Thm. 3.1]. At the same time, the concrete forms of the results for the Hamming space and the sphere are different: while for the sphere the quadratic discrepancy is expressed via the average geodesic distance in Z_N , in the Hamming case it is related to the average of the kernel μ_m and is not immediately connected to the average distance. Note that for quadratic discrepancy $D_2(G, Z_N)$ for the Hamming space defined above in (2), results of this form were previously established in [1].

Our final result in this section concerns a characterization of codes with zero discrepancy for hemispheres for the case of even N .

Theorem 3.4. *Let Z_N be a code of even size N . Then $D_2^{(m)}(Z_N) = 0$ if and only if the code Z_N is formed of $N/2$ antipodal pairs of points.*

Proof. The sufficiency part has been proved in Theorem 3.1. The proof in the other direction is a combination of the following steps.

Step 1. Since $\widehat{\mu}_k > 0$ for all odd k , expression (29) implies that a code $Z_N \subset \mathcal{X}_n$ has zero quadratic discrepancy for hemispheres if and only if its dual distance coefficients $A_k^\perp \neq 0$ only if k is even,

Step 2. A code Z_N is formed of antipodal pairs if and only if its distance distribution is symmetric, i.e., $A_w = A_{n-w}$ for all $w = 0, 1, \dots, m$.

Indeed, the distance distribution coefficients A_w , $w = 0, \dots, n$ can be written as

$$A_w = \sum_{z \in Z_N} A_w(z), \quad (31)$$

where $A_w(z) = \frac{1}{N} |\{y : d(z, y) = w\}|$ is the local distance distribution at the point $z \in Z_N$.

Suppose the code is formed of antipodal pairs. For every $y \in Z_N$ such that $d(z, y) = w$, the opposite point $\bar{y}$ satisfies $d(z, \bar{y}) = n - w$, and thus, the pair $(y, \bar{y})$ contributes to $A_w(z)$ and $A_{n-w}(z)$ in equal amounts. Therefore, from (31) also $A_w = A_{n-w}$.

Now suppose that the distance distribution is symmetric. For any code $A_0 = 1$, and then also $A_n = 1$, but this means that every code point has a diametrically opposite one, or otherwise (31) cannot be satisfied for $w = n$.

Step 3. The matrix

$$\Phi_m = \begin{pmatrix} K_1^{(n)}(0) & K_1^{(n)}(1) & \dots & K_1^{(n)}(m) \\ K_3^{(n)}(0) & K_3^{(n)}(1) & \dots & K_3^{(n)}(m) \\ \vdots & \vdots & \dots & \vdots \\ K_{2m+1}^{(n)}(0) & K_{2m+1}^{(n)}(1) & \dots & K_{2m+1}^{(n)}(m) \end{pmatrix}$$

has rank $m + 1$. This is shown as follows. Orthogonality of Krawtchouk polynomials [11], [13, Thm 5.16] implies that

$$\begin{aligned} \binom{n}{k} 2^n \delta_{j,k} &= \sum_{w=0}^{2m+1} K_k^{(n)}(w) K_j^{(n)}(w) \binom{n}{w} \\ &= \sum_{w=0}^m K_k^{(n)}(w) K_j^{(n)}(w) \binom{n}{w} + \sum_{w=m+1}^{2m+1} (-1)^{j+k} K_k^{(n)}(n-w) K_j^{(n)}(n-w) \binom{n}{n-w} \\ &= 2 \sum_{w=0}^m K_k^{(n)}(w) K_j^{(n)}(w) \binom{n}{w}. \end{aligned}$$

Here on the second line we used the relation

$$K_k^{(n)}(w) = (-1)^k K_k^{(n)}(n-w), \quad 0 \leq k, w \leq n. \quad (32)$$

which is immediate from (26). In other words, for odd j, k we have

$$\sum_{w=0}^m K_k^{(n)}(w) K_j^{(n)}(w) \binom{n}{w} = \delta_{k,j} 2^{n-1} \binom{n}{k}. \quad (33)$$

Rephrasing this relation, we obtain

$$\Phi_m B \Phi_m^T = 2^{n-1} \text{diag} \left(\binom{n}{1}, \binom{n}{3}, \dots, \binom{n}{2m+1} \right),$$

where $B = \text{diag}(\binom{n}{w}, w = 0, 1, \dots, m)$. This implies that $\text{rank}(\Phi_m) = m+1$.

Step 4. To complete the proof, suppose that $D_2^{(m)}(Z_N) = 0$ and thus from Step 1 above, $A_k^\perp = 0$ for all odd k . In particular, for $k = 1, 3, \dots, 2m+1$, using (25) and (32), we obtain

$$\sum_{w=0}^{2m+1} A_w K_k^{(n)}(w) = \sum_{w=0}^m (A_w - A_{n-w}) K_k^{(n)}(w) = 0. \quad (34)$$

Define the vector

$$\alpha = (A_w - A_{n-w}, w = 0, 1, \dots, m).$$

From (34) and the definition of Φ_m we obtain that $\Phi_m \alpha^T = 0$. From Step 3 we conclude that $\alpha = 0$ or $A_w = A_{n-w}, w = 0, 1, \dots, m$. Now Step 2 implies our claim. $\square$

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