A Closer Look at Teachers' Proportional Reasoning

Abstract

Teachers' mathematical knowledge has important consequences for the quality of the learning environment they create for their students to learn mathematics. Yet relatively little is known about how teachers reason proportionally, despite the fact that proportional reasoning is foundational for several mathematics concepts and that ratios and proportional relationships constitute a major component of the middle school mathematics curriculum. In this study, we investigated how teachers reasoned proportionally on a nonroutine ratio task and the extent to which their proportional reasoning was able to predict their overall understanding of the relevant concepts: ratios and proportional relationships. Using data collected from 238 U.S. mathematics teachers, we found that teachers' proportional reasoning could be grouped into four categories: incorrect, additive, relative, and proportional reasoning. Our results also indicated that teachers' overall knowledge of ratios and proportional relationships aligned with the way they reasoned proportionally, meaning that teachers who used incorrect reasoning on a separate task received the lowest scores on average on the ratios and proportional relationships measure, whereas those who reasoned proportionally had the highest mean scores on average. Implications of the study include the need to shift attention to the way teachers reason in relation to the two elements of proportional reasoning (covariance and invariance) to capture the nuances in their understanding of ratios and proportional relationships.

Keywords: mathematical knowledge for teaching; proportional reasoning; ratios and proportional relationships; teacher knowledge

Introduction

Proportional reasoning is at the heart of the middle school curriculum and is essential for learning a wide range of mathematics and secondary STEM concepts (e.g., Cai & Sun, 2002; Cramer & Post, 1993; Hoyles, Noss, & Pozzi, 2001; Langrall & Swafford, 2000; Lobato, Ellis, & Zbiek, 2010; Thompson & Saldanha, 2003). Yet, study after study has documented students' limited understanding of ratios and proportional relationships (Ayan & Isiksal-Bostan, 2018; Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998; Fernández, Llinares, & Valls, 2008; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Studies have found that teacher show struggles similar to those of students in terms of distinguishing proportional from nonproportional situations (e.g., Izsák & Jacobson, 2017; Nagar et al., 2015; Weiland et al., 2019).

Shedding light on why this concept is particularly challenging for students (as well as teachers) warrants investigating teachers' own proportional reasoning for several reasons. First, teachers' own understanding of mathematics concepts affects the learning opportunities they provide to their students (e.g., Copur-Gencturk, 2015; Borko et al., 1992). Prior work has documented that teachers' own understanding of mathematics and their approach to solving mathematics problems are closely related to how they explain concepts to their students and how they plan to solve ratio problems in class (e.g., Borko et al., 1992; Fisher, 1988). Thus, exploring teachers' reasoning could help researchers understand the learning opportunities teachers present to students to develop their proportional reasoning. Related to this issue, it is important to point out that much of the work in the area of teachers' understanding of ratios and proportional relationships has been conducted with preservice teachers (e.g., Arican, 2018; Berk et al., 2009; Izsák & Jacobson, 2017; Son, 2013). Yet in-service teachers are different from preservice teachers in that in-service teachers have opportunities to develop different levels of understanding through their teaching, which could cause their understanding of mathematics concepts to be drastically different from those of prospective teachers (e.g., Berk et al., 2009). Indeed, studies have provided empirical evidence that the way in-service teachers approach mathematics problems seems to be different from that of prospective teachers (Copur-Gencturk& Doleck, 2021). Taken together, understanding in-service teachers' proportional reasoning warrants investigation.

Second, prior work on teachers' proportional reasoning has focused heavily on the particular strategies teachers use in solving problems rather than the underlying reasoning behind their use of these strategies (Arican, 2018; Fisher, 1988; Livy & Vale, 2011). Whether teachers use the scaling-up (within-measure) approach or finding the unit rate (between-measures) approach tells us how they tackle a given problem or situation. However, it may also mask the fact that both approaches are similar in their use of multiplicative comparison. Additionally, studies have documented that the strategies used in solving ratio problems are dependent on the specific numbers involved in a given problem (Karplus et al., 1983; Tourniaire & Pulos, 1985; Van Dooren et al., 2010). For instance, it may be easier to use the scaling-up approach if the given values of the same quantity are multiples of each other but are not values of the different quantities. Thus, exploring teachers' proportional reasoning by looking beyond particular strategies may allow us to capture the nuances in teachers' understanding of the relevant concepts (i.e., ratios and proportional relationships).

Related to this point, a closer look at prior studies has underscored the importance of the tasks presented when exploring teachers' proportional reasoning (e.g., Izsák & Jacobson, 2017). Commonly used missing-value problems, in which three values for two quantities are given and the participant is asked to find the missing value, seem to overestimate teachers' understanding of proportional relationships (e.g., Berk et al., 2009; Izsák & Jacobson, 2017). In the present study, we purposefully selected a task with a greater potential to differentiate among different levels of reasoning, such as additive and multiplicative thinking (e.g., Lamon, 1993). This task has been used in mathematics methods courses over the years because it provides insights into how teachers reason proportionally.

Before describing the details of the study, we first elaborate on how we conceptualized proportional reasoning. We then review research related to teachers' proportional reasoning to highlight the contribution of this study to the literature. We continue with a description of the study context and the research method. Lastly, we end with a discussion of the results and the implications of the study for teacher education.

Conceptual Framework: Proportional Reasoning

Proportional reasoning is defined as one's "ability to understand, construct, and use the multiplicative relationship between the two co-varying measure spaces . . . or within the measure spaces" (Van Dooren et al., 2018, p. 14). Our Conceptual Framework was guided by the common themes that are highlighted across the different conceptualizations of proportional reasoning.

First, as noted in several conceptualizations of proportional reasoning (e.g., Lamon, 2007), we conceive of teachers recognizing *covariance*, which is identifying and relating quantities, as a first step in developing proportional reasoning. Recognizing covariance indicates that one understands how the change in one quantity is related to the change in the other (Lamon, 2007). The second important aspect of proportional reasoning is recognizing *invariance*, which is observing the invariant quotient of the quantities. Recognizing invariance is key to characterizing the nature of the proportional relationship (Lamon, 2007). For instance, understanding that the amount of water and amount of lemon concentrate will increase or decrease together depending on how large a batch one wants to make is an indicator that the teacher understands the covariance aspect of proportional reasoning. Yet realizing the amount of water per one unit of lemon concentrate is the same, no matter how large or small a batch one wants to make, indicates an understanding the invariance aspect of proportional reasoning. Thus, by using a constant unit of lemon concentrate for every unit of water, one could make different amounts of lemonade that taste the same.

Proportional reasoning does not develop in parallel with teachers' amount of formal education or age. Many adults, including teachers, show levels of proportional reasoning similar to those of children (e.g., Lamon, 2007). One developmental leap that is needed for teachers to reach the ability to reason proportionally is the ability to move from the more naturally developed or more commonly encountered characterization of the nature of the relationship between two quantities, that of additive reasoning (Sowder et al., 1998). Additive reasoning involves comparing two measurable quantities in relation to how much more or less of one quantity there is from the other and then maintaining the invariant nature of the relationship between quantities by keeping the difference between the quantities constant. Returning to the previous example, a person using additive reasoning would assume that the different mixtures of lemonade would taste the same if a constant amount of water was greater than the amount of lemon concentrate.

The ability to distinguish between additive and multiplicative reasoning in a given situation is instrumental in the development of proportional reasoning (Sowder et al., 1998). Transitioning from additive to proportional reasoning begins when one notices that the difference between two varying quantities does not stay constant as the size of the covarying quantities changes (Piaget et al., 1977). While at this intermediate stage, one might not notice the invariant quotient of the two covarying quantities, they might notice only that the difference between the quantities is not constant. As an example, in making multiple batches of lemonade that taste the same, one who is transitioning from additive to proportional thinking might notice that the constant difference between the amounts of water and lemon concentrate does not keep the taste the same for different batches, but one might fail to recognize that the amount of lemon concentrate per unit of water is constant across the different amounts of mixtures that taste the same.

Research on Teachers' Proportional Reasoning

Prior work has documented that teachers struggle with reasoning proportionally, particularly when they are not asked to solve ratio problems they might typically encounter (e.g., Berk et al., 2009; Izsák & Jacobson, 2017). Teachers' limited attention to the covariance and invariance aspects of proportional reasoning are more pronounced in studies documenting that teachers have difficulty distinguishing proportional from nonproportional situations (e.g., Izsák & Jacobson, 2017; Nagar et al., 2015; Weiland et al., 2019). Yet prior work has not considered how the two important indicators of proportional reasoning (i.e., covariance and invariance) might play a role in teachers' struggles with distinguishing proportional from nonproportional situations. (Izsák & Jacobson, 2017; Gillard et al., 2009a, 2009b; Modestou & Gagatsis, 2007; Van Dooren & Greer, 2010).

One common explanation for this pervasive error is that widespread encounters with proportional relationships in daily life and the strong emphasis on proportional relationships in the school curriculum could lead children and adults to use superficial cues (e.g., those in the missing-value structure of a word problem) when determining whether a situation is proportional (Gillard et al., 2009a, 2009b). Although these studies have provided insights into the inappropriate use of proportionality in missing-value problems, they have fallen short in explaining why some teachers do not use these cues or why, as documented in prior work, some teachers and students use additive reasoning in proportional situations (Fernandez, Llinares, & Valls, 2008; Valverde & Castro, 2012).

Scholars in mathematics teacher education focus more on the knowledge resources teachers invoke when identifying proportional situations (e.g., Burgos & Godino, 2020; Brown, Weiland, & Orrill, 2019; Izsak & Jacobson, 2017; Person et al., 2004). For instance, Brown and colleagues (2019) interviewed 32 U.S. middle school mathematics teachers as they were solving a two-part dynamic task in which a linear and a proportional situation were represented. The authors' analyses of the utterances of teachers suggested that the strategies the teachers used may not have been the most appropriate indicators of their proportional reasoning.

We argue that an investigation of teachers' reasoning behind their strategies with respect to the two key elements of proportional reasoning (i.e., covariance and invariance) could reveal additional information regarding how they reasoned proportionally. Our contention is supported by a recent study by Arican (2019), who examined the knowledge of proportional reasoning of 40 Turkish preservice teachers in a middle school mathematics program. Preservice teachers were given four problems for direct, indirect, and linear situations, one of which involved two linear function graphs. His analysis of the written solutions of teachers and his interviews with six of the participants later in the semester suggested that the participants focused only on whether quantities in the problem changed together (i.e., the covariance feature of proportional reasoning). Thus, this result suggests that teachers' lack of attention to the constant quotient of the quantities (i.e., invariance, the second feature of proportional reasoning) could be related to their struggle with differentiating proportional situations from nonproportional situations.

Present Study

Taken together, prior work has paid limited attention to the important indicators of proportional reasoning, which could help researchers identify why teachers struggle with understanding ratios and proportional relationships. Therefore, in this study we aimed to investigate teachers' proportional reasoning by using a task that had the potential to provide insights into the two indicators of proportional reasoning, namely, covariance (recognizing the covarying quantities in a ratio) and invariance (recognizing the invariant nature of the relationship through the constant quotient of the two quantities). To do so, we provided teachers with the dimensions of four rectangles with the same unit difference between their lengths and widths, and we asked them to identify and explain which of these rectangles looked most like a square. The task did not tell teachers which attributes might affect the appearance of a rectangle; therefore, teachers needed to recognize that the sizes of the lengths and widths were the two quantities that would affect the appearance of the rectangle (i.e., the covariance indicator of proportional reasoning). Thus, this task allowed us to investigate whether teachers were able to identify the quantities in a ratio. Additionally, all four of the rectangles had a constant difference between the sizes of the lengths and widths, which allowed us to investigate how teachers characterized the nature of the relationship between the two quantities (i.e., the invariance indicator of proportional reasoning), such as whether they focused on the constant differences between the quantities or realized that the quotient of these two quantities determined the appearance of the rectangle.

We hypothesized that because proportional reasoning is critical to one's understanding of ratios and proportional relationships, the way the teachers reasoned proportionally in this task could predict their overall knowledge of ratios and proportional relationships. To test this hypothesis, we asked teachers to solve a set of mathematics problems on ratios and proportional relationships, and we investigated the extent to which teachers' scores on this test were predicted by their reasoning categories on the aforementioned separate task.

Therefore, in this study, using data collected from 238 U.S. teachers of mathematics in Grades 3–7, we aimed to answer the following questions:

1. How do teachers reason proportionally when answering a nonroutine ratio task?
2. What is the association between teachers' professional background and their proportional reasoning?
3. To what extent can teachers' overall understanding of ratios and proportional relationships be predicted by their proportional reasoning as identified on a separate task?

Methods

Study Context

The data used in this study were collected for a research project funded by the U.S. National Science Foundation to investigate the development of content-specific expertise among mathematics teachers in Grades 3–7. To collect data from teachers with a wide range of educational backgrounds, we partnered with a large school district and an education research company[1] that offers services such as providing the contact and school information of teachers. In this way, we were able to collect data from teachers across the country so that our findings

[1] A few organizations in the United States maintain databases that contain information about teachers, including such variables as their email addresses, the subjects taught, and the grade level at which they are currently teaching, among others. These companies provide access to this information for a fee.

were not bound to teachers working in the same district or state or who had similar educational training.

The analytic sample consisted of 238 mathematics teachers in Grades 3–7 (i.e., students 8 to 13 years of age) in the United States who had completed the survey and whose background information was available.[2] The mathematics problems in the survey were presented to each teacher in a randomized order to avoid item order effects. Teachers were required to provide an answer to each question so that we could gather data on the items with which they were struggling. Teachers who completed the survey were compensated for their participation in the study.

As presented in Table 1, the majority of the study sample was female and White, which was similar to the characteristics of teachers in the U.S. (Snyder et al., 2019). Additionally, 71% of the teachers in the sample earned their teaching credentials through traditional teacher education programs, and 24% of them were teaching mathematics in Grades 6 or 7 during data collection. Finally, almost one-fifth of the sample held a credential to teach mathematics.

Table 1

Descriptive Statistics for Teacher-Level Variables

Variable	Sample (%)
Teacher background	
Gender (female)	84
Ethnicity (White)	68
Teaching level	
Elementary school (Grades 3–5)	76

[2] Teachers received an email inviting them to the study. Those who were eligible for the study (i.e., teachers who were teaching mathematics in Grades 3–7 at the time of data collection) completed an online survey that included questions on ratios and proportional relationships, along with other mathematics questions and background questions, such as their years of teaching experience and the number of mathematics courses they had taken.

Middle school (Grades 6 & 7)	24
Professional background	
Traditional certification	71
Master's degree (yes)	25
Credential in mathematics	19
Credential in multiple subjects	69
Credential in other (e.g., special education)	19

Measures

Proportional reasoning task. As mentioned earlier, because we wanted to focus on how teachers reason proportionally, we used the following problem, in which the dimensions of four rectangles were given and teachers were asked to decide and explain which one(s) looked like a square:

> The Science Club has four separate rectangular plots for experiments with plans. Which rectangle(s) looks more like a square? Explain your answer.
>
> a. 1 foot by 4 feet b. 17 feet by 20 feet
>
> c. 7 feet by 10 feet d. 27 feet by 30 feet

Teachers needed to recognize that the length and width of a rectangle were the two quantities that determined the appearance of the rectangle. Furthermore, they needed to recognize that the quotient of the length and width would determine the extent to which they looked more like a square, given that the quotient of the length and width of any square is 1. Given that prior research has indicated that additive reasoning (noticing the constant difference) tends to be used in proportional situations (e.g., Lamon, 1993; Misailidou & Williams, 2003), all the rectangles teachers were given had the same constant difference, to test whether any of the participants had used additive reasoning. This task had been used by the first author in a course for middle school mathematics teachers for several years and was known to provide insights into how preservice teachers reasoned proportionally.

Coding teachers' proportional reasoning responses. Our coding followed an iterative process. First, we created a set of categories based on the existing literature (e.g., additive and quantitative proportional reasoning; Ben Chaim et al., 1998; Cramer & Post, 1993; Lamon, 1993; Parish, 2010). Second, after compiling the categories of reasoning, which included definitions and examples from prior work, we used these categories to score a subsample of teachers' responses ($N = 20$) to test that the categories were comprehensive enough to capture teachers' reasoning. We also added categories based on the kinds of reasoning teachers used in their responses. Specifically, we observed that some teachers were not only noticing the constant difference between the length and width, but also how the constant difference affected the appearance of the rectangles differently depending on the dimensions. This category reminded us of the intermediate step Piaget and colleagues (1977) had identified when children moved from additive to proportional reasoning.

After the categories had undergone several rounds of revisions to capture teachers' reasoning, we noticed that the four categories were comprehensive enough to capture the nuances in teachers' reasoning. The first category, called incorrect reasoning, included responses in which teachers either failed to identify which quantities would determine which rectangles look more like a square or in which they used incorrect reasoning. As an example, one teacher whose response fell into the incorrect reasoning category said, "I would say that they all would be a rectangle for the fact that they are not equal sides of equal lengths." The teachers whose responses fell in this category also used flawed reasoning, such as, "[Rectangle] A because it is the closest in size to each other and can be stacked to be square."

The second category, called additive reasoning, included responses in which teachers were attending to the constant difference between the length and width. As one teacher explained, "They all have a difference of 3 feet between the length and width. I determined, then, that they would all have the same 'squareness.'" The third category, called relative reasoning, included responses indicating not only that teachers were paying attention to the three-unit difference but also that they took into consideration the impact of the size of the rectangles on the extent to which the rectangles looked more like a square. One such response in this category was, "Because, since the side lengths are longer than all of the other rectangles, the 3 feet difference is less noticeable."

Finally, the proportional relationships category included teachers who noticed that the ratio of the length to the width would determine which rectangle looked more like a square. Teachers in this category computed the ratios, fractions, or percentages to justify their selection. For instance, one teacher stated,

> "Rectangle D because 1 to 4 is just 25% as long as the other side, 7 to 10 is 70% as long as the other side, 17 to 20 is 85% as long as the other side and 27 to 30 is around 90% as long as the other side."

The first and second authors coded the data separately. Our agreement rate was 90%. Disagreements were resolved by discussing the disparities and reaching agreement. All responses were binary coded against the aforementioned four categories.

Ratios and Proportional Relationships Measure. To examine the relationships among different categories of proportional reasoning and teachers' overall understanding of ratios and proportional reasoning, we also used a separate measure to capture teachers' knowledge of ratios and proportional relationships (see Supplementary Materials for the items used in the study). The items included on the scale captured both theoretically important concepts, such as identifying proportional situations (e.g., Izsák, & Jacobson, 2017; Van Dooren et al., 2005), and those covered in the curriculum teachers were expected to teach, such as finding missing values in ratio tables (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). We adapted items from the existing literature (e.g., Beckmann, 2017; Schoenfeld, 2015; Van de Walle et al., 2013) to develop this measure, which included problems ranging from filling out a ratio table, to selecting an option and justifying the selection, to evaluating mathematical situations, to solving missing-value problems and word problems.

Constructed-response problems were scored by two raters to capture both the correctness and reasonableness of the teachers' strategies or explanations for their responses. Although the scoring rubric differed according to the item, the approach used to create each rubric was similar across items. Specifically, the items were coded based on the correctness of a final answer (0 for an incorrect answer and 1 for a correct answer) as well as on the reasoning provided or the strategy used (0 for incorrect reasoning or an incorrect strategy, 1 for partially correct reasoning or a partially correct strategy, and 2 for correct reasoning or a correct strategy) The total score for the ratio table item, which included three missing values, was computed by counting one point toward the total score for each correct answer. The item that asked teachers to evaluate the reasonableness of a given situation involving percentages was coded on a 3-point scale ranging from *incorrect reasoning* to *fully correct reasoning*.

The first author developed the scoring rubrics and trained the other authors to score the items. The initial training involved scoring a set of responses together as a team and discussing the assigned scores. After the team members reached a common understanding of how the rubrics were used, two raters scored each item independently. The interrater reliabilities for each item were beyond 90%. The items on which the raters disagreed were discussed as a team, and a final score was assigned after the team reached agreement.

Teachers' total scores were computed by calculating the percentage of points they received out of the maximum score they could have received if they had answered all the questions correctly. Thus, the minimum score a teacher could receive was 0, and the maximum score a teacher could receive was 100. The mean for the scale was 52.1 (*SD* = 16.0).

Teachers' background characteristics. Several variables were created to capture teachers' educational backgrounds. These included the grade level at which teachers were presently teaching (*elementary school and middle school mathematics teachers*), teachers' preparation route (*alternative*[3] *or traditional teacher education program*), and their certification in teaching mathematics. We also captured the number of courses completed in mathematics and mathematics methods courses (*two or fewer and three or more courses for each course type*).

Data Analysis

To investigate how teachers reasoned proportionally (i.e., Research Question 1), we computed the percentage of teachers whose responses fell into each reasoning category. To investigate whether there was a relationship between teachers' proportional reasoning and their background characteristics, we performed a set of Pearson chi-squared tests to determine whether teachers' background indicators were associated with their reasoning categories. We conducted this analysis by assuming that the reasoning categories were categorical. Finally, to investigate the extent to which teachers' proportional reasoning was associated with their knowledge of ratios and proportional relationships, we predicted teachers' total knowledge score by their reasoning categories, along with the aforementioned teacher background characteristics.

[3] An alternative teaching route indicates that teachers entered teaching without having a background in education. They had a degree in another field and usually received relatively short training in teaching before entering the profession.

Because we assumed that the reasoning categories were categorical, we entered three categories (additive, relative, and proportional) in the model, with the incorrect reasoning category being the reference category.

Findings

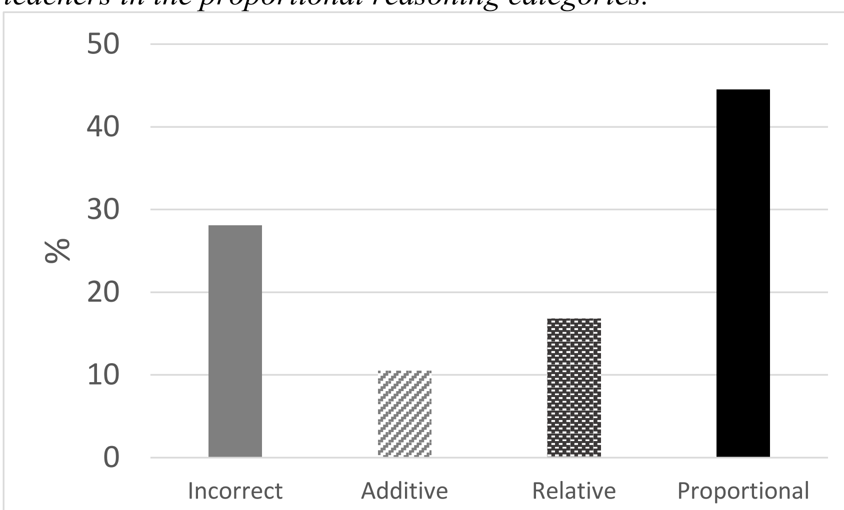
How Teachers Reasoned Proportionally

Teachers' responses to the item designed to capture their proportional reasoning indicated that 28% of the sample used incorrect reasoning, 11% reasoned additively, 17% reasoned relatively, and 45% reasoned proportionally (see Figure 1). The teachers who used incorrect reasoning either stated that they did not know the answer or focused on other attributes to determine the answer. For instance, one teacher chose the rectangle 7 feet by 10 feet because "they are closest to being a perfect square at 9 by 9."

The teachers who reasoned additively seemed to focus on the constant difference between the length and width of each given rectangle. A typical response for teachers focusing on the constant difference included reasoning such as, "They would all be the same. A square has all equal sides. Each of these rectangular plots has a width that is 3 more than the length, which would make them the same essentially." Similar to those who reasoned additively, teachers whose responses fell into the relative reasoning category also noticed the constant difference between the length and width. However, they also took into consideration that when the rectangle grew larger, the constant sum had a different impact. As one teacher explained, "I think [option] d is most square because although the difference in feet of the dimensions is the same, the larger number will make it harder to tell that there is 3 feet difference." As indicated in this response, teachers in this category were noticing that the same constant difference was having a differing impact, but their responses did not indicate that they recognized the proportional relationship between the length and width.

Finally, those who reasoned proportionally demonstrated the understanding that the quotient of the length and width would determine the extent to which these rectangles looked more like a square (see Figure 1). Teachers used a wide range of strategies in their responses; however, their responses included the fact that the rectangle would look more like a square the closer the ratio was to 1. For instance, one teacher explained his or her selection by stating it was "because the ratio of sides 27/30 is 0.9 and that is the closest ratio to 1."

Figure 1
Percentages of teachers in the proportional reasoning categories.



Relationship Between Teachers' Professional Background and How They Reasoned Proportionally

Our separate analysis of the association between each of the teacher educational background indicators and teachers' proportional reasoning indicated that the route through which teachers entered the teaching profession (alternative certification or traditional teacher education program), $\chi^2(3, N = 238) = 3.24$, $p = .36$, the number of mathematics courses taken, $\chi^2(3, N = 238) = 3.26$, $p = .35$, and the mathematics methods courses completed, ($\chi^2(3, N = 238)$

= 4.57, p = .21, were not related to teachers' proportional reasoning category. However, whether teachers taught upper elementary or middle school mathematics and whether they held mathematics teaching credentials were related to their reasoning categories. Specifically, teachers' reasoning categories appeared to be positively related to the grade band at which they taught, $\chi^2(3, N = 238) = 10.9$, $p = .01$. In fact, 62% of middle school mathematics teachers were reasoning proportionally compared with 39% of upper elementary school teachers. The incorrect reasoning rate was 22% among middle school mathematics teachers, whereas this rate was 30% among elementary school mathematics teachers. Similarly, we found a positive association between the reasoning categories and holding a mathematics teaching credential, $\chi^2(3, N = 238) = 10.2$, $p = .02$. Indeed, 65% of the teachers with a mathematics credential were reasoning proportionally, whereas this ratio was 40% for the teachers who did not have a credential in mathematics. Moreover, only 17% of teachers with a mathematics credential were reasoning incorrectly, whereas 31% of teachers without a mathematics credential showed incorrect reasoning.

Linking Teachers' Proportional Reasoning to Their Overall Performance

We examined how the differences in teachers' reasoning could be related to their overall knowledge of the relevant concepts, namely, ratios and proportional relationships. To do so, we conducted a linear regression analysis in which teachers' total scores on a separate measure of ratios and proportional relationships were predicted by the categories of proportional reasoning as identified on a separate task along with several indicators of their educational background.

Our regression analysis indicated that the estimated score of teachers who used incorrect reasoning, who did not hold a teaching credential in mathematics, and who were teaching elementary school mathematics was, on average, 47.2 points (see the intercept in Table 2).

Teachers who used additive reasoning received scores averaging 4.9 points higher than those who used incorrect reasoning; however, this difference was not statistically significant ($p = .18$). The scores of teachers who used relative reasoning were, on average, 7.4 points higher than those of teachers who used incorrect reasoning ($p = .01$). In addition, teachers who used proportional reasoning had scores, on average, 11.9 points higher than those who used incorrect reasoning ($p < .001$). We also found that middle school mathematics teachers performed significantly better than did elementary school teachers in Grades 3–5 ($p = .02$). The teachers' reasoning categories as well as the two indicators of their professional background (certification type and the grade band taught) explained 18.6% of the variation in their scores on the ratios and proportional relationships measure.

Table 2

Results of the Linear Regression Predicting Teachers' Knowledge of Ratios and Proportional Relationships by Their Reasoning Categories and Background Characteristics

Predictor	Regression coefficient (*SE*)
Reasoning category	
Additive	4.94 (3.66)
Relative	7.38* (2.95)
Proportional	11.87*** (2.43)
Background characteristic	
Mathematics certification	4.47 (2.89)
Middle school mathematics teachers	6.21* (2.60)
Intercept	47.19*** (3.48)

Note. $N = 238$ for all models. Standard errors are in parentheses. For the reasoning categories, incorrect reasoning is the reference category.
*$p < 0.05$. ***$p < 0.001$.

Discussion

In this study, we aimed to contribute to the literature by investigating teachers' proportional reasoning as it related to the two important indicators of proportional reasoning: covariance and invariance. Our findings indicated that the way teachers reasoned in a proportional situation could be divided into four categories: incorrect, additive, relative, and proportional reasoning. Before we discuss the study findings, we would like to point out the study limitations. First, we categorized teachers' reasoning based on their responses to a single item; thus, further research is needed with more items to explore whether the results would be similar with a larger set of items. Second, we captured teachers' proportional reasoning with a task situation situated in geometry. Thus, further research is needed to determine whether the context of the problem might lead teachers to reason differently, and if so, to what extent. Finally, although we collected data from teachers across the nation, our sample was not representative of middle grade mathematics teachers. Thus, further research is needed to determine whether our findings could be replicated when using different samples.

This study provides evidence supporting the categorization of teachers' proportional reasoning according to their understanding of the covariance and invariance components of proportional reasoning. The results will provide insights into teachers' knowledge of the relevant mathematical concepts (as in the case here of their knowledge of ratios and proportional relationships). The validity of our categorization was supported by showing that teachers' performance on a set of ratios and proportionality items and their reasoning categories were associated in the expected direction. Specifically, teachers who used incorrect reasoning received a lower score on average than did those in all the other reasoning groups, whereas teachers who correctly demonstrated proportional reasoning also performed statistically better on average than did those in the other reasoning categories.

Our study extends the theoretical discussion of teachers' understanding of ratios and proportional relationships by providing empirical evidence that attending to the two key features of proportional reasoning is useful in understanding how teachers think in proportional situations. Additionally, our study draws attention to the existence of relative reasoning—that intermediate category between additive (noticing a constant difference) and proportional reasoning. In fact, this category bears some similarity to Piaget and colleagues' transition category (Inhelder & Piaget, 1958; Karplus, 1980). Teachers in this category recognized that the constant difference did not adequately explain the situation, but they were not characterizing the nature of the relationship multiplicatively. We believe we were able to identify nuances in teachers' proportional reasoning because of the task chosen for this study. Prior work has consistently shown that using missing-value problems may not accurately assess teachers' proportional reasoning (Izsak & Jacobson, 2017; Lo, 2004; Weiland et al., 2019). Yet in this study, we used a task that provided insights into the two important indicators of proportional reasoning: to identify the quantities in a ratio and to characterize their relationship multiplicatively. Furthermore, the task included several situations with the same constant difference so that we could capture the two theoretically important ways one could reason (additive and relative reasoning).

One important implication of these study findings is that teachers need more opportunities to identify and relate the quantities in a proportional relationship (i.e., covariance) as well as more opportunities to characterize the nature of the relationship (i.e., invariance). Using tasks similar to the one used in this study would allow teacher educators to help teachers notice which quantities covary and assist them in moving away from additive toward proportional reasoning. This is particularly instrumental, given that some teachers struggled with identifying which quantities they needed to attend to, and some struggled with identifying the multiplicative nature of their relationship. Another important and relevant implication of this research finding is that it captures teachers shifting their attention from the strategies or knowledge pieces they used to solve the problems to the kinds of reasoning they used to arrive at their answers. We contend that these tasks should be used more often in studies to capture the nuances in teachers' reasoning. Given that teachers' own understanding of mathematical concepts and how they approach mathematics problems are related to the learning environment they create for their students to learn these concepts, further research is needed to investigate how teachers' proportional reasoning is associated with the learning environment they create to develop their students' learning.

Our results also suggest that U.S. teachers' background characteristics are not good indicators of the way they reason proportionally or their overall understanding of ratios and proportional relationships. This finding is somewhat in alignment with prior work (e.g., Weiland et al., 2019) in that the number of mathematics courses and the number of mathematics methods courses teachers had completed were not related to their understanding of proportionality. Given that we had a small number of teachers in each group, further research is needed to understand how teachers' educational background is related to their knowledge of proportionality. Nonetheless, it is important to highlight that those who were teaching middle school mathematics and who held teaching credentials in mathematics teaching seemed better able to reason proportionally compared with those who did not teach middle school mathematics or those who held multiple-subject teaching credentials or credentials in another field (e.g., special education). Taken together, these findings indicate that teachers, particularly those who did not have prior preparation in mathematics, may need more targeted opportunities in teacher education programs to enhance their proportional reasoning.

In conclusion, our study contributes to the literature by providing empirical evidence that teachers' proportional reasoning can be captured by how they identify and relate measurable quantities (i.e., covariance) and how they characterize the nature of the relationship (i.e., invariance), and that their responses can be categorized into four categories: incorrect, additive, relative, and proportional reasoning. Our study also provides evidence of how teachers reason proportionally by focusing on two indicators of proportional reasoning in their responses: (1) identifying the two quantities that change together and (2) characterizing the invariant nature of the relationship multiplicatively.

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