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Matching soulmates

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Abstract

We study iterated matching of soulmates (IMS), a recursive process of forming coalitions that are mutually preferred by members to *any* other coalition containing individuals as yet unmatched by this process. If all players can be matched this way, preferences are IMS-complete. A mechanism is a soulmate mechanism if it allows the formation of all soulmate coalitions. Our model follows Banerjee, Konishi, and Sönmez, except reported preferences are strategic variables. We investigate the incentive and stability properties of soulmate mechanisms. In contrast to prior literature, we do not impose conditions that ensure IMS-completeness. A fundamental result is that, (1) any group of players who could change their reported preferences and mutually benefit does not contain any players who were matched as soulmates and reported their preferences truthfully. As corollaries, (2) for any IMS-complete profile, soulmate mechanisms have a truthful strong Nash equilibrium, and (3) as long as all players matched as soulmates report their preferences truthfully, there is no incentive for any to deviate. Moreover, (4) soulmate coalitions are invariant core coalitions—that is, any soulmate coalition will be a coalition in every outcome in the core. To accompany our theoretical results, we present real-world data analysis and simulations that highlight the prevalence of situations in which many, but not all, players can be matched as soulmates. In the Appendix we relate IMS to other well-known coalition formation processes.

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1 | INTRODUCTION

This paper studies a coalition formation procedure we call *Matching Soulmates*. To introduce the framework of our model and the results, let us begin with a simple 7-person example.

Alice would rather be with Alex than anyone else. Alex feels the same way about Alice. They are soulmates. Since they would be willing to leave any other partners to be together, they are a threat to the stability of all matching that does not pair them.

Bertie and Ben are not so smitten; Ben would rather be with Alice than anyone, and Bertie with Alex. As long as Alice and Alex are paired, Bertie and Ben are soulmates in a conditional sense. As long as Alice and Alex are paired, Ben and Bertie threaten the stability of any matching in which they are not paired.

Casey, Devin, and Eddie would rather be matched with any of the previously introduced players. But suppose that among the three, Casey would prefer to be with Devin, Devin would prefer to be with Eddie, and Eddie would prefer to be with Casey; no pair of these three players are (conditional) soulmates.

We refer to the process by which Alice and Alex and then Bertie and Ben are matched by *iterated matching of soulmates* (IMS) and, when all players can be matched by IMS, we say that preferences are *IMS-complete*. While the example considers only a simple model matching pairs of players, our work applies to coalition formation models in which players have preferences over arbitrary sets of coalitions of which they are members.

Since Banerjee et al. (2001), this process has been used in a number of papers albeit without being named or with a different name.[1] Our work differs from prior literature in that we do not impose conditions on the model to ensure IMS-completeness.

Another seminal paper is Pápai (2004) which asks, in a coalition formation model, whether the core is unique, consisting of only one partition of the set of players.[2] Papai demonstrates that a necessary and sufficient condition on the environment for uniqueness of the core under any preference profile in that the environment is such that any two admissible coalitions have at most one member in common and permissible coalitions do not form a cycle, the *single-lapping property*. This property ensures that any preference profile in that environment is, in our words, IMS-complete. Papai continues to show that in a strategic form of her model, the single-lapping property ensures the existence of a strategy-proof equilibrium; players have no incentive to misrepresent their preferences.

[1]In fact, the process is even older. Banerjee et al. (2001, Section 6.5) show that in the context of a housing market (Shapley & Scarf, 1974), if the players are endowed with appropriate preferences over coalitions of players, then top trading cycle is equivalent to IMS (in our terminology).

[2]Appendix A.1 carefully relates several conditions in the literature ensuring nonemptiness of the core.

The results of Banerjee et al. (2001) and Pápai (2004) are beautiful and important to our understanding of coalition formation. Their results, however, require strong restrictions. These restrictions rule out our example above. It is not hard to imagine situations where some, but not all, players can be matched as soulmates, such as marriage models or roommate models with cycles.

Our paper breaks from the prior literature in that we consider the properties of mechanisms that match soulmates, without imposing further restrictions on the model. We consider both situations where preferences are IMS-complete and ones where preferences do not necessarily satisfy IMS-completeness. We next list a number of our results. The statements are somewhat informal, but each will give the reader a rough idea of the result.

A fundamental result (Proposition 1) is that: Given reported preferences and a mechanism that matches soulmates, *there does not exist a deviating coalition containing soulmates who have reported their true preferences*.

This result has the following important corollaries. First, if preferences are IMS-complete, then the preference revelation game has a truthful strong Nash equilibrium (Corollary 1). Indeed, we can go further: for any soulmates coalition C and any player $i \in C$, if all other players in C have reported their true preferences, then player i's best response is to report their true preferences (Corollary 2).

Note that Proposition 1 and Corollary 2 are positive results that do not require that all players can be matched as soulmates. However, for such situations, we also provide a negative result extending Rodrigues-Neto (2007). If true preferences contain a cycle of odd length, no mechanism that matches soulmates has a truthful strong Nash equilibrium (Proposition 2).

We also provide several results about the outcome properties of mechanisms that match soulmates. First, if the core of a coalition formation problem exists for a particular preference profile, then the coalitions formed by IMS are part of every core partition (Proposition 3 and Corollary 3). Further, for any profile, if there are blocking coalitions to a partition produced by a mechanism that matches soulmates, those coalitions do not contain soulmates (Proposition 5). Similarly, all those matched by IMS prefer their match to being alone (individual rationality, Corollary 5) and if a partition exists that some subset of players mutually prefers, that subset contains no soulmates (partial Pareto efficiency, Corollary 5). Again, these results pertain to preferences profiles that may be IMS-incomplete. On the other hand, if the profile is IMS-complete, then the set of coalitions produced by IMS is the unique core partition (Proposition 4), and is individually rational and Pareto optimal (Corollary 4). The core result is also implied by a result in Banerjee et al. (2001) and generalized to settings with restrictions to the allowable set of coalitions by İnal (2019). We also provide a number of informative examples.

Since our paper establishes that IMS provides desirable properties for those players that it matches, even when it cannot match all players, we also study the commonality of soulmates and conditional soulmates in an empirical analysis of real-world data sets and computational experiments.

Our empirical analysis studies three settings: a roommate's problem[3] using data from a university social network; a similar problem involving building work teams using data on

[3]The roommates problem was originally introduced by Gale and Shapley (1962) as an extension of the marriage problem. For a recent review of the literature related to the problem, see Manlove (2013).

connections within a consulting firm; and a two-sided matching problem using data from a speed-dating experiment. By studying these environments, we can understand how mechanisms implementing IMS can impact productivity in work environments, and compatibility between matched groups in social settings. In fact, a surprising number of people can be matched by IMS; about 25%–40% on average in the three environments and as many as 75% for particular instances of work teams, indicating that IMS may be particularly powerful in producing efficient teams in work settings.[4]

To study how the structure of preferences affects IMS, our computational experiments analyze how likely players are to be matched by IMS under various preference patterns. We consider unconstrained preference profiles as well as profiles that are a relaxation of reciprocal preferences,[5] and profiles that are a relaxation of common-ranking. While IMS-complete preferences are rare among unconstrained preferences, they are quite common when preferences exhibit strong reciprocity or are close to commonly ranked.

These results shed additional light on settings where IMS can be particularly effective. For instance, in work settings, strong complementarities in worker strengths and expertise may be reflected in highly reciprocal preferences that would lead many teams to be matched by IMS. This would provide both strong incentive properties to the mechanism and produce highly productive teams.

The structure of our paper is as follows. In Section 2 we present the general matching environment as well as several definitions used throughout the paper. In Section 3 we define IMS formally and provide several examples. We turn to our key results on incentive compatibility and stability in Sections 4 and 5, respectively. Section 6 contains the results of our computational and empirical analysis of soulmates.

2 | ENVIRONMENT

We closely follow the model of Banerjee et al. (2001), but do not assume that preferences are *known* to the mechanism designer. In contrast to Pápai (2004) we do not impose restrictions on the set of permissible coalitions. Instead, we consider unrestricted environments and focus on situations where preferences are such that some, possibly all, players can be matched as soulmates.

The *total player set* is given by $N = \{1, \ldots, n\}$. A *coalition* is a nonempty set of players $C \in 2^N \backslash \{\varnothing\}$. Each player $i \in N$ has a complete and transitive preference $\succ_i$ over the collection of coalitions to which she may belong, denoted $\mathcal{C}_i$ for player i. Coalition formation problems with this feature are often called “hedonic”, and this property may be understood as requiring that there are no externalities in the group-formation process. As the notation $\succ_i$ suggests, we assume that preferences over coalitions are *strict*.

The set of i's possible preferences is $\mathcal{D}_i$. A coalition $C \in \mathcal{C}_i$ is *acceptable* for i if $C = \{i\}$ or $C \succ_i \{i\}$.

A *profile* (of preferences) $\succ \in \mathcal{D} = \times_{i \in N} \mathcal{D}_i$ is a list of preferences, one for each player in N. Given profile $\succ \in \mathcal{D}$ and any subset $S \subseteq N$, the *subprofile* (of preferences) for players in S is denoted by $\succ_S \in \mathcal{D}_S = \times_{i \in S} \mathcal{D}_i$. As is customary, we let $\succ_{-i} = \succ_{N \backslash \{i\}}$.

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[4]The proportions depend on how indifferences are broken.

[5]We consider profiles where player i's top k partners also list i among their top k partners. We refer to these as k-reciprocal profiles.

The domain of possible *true profiles* is $\mathcal{R} \subseteq \mathcal{D}$. In general, the domain of true profiles $\mathcal{R}$ need not be equal to $\mathcal{D}$. Although a preference $\succ_i \in \mathcal{D}_i$ may be a "conceivable" preference for i, $\succ_i$ need not be player i's preference in any *true* profile $\succ \in \mathcal{R}$.[6] It is also possible that, although all preferences $\succ_i \in \mathcal{D}_i$ are i's true preference for *some* profile $\succ \in \mathcal{R}$, some profiles $(\succ_i, \succ'_{-i})$ with $\succ'_{-i} \neq \succ_{-i}$ are not elements of $\mathcal{R}$ because true preferences are *interdependent*. (That is, $\prec'_{-i}$ cannot be the subprofile for players in $N \backslash \{i\}$ when i's preference is $\succ_i$. See, for example, the domains of *k-reciprocal* profiles in Section 6.3.)

A *coalition structure* π is a partition of N. For any coalition structure π and any player $i \in N$, let π_i denote i's coalition of membership, that is, the coalition in π which contains i.

A (*direct coalition formation*) *mechanism* is a game form M that associates every *reported* profile $\succ' \in \mathcal{D}$ with a coalition structure $\pi \in \Pi$ (Π is the set of all coalition structures). For every $i \in N$, the set of preferences $\mathcal{D}_i$ is i's strategy space for the mechanism M. Because mechanisms are *simultaneous* game forms, $\mathcal{D}$ must be the Cartesian product of the sets $\mathcal{D}_i$; simultaneity makes it impossible for any player or group of players to condition their reports on the report of other players.[7]

Together, a pair $(M, \succ)$, where $\succ \in \mathcal{R}$ is a profile of true preferences, determines a *preference revelation game*. Again, the strategy space of a player i in this game is the set of her reported preferences $\mathcal{D}_i$. Once profile $\succ' \in \mathcal{D}$ is reported, the mechanism M determines a coalition structure, denoted $M(\succ')$. The coalition containing player i is denoted $M_i(\succ')$; we say that M *matches* i with $M_i(\succ')$. Each player i evaluates their assigned coalition according to their *true* preference $\succ_i$.

This model of direct coalition formation mechanisms generalizes many common matching environments. When mechanisms are individually rational, restrictions on the collection of feasible coalitions can often be translated into restrictions on the domain of preferences by forcing infeasible coalitions that contain i to be "*un*acceptable" for i (i.e., ordered strictly below i given $\succ_i$). As an example, in our environment, the roommates problem is obtained by restricting $\mathcal{D}$ to the set of profiles in which only singletons and pairs of players are acceptable. In the marriage problem (Gale & Shapley, 1962), only singletons and pairs of players with a player from each side of the market are acceptable. In the college admission problem (Roth, 1985), $\mathcal{D}$ is such that only coalitions containing a single college and some (or no) students are acceptable (and that students are indifferent between any two coalitions with the same college).

3 | ITERATED MATCHING OF SOULMATES

Given $\succ \in \mathcal{D}$, a coalition $C \in 2^N$ is a *first-order soulmate coalition* if

$$C \succ_i C' \quad \text{for all } i \in C \text{ and all } C' \in \mathcal{C}_i \backslash C. \tag{1}$$

The process of IMS involves repeatedly forming (first-order) soulmates coalitions from player sets decreasing in size as coalitions of soulmates are formed.

[6]For example, a preference $\succ_i$ in which only $\{i\}$ is acceptable for i is conceivable. However, the mechanism designer may believe that i's domain of true preferences $\mathcal{R}$ contains preferences in which at least one pair $\{i, j\}$ is acceptable for i.
[7]Such conditioning would require M to be a sequential mechanism, which is *not* allowed in this paper. Recall that, unlike the space of strategy profiles $\mathcal{D}$, the domain of *true* profiles $\mathcal{R}$ needs *not* be Cartesian (see above).

While no ordering is required in the formation of soulmates coalitions, it is easiest to describe IMS as if it were a dynamic process. In the first round, given a profile of reported preferences, the mechanism forms soulmates coalitions. The members of these coalitions prefer their assigned coalition to all others. In the second round, the mechanism forms soulmates coalitions among the players who are not assigned to coalitions in the first round, and so forth. Formally, the process of IMS is defined as follows.

Round 1. Form first-order soulmates coalitions (i.e., coalitions satisfying 1). Denote the collection of these coalitions by $\mathcal{S}_1(\succ)$. The set of players who belong to a coalition in $\mathcal{S}_1(\succ)$ is denoted by $N_1(\succ)$. These players are called *first-order soulmates*.

⋮

Round $\boldsymbol{r}$. Form coalitions of first-order soulmates among the players who are *not* part of a coalition that forms in any round preceding round r. Call these coalitions rth-*order soulmates coalitions* and denote the collection of these coalitions by $\mathcal{S}_r(\succ)$. The set of players who belong to a coalition in $\mathcal{S}_r(\succ)$ is denoted by $N_r(\succ)$. These players are called rth-*order soulmates*.

Formally, given any integer r, the rth-order soulmates coalitions are the coalitions C that contain no players from $\cup_{j=1}^{r-1} N_j(\succ)$ and are such that

$$C \succ_i C' \quad \text{for all } i \in C \text{ and all } C' \in \mathcal{C}_i \backslash C \text{ with } C' \cap \left(\cup_{j=1}^{r-1} N_j(\succ)\right) = \varnothing.$$

End. The process ends when no coalition forms in some round r^*, that is,

$$\mathcal{S}_{r^*}(\succ) = \varnothing.$$

For convenience, we will denote the collection of coalitions $\cup_{j=1}^{r^*-1} \mathcal{S}_j(\succ)$ formed by this process as $IMS(\succ)$. We refer to any player who is matched by IMS as a *soulmate*, and to every coalition that forms under IMS as a *soulmates coalition* (or *coalition of soulmates*).

A mechanism M is a *first-order soulmate mechanism* if for every $\succ \in \mathcal{D}$, every first-order soulmates coalition forms under M (i.e., $\mathcal{S}_1(\succ) \subseteq M(\succ)$). Similarly, a mechanism M is a *soulmate mechanism* if for every $\succ \in \mathcal{D}$, the coalitions that form under IMS form under M (i.e., $IMS(\succ) \subseteq M(\succ)$).

3.1 | Examples

Our next examples illustrate the process of IMS for particular profiles. Though preferences in our environment are over *coalitions* to which a player might belong, to save space in examples below we will often express these as the player's preferences over partners.

Example 1 (Formation of parliamentary groups). A Left (L), a Center (C), a Right (R), and a Green (G) party have to form parliamentary groups. Their preferences form a *roommate's profile*: (i) every party prefers a coalition of two to being alone, and (ii) every party prefers being alone to being in a coalition of more than two players. The parties have the following preferences over partners

$$\begin{aligned} R&: C \succ_R G \succ_R L, \\ C&: R \succ_C G \succ_C L, \\ G&: L \succ_G C \succ_G R, \\ L&: C \succ_L G \succ_L R. \end{aligned}$$

If we apply IMS to this profile, coalition $\{R, C\}$ forms in the first round, and coalition $\{G, L\}$ forms in the second round.

Example 2 (Marriage, aligned women and cyclic men). In a *marriage profile* (i) $N = M \cup W$, (ii) every woman $w \in W$ (resp., man $m \in M$) prefers being in a pair with a man m (resp., woman w) to being alone, and (iii) every woman $w \in W$ (resp., man $m \in M$) prefers being alone to being in any coalition different from a pair with a man m (resp., woman w). Consider the following profile of preferences over partners

$$\begin{aligned} w_1&: m_1 \succ_1^w m_2 \succ_1^w m_3, & m_1&: w_1 \succ_1^m w_2 \succ_1^m w_3, \\ w_2&: m_1 \succ_2^w m_2 \succ_2^w m_3, & m_2&: w_2 \succ_2^m w_3 \succ_2^m w_1, \\ w_3&: m_1 \succ_3^w m_2 \succ_3^w m_3, & m_3&: w_3 \succ_3^m w_1 \succ_3^m w_2. \end{aligned}$$

In the first round of IMS, $\{m_1, w_1\}$ forms. In the second round, given that m_1 has already been matched, $\{m_2, w_2\}$ is a coalition of soulmates and forms. In the third round, given that m_1 and m_2 have already been matched $\{m_3, w_3\}$ is a coalition of soulmates and forms. It is easy to see how this example extends to larger sets of players.

Some famous coalition formation mechanisms are soulmate mechanisms. This is the case, for example, for the *deferred acceptance* (DA) mechanism in two-sided matching. As we show in Section 5, this follows from the well-known fact that DA always selects a core partition. In contrast, despite being a first-order soulmate mechanism, the *immediate acceptance* (IA) mechanism (or *Boston* mechanism; Abdulkadiroğlu & Sönmez, 2003) also used in two-sided matching is *not* a soulmate mechanism.

Example 3 (IA is not a soulmate mechanism). Consider the following profile of preferences over partners in a marriage profile.

$$\begin{aligned} w_1&: m_1 \succ_1^w m_2, \\ w_2&: m_1 \succ_2^w m_2, \\ w_3&: m_2 \succ_3^w m_1, \end{aligned} \qquad \begin{aligned} m_1&: w_1 \succ_1^m w_2 \succ_1^m w_3, \\ m_2&: w_1 \succ_2^m w_2 \succ_1^m w_3. \end{aligned}$$

In the first round of IA, women propose to their favorite man, and men immediately form a coalition with the woman they like best among the women from whom they receive a proposal (hence the name "*immediate* acceptance"). Thus, at the end of the first round, coalitions $\{w_1, m_1\}$ and $\{w_3, m_2\}$ have formed. In the subsequent round, there are no more men available to form a coalition with w_2. Thus, the coalition structure selected by IA is $\{\{w_1, m_1\}, \{w_3, m_2\}, \{w_2\}\}$.

In the first round of IMS, only coalition $\{w_1, m_1\}$ forms. Coalition $\{w_2, m_2\}$ forms in the second round followed by coalition $\{w_3\}$ in the last round. Hence, the coalition structure selected by IMS is $\{\{w_1, m_1\}, \{w_2, m_2\}, \{w_3\}\}$ which differs from that selected by IA.

3.2 | IMS-complete profiles

A profile $\succ \in \mathcal{D}$ is *IMS-complete* if all players match through IMS. All the profiles in Examples 1–3 are IMS-complete, although it is not hard to construct IMS-*in*complete profiles (see Example 5). In the literature, two important classes of IMS-complete profiles are: (1) profiles satisfying the common-ranking property (Farrell & Scotchmer, 1988)[8]; and (2) profiles satisfying the top-coalition property (Banerjee et al., 2001). A profile satisfies the *common-ranking property* if, for any two coalitions, the players in the two coalitions have the same preferences over these two coalitions.[9] A profile satisfies the *top-coalition property* if for every subset $S \subseteq N$, there exists a coalition $C^* \subseteq S$ which is preferred by all its members to any other coalition made of players from S.[10] The common-ranking property implies the top-coalition property, which itself implies IMS-completeness. We illustrate the relationship between the three conditions in Figure 1. See Appendix A.1 for a more complete analysis of the relationship between IMS-completeness and other profile conditions in the literature.[11]

While profiles satisfying the top-coalition property are IMS-complete the converse is not necessarily true.[12] To gain intuition why, consider a profile in which IMS completes in two rounds. This implies that there is *a* coalition C_1 and *a* coalition C_2 such that (i) C_1 is a coalition of first-order soulmates in N, (ii) C_2 is a coalition of first-order soulmates in $N \backslash C_1$, and (iii) $C_1 \cup C_2 = N$. The top-coalition property is much stronger as it requires that, for *any* coalition C, there be a coalition of first-order soulmates in $N \backslash C$. This is illustrated more concretely in Example 1, where the profile is IMS-complete but does not satisfy the top-coalition property because there is no coalition of first-order soulmates in $\{C, G, L\}$.[13]

[8]Pycia (2012) leverages the fact that a condition similar to common-ranking (and thus IMS-completeness in our terminology) is implied when all preference profiles in a domain are pairwise-aligned and the domain is sufficiently rich. We discuss this further in Appendix A.1.3.

[9]Formally, there exists an ordering $\triangleright$ of 2^N such that for all $i \in N$ and any $C, C' \in \mathcal{C}_i$, we have $C \succ_i C'$ if and only if $C \triangleright C'$.

[10]See Banerjee et al. (2001) for examples of games from the literature that feature profiles satisfying the common-ranking and top-coalition properties.

[11]Appendix A.1 discusses several other conditions studied in Bogomolnaia and Jackson (2002) that guarantee the existence of a core coalition structure in our environment (including a weakening of top-coalition introduced in Banerjee et al., 2001). Of these conditions only *weak consecutiveness* is implied by IMS-completeness. Appendix A.1 also studies four additional conditions for the existence of a core coalition structure introduced by Alcalde and Romero-Medina (2006) and the acyclicity condition introduced by Rodrigues-Neto (2007). IMS-completeness is independent of any of these last conditions.

[12]In Section 6, we present computational results on the size of the overlap between IMS-complete profiles and profiles with the top-coalition property.

[13]Another approach to guarantee that IMS matches all the players is to constrain the set of feasible coalitions. Pápai (2004) shows that if the collection of feasible coalitions satisfies a property she calls *single-lapping*, then IMS matches all players.

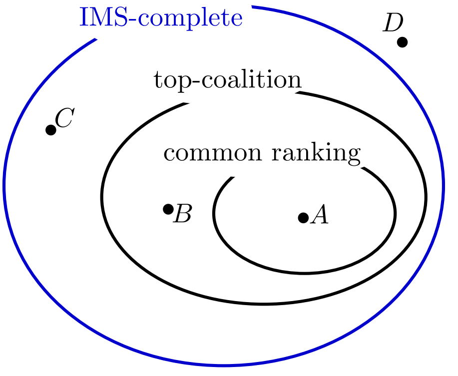

FIGURE 1 Venn diagram of the profile conditions. A dot indicates that the section of the Venn diagram is nonempty. The inclusion relationship is trivial. For examples of profiles of type A see Banerjee et al. (2001, Section 6). For an example of a profile of type B, see Banerjee et al. (2001, Game 4). Example 1 in this paper is a profile of type C. Example 5 in this paper is a profile of type D (any other profile for which the core is empty would also be an example). IMS, iterated matching of soulmates

4 | INCENTIVE PROPERTIES OF SOULMATE MECHANISMS

4.1 | Introduction

In this section, we introduce the desirable properties of IMS with respect to players' incentive to report preferences truthfully. There are at least two reasons to favor mechanisms that provide incentives to report preferences truthfully. First, if players do not report preferences truthfully, then desirable properties that the mechanism satisfies with respect to the *reported* preferences might not be satisfied with respect to the *true* preferences.[14] Second, mechanisms with good incentives to report preferences truthfully "level the playing field" (Pathak & Sönmez, 2008) by protecting naive players who report preferences truthfully against the manipulations of more strategically skilled players.[15]

4.2 | Main result

Any player who reports her preference truthfully and is matched by IMS can find no alternative preference report that makes her better off. In fact, as Proposition 1 shows, no group of players containing a truth-telling soulmate can collude to simultaneously make themselves better off. Moreover, it follows that the reduced game with player set consisting only of players that are matched as soulmates has a truthful strong Nash equilibrium.

[14]As illustrated at the end of Example 5, a mechanism can, for example, produce a *core* outcome with respect to the reported preferences that is not Pareto optimal with respect to the true preferences.

[15]For empirical evidence on the loss incurred by naive players in mechanisms with low incentives to be truthful, see the school choice laboratory experiments in Basteck and Mantovani (2016a) and Basteck and Mantovani (2016b). See also Pathak and Sönmez (2008) for a theoretical argument in the case of school choice.

Proposition 1 (No soulmates among deviators). *Suppose that M is a soulmate mechanism. For any reported preference profile $\succ \in \mathcal{D}$ and set of players W that contains some soulmates, all of whom report their true preferences, there does not exist a joint deviation $\succ'_W$ by the members of W that makes every player in W better-off than reporting $\succ_W$.*

Proof. Consider a player matched in the first round of IMS. That is, $i \in N_1(\succ)$. If i is in W then $\succ_i$ is the true preference of i and they cannot be made better off since $N_1(\succ)$ must contain their favorite coalition.

Now suppose i is matched in some other round of IMS. $i \in N_k(\succ)$ for $k > 1$. If i is in W then $\succ_i$ is their true preference and they can only be made better off being matched with a coalition that contains at least some players matched earlier in the process of IMS.

Thus, to make i better off, there must be some player i' matched in an earlier round of IMS under reported preferences $\succ$ that is also in W. But, this requires that $\succ_{i'}$ is also the true preference of i' and that i' be made better off. Either $i' \in N_1(\succ)$, in which case i' cannot be made better off (by the logic above), or $i' \in N_{k'}(\succ)$ for $k' < k$ in which case this logic applies recursively—there must be an i'' matched even earlier in IMS who is in W and who has reported truthfully.

Since the number of players (and thus the number of possible rounds of IMS) is finite, induction leads to the requirement that for players in W matched in some round k of IMS for $k > 1$ to be made better off, W must contain a first-order soulmate who, by the truthful reporting requirement, *cannot be made better off*. Thus, either W contains no players who have truthfully reported and are matched in some round of IMS, or not every player in W is made better off. □

4.3 | Corollaries of Proposition 1 and strong Nash equilibrium

Proposition 1 implies some remarkable incentive properties for IMS-complete profiles. Perhaps more significantly, soulmate mechanisms retain these properties *in general* for the players matched by IMS, even in profiles that are *not* IMS-complete. These properties are presented more formally in Corollaries 1 and 2.

Before presenting these corollaries, we provide the formal definition of strong Nash equilibrium. Given a true profile $\succ \in \mathcal{R}$, game $(M, \succ)$ has a *truthful strong Nash equilibrium* if there exists no group of players $S \subseteq N$ and no reported subprofile $\succ'_S \in \mathcal{D}_S$ different from $\succ_S$ such that

$$M_i(\succ'_S, \succ_{N\setminus S}) \succ_i M_i(\succ_S, \succ_{N\setminus S}) \quad \text{for all } i \in S. \tag{2}$$

Often,[16] we care about whether a mechanism M induces games that have a truthful strong Nash equilibrium for *every* profile in the domain of true profiles $\mathcal{R}$. Mechanism M is said to have a *truthful strong Nash equilibrium on domain*$\mathcal{R}$ if $(M, \succ)$ has a truthful strong Nash equilibrium for all $\succ \in \mathcal{R}$.

For IMS-complete profiles, the following is a corollary of Proposition 1.

[16]The terminology *strong Nash equilibrium* was introduced by Aumann (1959).

Corollary 1 (Truthful strong Nash equilibrium). *Suppose that M is a soulmate mechanism. For any IMS-complete profile $\succ$, the preference revelation game $(M, \succ)$ has a truthful strong Nash equilibrium.*

In particular, a soulmate mechanism M always has a truthful strong Nash equilibrium on a domain $\mathcal{R}$ containing only IMS-complete profiles. However, the incentive compatibility properties extend beyond IMS-complete profiles. Proposition 1 implies that even if a profile is not IMS-complete, soulmate mechanisms are incentive compatible for all soulmates under the mutual belief that every soulmate tells the truth.

Corollary 2 (Mutual truthfulness among soulmates). *Suppose that M is a soulmate mechanism. Let S be the set of players who are part of a soulmates coalition in $\succ$. For all $i \in S$ there is no profitable deviation from truth-telling under the condition that every other $j \in S (j \neq i)$ tells the truth.*

4.4 | Impossibilities

The incentive compatibility implied by Proposition 1 does *not* generally extend to players who are not matched by IMS, as we illustrate in the following modification of Example 1:

Example 4 (Potential deviation for nonsoulmates). A Left (L), a Right (R), and a Center (C) party have to form parliamentary groups. All parties prefer being in a coalition to being alone. The parties have the following preferences over partners:

$$\begin{aligned} R&: \; C \succ_R L, \\ C&: \; L \succ_C C, \\ L&: \; R \succ_L C. \end{aligned}$$

Consider for instance IMS-serial dictatorship (IMS^{SD}) which consists in first applying IMS and then using the serial dictatorship mechanism to determine the assignment of the players who do not match under IMS (any such mechanism is therefore a soulmate mechanism). Suppose that the series of dictators is R, C, L. Because $IMS(\succ) = \emptyset$, we have $IMS^{SD}(\succ) = SD(\succ)$, the outcome of the serial dictatorship mechanism given profile $\succ$. Thus, $IMS^{SD}(\succ) = \{\{R, C\}, \{L\}\}$. L can deviate by reporting C as their most preferred partner. This forces IMS^{SD} to form $\{C, L\}$ as a coalition of soulmates, and both C and L prefer to their match in $IMS^{SD}(\succ)$.

Our next result shows that the kind of deviation in Example 4 implies that the remarkable incentive properties identified in Proposition 1 cannot *generally* be strengthened much further. Proposition 2 shows that if $\mathcal{R}$ is sufficiently rich, with the possibility of containing the kind of preference structure in Example 4, *any* soulmates mechanism M will fail to induce a truthful Nash equilibrium in game $(M, \succ)$ for some IMS-*in*complete $\succ \in \mathcal{R}$.

A similar incompatibility can be found in Takamiya (2012, Proposition 3), which shows that, without further restrictions, no two-sided matching mechanism that (in our terminology) is also a first-order soulmate mechanism is strategy-proof. As we demonstrate below, this impossibility extends to many settings outside of two-sided matching.

Notice that there is a cycle of preferences among the three parties in Example 4. Suppose that M is a first-order soulmate mechanism. Mechanism M can form at most one of the three pairs $\{L, R\}$, $\{L, C\}$, $\{R, C\}$. But then, any party who is not in one of these pairs can manipulate by reporting that they are the soulmate of one of the others. Hence, no first-order soulmate mechanism M has a truthful Nash equilibrium on a domain that includes the type of cyclic profile in Example 4 (and that allows the aforementioned deviations).

The reason the above profile prevents first-order soulmate mechanisms to have a truthful Nash equilibrium is that it contains a cycle of the type: 1 likes 2 best, 2 likes 3 best, and 3 likes 1 best. In roommates problems, this impossibility can *only* occur in profiles featuring such cycles. As Rodrigues-Neto (2007) argued, roommates profiles that do not feature cycles of any length are IMS-complete and therefore have a truthful strong Nash equilibrium by Corollary 1.

It is possible to extend the cycle condition from Rodrigues-Neto (2007) to general coalition formation environments. In a general coalition formation environment, soulmate mechanisms may fail to have a truthful Nash equilibrium for reasons other than cycles, but the presence of a cycle of odd size is sufficient to induce the impossibility.

Our generalized cycle conditions are defined formally in Appendix A.2. Intuitively, *individually cyclic* domains have cycles in which coalitions $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$ are replaced by coalitions of the form $\{1, 2\} \cup O_{12}$, $\{2, 3\} \cup O_{23}$, and $\{3, 1\} \cup O_{31}$, where the O_{jh} are set of players that are allowed to rank coalition $\{j, h, O_{jh}\}$ as their best coalition. In Appendix A.2, we also define *cyclic* domains in which 1, 2, and 3 are replaced by groups of players N_1, N_2, and N_3 that can jointly deviate. *Individually odd-cyclic* and *odd-cyclic domains* have cycles of the corresponding type that involve an odd number of coalitions.

Proposition 2 (Impossibilities). *(i) If $\mathcal{R}$ is an odd-cyclic domain, no first-order soulmate mechanism M has a truthful* strong *Nash equilibrium on $\mathcal{R}$. (ii) If $\mathcal{R}$ is an individually odd-cyclic domain, no first-order soulmate mechanism M has a truthful Nash equilibrium on $\mathcal{R}$.*

The proof of Proposition 2 generalizes the logic of roommates problem cycle discussed above and can be found in Appendix A.2.

5 | OUTCOME PROPERTIES OF SOULMATE MECHANISMS

We now introduce properties of the *outcomes* of a mechanism, where these properties are evaluated with respect to the *reported* preferences. As discussed above, properties with respect to reported preferences will only reflect properties with respect to true preferences when and for players who have the incentive to report preferences truthfully. In each case below, either our results pertain to IMS-complete profiles, in which case truth-telling is a strong Nash equilibrium by Corollary 1, or the properties are focused on the outcomes of those players who match as soulmates. By Proposition 1, there are strong incentives for truth-telling among those who would match as soulmates when reporting truthfully. Thus, for soulmates, the properties with respect to reported preferences can be seen as reliable with respect to true preferences.

Properties with respect to reported preferences may also be relevant per se to the mechanism designer. For example, in school choice, a mechanism that selects a *core* matching with respect to the *reported* preferences provides a protection against challenges of the matching in court.[17]

Before presenting the results, we first define some relevant concepts.

5.1 | Definitions

Given any profile $\succ \in \mathcal{D}$, a core partition is a coalition structure π^* in which no subset of players strictly prefers matching with each other rather than matching with their respective coalitions in π^*. Formally, π^* is a *core partition* if there does not exist a *blocking coalition* to π^*, that is, a coalition $C \in 2^N$ such that $C \succ_i \pi_i^*$ for all $i \in C$.

A particular kind of blocking coalitions is singletons coalitions $\{i\}$, where i prefers being alone to being in the coalition to which she is matched by the mechanism. Given any profile $\succ \in \mathcal{D}$, a partition π^* is *individually rational* if, for all $i \in N$, $\pi_i^* = \{i\}$ or $\pi_i^* \succ_i \{i\}$. Mechanism M is *individually rational* if $M(\succ)$ is individually rational for all $\succ \in \mathcal{D}$.

Blocking coalitions also exist if the outcome of a mechanism fails to be Pareto optimal. Given any profile $\succ \in \mathcal{D}$, a partition π^* is *Pareto optimal* if there exists no other coalition structure that is preferred by every player to π^*. Clearly, a core partition π is Pareto optimal, because any coalition in a partition π' that is preferred by every player to π is a blocking coalition to π. *Mechanism* $\boldsymbol{M}$ *is Pareto optimal* if $M(\succ)$ is Pareto optimal for all $\succ \in \mathcal{D}$.

5.2 | Soulmates and the core

Our first result pertains to the relationship between coalitions formed by IMS and the core of a matching problem. Soulmates coalitions are coalitions in any core partition. The logic of the proof is similar to that of Proposition 1.

Proposition 3 (Soulmate coalitions are core coalitions). *Given any $\succ \in \mathcal{D}$, let π^* be a partition in the core and let C be a coalition formed by IMS$(\succ)$. Then $C \in \pi^*$.*

Proof. Again, we use the notation in the definition of IMS. Let π^* be a partition in the core and suppose that $C^1 \in \mathcal{S}_1(\succ)$, the set of first-order soulmates coalitions under $IMS(\succ)$. Suppose that $C^1 \notin \pi^*$. Then, from the definition of first-order soulmates, since each member of C^1 would strictly prefer to be in C^1 rather than in any other coalition, C^1 can improve upon π^*; that is, $C^1 \succ_i \pi_i^*$ for all $i \in C^1$. But since π^* is in the core, this is a contradiction; therefore $C^1 \in \pi^*$. Now consider the player set $N \backslash C^1$ and suppose that $C^2 \in \mathcal{S}_2(\succ)$. The coalition C^2 can improve upon any partition of $N \backslash C^1$ that does not contain C^2. This process can be continued until no more players can be matched by IMS. □

Proposition 3 does not rule out the possibility that the core is empty. If the core is empty, then there are no core partitions π^* and Proposition 3 is trivially true. When the core is nonempty,

[17] If the selected matching is not a core matching, it could be challenged in courts on the basis that students' priorities at schools have not been respected. It seems plausible that courts will rule based on reported preferences rather than true preferences. It is harder to imagine a court ruling in favor of a student who complains about a mechanism's outcome based on unreported true preferences.

however, the following is a corollary of Proposition 3. Let $I(\succ)$ be the *invariant portion of the core*, that is, the collection of coalitions that belong to every core partition given $\succ$. Then, because every player belongs to *at most* one coalition in $IMS(\succ)$, we have the next result.

Corollary 3 (Soulmate coalitions are invariant core coalitions). *Given any* $\succ \in \mathcal{D}$, *if a core partition exists, then the collection of coalitions* $IMS(\succ) \subseteq I(\succ)$.

In this sense, $IMS(\succ)$ captures *a part* of the invariant portion of the core. Observe that, by Corollary 3, if there are multiple core partitions but $I(\succ) = \varnothing$ (i.e., no player matches with the same coalition in every core partition), then $IMS(\succ) = \varnothing$.[18] Also observe that, if mechanism M always selects a core outcome when one exists, $I(\succ) \subseteq M(\succ)$ for all $\succ \in \mathcal{D}$. Thus, by Corollary 3, $IMS(\succ) \subseteq M(\succ)$ for any such mechanism M (examples include the famous *DA* mechanism in two-sided matching).

Our next proposition can be viewed as a consequence of Corollary 3: If IMS successfully matches all players, the entire match must be a part of any core coalition structure, implying that it is the unique core coalition structure. Obviously, this implies that any soulmate mechanism M selects the unique core coalition structure for every IMS-complete profile. This also implies that, for any profile $\succ \in \mathcal{D}$, mechanism M selects the unique core coalition in the reduced game in which the player set is shrunk to $IMS(\succ)$ itself.

This result is also implied by the proof of Theorem 2 in Banerjee et al. (2001) which shows that the top-coalition property is sufficient to guarantee the existence of a unique core coalition structure. As noted by Banerjee et al. (2001), their proof can be generalized to IMS-complete profiles (in our terminology).[19] The result is also similar to Theorem 1 of İnal (2019), which strengthens the implied extension of Theorem 2 of Banerjee et al. (2001) by allowing restrictions on the set of allowable coalitions while preferences are defined over all coalitions.

Proposition 4 (Unique core under IMS-complete Profiles). *For every IMS-complete profile* $\succ \in \mathcal{D}$, *the coalition structure* $IMS(\succ)$ *is the unique core coalition structure.*

Example 1 together with Proposition 4 proves that the top-coalition condition is sufficient but not necessary for the core to be nonempty. The same is true of IMS-completeness. Although, by Proposition 4, IMS-completeness guarantees that the core is nonempty and unique, IMS-completeness is not a necessary condition for the core to be nonempty or unique, as the next example shows.

Example 5. Consider the following profile of preferences over partners in a roommate's profile.

$$\begin{aligned} 1&: 3 \succ_1 2 \succ_1 4,\\ 2&: 4 \succ_2 1 \succ_2 3,\\ 3&: 2 \succ_3 1 \succ_3 4,\\ 4&: 1 \succ_4 2 \succ_4 3. \end{aligned}$$

If 1 matches with 2 then 3 matches with 4. In this case 2 and 4 form a blocking coalition. If 1 matches with 4 then 2 matches with 3. In this case 1 and 2 form a blocking

[18]See, for example, the *Latin Square profile* (Van der Linden, 2016) in Klaus and Klijn (2006, Example 3.7).

[19]Theorem 2 as stated in Banerjee et al. (2001) does not imply Proposition 4. However, the proof of Theorem 2 as stated in Banerjee et al. (2001) also proves Proposition 4 as the authors underline.

coalition. Thus, 1 must match with 3 in any core partition. It is easy to check that $\{\{1, 3\}, \{2, 4\}\}$ is indeed the unique core coalition structure (because 1 and 2 match with their favorite coalition, they cannot be part of a blocking coalition, and 3 and 4 can only benefit by joining a coalition including 2 or 1, respectively). Clearly, IMS does not match all the players under this profile as there are no soulmates in N.

5.3 | Soulmates, blockers, and Pareto efficiency

As for incentives to misrepresent preferences, soulmate mechanisms retain part of their stability on IMS-incomplete profiles. Although coalitions can block the outcome of a soulmate mechanism in IMS-*in*complete profiles, these coalitions can only consist of players that do not match in IMS.

Proposition 5 (No soulmates among blockers). *Suppose that M is a soulmate mechanism. For any profile $\succ \in \mathcal{D}$, any blocking coalition to $M(\succ)$ contains only players who are* not *soulmates.*

Proof. Again, we use the notation in the definition of IMS. Clearly, no coalition W_1 that blocks $M(\succ)$ contains any players from $N_1(\succ)$, the set of first-order soulmates.

Now suppose that a coalition W_2 that blocks $M(\succ)$ contains a player from $N_2(\succ)$, but contains no player from $N_1(\succ)$. Then, for any player $i^* \in W_2 \cap N_2(\succ)$, there must exist a coalition $C^* \in \mathcal{C}_{i^*}$ such that (i) $C^* \subseteq W_2$, and (ii) $C^* \succ_{i^*} M_{i^*}(\succ)$. However, by definition of IMS, coalition $M_{i^*}(\succ)$ is i^*'s most preferred coalition among the coalitions made of players in $N \backslash N_1(\succ)$. Hence, by (ii), C^* must contain at least one player from $N_1(\succ)$, which contradicts (i). The same logic extends by induction to soulmates of any order. □

The following are corollaries of Propositions 5.

Corollary 4 (Individual rationality and Pareto optimal). *Given any IMS-complete profile $\succ$, the coalition structure $IMS(\succ)$ is individually rational and Pareto optimal.*

Corollary 5 (Partial individual rationality and Pareto efficiency). *Suppose that M is a soulmate mechanism. Given any profile $\succ \in \mathcal{D}$,*

(i) *any player i who is matched with a coalition that she likes less than $\{i\}$ is not a soulmate, and*
(ii) *for any subset of players $S \subseteq N$ such that there exists a partition π^S of S with $\pi_i^S \succ_i M_i(\succ)$ for all $i \in S$, subset S contains no soulmates.*

6 | HOW COMMON ARE SOULMATES?

How common are IMS-complete profiles? How likely are players to be matched into a coalition by IMS? In this section, we study these questions using empirical analysis of real-world data and computational experiments. We focus primarily on the roommates environment (with an even number of players) where every player prefers being in any coalition of two better to being alone.

While IMS-complete profiles are relatively rare (in large player sets) when the set of profiles is unconstrained, there is often some structure to preference profiles encountered in real-world problems. Two particularly natural properties of preference profiles are reciprocity, where individual preferences for others are mutually correlated, and common-ranking, in which individual preferences over coalitions are correlated.

We demonstrate that IMS matches many players in three real-world data sets concerning environments where reciprocal or commonly ranked preferences are at least intuitively likely. Our computational experiments indicate that when preferences are highly reciprocal, or close to commonly ranked, IMS matches many players on average and profiles are often IMS-complete.

6.1 | Soulmates in the field

Here we present empirical results on soulmates in applied problems using real-world data sets in three different environments.

We first consider a roommates problem using social-network data from 1350 students at a university. We next consider a similar problem involving matching coalitions of no more than two in a work setting using data from 44 consultants. Finally, we consider a two-sided matching problem using data from 551 people attending "speed-dating" events.

Each data set includes information we use as a surrogate for preferences.[20] In each case, ties occur in the preferences we derive from the data. Because of this, for each set of data, we have run IMS 1000 times with random tie-breaking each time and recorded the proportion of players matched in each case. More details about the data sets and the precise assumptions used in deriving preferences are given below.

In each environment IMS was able to match about a third of players on average and sometimes substantially more depending on the tie-breaking—up to three-quarters in one instance of the work teams data. This is far more than would be predicted by our results on unstructured preferences in Section 6.2.

To illustrate better how IMS is operating in these environments, Table 1 details the number of individuals left after each round of IMS (for a single random tie-breaking of preferences). This demonstrates that the interative nature of IMS is far from trivial in practice. In each case, IMS is able to match at least sixth-order soulmates and, for example, in the roommates data there are 34 fifth-order soulmates for this particular tie-breaking of preferences.

6.1.1 | University roommates

Our roommates data set comes from a network of 1350 users[21] of a "Facebook-Like" social network at the University of California Irvine. The data are provided by and described in Panzarasa et al. (2009). The data include, for each user, the number of characters sent in private messages to each other user. We use this information as a surrogate for the preference data of each user, assuming that if a user sends more characters to i than to j than the user would prefer to be matched with i over j. We assume that a user would prefer to be matched with any random partner than to remain alone.

[20]The data in this section were not used to produce an actual matching. Thus, it is less likely that the derived preferences are already strategic.

[21]The data set contains 1899 users but only 13,500 have the message data we use to produce preferences.

TABLE 1 Group size remaining after each round of IMS

	Roommates	Work teams	Dating (overall)	Dating (shared interest)
Start	1350	44	551	551
1	1116	36	469	471
2	1000	34	431	435
3	936	32	407	413
4	896	30	395	397
5	862	28	387	385
6	840	**26**	385	375
7	826		**383**	371
8	824			**369**
9	**822**			

Note: Bold numbers indicate the final size at the end of IMS.
Abbreviation: IMS, iterated matching of soulmates.

Over 1000 trials, an average of 39.0% of users is matched by IMS. The maximum was 39.4%. Since the data measure characters sent—a relatively fine-grained measure—most of the ties occur where the users have sent zero characters to each other. This has the effect of randomizing the "bottom" of each user's preference list and does not substantially affect IMS. The success of IMS in this case is likely due to the reciprocity in the derived preferences. If i sends many messages to j, then it is likely that j sends many to i.

6.1.2 | Work teams

The work coalitions data set comes from a study of 44 consultants within a single company. The data are provided by and described in Cross et al. (2004). The consultants responded on a 1–5 scale for each of the other consultants to the question "In general, this person has expertise in areas that are important in the kind of work I do." Here, we assume that if a consultant gave a higher score to i than to j, the consultant would rather be on a coalition with i. (Again, ties in preferences are broken randomly.) We again assume that a consultant would prefer to be matched with any random partner than to remain alone.

Over 1000 trials, an average of 31.3% of consultants is matched by IMS. The maximum was 77.3%. Since the preference measure is less fine-grained in this case, tie-breaking randomization has a stronger effect. Here, it is likely that elements of reciprocity and common-ranking are present in the data. Expertise is a relatively objective measure, while the fact that the question asks about "work I do" makes two consultants with the same focus likely to give each other higher scores.

6.1.3 | Speed dating

The speed-dating data set comes from a study of 551 students at Columbia University invited to participate in a speed-dating experiment. The data are provided by and described in Fisman et al. (2006). Each participant had a 4-min conversation with roughly 10–20 partners and was then asked to rate each partner on a 1–10 scale in various aspects. Here, we focus on two ratings: "Overall, how much do you like this person?" and a shared interest rating. For both, we assume that if a participant gave a higher ranking to i than to j this participant would rather be matched with i than with j.

The proportion matched depends on the question. On average 26.2% can be matched by IMS based on the overall ranking but 31.0% can be matched using the shared interest rating. The maximum for the overall rating was 36.6% and 43.2% for shared interest. The improvement of using shared interest is likely due to the additional reciprocal structure in shared interest data.

6.2 | Unconstrained preferences

In this section, we compare the proportion of IMS-complete profiles to the proportions of other types of profiles that imply IMS-completeness, some of which are discussed in the literature: top-coalition, common-ranking, and reciprocal profiles.

Recall that common-ranking profiles are those in which if player i prefers to be matched with j over k then every other player prefers to be matched with j over k.[22]

Reciprocal preferences are preferences such that, for any player i, if i prefers coalition T to all other coalitions, then for all $j \in T$, j also prefers coalition T to all other coalitions.[23]

To do our analysis, we randomly generated 10, 000 preference profiles for total player set sizes $n = 4, 6, 8$, and 10 and tested the IMS-completeness and the top-coalition property. We have limited the size of player sets since it is computationally hard to test every subset of players to determine whether even a particular profile has the top-coalition property. Also, even the simple problem of counting preference profiles that contain *at least some* soulmates is a complex problem, not conducive to standard counting techniques. For some analytical results on this problem see Appendix A.3.

The proportion of common-ranking and reciprocal roommates profiles is much easier to count than IMS-complete and top-coalition profiles. The number of common-ranking profiles is $n!$. Hence, the proportion $\frac{n!}{(n-1)!^n}$ of common-ranking profiles is very small. Even for total player sets of size 6 the proportion is 2.41×10^{-10}. There are $n!((n-2)!)^n/2^{\frac{n}{2}}$ reciprocal profiles. In comparison to common-ranking, reciprocal profiles are far more abundant. When $n = 6$ about a half of 1% of all profiles are reciprocal. However, each of these accounts for only a small portion of the IMS-complete profiles.

[22]Common-ranking profiles are IMS-complete since they satisfy the top-coalition property. For any subset of the players, the two most highly ranked players form a top-coalition.

[23]Reciprocal preferences are always IMS-complete since reciprocity partitions the set of players into coalitions of soulmates. Because reciprocity requires no structure on preferences other than each player's top choice of coalition, reciprocal preferences need not have the top-coalition property. Furthermore, the top-coalition property does not require that every player is in a top-coalition within the entire set of players but only that there is at least *one* top-coalition. Also, common-ranking and reciprocal preferences are incompatible since common-ranking requires that everyone has the same favorite partners while reciprocal preferences require players have unique favorite partners.

TABLE 2 Approximate proportions of unconstrained profiles meeting each of the conditions from 10,000 randomly generated test-cases

	IMS-complete	Top-coalition	Common-ranking	Reciprocal
$n = 4$	0.6249	0.3442	0.0185	0.0741
6	0.3064	0.0087	0.0000	0.0057
8	0.1219	0.0000	0.0000	0.0004
10	0.0460	0.0000	0.0000	0.0000

Table 2 compares the approximate proportions of IMS-complete, top-coalition, common-ranking, and reciprocal profiles. Top-coalition profiles are only a small portion of IMS-complete profiles as well. In fact, when the number of players is greater than or equal to 8, there were no top-coalition profiles among the 10,000 test cases.

Even[24] though IMS-complete profiles are far more abundant, they still form a vanishing subset of profiles under unconstrained preferences. In Sections 6.3 and 6.4, we consider preference domains endowed with natural structure.

6.3 | Soulmates under relaxed reciprocity

Relaxed forms of reciprocity are natural in some environments: if Alice prefers Alex to all others, it may be that Alex ranks Alice highly as well. If preferences are purely reciprocal then the preference profile is IMS-complete, but what happens under relaxed reciprocity?

To study IMS under different degrees of reciprocity, we now introduce a generalization of reciprocal preferences, *k-reciprocal* preferences. For the roommates problem, a profile $\succ$ is *k-reciprocal* if for any two players i and j, j is in i's top k most preferred players *if and only if* i is in j's top k most preferred players.

Example 6. Consider the following profile of preferences over partners in a roommate's profile.

$$\begin{aligned} 1&: \ 3 \succ_1 2 \succ_1 4, \\ 2&: \ 1 \succ_2 4 \succ_2 3, \\ 3&: \ 4 \succ_3 1 \succ_3 2, \\ 4&: \ 2 \succ_4 3 \succ_4 1. \end{aligned}$$

The profile is trivially 3-reciprocal, as there are only three possible partners for each player. In general, $n - 1$ reciprocal preferences are unconstrained preferences.

The profile is also 2-reciprocal. For example, player 1's two most preferred partners are players 3 and 2, and 1 is the most preferred partner for player 2, and the second most preferred for player 3.

[24]This proportion can be calculated analytically using the results in Appendix A.3. The true value is $\frac{816}{1296} \approx 0.6296$.

As player 3 is player 1's most preferred partner and 1 is not 3's most preferred partner, the profile is not 1-reciprocal.

We randomly generated 10, 000 k-reciprocal profiles for each combination of total number of players $n \in \{4, 6, 8, 10\}$ and $k \in \{2, 4, 6, n - 1\}$ (recall that $k = n - 1$ are unconstrained profiles), tested each profile for IMS-completeness and recorded the number of players matched by IMS. Results for $N = 10$ are shown in Figure 2 and full results are reported in Table A2.

Most 2-*reciprocal* profiles are IMS-complete. While, as expected, the percentage of IMS-complete profiles decreases k, a large portion of players can be matched by IMS even with moderate amounts of commonality. For instance, on average in groups of 10, about half of players can be matched by IMS. For comparison, on average only about 16% can be matched in unconstrained profiles. Interestingly, for $k = 2,\ 4$, and 6, the average proportion of players that can be matched by IMS is nearly constant as n increases.

6.4 | Soulmates under relaxed common-ranking

We now consider a relaxed form of common-ranking where players' preferences are partially correlated. Unlike in Section 6.3, where we study a class of preference profiles, here we will instead study distributions over preference profiles that can be thought of as "noisy" common-rankings.

In this experiment, we randomly generated 10,000 noisy common-ranking profiles for $N = 6, 8, 10$ using the following procedure. Each player's preference starts with an underlying common cardinal vector of utilities over partners. Players' utilities are then perturbed via a normally distributed "noise" term. A player's new ordinal ranking becomes the player's preference. The larger the variance of the noise, the larger the deviation of the expected perturbed preference from the original common-ranking profile.

To make the variance of the noise term informative of the extent of commonality of rankings, we calibrated variances to target specific amounts of ordinal distance between the rankings. We measure the distance between any two players' ordinal rankings using the Kemeny distance (swap-distance; Kemeny, 1959). The Kemeny distance counts the number of pairwise components which have a different ordering in two preference lists.

In the experiment, we chose noise amounts that correspond to an average Kemeny distance between any two players ranging from 0.2 to 2 (in 0.2 increments). For instance, $d = 1$ indicates that, on average, it takes a "swap" of a single pairwise preference component to transform one player's preference list into another.[25]

Figure 3 reports the results for $N = 10$ and the full results are reported in Table A3. Even with a moderate commonality, a large proportion of profiles is IMS-complete. For example, with $N = 10$, approximately two-thirds of profiles are IMS-complete when the variance in common-ranking is such that player's preference lists difference by on-average one swap ($d = 1$). Approximately one-half of profiles are IMS-complete for $d = 2$. Interestingly, the likelihood of a profile being IMS-complete does not decrease much with the size of the total player set, holding the average distance constant.

[25]For example, if the two preferences are $1 \succ 2 \succ 3$ and $2 \succ' 1 \succ' 3$, then the Kemeny distance is one, as swapping player 2 for player 1 in one of the preferences makes the two preferences identical.

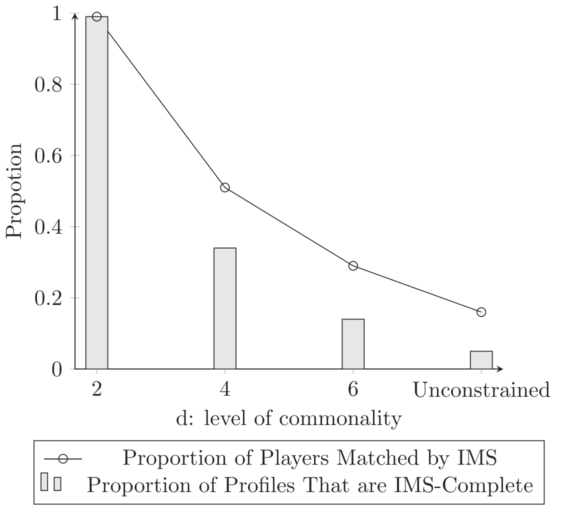

FIGURE 2 Proportion of players matched and IMS-complete profiles by level of commonality (D) for $N = 10$. IMS, iterated matching of soulmates

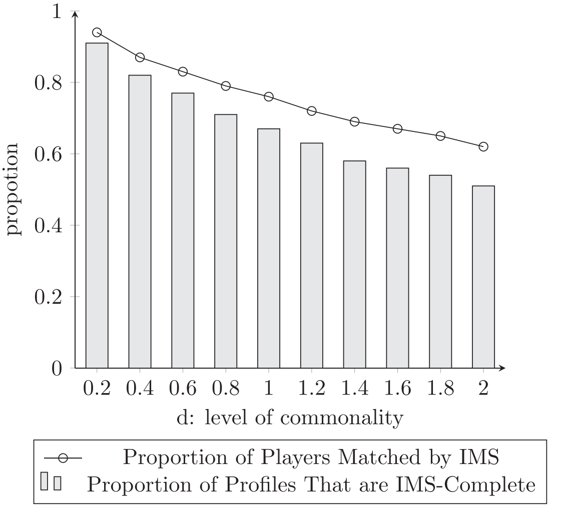

FIGURE 3 Proportion of players matched and IMS-complete profiles for by level of commonality (D) for $N = 10$. IMS, iterated matching of soulmates

7 | CONCLUSION

In this paper, we have provided the first detailed study of IMS. Our paper demonstrates that IMS is a relevant property regardless of whether IMS can match all players, and even when the core is empty, as is the case in many environments studied in the literature. For instance, in a recent paper, Graziano et al. (2020) study housing market scenarios where the core may be empty, Liu (2019) studies the potential emptiness of the core in matching problems with participation constraints, and Choi (2021) studies international matching markets between workers and entrepreneurs. What sort of results can be obtained if soulmate coalitions are formed before applying other solution concepts in such models?

It may also be fruitful to extend results about IMS beyond the hedonic matching environments studied in this paper. For instance, how can IMS be applied to matching environments with externalities such as those studied in Stamatopoulos (2021) and Gonzalez et al. (2019)?

Beyond coalition formation, IMS may also have implications for concepts that can be mapped into problems about the existence or structure of the core in a cooperative game. For instance, Jackson and Van den Nouweland (2005) study the concept of "strongly stable networks" through the use of cooperative game theory and the core. How can the invariant core properties of IMS be used in studying similar concepts?[26]

Club economies seem like another situation in which IMS may provide interesting, new results. For instance, an innovative paper by Windsteiger (2021) introduces the idea of a sorting technology, such as clubs, that may facilitate segregation. Do exclusive clubs, for example, London social clubs (once called "gentlemen's clubs") match soulmates in contexts for which there may be discrimination both by price and by personal characteristics of club members?

While this is all speculative, we conjecture that there will be future research advancing the application of IMS. We conclude by noting two classic contributions on coalition formation which may be inspiring: see Gamson (1961) for a discussion of coalitions from a sociological perspective, and Ray (2007) who provides a broad more recent, game-theoretic approach to coalition formation problems (See also Chung, 2000; Fripertinger & Schöpf, 1999; Greenberg & Weber, 1986; Kaneko & Wooders, 1982, discussed in the Appendix).

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APPENDIX A

A.1 | Relationship between IMS-completeness and other conditions

A.1.1 | Weak consecutiveness, weak top-coalition, and ordinal balancedness

In this appendix, we analyze the relationships between IMS-completeness and three additional profile conditions notably studied in Bogomolnaia and Jackson (2002) and Banerjee et al. (2001). These relationships are represented in Figure A1 and Figure A2. Let us first define these additional profile conditions.

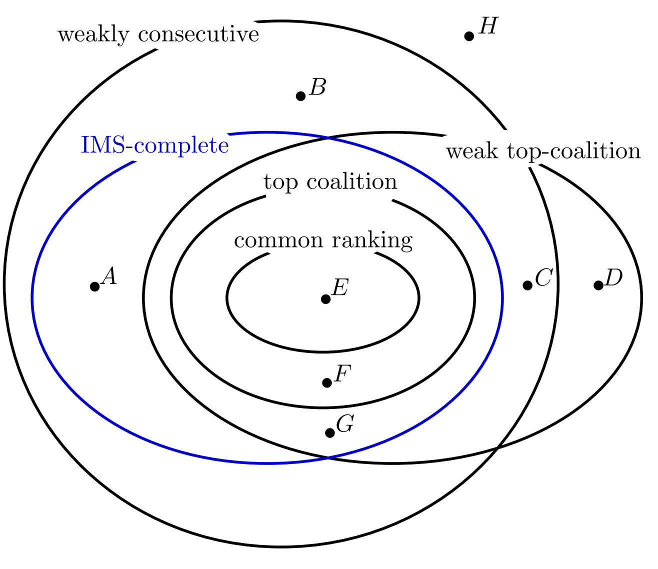

FIGURE A1 Venn diagram of the relationship between IMS-complete, weak top-coalition, and other profile conditions in the literature (when preferences are strict). A dot indicates that the section of the Venn diagram is nonempty. IMS, iterated matching of soulmates

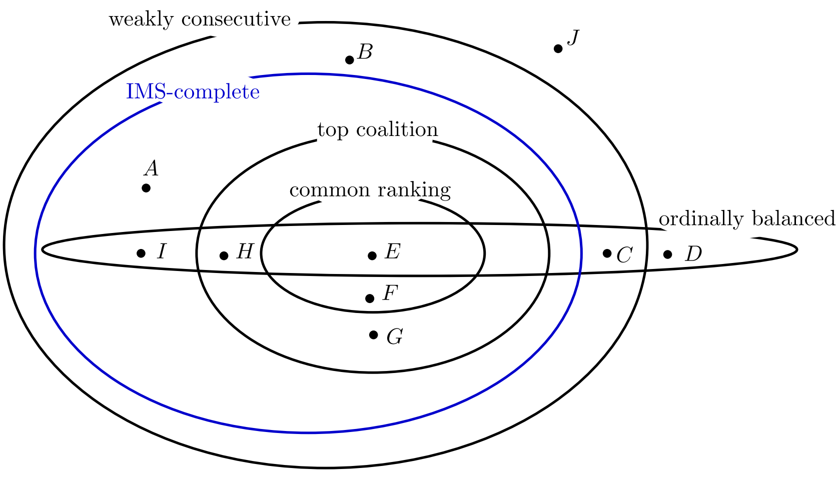

FIGURE A2 Venn diagram of the relationship between IMS-completeness, ordinal balancedness, and other profile conditions in the literature (when preferences are strict). A dot indicates that the section of the Venn diagram is nonempty. IMS, iterated matching of soulmates

Greenberg and Weber (1986) introduced the concept of a consecutive profile (or game) for transferable utility games.[27] Bogomolnaia and Jackson (2002) adapt it to the nontransferable case. The following follows Bogomolnaia and Jackson (2002). A coalition $C \in 2^N$ is *consecutive with respect to an ordering* $\trianglerighteq$ of the players in N if for all $j, k, h \in N$, $[j \in C, h \in C$, and $j \triangleright k \triangleright h]$ implies $k \in C$. A profile $\succ$ is *weakly consecutive* if there exists an ordering $\trianglerighteq$ of the players such that for every coalition structure π, whenever a coalition S blocks π, there exists a consecutive coalition S' that also blocks π.

Shapley and Scarf (1974) introduced the concept of an ordinally balanced profile, again in the context of games of transferable utility. Bogomolnaia and Jackson (2002) and Banerjee et al. (2001) propose equivalent adaptations to the nontransferable case. Here we follow Bogomolnaia and Jackson (2002) in terms of exposition. A collection of coalitions $\mathcal{C} \subseteq 2^N$ is *balanced* if there exists a vector d of positive weights d_C such that for each player $i \in N$, we have $\sum_{\{C \in \mathcal{C} | i \in C\}} d_C = 1$. As Bogomolnaia and Jackson (2002) put it, a profile is ordinally balanced if for each balanced collection of coalitions, there exists a coalition structure such that each player weakly prefers her coalition in the coalition structure to her *worst* coalition in the balanced collection. Formally, a profile $\succ$ is *ordinally balanced* if for each balanced collection of coalitions $\mathcal{C} \subseteq 2^N$, there exists a coalition structure π such that for each $i \in N$, there exists $C \in \mathcal{C}$ with $i \in C$ such that $\pi_i = C$ or $\pi_i \succ_i C$.

Finally, Banerjee et al. (2001) introduce a weaker version of the top-coalition property. Given a set of players $S \in 2^N$, a coalition $C \subseteq S$ is a *weak top-coalition* of S if and only if C has an ordered coalition structure $\{C^1, \ldots, C^l\}$ such that (i) for any $i \in C^1$ and any $T \subseteq S$ with $i \in T$, we have $C \succ_i T$ and (ii) for any $k > 1$, any $i \in C^k$ and any $T \subseteq S$ with $i \in T$, we have $[T \succ_i C] \Rightarrow [T \cap (\bigcup_{m<k} S^m) \neq \varnothing]$. A profile $\succ$ satisfies the *weak top-coalition property* if and only if for any nonempty set of player $S \subseteq N$, there exists a weak top-coalition C.

[27]Games with consecutive coalitions have nonempty cores independently of the payoff functions. Kaneko & Wooders (1982) provide a characterization of conditions on admissible coalition structures for transferable and nontransferable games to have this property.

Any of these three additional properties is sufficient for the existence of a core coalition structure (Banerjee et al., 2001; Bogomolnaia & Jackson, 2002). We now prove every relationship in Figure A1.

IMS-complete ⇒ weakly consecutive. See the first part of the proof of Proposition 1 in Bogomolnaia and Jackson (2002, p. 211) which proves that the top-coalition property implies the weakly consecutive property. The proof of this result is easily adapted to show that IMS-completeness implies the weakly consecutive property.

A. *IMS-complete, not weak top-coalition*

Consider the following profile of preferences over partners in a roommate's profile

$$\begin{array}{llllll} 1: & 2 & \succ_1 & 3 & \succ_1 & 4 \\ 2: & 1 & \succ_2 & 3 & \succ_2 & 4 \\ 3: & 1 & \succ_3 & 4 & \succ_3 & 2 \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 \end{array}$$

The profile is IMS-complete with {{1, 2}{3, 4}} as the outcome of IMS, but it does not satisfy any of the properties (i) common ranking, (ii) top-coalition, or (iii) weak top-coalition. The fact that the profile satisfies neither the top-coalition property nor the weak top-coalition property follows from {2, 3, 4} not having a top-coalition or a weak top-coalition.

B. *Weakly consecutive, not IMS-complete, not weak top-coalition*

See the second part of the proof of Proposition 1 in Bogomolnaia and Jackson (2002, p. 212) which proves that the weakly consecutive property does not imply the top-coalition property. The example given in that proof is weakly consecutive, but fails (i) to be IMS-complete, (ii) to be ordinally balanced, and (iii) to satisfy the weak top-coalition property.

C. *Weakly consecutive, weak top-coalition, not IMS-complete*

Consider the following profile of preferences over coalitions, where ⋯ indicates that the rest of the preferences are arbitrary

$$\begin{array}{llllllll} 1: & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \dots & & \\ 2: & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \dots & & \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 & & \end{array}$$

Let the ordering of the players be $1 \triangleright 2 \triangleright 3$. Every coalition structure except {{1, 3}, {2}} is blocked by {1, 2, 3}, which is consecutive according to $\triangleright$. Partition {{1, 3}, {2}} is blocked by {1} which is consecutive too. Hence every coalition structure that is blocked is also blocked by a consecutive coalition and the profile is weakly consecutive (the core coalition structure is N).

Also, {1, 2, 3} is a weak top-coalition for {1, 2, 3}, and every other set of players admits one of its singletons as a weak top-coalition (e.g., {1, 3} has {1} as a weak top-coalition). Hence the profile satisfies the weak top-coalition property.

However, N does not have a top-coalition, and the profile is therefore not IMS-complete.

D. *Weak top-coalition, not weakly consecutive*

See the second profile on Bogomolnaia and Jackson (2002, p. 212). As the authors show, the profile satisfies the weak top-coalition property, but is not weakly consecutive. As N does not have a top-coalition, the profile satisfies neither the top-coalition property nor IMS-completeness.

E. *Common ranking*
See profiles of type *A* in Figure 1.
F. *Top-coalition, not common ranking*
See profiles of type *B* in Figure 1.
G. *IMS-complete, weak top-coalition, not top-coalition* Consider the following profile of preferences over coalitions, where ⋯ indicates that the rest of the preference is arbitrary

$$
\begin{array}{llllllllll}
1: & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \dots \\
2: & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\
3: & \{3,4\} & \succ_3 & \{2,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \\
4: & \{3,4\} & \succ_4 & \{4\} & \succ_4 & \dots
\end{array}
$$

Coalition $\{1, 2, 3\}$ is a weak top-coalition for $\{1, 2, 3\}$. Coalitions $\{1\}$ or $\{2\}$ are weak top-coalitions for any other set of players containing 1 or 2. Finally, $\{3, 4\}$ is a weak top-coalition for $\{3, 4\}$, and $\{3\}$ and $\{4\}$ are weak top-coalitions for $\{3\}$ and $\{4\}$ respectively. Hence, the profile satisfies the weak top-coalition property.

The profile is IMS-complete, with IMS yielding coalition structure $\{\{1\}, \{2\}, \{3, 4\}\}$. However there is no top-coalition for 1, 2, 3.
H. *Not weakly consecutive, not weak top-coalition*
See profiles of type *D* in Figure 1 (any other profile for which the core is empty would also be an example).

We now turn to the elements of Figure A2.

IMS-complete ⇒ *weakly consecutive*. See above.
A. *IMS-complete, not ordinally balanced, not top-coalition*
Consider the following profile of preferences over coalitions, where ⋯ indicates that the rest of the preference is arbitrary

$$
\begin{array}{lllllll}
\{1,2\} & \succ_1 & \{1,3\} & \succ_1 & \{1\} & \succ_1 & \dots \\
\{1,2\} & \succ_2 & \{2,4\} & \succ_2 & \{2\} & \succ_2 & \dots \\
\{3,5\} & \succ_3 & \{3,4\} & \succ_3 & \{3\} & \succ_3 & \dots \\
\{1,4\} & \succ_4 & \{4,5\} & \succ_4 & \{4\} & \succ_4 & \dots \\
\{3,5\} & \succ_5 & \{4,5\} & \succ_5 & \{5\} & \succ_5 & \dots
\end{array}
\tag{A1}
$$

The profile does not satisfy the top-coalition property because $\{1, 3, 4\}$ does not have a top-coalition.

The profile does not satisfy ordinal balancedness either, with respect to the balanced collection $BC = \{\{1, 2\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$. Any partition satisfying the balancedness condition with respect to BC must match $\{1, 2\}$ and cannot match $\{3, 4, 5\}$. But then the player who is left alone among $\{3, 4, 5\}$ cannot be better-off than in any coalition in BC.

However, the profile is IMS-complete with IMS yielding coalition structure $\{\{1, 2\}, \{3, 5\}, \{4\}\}$.

B. *Weakly consecutive, not IMS-complete, not ordinally balanced*
See the first profile on Bogomolnaia and Jackson (2002, p. 212). As the authors show, the profile is weakly consecutive, but violates ordinal balance. The profile fails to be IMS-complete as N does not have a top-coalition.

C. *Weakly consecutive, ordinally balanced, not IMS-complete*
Consider the following profile of preferences over coalitions, where $\cdots$ indicates that the rest of the preference is arbitrary

$$\begin{array}{lccccccc} 1: & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \dots \\ 2: & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 \end{array}$$

For every balanced collection that contains $\{1, 2, 3\}$, coalition structure $\{1, 2, 3\}$ is such that every player likes a coalition in the balanced collection (namely $\{1, 2, 3\}$) at least as much as the coalition structure. The remaining balanced collection are (i) $\{\{1\}, \{2\}, \{3\}\}$, (ii) $\{\{1, 2\}, \{3\}\}$, (iii) $\{\{1\}, \{2, 3\}\}$, (iv) $\{\{1, 3\}, \{2\}\}$, and (v) $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. For (i)–(iv), the balanced collection is itself a coalition structure. For (v), coalition structure $\{1, 2, 3\}$ is again such that every player likes $\{1, 2, 3\}$ better than some coalition in $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Hence, the profile is balanced.

However, N does not have a top-coalition and therefore, the profile is IMS- *in*complete.

D. *Ordinally balanced, not weakly consecutive*
Consider the profile at the top of Bogomolnaia and Jackson (2002, p. 214). As the authors show, the profile is ordinally balanced and is not weakly consecutive.

E. *Common ranking, ordinally balanced*
Profile

$$\begin{array}{lccccc} 1: & \{1,2\} & \succ_1 & \{1\} & \succ_1 & \dots \\ 2: & \{1,2\} & \succ_2 & \{2\} & \succ_2 & \dots \end{array}$$

is trivially both a common-ranking profile and an ordinally balanced profile.

F. *Common ranking, not ordinally balanced*
See Game 5 in Banerjee et al. (2001).

G. *Top-coalition, not common ranking, not ordinally balanced*
Consider the following profile of preferences over coalitions which is adapted from profile A1.

$$\begin{array}{ccccccc} \{1,2\} & \succ_1 & \{1,3\} & \succ_1 & \{1\} & \succ_1 & \dots \\ \{1,2\} & \succ_2 & \{2,4\} & \succ_2 & \{2\} & \succ_2 & \dots \\ \{3,4\} & \succ_3 & \{3,5\} & \succ_3 & \{3\} & \succ_3 & \dots \\ \{3,4\} & \succ_4 & \{4,5\} & \succ_4 & \{4\} & \succ_4 & \dots \\ \{3,5\} & \succ_5 & \{4,5\} & \succ_5 & \{5\} & \succ_5 & \dots \end{array}$$

The profile satisfies the top-coalition property. The profile however violates ordinal balancedness for the same reason profile A1 violates ordinal balancedness in A.

H. Ordinally balanced, top-coalition, not common ranking

Consider the following profile of preferences over coalitions, where ⋯ indicates that the rest of the preference is arbitrary

$$\begin{aligned}
1:&\ \{1,2\} \succ_1 \{1\} \succ_1 \ldots\\
2:&\ \{1,2\} \succ_2 \{2\} \succ_2 \ldots\\
3:&\ \{1,3\} \succ_3 \{3\} \succ_3 \ldots
\end{aligned}$$

The profile satisfies the top-coalition property with $\{1, 2\}$ as the top-coalition for both $\{1, 2, 3\}$ and $\{1, 2\}, \{3\}$ as top-coalition for $\{3\}, \{2\}$ as top-coalition for $\{2, 3\}$ and $\{2\}$, and $\{1\}$ as top-coalition for $\{1, 3\}$ and $\{1\}$.

I. Ordinally balanced, IMS-complete, not top-coalition

Consider the following profile of preferences over coalitions, where ⋯ indicates that the rest of the preference is arbitrary

$$\begin{aligned}
1:&\ \{1,2\} \succ_1 \{1\} \succ_1 \ldots\\
2:&\ \{1,2\} \succ_2 \{2,3\} \succ_2 \{2\} \succ_2 \ldots\\
3:&\ \{3,4\} \succ_3 \{3\} \succ_3 \ldots\\
4:&\ \{2,4\} \succ_4 \{3,4\} \succ_4 \{4\} \succ_4 \ldots
\end{aligned}$$

The profile is ordinally balanced. For most balanced collections of coalitions, the balancedness condition is satisfied with respect to the core coalition structure $\pi^* = \{\{1, 2\}, \{3, 4\}\}$. If this is not the case for some balanced collections of coalitions, then this collection must contain $\{2, 4\}$ and cannot contain any other coalition that includes player 4. But then $\{2, 4\}$ must have weight 1 which means the collection does not contain any other coalition that includes player 2 either. Any such balanced collection satisfies the balancedness condition with respect to partition $\pi^{**} = \{\{2, 4\}, \{1\}, \{3\}\}$.

The profile does not satisfy the top-coalition property because $\{2, 3, 4\}$ does not have a top-coalition.

The profile is IMS-complete with IMS yielding coalition structure $\{\{1, 2\}, \{3, 4\}\}$.

J. Not weakly consecutive, not ordinally balanced

See Example 5 (any other profile for which the core is empty would also be an example).

Let us finally note that ordinal balancedness and the weak top-coalition property are also independent of one another, in the sense that there exists profile satisfying one of the properties but not the other, as proven in Bogomolnaia and Jackson (2002, Proposition 1).

A.1.2 | Alcalde and Romero-Medina (2006)

Alcalde and Romero-Medina (2006) present four conditions that guarantee the existence of a core allocation. Below, we demonstrate there is no containment relation between IMS-complete and any of these four conditions. The definitions below follow Alcalde and Romero-Medina (2006) and are, again, for profiles of strict preferences.

A profile is *union responsive* if for every $i \in N$ and any two coalitions C, C'^N such that $C' \subset C$, and C' is not the most preferred coalition for i in 2^C, we have $C \succ_i C'$.

The following profile is IMS-complete but violates union responsiveness

$$\begin{array}{llllll} 1: & \{1\} & \succ_1 & \dots & & \\ 2: & \{1,2\} & \succ_2 & \{2,3\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{2,3\} & \succ_3 & \dots \end{array} \tag{A2}$$

For example, $\{2,3\} \subset \{1,2,3\}, \{2,3\}$ is not the most preferred coalition for 2 in $2^{\{1,2,3\}}$ and $\{2,3\} \succ_2 \{1,2,3\}$

The following profile is union responsive but not IMS-complete.

$$\begin{array}{llllllll} 1: & \{1,2\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1,3\} & \succ_1 & \{1\}, \\ 2: & \{2,3\} & \succ_2 & \{1,2,3\} & \succ_2 & \{1,2\} & \succ_2 & \{2\}, \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{2,3\} & \succ_3 & \{3\}. \end{array} \tag{A3}$$

A profile is *intersection responsive* if for every $i \in N$ and any two coalitions $C, C'^N, C \succ_i C'$ implies $C \cap C' \succ_i C'$.

Again, profile A2 is IMS-complete but violate intersection responsiveness. For example, $\{1,2\} \cap \{2,3\} = \{2\}, \{1,2\} \succ_2 \{2,3\}$ but $\{2,3\} \succ_2 \{2\}$). The following variant of profile A3 is intersection responsive but not IMS-complete.

$$\begin{array}{llllllll} 1: & \{1,2\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \{1,3\}, \\ 2: & \{2,3\} & \succ_2 & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \{1,2\}, \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \{2,3\}. \end{array}$$

A profile is *singular* if for every $i \in N$ there is a unique acceptable coalition $C \in 2^N$ (recall that an acceptable coalition C is a coalition for which $C \succ_i \{i\}$).

Profile A2 is IMS-complete but not singular. The following profile is singular but not IMS-complete.

$$\begin{array}{llllll} 1: & \{1,2\} & \succ_1 & \{1\} & \succ_1 & \dots \\ 2: & \{2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{3\} & \succ_3 & \dots \end{array}$$

Finally, a profile is *essential* if for every $i \in N$, there is an essential coalition $C^i \in 2^N$, that is, a coalition such that

(i) if $C^i = \{i\}$, then $\{i\} \succ_i C$ for any $C \neq \{i\}$, and
(ii) if $C^i \neq \{i\}$, then
  (a) $\{i\} \succ_i S$ if and only if S is not a superset of C^i, and
  (b) for any two coalitions C, C'^N, if $C^i \subseteq C \subset C'$, then $C \succ_i C'$.

The following variant of profile A2 is IMS-complete but not essential. For example, neither {1, 2} nor {2, 3} are a superset of one another, but both are acceptable for 2.

$$\begin{aligned}
&1:\ \{1\} \succ_1 \ldots\\
&2:\ \{1,2\} \succ_2 \{2,3\} \succ_2 \{2\} \succ_2 \ldots\\
&3:\ \{1,3\} \succ_3 \{2,3\} \succ_3 \{3\} \succ_3 \ldots
\end{aligned}$$

The following profile is essential but not IMS-complete

$$\begin{aligned}
&1:\ \{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\} \succ_1 \ldots\\
&2:\ \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{2\} \succ_2 \ldots\\
&3:\ \{1,3\} \succ_3 \{1,2,3\} \succ_3 \{3\} \succ_3 \ldots
\end{aligned}$$

A.1.3 | Pycia (2012)

Pycia (2012) studies rules for sharing coalition surplus that guarantee the existence of a stable outcome in a generalized many-to-one matching environment. The relevant condition that Pycia identifies in the induced preferences is that of *pairwise alignment*. A profile is said to be pairwise aligned if for any two $i, j \in N$ and coalitions C and C' both containing i and j, $C \succeq_i C' \Leftrightarrow C \succeq_j C'$

Theorem 1 of Pycia (2012) states that, in this environment, when all potential preference profiles induced by a sharing rule are *pairwise aligned* and the domain of induced preference profiles is sufficiently rich, there is a unique stable outcome when preferences are strict. The proof of uniqueness (see Pycia, 2012, *lemma 5*) utilizes the logic of matching soulmates, leveraging the fact that under these conditions, a relaxed version of common ranking holds which implies that the profile is IMS-complete.

We note that pairwise alignment (in isolation of the additional domain richness requirement) is logically distinct from IMS-completeness. The following profile is IMS-complete (and Top-Coalition) but not pairwise aligned (due to the preferences of 1 and 2 over {1,2,3} and {1,2}).

$$\begin{aligned}
&1:\ \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1,2\} \succ_1 \{1\},\\
&2:\ \{2,3\} \succ_2 \{1,2\} \succ_2 \{1,2,3\} \succ_2 \{2\},\\
&3:\ \{2,3\} \succ_3 \{1,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}.
\end{aligned}$$

The following profile is pairwise aligned but not IMS-complete (due to the top-coalition cycle):

$$\begin{aligned}
&1:\ \{1,3\} \succ_1 \{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\},\\
&2:\ \{1,2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{2\},\\
&3:\ \{2,3\} \succ_3 \{1,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}.
\end{aligned}$$

This profile demonstrates the importance of the further condition that Pycia's theorem imposes: that the profile is embeddable in a rich-enough domain of pairwise-aligned preferences (see Pycia's assumption *R1*).

A.1.4 | Cyclical roommates profiles Rodrigues-Neto (2007)

For roommates profiles, Rodrigues-Neto (2007) defines an acyclicity condition that strengthens the "no odd rings" condition from Chung (2000). A roommate's profile is *acyclic* if there exists no subset of agents $\{i(1), \ldots, i(k)\}$ with $k \geq 3$ and $i(j) \neq i(j+1)$ for all $j \in \{1, \ldots, k-1\}$ such that

$$\begin{aligned} & i(j+1) \succ_{i(j)} i(j-1) \qquad \text{for all } j \in \{2, \ldots, k-1\}, \\ & i(1) \succ_{i(k)} i(j-1), \text{and} \\ & i(2) \succ_{i(1)} i(k). \end{aligned} \tag{A4}$$

As Chung (2000) shows, the "no odd rings" condition is sufficient for the nonemptiness of the core. Because the acyclicity condition from Rodrigues-Neto (2007) strengthens the "no odd rings" condition, it is also sufficient for the core to be nonempty. In fact, as Rodrigues-Neto (2007) argues, if a roommate's profile is acyclic, then the profile is also IMS-complete. The next example shows that the converse is not true. Consider the following profile of preferences over partners in a roommate's profile

$$\begin{array}{l} 1:\ 4 \succ_1 2 \succ_1 3 \\ 2:\ 4 \succ_2 3 \succ_2 1 \\ 3:\ 4 \succ_3 1 \succ_3 2 \\ 4:\ 1 \succ_4 2 \succ_4 3 \end{array}$$

Players 1–3 form a cycle but because 1 and 4 are soulmates, the cycle is "broken" after soulmates have been matched together and the profile is IMS-complete with IMS producing partition $\{\{1, 4\}, \{2, 3\}\}$.

A.2 | Cyclical domains and proof of Proposition 2

In this appendix, we provide generalizations to the general coalition formation environment of cycle notion defined by Rodrigues-Neto (2007) for roommates profiles (see A4). We then use these generalizations to prove Proposition 2.

A *coalition k-cycle* is a collection of coalitions

$$\begin{aligned} & \{C_{12}, C_{23} \ldots, C_{k1}\} \text{ such that } C_{ij} = N_i \cup N_j \cup O_{ij}, \\ & \text{for some nonempty } N_1, \ldots, N_k \subset N \text{ with } N_1 \cap N_2 = \cdots = N_{k-1} \cap N_k = \varnothing \\ & \text{and some (possible empty) } O_{12}, O_{23}, \ldots, O_{k1} \subset N \backslash (N_1 \cup \cdots \cup N_k) \\ & \text{with } O_{12} \cap O_{23} = O_{23} \cap O_{31} = \cdots = O_{k1} \cap O_{12} = \varnothing. \end{aligned}$$

For instance, a *coalition 3-cycle* is a triple of coalitions C_{12}, C_{23}, C_{31} such that

$$C_{12} = N_1 \cup N_2 \cup O_{12}, C_{23} = N_2 \cup N_3 \cup O_{23}, \text{ and } \quad C_{31} = N_3 \cup N_1 \cup O_{31}$$

with N_1, N_2, N_3 and O_{12}, O_{23}, O_{31} satisfying the above conditions. In the roommates profile A2, the coalition 3-cycle corresponds to the situation in which N_1, N_2, and N_3 are singletons, and $O_{12} = O_{23} = O_{31} = \emptyset$.

Let $[C \succ C' \succ \ldots]$ represent any preference in which C is ranked first, C' second, and the rest of the ranking is arbitrary. A domain $\mathcal{R}$ is *odd-cyclic* if there exists a coalition k-cycle *with k odd* such that

$$\begin{aligned}
&\mathcal{R}_i \supseteq \left\{\left[C_{j(j+1)} \succ_i^1 C_{(j-1)j} \succ_i^1 \ldots\right], \left[C_{(j-1)j} \succ_i^2 C_{j(j+1)} \succ_i^2 \ldots\right]\right\} \\
&\text{for all } i \in N_j \text{ and all } j \in \{2, \ldots, k-1\}, \\
&\mathcal{R}_i \supseteq \left\{\left[C_{12} \succ_i^1 C_{k1} \succ_i^1 \ldots\right], \left[C_{k1} \succ_i^2 C_{12} \succ_i^2 \ldots\right]\right\} \\
&\text{for all } i \in N_1, \\
&\mathcal{R}_i \supseteq \left\{\left[C_{k1} \succ_i^1 C_{(k-1)k} \succ_i^1 \ldots\right], \left[C_{(k-1)k} \succ_i^2 C_{k1} \succ_i^2 \ldots\right]\right\} \\
&\text{for all } i \in N_k, \text{ and} \\
&\mathcal{R}_i \supseteq \left\{\left[, C_{jh} \succ_i^* \ldots ,\right]\right\} \\
&\text{for all } i \in O_{jh} \text{ and all } O_{jh} \in \{O_{12}, \ldots, O_{k1}\}.
\end{aligned} \tag{A5}$$

For example, a domain $\mathcal{R}$ is *odd-cyclic* if there exists a coalition 3-cycle such that

$$\begin{aligned}
&\mathcal{R}_i \supseteq \left\{\left[C_{12} \succ_i^1 C_{31} \succ_i^1 \ldots\right], \left[C_{31} \succ_i^2 C_{12} \succ_i^2 \ldots\right]\right\} && \text{for all } i \in N_1, \\
&\mathcal{R}_i \supseteq \left\{\left[C_{23} \succ_i^1 C_{12} \succ_i^1 \ldots\right], \left[C_{12} \succ_i^2 C_{23} \succ_i^2 \ldots\right]\right\} && \text{for all } i \in N_2, \\
&\mathcal{R}_i \supseteq \left\{\left[C_{31} \succ_i^1 C_{23} \succ_i^1 \ldots\right], \left[C_{23} \succ_i^2 C_{31} \succ_i^2 \ldots\right]\right\} && \text{for all } i \in N_3, \\
&\mathcal{R}_i \supseteq \left\{\left[C_{12} \succ_i^* \ldots\right],\right\} && \text{for all } i \in O_{12}, \\
&\mathcal{R}_i \supseteq \left\{\left[C_{23} \succ_i^* \ldots\right],\right\} && \text{for all } i \in O_{23}, \text{ and} \\
&\mathcal{R}_i \supseteq \left\{\left[C_{31} \succ_i^* \ldots\right],\right\} && \text{for all } i \in O_{31}.
\end{aligned}$$

In the roommates domain described in the text we have

$$\begin{aligned}
&\mathcal{R}_1 \supseteq \left\{\left[\{1,2\} \succ_1^1 \{3,1\} \succ_1^1 \ldots\right], \left[\{3,1\} \succ_1^2 \{1,2\} \succ_1^2 \ldots\right]\right\}, \\
&\mathcal{R}_2 \supseteq \left\{\left[\{2,3\} \succ_2^1 \{1,2\} \succ_2^1 \ldots\right], \left[\{1,2\} \succ_2^2 \{2,3\} \succ_2^2 \ldots\right]\right\}, \text{ and} \\
&\mathcal{R}_3 \supseteq \left\{\left[\{3,1\} \succ_3^1 \{2,3\} \succ_i^1 \ldots\right], \left[\{2,3\} \succ_3^2 \{3,1\} \succ_i^2 \ldots\right]\right\}.
\end{aligned}$$

A domain is *individually odd-cyclic* if it is odd-cyclic for some $N_1, \ldots, N_k$ with $\#N_1 = \cdots = \#N_k = 1$.

Proof of Proposition 2. The proof follows the same logic the argument in example 4 in the text. The proof is for (i). For (ii), simply replace any joint deviation by players N_i by a deviation from the only $i \in N_i$ (and truthful *strong* Nash equilibrium by truthful Nash equilibrium). In order to derive a contradiction, suppose that M has a truthful strong Nash equilibrium on $\mathcal{R}$ and that M is a soulmate mechanism.

Consider profile $\tilde{\succsim}$ in which (a) players in $O_{12}, \ldots, O_{k1}$ have preference $\succ_i^*$, and (b) players in $i \in (N_1 \cup \cdots \cup N_3)$ have preference $\succ_i^1$ (see the definition of an odd cyclic profile above). Because $\mathcal{R}$ is an odd cyclic domain, there exists such an $\tilde{\succsim} \in \mathcal{R}$. Hence, because M has a truthful strong Nash equilibrium on $\mathcal{R}$, no coalition of players can deviate when players report profile $\tilde{\succsim}$.

This implies that either C_{12} or C_{k1} must form. Otherwise, N_1 can jointly deviate by reporting $\succ_{N_1}^2$, that is pretending they are the soulmates of players in N_k.

Suppose that C_{12} forms. If players in N_3 are matched with their second-ranked coalition, they could deviate by reporting $\succ_{N_3}^2$, that is pretending they are the soulmates of players in N_2. To prevent this profitable joint deviation, the players in N_3 must be matched with their best coalition C_{34}.

Extending the argument by induction, all coalitions $C_{12}, C_{34}, \ldots, C_{j(j+1)}$ must form, for any odd $j \leq k$. But because k is odd this implies that neither C_{1k} nor $C_{(k-1)k}$ form (because players in N_1 are already in C_{12} and players in N_{k-1} are already in $C_{(k-2)(k-1)}$). However, players in N_{k-1} are matched with their second-ranked coalition, and the players in N_k can therefore jointly deviate by reporting preference $\succ_k^2$ in which they are the soulmates of the players in N_{k-1}.

Similarly, if C_{23} forms, the players in N_1 can jointly deviate by reporting preference $\succ_1^2$ in which they are the soulmates of the players in N_k. In both cases, a coalition of players can deviate when players report $\tilde{\succsim}$, a contradiction. □

A.3 | The difficulty of counting IMS-complete profiles

In the roommates problem, the players' favorite partners can be represented by an endofunction $f : N \rightarrow N$ or its equivalent functional graph (directed 1-forest). A pair of soulmates is a fixed point of $f(f(i))$ or alternatively a 2-cycle of its functional graph.

These objects are too complex to be directly counted using standard methods. However, Fripertinger and Schöpf (1999) provide useful results by using Polya enumeration theory. Their Corollary 6 provides a count of functional graphs without 1 or 2-cycles. Dividing this count by the number of functional graphs with no 1-cycles (for n players this number is $(n-1)^n$) provides the proportion of preference profiles (of those where being matched is better than being alone) for n players in which there are no soulmates. Subtracting this from 1 provides $N(n)$, the probability that at least some players can be matched during IMS:

$$N(n) = 1 - \frac{1}{(n-1)^n} \sum_{j=0}^{n-1} j! \binom{n-s}{j} n^{n-s-j} Z\left(S_j, j | x_k = \begin{cases} -1 & k \in L \\ 0 & k \notin L \end{cases}\right),$$

where Z is the symmetric group cycle index polynomial.

It is possible to use $N(n)$ to find an upper bound on the probability that IMS matches everyone when preferences are uniform by using the approximation that the preferences players have over the remaining players when a pair of soulmates are removed is also uniform. This overestimates the probability that there will be a set of soulmates in the remaining players since the fact that a particular player was not just matched with a soulmate implies it is more likely that her favorite among the remaining players does not also like her best.

Using this bound however, the approximate probability that there are more soulmates to match after removing the first pair is $N(n-2)$. Continuing this, the approximate probability that IMS-completes is the probability that there continue to be "more soulmates to remove" as the group dwindles from N to 2. This is the product:

$$N(n) \cdot N(n-2) \cdot N(n-4) \cdots N(2).$$

This yields the following bounds,[28] which are compared to the computed proportions from Section 6.

A.4 | Computational tables

TABLE A1 Bound and computed proportion of IMS-complete profiles

	Upper bound	Computed
$n = 4$	0.6296	0.6249
6	0.3330	0.3064
8	0.1624	0.1219
10	0.0755	0.0460

Abbreviation: IMS, iterated matching of soulmates.

TABLE A2 Proportion of players matched and IMS-complete profiles by level of reciprocity (r) and group size (N)

	Proportion of players matched				IMS-complete proportion			
	$r = 2$	$r = 4$	$r = 6$	$r = n - 1$	$r = 2$	$r = 4$	$r = 6$	$r = n - 1$
$N = 8$	0.99	0.51	0.30	0.24	0.99	0.39	0.18	0.12
10	0.99	0.51	0.29	0.16	0.99	0.34	0.14	0.05
12	0.99	0.50	0.29	0.12	0.99	0.28	0.11	0.01
14	0.99	0.49	0.29	0.09	0.99	0.23	0.08	0.00
16	0.99	0.49	0.28	0.08	0.97	0.17	0.06	0.00
18	0.99	0.49	0.28	0.07	0.94	0.15	0.04	0.00
20	0.98	0.49	0.27	0.06	0.92	0.12	0.03	0.00

Abbreviation: IMS, iterated matching of soulmates.

[28]For $n = 4$ this procedure yields the true proportion since there are always soulmates when $n = 2$.

TABLE A3 Proportion of players matched and IMS-complete profiles by group size (N) and level of commonality (k)

	Proportion of players matched			**IMS-complete proportion**		
	$N = 6$	**$N = 8$**	**$N = 10$**	**$N = 6$**	**$N = 8$**	**$N = 10$**
$k = 0.2$	0.93	0.93	0.94	0.91	0.91	0.91
0.4	0.86	0.88	0.87	0.84	0.84	0.82
0.6	0.82	0.82	0.83	0.78	0.77	0.77
0.8	0.77	0.78	0.79	0.73	0.72	0.71
1.0	0.73	0.74	0.76	0.69	0.67	0.67
1.2	0.70	0.72	0.72	0.65	0.64	0.63
1.4	0.67	0.69	0.69	0.62	0.61	0.58
1.6	0.65	0.66	0.67	0.60	0.58	0.56
1.8	0.62	0.64	0.65	0.57	0.55	0.54
2.0	0.59	0.62	0.62	0.54	0.53	0.51

Abbreviation: IMS, iterated matching of soulmates.