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ARTICLE

A blurred view of Van der Waerden type theorems

Vojtech Rödl and Marcelo Sales*

Department of Mathematics, Emory University, Atlanta, GA 30322, USA
*Corresponding author. Email: mtsales@emory.edu

Dedicated to the memory of Ronald Graham

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Abstract

Let $\mathrm{AP}_k = \{a, a+d, \ldots, a+(k-1)d\}$ be an arithmetic progression. For $\varepsilon > 0$ we call a set $\mathrm{AP}_k(\varepsilon) = \{x_0, \ldots, x_{k-1}\}$ an ε-approximate arithmetic progression if for some a and d, $|x_i - (a+id)| < \varepsilon d$ holds for all $i \in \{0, 1 \ldots, k-1\}$. Complementing earlier results of Dumitrescu (2011, *J. Comput. Geom.* **2**(1) 16–29), in this paper we study numerical aspects of Van der Waerden, Szemerédi and Furstenberg–Katznelson like results in which arithmetic progressions and their higher dimensional extensions are replaced by their ε-approximation.

Keywords: Combinatorics; Arithmetic progressions; Ramsey theory

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1. Introduction

For a natural number N we set $[N] = \{1, 2 \ldots, N\}$. Assume that $[N]$ is coloured by r colours. We denote by

$$N \to (\mathrm{AP}_k)_r$$

the fact that any such r-colouring yields a monochromatic arithmetic progression AP_k of length k. With this notation the well known Van der Waerden's theorem can be stated as follows.

Theorem 1.1. *For every positive integers r and k, there exists a positive integer N such that $N \to (\mathrm{AP}_k)_r$.*

The minimum N with the property of Theorem 1.1 is called the Van der Waerden number of r, k and is denoted by $W(k, r)$. In other words, $W(k, r)$ is the minimum integer N such that any r-colouring of $[N]$ contains a monochromatic arithmetic progression of length k. Much effort was put to determine lower and upper bounds for $W(k, r)$, but the problem remains widely open. As an illustration, the best known bounds for $W(k, 2)$ are

$$\frac{2^k}{k^{o(1)}} \le W(k, 2) \le 2^{2^{2^{2^{2^{k+9}}}}},$$

where $o(1) \to 0$ as $k \to \infty$. The lower bound is due to Szabo [25] while the upper bound is a celebrated result of Gowers on Szemerédi's theorem [10]. It is good to remark that when k is a prime the lower bound can be improved to $W(k+1, 2) \ge k2^k$ by a construction of Berlekamp [2].

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Ron Graham was keenly interested in the research leading to improvements of the upper bound of $W(k, 2)$ and motivated it by monetary prizes. Currently open is his \$1000 award for the proof that $W(k, 2) < 2^{k^2}$ (see [14]). During his career he also contributed to related problems in the area (see [3, 12, 13]). For instance, together with Erdös [6], Graham proved a canonical version of Van der Waerden: Every colouring of $\mathbb{N}$, not necessarily with finitely many colours, contains either an monochromatic arithmetic progression or a rainbow arithmetic progression, i.e., a progression with every element of distinct colour.

Inspired by the works of [4] and [16], we are interested in the related problem where we replace an arithmetic progression by an perturbation of it.

Definition 1.2. Given $\varepsilon > 0$, a set $X = \{x_0, \ldots, x_{k-1}\} \subseteq [N]$ is an ε-approximate $\mathrm{AP}_k(\varepsilon)$ of an arithmetic progression of length k if there exists $a \in \mathbb{R}$ and $d > 0$ such that $|x_i - (a + id)| < \varepsilon d$.

In other words, an $\mathrm{AP}_k(\varepsilon)$ is just a transversal of $\bigcup_{i=0}^{k-1} B(a + id; \varepsilon d)$, where $B(a + id; \varepsilon d)$ is the open ball centred at $a + id$ of radius εd. Depending on the choice of ε, an $\mathrm{AP}_k(\varepsilon)$ can be different from an AP_k. For example, if $\varepsilon = 1/3$, then $a = 0.8$ and $d = 2.4$ testifies that $\{1, 3, 6\}$ is an ε-approximate arithmetic progression of length 3, but it is not an arithmetic progression itself.

For integers r, k and $\varepsilon > 0$, let

$$W_\varepsilon(k, r) = \min\{N : N \to (\mathrm{AP}_k(\varepsilon))_r\}.$$

That is, $W_\varepsilon(k, r)$ is the smallest N with the property that any colouring of $[N]$ by r colours yields a monochromatic $\mathrm{AP}_k(\varepsilon)$. Our first result shows that one can obtain sharper bounds to the Van der Waerden problem by replacing AP_k to $\mathrm{AP}_k(\varepsilon)$.

Theorem 1.3. *Let $r \geq 1$. There exists a positive constant ε_0 and a real number c_r depending on r such that the following holds. If $0 < \varepsilon \leq \varepsilon_0$ and $k \geq 2^r r! \varepsilon^{-1} \log^r(1/5\varepsilon)$, then*

$$c_r \frac{k^r}{\varepsilon^{r-1} \log(1/\varepsilon)^{\binom{r+1}{2}-1}} \leq W_\varepsilon(k, r) \leq \frac{2k^r}{\varepsilon^{r-1}}.$$

Similar as in the previous discussion we will write $N \to_\alpha \mathrm{AP}_k$ (or $N \to_\alpha \mathrm{AP}_k(\varepsilon)$) to denote that any subset $S \subseteq [N]$ with $|S| \geq \alpha N$ necessarily contains an arithmetic progression AP_k (or $\mathrm{AP}_k(\varepsilon)$, respectively). Answering a question of Erdös and Turan [7], Szemerédi proved the following celebrated result:

Theorem 1.4. *For any $\alpha > 0$ and a positive integer k, there exists an integer N_0 such that for every $N \geq N_0$ the relation $N \to_\alpha \mathrm{AP}_k$ holds.*

Basically Szemerédi theorem states that any positive proportion of $\mathbb{N}$ contains an arithmetic progression of length k. Not much later Furstenberg [9] gave an alternative proof of Theorem 1.4 using Ergodic theory. Extending [9], Furstenberg and Katznelson [8] were able to prove a multidimensional version of Szemerédi's theorem:

An m-dimensional cube $C(m, k)$ is a set of k^m points in m-dimensional Euclidean lattice $\mathbb{Z}^m$ such that

$$C(m, k) = \{\vec{a} + d\vec{v} : \vec{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m \text{ and } \vec{v} = (v_1, \ldots, v_m) \in \{0, 1, \ldots, k-1\}^m\}.$$

That is, $C(m, k)$ is a homothetic translation of $[k]^m$. As in the one dimensional case, for $\alpha > 0$ and integers m, k and N we will write $[N]^m \to_\alpha C(m, k)$ to mean that any subset $S \subseteq [N]^m$ with $|S| \geq \alpha N^m$ contains a cube $C(m, k)$. The following is the multidimensional version of Theorem 1.4 proved in [8].

Theorem 1.5. *For any $\alpha > 0$ and positive integers k and m, there exists an integer N_0 such that for every $N \geq N_0$ the relation $[N]^m \to_\alpha C(m, k)$ holds*

Define $f(N, m, k)$ as the maximum size of a subset $A \subseteq [N]^m$ without a cube $C(m, k)$. Note that $f(N, 1, k)$ corresponds to the maximal size of a subset $A \subseteq [N]$ without an arithmetic progression AP_k. Theorems 1.4 and 1.5 give us that $f(N, m, k) = o(N^m)$. Determining bounds for $f(N, m, k)$ is a long standing problem in additive combinatorics. For $m = 1$ the best current bounds are

$$N \exp\left(-c_k(\log N)^{1/\lceil \log_2 k \rceil}\right) \leq f(N, 1, k) \leq \frac{N}{(\log\log N)^{2^{-2^{k+9}}}}$$

where c_k is a positive constant depending only on k. The upper bound is due to Gowers [10], while the lower bound with best constant c_k is due to O'Bryant [21].

For larger m it is worth mentioning that Furstenberg–Katznelson proof of Theorem 1.5 uses Ergodic theory and gives us no quantitative bounds on $f(N, m, K)$. Purely combinatorial proofs were given later based on the hypergraph regularity lemma in [11] and [20, 23]. Those proofs give quantitative bounds which are incomparably weaker than the one for $m = 1$. For instance, in [19] Moshkovitz and Shapira proved that the hypergraph regularity lemma gives a bound of the order of the k-th Ackermann function.

Now we consider ε-approximate versions of Theorems 1.4 and 1.5.

Definition 1.6. Given $\varepsilon > 0$, a set $X = \{x_{\vec{v}} : \vec{v} \in \{0, 1, \ldots, k-1\}^m\} \subseteq [N]^m$ is an ε-approximate cube $C_\varepsilon(m, k)$ if there exists $\vec{a} \in \mathbb{R}^m$ and $d > 0$ such that $||x_{\vec{v}} - (\vec{a} + d\vec{v})|| < \varepsilon d$.

For integers N, m, k and $\varepsilon > 0$, let $f_\varepsilon(N, m, k)$ be the maximal size of a subset $A \subseteq [N]^m$ without an $C_\varepsilon(m, k)$. Dimitrescu showed an upper bound for $f_\varepsilon(N, m, k)$ in [4]. We complement his result by also providing a lower bound to the problem.

Theorem 1.7. *Let $m \geq 1$ and $k \geq 3$ be integers and $0 < \varepsilon < 1/125$. Then there exists an integer $N_0 := N_0(k, \varepsilon)$ and positive constants c_1 and c_2 depending only on k and m such that*

$$N^{m - c_1(\log(1/\varepsilon))^{\frac{1}{\ell} - 1}} \leq f_\varepsilon(N, m, k) \leq N^{m - c_2(\log(1/\varepsilon))^{-1}},$$

for $N \geq N_0$ and $\ell = \lceil \log_2 k \rceil$.

The paper is organised as follows. In Section 2, we present a proof of Theorem 1.3. The upper bound is an iterated blow-up construction, while the lower bound is given by an ad-hoc inductive colouring. We prove Theorem 1.7 in Section 3. The lower bound uses the current lower bounds for $f(N, 1, k)$, while the upper bound is given by an iterated blow-up construction combined with an averaging argument.

2. Proof of Theorem 1.3

2.1 Upper bound

We start with the upper bound. Given $r \geq 1$ colours, we consider the following r-iterated blow-up of an AP_k given by the set of integers

$$B_r = \left\{b_0 + tb_1 + \ldots + t^{r-1}b_{r-1} : (b_0, \ldots, b_{r-1}) \in \{0, 1 \ldots, k-1\}^r,\ t = \lceil k/\varepsilon \rceil \right\}.$$

Note that B_r is a set of size $|B_r| = k^r$ and $\mathrm{diam}(B_r) \leq (k-1)(1 + t + \ldots + t^{r-1}) < 2(k-1)t^{r-1}$. It turns out that any r-colouring of B_r contains a monochromatic $\mathrm{AP}_k(\varepsilon)$. In particular, this implies that $W_\varepsilon(k, r) \leq \mathrm{diam}(B_r) + 1 \leq 2k^r/\varepsilon^{r-1}$.

Proposition 2.1. *Any r-colouring of B_r has a monochromatic* $\mathrm{AP}_k(\varepsilon)$.

Proof. We prove the proposition by induction on the number of colours r. For $r = 1$, one can see that $B_1 = [k]$, which is an arithmetic progression of length k and in particular a $\mathrm{AP}_k(\varepsilon)$.

Now suppose that any $(r-1)$-colouring of B_{r-1} contains a monochromatic $\mathrm{AP}_k(\varepsilon)$. Consider an r-colouring of B_r. Note that we can partition $B_r = \bigcup_{i=0}^{k-1} B_{r,i}$ where

$$B_{r,i} = \left\{ b_0 + \ldots + t^{r-2} b_{r-2} + i t^{r-1} : (b_0, \ldots, b_{r-2}) \in \{0, 1, \ldots, k-1\}^{r-1},\ t = \lceil k/\varepsilon \rceil \right\}.$$

That is, for every $0 \le i \le k-1$, the set $B_{r,i}$ is a translation of B_{r-1} by it^{r-1}.

Consider a transversal $X = \{x_0, \ldots, x_{k-1}\}$ of $B_r = \bigcup_{i=0}^{k-1} B_{r,i}$ with $x_i \in B_{r,i}$ for every $0 \le i \le k-1$. Let $a = \mathrm{diam}(B_{r-1})/2$ and $d = t^{r-1}$. Since $x_i \in B_{r,i}$ implies that $it^{r-1} \le x_i \le it^{r-1} + \mathrm{diam}(B_{r-1})$, we obtain that

$$|x_i - (a + id)| \le \frac{\mathrm{diam}(B_{r-1})}{2} \le \frac{k^{r-1}}{\varepsilon^{r-2}} \le \varepsilon d$$

and X is an ε-approximate $\mathrm{AP}_k(\varepsilon)$. Therefore, if some colour c is present in each of the sets $B_{r,i}$ for $0 \le i \le r-1$, we could select X to be a monochromatic $\mathrm{AP}_k(\varepsilon)$. Consequently we may assume that there is no monochromatic transversal in B_r, which means that there exists an index i such that $B_{r,i}$ is coloured with at most $(r-1)$ colours. Since $B_{r,i}$ is just a translation of B_{r-1}, by induction hypothesis we conclude that there exists a monochromatic $\mathrm{AP}_k(\varepsilon)$ inside $B_{r,i}$. □

2.2 Lower bound

In order to construct a large set avoiding ε-approximate $\mathrm{AP}_k(\varepsilon)$ we need some preliminary results. Given a real number $D > 0$, we define an $(r-1, 1; D)$-alternate labelling of $\mathbb{R}$ to be an labelling $\chi : \mathbb{R} \to \{-1, +1\}$ such that

$$\chi(x) = \begin{cases} +1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left(irD + mD, \left(i + \frac{r-1}{r}\right) rD + mD \right], \\ -1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left(\left(i + \frac{r-1}{r}\right) rD + mD, (i+1)rD + mD \right], \end{cases}$$

for some $m \in \mathbb{Z}$. That is, χ is a periodic labelling of $\mathbb{R}$ with period rD, where we partition $\mathbb{R}$ into disjoint intervals of length D and label them alternating between $r-1$ consecutive intervals of label $+1$ and one of label -1. The restriction of an $(r-1, 1; D)$-alternate labelling to $\mathbb{Z}$ will be of great importance for us. The following lemma roughly characterises the common difference of any large monochromatic approximate arithmetic progression in such a labelling.

Lemma 2.2. *Let $D, \delta > 0$, m be a positive integer with $\delta \le \frac{1}{2r(r+1)}$ and $\chi : \mathbb{R} \to \{-1, +1\}$ be an $(r-1, 1; D)$-alternate labelling of $\mathbb{R}$. If there exist $a, d \in \mathbb{R}$ and an integer ℓ such that*

$$d \notin \bigcup_{i \in \mathbb{Z}} \bigcup_{q=1}^{r} \left(\left(\frac{i}{q} - \delta \right) rD, \left(\frac{i}{q} + \delta \right) rD \right),$$

and that $B = \bigcup_{i=0}^{\ell-1} B(a + id, \delta rD)$ has a monochromatic transversal of label $+1$, then $\ell \le 3r/\delta$.

Proof. We may assume without loss of generality that χ is the following labelling of $\mathbb{R}$:

$$\chi(x) = \begin{cases} +1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left(irD, \left(i + \frac{r-1}{r}\right) rD \right], \\ -1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left(\left(i + \frac{r-1}{r}\right) rD, (i+1)rD \right], \end{cases}$$

That is, we may assume that $m = 0$ in the definition of an alternate labelling. Also, during the proof we shall write $\overline{x}$ to be the representative of x modulo rD in the interval $(0, rD]$, i.e., the number $0 < \overline{x} \le rD$ such that $x - \overline{x} = brD$ for some integer $b \in \mathbb{Z}$.

We start by claiming that there exists $1 \le s \le r$ such that

$$\overline{sd} \in \left[\delta rD, \frac{rD}{r+1} \right] \cup \left[\left(1 - \frac{1}{r+1} \right) rD, (1-\delta)rD \right]. \tag{1}$$

First note by our hypothesis that

$$d \notin \left(\left(\frac{i}{q} - \delta\right) rD, \left(\frac{i}{q} + \delta\right) rD\right)$$

for every $i \in \mathbb{Z}$ and $1 \leq q \leq r$. Therefore,

$$qd \notin ((i-\delta)rD, (i+\delta)rD) \subseteq ((i-q\delta)rD, (i+q\delta)rD) \tag{2}$$

for every $i \in \mathbb{Z}$ and $1 \leq q \leq r$.

Now consider the partition $(0, rD] = \bigcup_{j=0}^{r} \left(\frac{jrD}{r+1}, \frac{(j+1)rD}{r+1}\right]$. If there exists $1 \leq s \leq r$ such that $\overline{sd}$ is in the two outer intervals above, i.e., in either $\left(0, \frac{rD}{r+1}\right]$ or $\left(\left(1 - \frac{1}{r+1}\right) rD, rD\right]$, then by (2) we obtain that s satisfies (1). Otherwise, assume that there is no $1 \leq s \leq r$ with $\overline{sd}$ in the two outer intervals. Then by the pigeonhole principle there exist $1 \leq p < q \leq r$ and an index j such that $\overline{pd}, \overline{qd} \in \left(\frac{jrD}{r+1}, \frac{(j+1)rD}{r+1}\right]$. Consequently, we have that $\overline{qd} - \overline{pd} \in \left(-\frac{rD}{r+1}, \frac{rD}{r+1}\right)$. By letting $s = q - p$ we obtain that

$$\overline{sd} \in \left(0, \frac{rD}{r+1}\right] \cup \left(\left(1 - \frac{1}{r+1}\right) rD, rD\right],$$

for $1 \leq s \leq r$, which is a contradiction. Therefore, condition (1) is always satisfied for some s.

Let $1 \leq s \leq r$ be the number satisfying (1) and consider the subset

$$B' = \bigcup_{i=0}^{\ell'} B(a + isd, \delta rD) \subseteq B,$$

where $\ell' = \lfloor (\ell - 1)/s \rfloor$. That is, if we see B as the arithmetic progression of intervals of length δrD, size ℓ and common difference d, then B' is a subarithmetic progression of B with common difference sd. Since B has a monochromatic transversal labeled $+1$, then B' also has a monochromatic transversal labeled $+1$. Hence, because $\bigcup_{i\in\mathbb{Z}} \left(irD, (i + \frac{r-1}{r})rD\right]$ are the elements of label $+1$ in our $(r-1, 1; D)$-alternate labelling, we have that

$$\{a, a + sd, \ldots, a + \ell' sd\} \subseteq \bigcup_{i\in\mathbb{Z}} \left((i-\delta)rD, \left(i + \frac{r-1}{r} + \delta\right) rD\right).$$

Suppose that $\overline{sd} \in \left[\delta rD, \frac{1}{r+1} rD\right]$. Since the colouring χ is periodic modulo rD, we may assume without loss of generality that $sd \in \left[\delta rD, \frac{rD}{r+1}\right]$. We claim that there exists an integer p such that $\{a, a + sd, \ldots, a + \ell' sd\} \subseteq \left((p-\delta)rD, \left(p + \frac{r-1}{r} + \delta\right) rD\right)$. Suppose that this is not the case. Because $sd > 0$ there exist integers $p < q$ and $0 \leq i \leq \ell' - 1$ such that $a + isd \in \left((p-\delta)rD, \left(p + \frac{r-1}{r} + \delta\right) rD\right)$ and $a + (i+1)sd \in \left((q-\delta)rD, \left(q + \frac{r-1}{r} + \delta\right) rD\right)$. A computation shows that

$$sd = a + (i+1)sd - (a + isd) > (q-\delta)rD - \left(p + \frac{r-1}{r} + \delta\right) rD \geq (1 - 2\delta r)D \geq \frac{rD}{r+1}$$

for $\delta \leq \frac{1}{2r(r+1)}$, which contradicts our assumption on sd.

Hence, there exists p such that $a, a + \ell' sd \in \left((p-\delta)rD, \left(p + \frac{r-1}{r} + \delta\right) rD\right)$, which implies that

$$\ell' sd = (a + \ell' sd) - a \leq \left(p + \frac{r-1}{r} + \delta\right) rD - (p-\delta)rD = (r-1)D + 2\delta rD.$$

Since $sd \geq \delta rD$, we obtain that

$$\ell' sd \geq \left\lfloor \frac{\ell-1}{s} \right\rfloor \delta rD \geq \frac{\ell\delta rD}{2s} \geq \frac{\delta\ell D}{2}$$

for $\ell > r \geq s$. The last two computations combined with the fact that $\delta \leq \frac{1}{2r(r+1)} \leq \frac{1}{4}$ gives us that

$$\ell \leq \frac{2(r-1)D + 4\delta rD}{\delta D} \leq \frac{2(r-1)}{\delta} + 4r \leq \left(\frac{2}{\delta} + 4\right) r \leq \frac{3r}{\delta}$$

Now assume that $\overline{sd} \in \left[\left(1 - \frac{1}{r+1}\right) rD, (1-\delta)rD\right]$. By the periodicity of χ, we may assume without loss of generality that $sd \in [-\frac{rD}{r+1}, -\delta rD]$. By rewriting $\{a, a+sd, \ldots, a+\ell' sd\}$ as $\{a', a'+sd', \ldots, a'+\ell' sd'\}$ with $a' = a + \ell' sd$ and $d' = -d$, we are back to the previous case and again $\ell \leq 3r/\delta$. □

Although it is convenient to prove Lemma 2.2 using an alternate labelling of $\mathbb{R}$, the lower bound construction will use alternate labellings of set of integers. With this in mind, we give the following companion definition.

Given positive integers D, r and t, an $(r-1, 1; D)$-alternate labelling of the set $[rtD]$ is a labelling $\chi' : [rtD] \to \{-1, +1\}$ such that $\chi'(x) = \chi(x)$, where χ is an $(r-1, 1; D)$-alternate labelling of $\mathbb{R}$. In other words, an alternate labelling of a set of integers is just the restriction of an alternate labelling of $\mathbb{R}$ to the set. Note that by this definition, there exists r distinct $(r-1, 1; D)$-alternate labellings of $[rtD]$. A D-block of $[rtD]$ is a block of D consecutive integers of the form $[iD+1, (i+1)D]$. One can note that the D-blocks form a partition of $[rtD]$ and each D-block is monochromatic in an $(r-1, 1; D)$-alternate labelling of $[rtD]$.

Finally, note that given an alternate labelling χ' of a set $[rtD]$ we can extend back to a an alternate labelling of $(0, rtD]$ by labelling the entire interval $(iD, (i+1)D]$ with the same label as the D-block of integers $[iD+1, (i+1)D]$. Since the labelling is periodic, it is now easy to extend back to a labelling χ of $\mathbb{R}$.

The next result is a consequence of the proof of Lemma 2.2.

Proposition 2.3. *Let D, r, t and ℓ be positive integers with $\ell \geq t(r+1)+2$ and $0 < \varepsilon < 1/2r$ be a real number. If $[rtD]$ is coloured by an $(r-1, 1; D)$-alternate labelling and $X \subseteq [rtD]$ is a monochromatic $\mathrm{AP}_\ell(\varepsilon)$ of label $+1$, then there exists $0 \leq i \leq rt-1$ such that the D-block $[iD+1, (i+1)D]$ satisfies $|X \cap [iD+1, (i+1)D]| \geq \ell/(r-1)$.*

Proof. Write $X = \{x_0, \ldots, x_{\ell-1}\}$. Since X is an $\mathrm{AP}_\ell(\varepsilon)$, there exists $a \in \mathbb{R}$, $d > 0$ such that $|x_i - (a+id)| < \varepsilon d$. Therefore, a computation shows that

$$rtD > |x_{\ell-1} - x_0| \geq a + (\ell-1)d - a - 2\varepsilon d = (\ell - 1 - 2\varepsilon)d,$$

which implies that

$$d \leq \frac{rtD}{\ell-2} \leq \frac{rD}{r+1} \tag{3}$$

for $\ell \geq t(r+1)+2$.

Similarly as in the proof of Lemma 2.2, we will show that all the elements of X are inside an interval of $(r-1)$ consecutive D-blocks of label $+1$.

Suppose that this was not the case. Since non-consecutive D-blocks of label $+1$ are at a distance of at least D elements, then there exists x_i and x_{i+1} such that $|x_{i+1} - x_i| \geq D$. However, in view of $\varepsilon < 1/2r$ and (3), we obtain

$$|x_{i+1} - x_i| \leq |x_{i+1} - (a+(i+1)d)| + |a+(i+1)d - (a+id)| + |x_i - (a+id)| \leq (1+2\varepsilon)d < D,$$

which is a contradiction. The result now follows by an application of the pigeonhole principle. □

Note that Proposition 2.3 already gives us a lower bound for the case $r=2$. Indeed, we will prove that an $(1,1;k-1)$-alternate labelling of $\left[\frac{2(k-1)(k-2)}{3}\right]$[1] does not contain a monochromatic $\mathrm{AP}_k(\varepsilon)$ for $\varepsilon < 1/4$ and sufficiently large k.

Suppose that this is not the case. Since an $(1,1;k-1)$-alternate labelling is symmetric, we may assume that there is a monochromatic $\mathrm{AP}_k(\varepsilon)$ of label $+1$. Applying Proposition 2.3 with $r=2$, $t=(k-2)/3$, $D=k-1$ and $\ell=k$ gives us that there exists a $(k-1)$-block of the form $[i(k-1)+1,(i+1)(k-1)]$ such that $|X\cap[i(k-1)+1,(i+1)(k-1)]|\geq k$, which contradicts the size of the block.

Unfortunately, the argument above does not give a lower bound depending on ε. To achieve such a bound we will need to refine the previous construction, but first we need one more preliminary result.

The second Chebyshev function $\psi(x)$ is defined to be the logarithm of the least common multiple of all positive integers less or equal than x. The following bound on $\psi(x)$ will be useful for us.

Theorem 2.4. ([24], Theorem 7). *If $x\geq 10^8$, then $|\psi(x)-x|<cx/\log x$ for some positive constant c.*

In particular, Theorem 2.4 asserts that for sufficiently large n we have

$$\mathrm{lcm}(1,\ldots,n)=e^{n+O(n/\log n)}. \tag{4}$$

We are now ready to prove the lower bound of Theorem 1.3.

Theorem 2.5. *Let $r\geq 1$. There exists a positive constant ε_0 and a real number c_r depending on r such that the following holds. If $0<\varepsilon\leq\varepsilon_0$ and $k\geq 2^r r!\varepsilon^{-1}\log^r(1/5\varepsilon)$ is a integer, then there exist an integer $N:=N(\varepsilon,k,r)$ satisfying*

$$N\geq c_r\frac{k^r}{\varepsilon^{r-1}\log(1/\varepsilon)^{\binom{r+1}{2}-1}},$$

so that $[N]$ admits an r-colouring without monochromatic $\mathrm{AP}_k(\varepsilon)$.

Proof. The proof is by induction on the number of colours r. For $r=1$, the result clearly holds for $N(\varepsilon,k,1)=k-1$ since there is no $\mathrm{AP}_k(\varepsilon)$, or even AP_k, on $(k-1)$ terms. Now suppose that for any ε and k such that $0<\varepsilon\leq\varepsilon_0$ and $k\geq 2^{r-1}(r-1)!\varepsilon^{-1}\log^{r-1}(1/5\varepsilon)$, there exists $N(\varepsilon,k,r-1)$ and a $(r-1)$-colouring of $[N(\varepsilon,k,r-1)]$ satisfying the conclusion of the statement. We want to find an integer N_1 so that $[N_1]$ has a r-colouring without monochromatic $\mathrm{AP}_k(\varepsilon)$.

To do that we start with some choice of variables. Let

$$N_0=N\left(\varepsilon,\frac{k}{rs},r-1\right),\quad s=\frac{1}{0.9}\log(1/5\varepsilon),\quad w=\frac{e^{0.9s}}{s(r-1)!},\quad t=\frac{k}{2rs},\quad D_j=\frac{s-j+1}{s}N_0 \tag{5}$$

be integers for $1\leq j\leq s/2$. Note that although s, w, t and $\{D_j\}_{1\leq j\leq s/2}$ might not be integers, we prefer to write in this way, since it simplifies the exposition and has no significant effect on the arguments. Moreover, the integer N_0 always exists since by hypothesis

$$\frac{k}{rs}\geq\frac{2^r r!\varepsilon^{-1}\log^r(1/5\varepsilon)}{rs}\geq 2^{r-1}(r-1)!\varepsilon^{-1}\log^{r-1}(1/5\varepsilon).$$

[1]Strictly speaking we should use the set $\left[2\left\lfloor\frac{k-2}{3}\right\rfloor(k-1)\right]$, since $\frac{k-2}{3}$ is not necessarily an integer. However, during our exposition we will not bother with this type of detail since it has no significant effect on arguments or results.

Let $N_1 = rwt(D_1 + \ldots + D_{s/2})$. We are going to define a colouring $\varphi:[N_1] \to [r]$ not admitting monochromatic $\mathrm{AP}_k(\varepsilon)$. To this end we partition $[N_1]$ into consecutive intervals following the four steps below:

- First we partition $[N_1]$ into $[N_1] = Y_1 \cup \ldots \cup Y_w$, where Y_i are consecutive intervals and $|Y_i| = rt(D_1 + \ldots + D_{s/2})$ for every $i = 1, \ldots, w$.
- Each Y_i is partitioned into $Y_i = Y_{i,1} \cup \ldots \cup Y_{i,s/2}$, where $Y_{i,j}$'s are consecutive intervals and $|Y_{i,j}| = rtD_j$ for every $j = 1, \ldots, s/2$.
- Each $Y_{i,j}$ is partitioned into $Y_{i,j} = Z_1^{i,j} \cup \ldots \cup Z_t^{i,j}$, where $Z_u^{i,j}$'s are consecutive intervals and $|Z_u^{i,j}| = rD_j$ for every $u = 1, \ldots, t$.
- Each $Z_u^{i,j}$ is partitioned into $Z_u^{i,j} = Z_{u,1}^{i,j} \cup \ldots \cup Z_{u,r}^{i,j}$, where $Z_{u,v}^{i,j}$'s are consecutive intervals and $|Z_{u,v}^{i,j}| = D_j$ for every $v = 1, \ldots, r$.

More explicitly, we define

$$\begin{aligned}
\alpha_i &= (i-1)rt(D_1 + \ldots + D_{s/2}), \quad i \in [w] \\
\beta_{i,1} &= \alpha_i, \quad i \in [w] \\
\beta_{i,j} &= rt(D_1 + \ldots + D_{j-1}) + \alpha_i, \quad (i,j) \in [w] \times [2, s/2] \\
\gamma_{i,j,u} &= (u-1)rD_j + \beta_{i,j}, \quad (i,j,u) \in [w] \times [s/2] \times [t] \\
\sigma_{i,j,u,v} &= (v-1)D_j + \gamma_{i,j,u} \quad (i,j,u,w) \in [w] \times [s/2] \times [t] \times [r].
\end{aligned}$$

Therefore, our intervals can be written as:

$$\begin{aligned}
Y_i &= [\alpha_i + 1, \alpha_i + rt(D_1 + \ldots + D_{s/2})], \quad i \in [w] \\
Y_{i,j} &= [\beta_{i,j} + 1, \beta_{i,j} + rtD_j], \quad (i,j) \in [w] \times [s/2] \\
Z_u^{i,j} &= [\gamma_{i,j,u} + 1, \gamma_{i,j,u} + rD_j], \quad (i,j,u) \in [w] \times [s/2] \times [t] \\
Z_{u,v}^{i,j} &= [\sigma_{i,j,u,v} + 1, \sigma_{i,j,u,v} + D_j], \quad (i,j,u,v) \in [w] \times [s/2] \times [t] \times [r]
\end{aligned}$$

Finally, we describe the colouring $\varphi:[N_1] \to [r]$ on the intervals $Z_{u,v}^{i,j}$. By induction hypothesis, given any set C of $r-1$ colours there exists a colouring $\varphi_C:[N_0] \to C$ with no monochromatic $\mathrm{AP}_{k/rs}(\varepsilon)$. Fix $Z_{u,v}^{i,j}$ with $(i,j,u,v) \in [w] \times [s/2] \times [t] \times [r]$. We colour $Z_{u,v}^{i,j}$ by the same colouring as the first D_j elements of $[N_0]$ when $[N_0]$ is coloured by $\varphi_{[r]\setminus\{v\}}$. That is, the colouring φ restricted to $Z_{u,v}^{i,j}$ only uses $r-1$ colours and does not contain a monochromatic $\mathrm{AP}_{k/rs}(\varepsilon)$.

To prove that the colouring φ is free of $\mathrm{AP}_k(\varepsilon)$ we are going to show that there is no $a \in \mathbb{R}$ and $d > 0$ such that $\bigcup_{i=0}^{k-1} B(a + id, \varepsilon d)$ has a monochromatic transversal in $[N_1]$. Suppose the opposite and assume that there exists a and d such that $\bigcup_{i=0}^{k-1} B(a + id, \varepsilon d)$ has a monochromatic transversal $X = \{x_0, \ldots, x_{k-1}\} \subseteq [N_1]$ of colour $c \in [r]$. Since all the balls have radius εd, we obtain that $\{a, a+d, \ldots, a+(k-1)d\} \subseteq (1 - \varepsilon d, N_1 + \varepsilon d)$, which gives that $(k-1)d \leq (N_1 - 1) + 2\varepsilon d$. By (5) and by the fact that $\varepsilon \leq \varepsilon_0$ we have that

$$d \leq \frac{N_1 - 1}{k - 1 - 2\varepsilon} \leq \frac{2N_1}{k} = \frac{2rwt(D_1 + \ldots + D_{s/2})}{k} = \frac{wN_0}{s^2}\left(s + \ldots + \left(\frac{s}{2} + 1\right)\right) \leq \frac{wN_0}{2}, \tag{6}$$

for sufficiently small ε_0.

For a fixed $Y_{i,j} = \bigcup_{u=1}^{t} \bigcup_{v=1}^{r} Z^{i,j}_{u,v}$ we define an auxiliary labelling $\chi_{i,j} : Y_{i,j} \to \{-1, +1\}$ of $Y_{i,j}$ such that every D_j-block $Z^{i,j}_{u,v}$ is monochromatic and

$$\chi_{i,j}(Z^{i,j}_{u,v}) = \begin{cases} +1, & \text{if } v \neq c, \\ -1, & \text{if } v = c. \end{cases}.$$

In other words, every element of a D_j-block $Z^{i,j}_{u,v}$ is of label -1 if the colouring φ restricted to $Z^{i,j}_{u,v}$ has the same colouring of the first D_j elements of $\varphi_C : [N_0] \to C$, where $C = [r] \setminus \{c\}$, i.e., the set of colours missing the colour c. Otherwise, we label all the elements in $Z^{i,j}_{u,v}$ by $+1$. It is not difficult to check that $\chi_{i,j}$ is an $(r-1, 1; D_j)$-alternate labelling of $Y_{i,j}$. Moreover, since X is monochromatic of colour c and $Z^{i,j}_{u,c}$ is coloured by $\varphi_{[r]\setminus\{c\}}$, we obtain that $X \cap Z^{i,j}_{u,c} = \emptyset$. This implies that every element of $X \cap Y_{i,j}$ is labeled $+1$. Finally, in order to apply Lemma 2.2, we extend the labelling $\chi_{i,j}$ to the set of real numbers $(\beta_{i,j}, \beta_{i,j} + rtD_j]$ by labelling the entire interval $(\sigma_{i,j,u,v}, \sigma_{i,j,u,v} + D_j]$ by colour $\chi_{i,j}(Z^{i,j}_{u,v})$ for every $u, v \in [t] \times [r]$.

The main idea of the proof is based on the fact that for d not too small, there exists an index j_0 such that d is far from certain fractions involving D_{j_0}. We will then imply by Lemma 2.2 that the number of elements of X in Y_{i,j_0} is 'small'. It turns out that this fact is enough to restrict the entire location of X to just a few $Y_{i,j}$'s. Then by the pigeonhole principle and Proposition 2.3 we can show that there exists a D_j-block $Z^{i,j}_{u,v}$ with large intersection with X, which contradicts the inductive colouring of $Z^{i,j}_{u,v}$.

The next proposition elaborates more on the existence of such a j_0.

Proposition 2.6. *If $d > \frac{N_0}{s(r-1)!}$, then there exists index $1 \leq j_0 \leq s/2$ such that*

$$\left| d - \frac{mD_{j_0}}{(r-1)!} \right| \geq \frac{N_0}{2s(r-1)!}$$

for every $m \in \mathbb{Z}$.

Proof. Let $M_0 = \frac{N_0}{s(r-1)!}$. Note that by (5) we can write

$$\frac{D_j}{(r-1)!} = (s-j+1)\frac{N_0}{s(r-1)!} = (s-j+1)M_0,$$

for every $1 \leq j \leq s/2$. Therefore, every number of the form $\frac{mD_j}{(r-1)!}$ for $m \in \mathbb{Z}^+$ and $1 \leq j \leq s/2$ is a multiple of M_0. Moreover, the least non-zero common term among the sequences $\left\{\frac{mD_j}{(r-1)!}\right\}_{m \in \mathbb{Z}^+}$ for $1 \leq j \leq s/2$, i.e.,

$$\min \bigcap_{1 \leq j \leq s/2} \left\{ \frac{mD_j}{(r-1)!} : m \in \mathbb{Z}^+ \right\} = \min \bigcap_{1 \leq j \leq s/2} \left\{ m(s-j+1)M_0 : m \in \mathbb{Z}^+ \right\}$$

is equal to LM_0, where $L = \operatorname{lcm}(s/2+1, \ldots, s)$.

Since every number in $\{1, \ldots, s/2\}$ has a non-trivial multiple inside $\{s/2+1, \ldots, s\}$ we obtain by (4) that

$$L = \operatorname{lcm}(s/2+1, \ldots, s) = \operatorname{lcm}(1, \ldots, s) = e^{s+O(s/\log s)} \geq e^{0.9s},$$

for $s = \frac{1}{0.9}\log(1/5\varepsilon) \geq \frac{1}{0.9}\log(1/5\varepsilon_0)$ and ε_0 sufficiently small. Hence, by (5) and (6) we have

$$d \leq \frac{wN_0}{2} = \frac{N_0 e^{0.9s}}{2s(r-1)!} \leq \frac{LN_0}{2s(r-1)!} = \frac{L}{2}M_0.$$

Let pM_0 be the multiple of M_0 closest to d. Since $d > M_0$, we clearly have that $p \neq 0$. By definition,

$$pM_0 = \frac{pN_0}{s(r-1)!} \leq d + \left| \frac{pN_0}{s(r-1)!} - d \right| \leq d + \frac{M_0}{2} < LM_0.$$

Therefore, by the minimality of LM_0, there exists an index $1 \leq j_0 \leq s/2$ such that pM_0 is not a multiple of $\frac{D_{j_0}}{(r-1)!} = (s - j_0 + 1)M_0$. Since, by the definition of p, all the other numbers of the form mM_0 have distance at least $\frac{M_0}{2} = \frac{N_0}{2s(r-1)!}$ to d, Proposition 2.6 follows. □

We now prove that there exists a set $Y_{i,j}$ with a large proportion of elements of X.

Proposition 2.7. *There exist indices* $(i_1, j_1) \in [w] \times [s/2]$ *such that* $|X \cap Y_{i_1,j_1}| \geq k/s$.

Proof. Let $I \subseteq [w] \times [s/2]$ be set of pair of indices defined by

$$I = \left\{ (i,j) \in [w] \times [s/2] : X \cap Y_{i,j} \neq \emptyset \right\},$$

and let $\mathcal{Y} = \bigcup_{(i,j)\in I} Y_{i,j}$. By (5) and (6) we obtain that the difference between two consecutive terms of X is bounded by

$$|x_{h+1} - x_h| \leq (1+2\varepsilon)d \leq (1+2\varepsilon)\frac{e^{0.9s}N_0}{2s(r-1)!} < \frac{kN_0}{4s} \leq \frac{k(s-j+1)N_0}{2s^2} = rtD_j = |Y_{i,j}|,$$

for $k \geq 2^r r! \varepsilon^{-1} \log^r (1/5\varepsilon) \geq \varepsilon^{-1}/(r-1)!$. That is, the difference between two consecutive terms of X is smaller than the size of an interval $Y_{i,j}$ for $(i,j) \in [w] \times [s/2]$. This implies that all intervals in $\mathcal{Y}$ must be consecutive. Recall that by construction two intervals $Y_{i,j}$ and $Y_{i',j'}$ are consecutive if (i,j) and (i',j') are consecutive in the lexicographical ordering of $[w] \times [s/2]$.

If $|I| \leq 2$, then by the pigeonhole principle there exist indices (i_1, j_1) such that $|X \cap Y_{i_1,j_1}| \geq k/2 \geq k/s$ for ε_0 sufficiently small. Thus we may assume that $|I| > 3$. This implies that there exists at least one pair of indices (i', j') such that $Y_{i',j'}$ is neither the first or last interval of $\mathcal{Y}$.

Let $X \cap Y_{i',j'} = \{x_h, \ldots, x_{h+b-1}\}$, where $b = |X \cap Y_{i',j'}|$. Since $Y_{i',j'}$ is not one of intervals in the extreme of $\mathcal{Y}$, we obtain that $2 \leq h \leq h+b-1 \leq k-1$ and in particular there exists points x_{h-1} and x_{h+b} outside of $Y_{i',j'}$. Then a simple computation gives us that

$$|Y_{i',j'}| \leq |x_{h+b} - x_{h-1}| \leq (b+1+2\varepsilon)d < 2bd$$

and consequently

$$|X \cap Y_{i',j'}| = b > \frac{|Y_{i',j'}|}{2d} \tag{7}$$

for any $Y_{i',j'}$ not on the extremes of $\mathcal{Y}$.

We split the proof into two cases depending on the size of d. If $d \leq \frac{N_0}{s(r-1)!}$, then (5) and (7) give that

$$|X \cap Y_{i',j'}| > \frac{|Y_{i',j'}|}{2d} = \frac{rtD_{j'}}{2d} \geq \frac{k(s-j'+1)(r-1)!}{4s} \geq \frac{k(r-1)!}{8} \geq \frac{k}{s}$$

for every $Y_{i',j'}$ not on the extremes and sufficiently large s. Taking (i_1, j_1) as one such (i', j') gives the desired result.

Now suppose that $d > \frac{N_0}{s(r-1)!}$. Let j_0 be the index provided by Proposition 2.6. In particular, it holds that

$$\left| d - \frac{mrD_{j_0}}{q} \right| \geq \frac{N_0}{2s(r-1)!} \tag{8}$$

for every $m \in \mathbb{Z}$ and $1 \le q \le r$. Suppose that $X \cap Y_{i,j_0} \neq \emptyset$ for some $1 \le i \le w$. Our goal is to apply Lemma 2.2 with $D = D_{j_0}$, $\delta = 1/4sr!$ to the interval $(\min(Y_{i,j_0}) - 1, \max(Y_{i,j_0})] = (\beta_{i,j_0}, \beta_{i,j_0} + rtD_j]$ labeled with our extension of χ_{i,j_0}. In order to verify the assumptions of the lemma note that

$$\frac{N_0}{2s(r-1)!} = \frac{D_{j_0}}{2(s-j_0+1)(r-1)!} \ge \frac{D_{j_0}}{2s(r-1)!} > \delta r D_{j_0}$$

and therefore by (8) we have

$$d \notin \bigcup_{m\in\mathbb{Z}} \bigcup_{q=1}^{r} \left(\left(\frac{m}{q} - \delta\right) rD_{j_0}, \left(\frac{m}{q} + \delta\right) rD_{j_0} \right).$$

Consequently, the conclusion of the lemma gives to us that any arithmetic progression of intervals of radius $\delta r D_{j_0}$ with common difference d and a monochromatic transversal of label $+1$ inside the interval $(\min(Y_{i,j_0}) - 1, \max(Y_{i,j_0})]$ has length bounded by $3r/\delta$. This is true in particular for $\bigcup_{i=0}^{k-1} B(a + id, \varepsilon d)$, since by (5) and (6) we have

$$\varepsilon d \le \frac{\varepsilon w N_0}{2} = \frac{N_0}{10s(r-1)!} = \frac{D_{j_0}}{10(s-j_0+1)(r-1)!} \le \frac{D_{j_0}}{5s(r-1)!} < \delta r D_{j_0}.$$

Hence, because X is transversal of label $+1$ of $\bigcup_{i=0}^{k-1} B(a + id, \varepsilon d)$, the conclusion of Lemma 2.2 gives for $k \ge 2^r r! \varepsilon^{-1} \log^r(1/5\varepsilon) > \frac{32}{3} r^2 \varepsilon^{-1} \log(1/5\varepsilon)$ that

$$|X \cap Y_{i,j_0}| \le \frac{3r}{\delta} = 12sr!r = \frac{40}{3} r!r \log(1/5\varepsilon) < \frac{5}{4}\varepsilon(r-1)!k. \tag{9}$$

However, by (5), (6) and (7) we have

$$|X \cap Y_{i',j'}| > \frac{|Y_{i',j'}|}{2d} = \frac{rtD_{j'}}{2d} \ge \frac{1}{wN_0} \cdot \frac{k(s-j'+1)N_0}{2s^2} \ge \frac{k}{4ws} = \frac{5}{4}\varepsilon(r-1)!k \tag{10}$$

for any $Y_{i',j'}$ in the middle of $\mathcal{Y}$. Comparing (9) and (10) yields that $|X \cap Y_{i,j_0}| < |X \cap Y_{i',j'}|$ for any interval $Y_{i',j'}$ in the middle of $\mathcal{Y}$. Thus Y_{i,j_0} cannot be a middle interval and we obtain that if $(i, j_0) \in I$, then Y_{i,j_0} is either the first or last interval of $\mathcal{Y}$. Therefore, we can have at most two occurrences of j_0 in I and consequently the entire location of I is contained between those two occurrences, i.e., $I \subseteq \{(i, j_0), (i, j_0 + 1), \ldots, (i + 1, j_0 - 1), (i + 1, j_0)\}$ for some $1 \le i \le w - 1$. Hence, the set I has at most $s/2 + 1$ elements and by the pigeonhole principle there exists a pair of indices $(i_1, j_1) \in I$ such that $|X \cap Y_{i_1,j_1}| \ge k/(s/2 + 1) \ge k/s$. □

Let (i_1, j_1) be the indices given by Proposition 2.7. Next we apply Proposition 2.3 to the set Y_{i_1,j_1} labeled by χ_{i_1,j_1} with $D = D_{j_1}$, $\ell = k/s$ and ε-approximate arithmetic progression $X \cap Y_{i_1,j_1}$. Note that by (5) the hypothesis concerning r, t and ℓ in the statement holds since

$$t(r+1) + 2 = \frac{(r+1)k}{2rs} + 2 < \frac{k}{s} = \ell$$

for $r \ge 2$ and $k \ge 2^r r! \varepsilon^{-1} \log^r(1/5\varepsilon) \ge 80\log(1/5\varepsilon)/9$. Also a D_j-block of Y_{i_1,j_1} is an interval of the form $Z_{u,v}^{i_1,j_1}$. Hence, by the conclusion of the proposition, there exists $Z_{u,v}^{i_1,j_1}$ such that $|X \cap Z_{u,v}^{i_1,j_1}| \ge \ell/(r-1) > k/rs$. Since each set $Z_{u,v}^{i,j}$ was $(r-1)$-coloured inductively not to contain an $AP_{k/rs}(\varepsilon)$, we reach a contradiction. Thus there is no monochromatic $AP_k(\varepsilon)$ in $[N_1]$. In view of (5) we have

$$N_1 = rwt(D_1 + \ldots + D_{s/2}) = \frac{ke^{0.9s}N_0}{2s^3(r-1)!}\left(s + \ldots + \left(\frac{s}{2}+1\right)\right)$$
$$\geq \frac{kN_0}{40\varepsilon s(r-1)!} \geq \frac{kN_0}{50(r-1)!\varepsilon \log(1/\varepsilon)}.$$

Consequently, in view of $s = O(\log(1/5\varepsilon))$ we obtain by induction that

$$N_1 \geq \frac{k}{50(r-1)!\varepsilon \log(1/\varepsilon)} \cdot \frac{c'_r\left(\frac{k}{rs}\right)^{r-1}}{\varepsilon^{r-2}\log(1/\varepsilon)^{\binom{r}{2}-1}} \geq c_r \frac{k^r}{\varepsilon^{r-1}\log(1/\varepsilon)^{\binom{r+1}{2}-1}}. \qquad \square$$

3. Proof of Theorem 1.7

3.1 Lower bound

For positive integers k and N, recall that $f(N, 1, k)$, sometimes denoted by $r_k(N)$, is defined to be the size of the largest set $A \subseteq [N]$ without an arithmetic progression of length k. A classical result of Behrend [1] shows that,

$$f(N, 1, 3) > N \exp\left(-c\sqrt{\log N}\right),$$

for a positive constant c (see [5, 15] for slightly improvements). In [22] (See also [18]) the argument was generalised to yield that

$$f(N, 1, k) > N \exp\left(-c(\log N)^{1/\ell}\right), \tag{11}$$

where $\ell = \lceil \log_2 k \rceil$ and $k \geq 3$ and c is a constant depending only on k. We will use the last result as a building block for our construction.

Before we turn our attention to the lower bound construction, we will state a preliminary result about ε-approximate arithmetic progressions. Given a set of k integers, one can identify them as an AP_k by the common difference between the elements. Unfortunately, the same is not true for an $AP_k(\varepsilon)$. On the positive side, the next result shows that if a set of k elements is an $AP_k(\varepsilon)$, then the differences of consecutive terms are almost equal.

Proposition 3.1. *Given $0 < \varepsilon < 1/10$, let $X = \{x_0, \ldots, x_{k-1}\}$ be an $\mathrm{AP}_k(\varepsilon)$. Then for every pair of indices $0 \leq i, j \leq k-2$ the following holds*

$$\left|\frac{|x_{j+1} - x_j|}{|x_{i+1} - x_i|} - 1\right| < 5\varepsilon.$$

Proof. Since X is an $\mathrm{AP}_k(\varepsilon)$, there exist a and d such that $|x_i - (a + id)| < \varepsilon d$ for $0 \leq i \leq k-1$. Therefore, a simple computation shows that

$$1 - 5\varepsilon < \frac{(1-2\varepsilon)d}{(1+2\varepsilon)d} < \frac{|x_{j+1} - x_j|}{|x_{i+1} - x_i|} < \frac{(1+2\varepsilon)d}{(1-2\varepsilon)d} < 1 + 5\varepsilon$$

for $0 < \varepsilon < 1/10$ and $0 \leq i, j \leq k-2$. □

We now prove the lower bound of Theorem 1.3 for one dimension.

Lemma 3.2. *Let $k \geq 3$ and $0 < \varepsilon \leq 1/125$. Then there exists a positive constant c_1 depending only on k and an integer $N_0 := N_0(k, \varepsilon)$ such that the following holds. If $N \geq N_0$, then there exists a set $A \subseteq [N]$ without $AP_k(\varepsilon)$ such that*

$$|A| \geq N^{1-c_1(\log(1/\varepsilon))^{\frac{1}{\ell}-1}}$$

for $\ell = \lceil \log_2 k \rceil$.

Proof. For integers a, b, let $S_k([a, b])$ be the largest subset in the interval $[a, b]$ without any arithmetic progression AP_k of length k. By a simple translation, one can note that $S_k([a, b])$ has the same size as $S_k([b-a+1])$ and by (11) we have

$$|S_k([a, b])| = f(b-a+1, 1, k) \geq (b-a+1)\exp\left(-c(\log(b-a+1))^{1/\ell}\right), \tag{12}$$

for a positive constant c and $\ell = \lceil \log_2 k \rceil$.

Let $q = \frac{1}{25\varepsilon} \geq 5$ be an integer and h be largest exponent such that $q^h \leq N < q^{h+1}$. For such a choice of q and h, we construct the set

$$A = \left\{s \in [N] : a = s_0 + s_1 q + \ldots + s_{h-1}q^{h-1}\right\},$$

where $s_{h-1} \in S_k([0, q-1])$ and $s_i \in S_k\left(\left[2q/5, 3q/5\right]\right)$ for $0 \leq i \leq h-2$. Our goal is to show that A satisfies the conclusion of Lemma 3.2.

First note by (12) that

$$\begin{aligned}
|A| &= |S_k([0, q-1])| \cdot \left|S_k\left(\left[2q/5, 3q/5\right]\right)\right|^{h-1} \\
&\geq \frac{q}{\exp(c(\log q)^{1/\ell})} \cdot \left(\frac{q}{5\exp(c(\log q/5)^{1/\ell})}\right)^{h-1} \\
&\geq \frac{q^h}{\exp(c(\log q)^{1/\ell})(5\exp(c(\log q)^{1/\ell}))^{h-1}} \\
&\geq \frac{N}{5^{h-1}q\exp(c(\log q)^{1/\ell})^h} \geq \frac{N}{q\exp(c'h(\log q)^{1/\ell})},
\end{aligned}$$

and in view of $h \leq \frac{\log N}{\log q}$ and our choice of q we obtain that

$$|A| \geq \frac{20\varepsilon N}{\exp\left(c'\log N(\log q)^{1/\ell-1}\right)} \geq N^{1-c_1\log(1/\varepsilon)^{1/\ell-1}}$$

for sufficiently large N and appropriate constant c_1 depending only on k. Therefore the set A has the desired size. It remains to prove that A is $AP_k(\varepsilon)$-free.

Suppose that there exists an ε-approximate arithmetic progression $X = \{x_0, \ldots, x_{k-1}\}$ in A. For each $0 \leq i \leq k-1$, write $x_i = \sum_{j=0}^{h-1} x_{i,j}q^j$. Since all x_i's are distinct, there exists a maximal index j_0 such that the elements of $X_{j_0} = \{x_{i,j_0}\}_{0 \leq i \leq k-1}$ are not all equal. By construction of A the set X_{j_0} fails to be an AP_k. Therefore there exists two indices $0 \leq i_1, i_2 \leq k-2$ such that

$$|x_{i_1+1,j_0} - x_{i_1,j_0}| \neq |x_{i_2+1,j_0} - x_{i_2,j_0}|. \tag{13}$$

For $0 \leq i \leq k-1$, note that

$$|x_{i+1} - x_i| = \left|\sum_{j=0}^{h-1}(x_{i+1,j} - x_{i,j})q^j\right| = \left|\sum_{j=0}^{j_0}(x_{i+1,j} - x_{i,j})q^j\right|$$

by the maximality of j_0. Thus by the triangle inequality we obtain that

$$\left||x_{i+1} - x_i| - |x_{i+1,j_0} - x_{i,j_0}|q^{j_0}\right| \leq \sum_{j=0}^{j_0-1}|x_{i+1,j} - x_{i,j}|q^j. \tag{14}$$

Moreover, recalling that $x_{i,j} \in \left[2q/5, 3q/5\right]$ for $0 \le j \le h-2$ we infer that

$$\sum_{j=0}^{j_0-1} |x_{i+1,j} - x_{i,j}| q^j \le \sum_{j=0}^{j_0-1} \frac{q^{j+1}}{5} \le \frac{2q^{j_0}}{5}$$

for $q \ge 2$. The last inequality combined with (14) gives us that

$$\Big| |x_{i+1} - x_i| - |x_{i+1,j_0} - x_{i,j_0}| q^{j_0} \Big| \le \frac{2}{5} q^{j_0}, \tag{15}$$

for $0 \le i \le k-2$. Hence by (13) we have that

$$\begin{aligned} \Big| |x_{i_2+1} - x_{i_2}| - |x_{i_1+1} - x_{i_1}| \Big| &\ge \Big| |x_{i_2+1,j_0} - x_{i_2,j_0}| - |x_{i_1+1,j_0} - x_{i_1,j_0}| \Big| q^{j_0} - \frac{4}{5} q^{j_0} \\ &\ge q^{j_0} - \frac{4}{5} q^{j_0} = \frac{q^{j_0}}{5} \end{aligned} \tag{16}$$

On the other hand, Proposition 3.1 for i_1 and i_2 together with (15) gives us that

$$\Big| |x_{i_2+1} - x_{i_2}| - |x_{i_1+1} - x_{i_1}| \Big| < 5\varepsilon |x_{i_1+1} - x_{i_1}| < 5\varepsilon \left(|x_{i_1+1,j_0} - x_{i_1,j_0}| + \frac{2}{5} \right) q^{j_0}.$$

Since $x_{i,j_0} \in [0, q-1]$ for every $0 \le i \le k-1$ and $\varepsilon q = 1/25$ we have

$$\Big| |x_{i_2+1} - x_{i_2}| - |x_{i_1+1} - x_{i_1}| \Big| < 5\varepsilon q^{j_0+1} = \frac{q^{j_0}}{5},$$

which contradicts (16). □

For higher dimensions the result follow as a corollary of Lemma 3.2. Recall by Definition 1.6 that an ε-approximate cube $C_\varepsilon(m,k)$ is just an multidimensional version of an $\mathrm{AP}_k(\varepsilon)$

Corollary 3.3. *Let $k \ge 3$ and $m \ge 1$ be integers and $0 < \varepsilon \le 1/125$. Then there exists an integer $N_0 := N_0(k, \varepsilon)$ and a positive constant c_1 depending on k such that the following holds. If $N \ge N_0$, then there exists a set $S \subseteq [N]^m$ without $C_\varepsilon(m,k)$ such that*

$$|S| \ge N^{m - c(\log(1/\varepsilon))^{\frac{1}{\ell} - 1}}$$

for $\ell = \lceil \log_2(k-1) \rceil$.

Proof. Let N_0 be the integer given by Lemma 3.2 and let $A \subseteq [N]$ be the set such that A has no $\mathrm{AP}_k(\varepsilon)$ for $N \ge N_0$. Set $S = A \times [N]^{m-1}$, i.e., $S = \{(s_1, \ldots, s_m) : s_1 \in A,\ s_2, \ldots, s_m \in [N]\}$. Note that S has the desired size since

$$|S| = N^{m-1}|A| \ge N^{m - c(\log(1/\varepsilon))^{\frac{1}{\ell} - 1}}.$$

We claim that S is free of $C_\varepsilon(m,k)$.

Suppose that the claim is not true and let $X = \{x_{\vec{v}} : \vec{v} \in \{0, \ldots, k-1\}^m\}$ be an $C_\varepsilon(m,k)$ in S. By definition there exists $\vec{a} \in \mathbb{R}^m$ and $d > 0$ such that $||x_{\vec{v}} - (\vec{a} + d\vec{v})|| < \varepsilon d$ for every $\vec{v} \in \{0, \ldots, k-1\}^m$. In particular, when applied to $\{te_1 = (t, 0, \ldots, 0) : 0 \le t \le k-1\}$ the observation gives us that

$$|x_{te_1,1} - (a_1 + td)| \le \left((x_{te_1,1} - (a_1 + dt))^2 + \sum_{i=2}^{m} (x_{te_1,i} - a_i)^2 \right)^{1/2} = ||x_{te_1} - (\vec{a} + dte_1)|| < \varepsilon d$$

Therefore, the set $\{x_{te_1,1}\}_{0 \le t \le k-1} \subseteq A$ is an $\mathrm{AP}_k(\varepsilon)$, which contradicts our choice of A. □

3.2 Upper bound

As in the upper bound of $W_\varepsilon(k, r)$, our proof of the upper bound of $f_\varepsilon(N, m, k)$ will use an iterative blow-up construction. It is worth to point out that a similar proof was obtained independently by Dumitrescu in [4]. While both proofs use a blow-up construction, the author of [4] finishes the proof with a packing argument. Here we will follow the approach of [16, 17], which uses an iterative blow-up construction combined with an average argument to estimate the largest subset of a grid without a class of configurations of a given size. This approach allows us to slightly improve the constants in the result.

The proof is split into two auxiliary lemmas.

Lemma 3.4. *Given positive real numbers* α, $\varepsilon > 0$ *and integers* $m \geq 1$ *and* $k \geq 3$, *there exists* $N_0 := N_0(\alpha, \varepsilon, m, k) \leq \left(k\sqrt{m}/\varepsilon\right)^{2k^m \log(1/\alpha)}$ *and a subset* $A \subseteq [N]^m$ *with the property that any* $X \subseteq A$, $|X| \geq \alpha|A|$ *contains a* $C_\varepsilon(m, k)$.

Proof. For m and k, let Δ be the standard cube $C(m, k)$ of dimension m over $\{0, \ldots, k-1\}$, i.e., Δ is the set of all m-tuples $\vec{v} = \{v_1, \ldots, v_m\} \in \{0, \ldots, k-1\}^m$. Viewing Δ as an m-dimensional lattice in the Euclidean space, we note that $\mathrm{diam}(\Delta) = (k-1)\sqrt{m}$, while the minimum distance between two vertices in Δ is one.

Similarly as in the proof of the upper bound of Theorem 1.3, we consider an iterated blow-up of the cube. For integers r and $t = k\sqrt{m}/\varepsilon$, let A_r be the following r-iterated blow-up of a cube

$$A_r = \left\{\vec{v}_0 + t\vec{v}_1 + \ldots + t^{r-1}\vec{v}_{r-1} : \vec{v}_0, \ldots, \vec{v}_{r-1} \in \Delta,\ t = \frac{k\sqrt{m}}{\varepsilon}\right\}.$$

Alternatively, we can view A_r as the product $\prod_{i=1}^m B_r^{(i)}$ of m identical copies of

$$B_r = \left\{b_0 + tb_1 + \ldots + t^{r-1}b_{r-1} : (b_0, \ldots, b_{r-1}) \in \{0, 1 \ldots, k-1\}^r,\ t = \frac{k\sqrt{m}}{\varepsilon}\right\},$$

an r-iterated blow-up of the standard AP_k. Note by the construction that $|A_r| = k^{rm}$. The next proposition shows that fixed $\alpha > 0$, for a sufficiently large r any α-proportion of A_r will contain a $C_\varepsilon(m, k)$.

Proposition 3.5. *Let* $0 < \alpha < 1$ *be a real number and* r *a positive integer such that* $\alpha > \left(\frac{k^m-1}{k^m}\right)^r$. *Then every* $X \subseteq A_r$ *with* $|X| \geq \alpha|A_r|$ *contains a* $C_\varepsilon(m, k)$.

Proof. The proof is by induction on r. If $r = 1$, then $A_1 = \Delta$ and $\alpha > \frac{k^m-1}{k^m}$. Let $X \subseteq A_1$ with $|X| \geq \alpha|A_1|$. Thus

$$|X| \geq \alpha|A_1| > \frac{k^m - 1}{k^m} \cdot k^m = k^m - 1,$$

which implies that $X = \Delta$. So X contains a cube $C(k, m)$ and in particular an ε-approximate cube.

Now suppose that the proposition is true for $r - 1$ and we want to prove it for r. First, we partition A_r into $\bigcup_{\vec{u}\in\Delta} A_{r,\vec{u}}$, where

$$A_{r,\vec{u}} = \left\{\vec{v}_0 + t\vec{v}_1 + \ldots + t^{r-2}\vec{v}_{r-2} + t^{r-1}\vec{u} : \vec{v}_0, \ldots, \vec{v}_{r-2} \in \Delta,\ t = \frac{k\sqrt{m}}{\varepsilon}\right\}.$$

Note that by definition $A_{r,\vec{u}}$ is a translation of A_{r-1} by $t^{r-1}\vec{u}$. In particular, this implies that $|A_{r,\vec{v}}| = k^{(r-1)m}$. Let $X \subseteq A_r$ with $|X| \geq \alpha|A_r|$ be given. We will distinguish two cases:

Case 1: $X \cap A_{r,\vec{u}} \neq \emptyset$ for all $\vec{u} \in \Delta$.

For each $\vec{u} \in \Delta$ choose an arbitrary vector $w(\vec{u}) \in X \cap A_{r,\vec{u}}$. We will observe that $\{w(\vec{u})\}_{\vec{u}\in\Delta}$ forms a $C_\varepsilon(m,k)$. To testify that, set $\vec{a} = (0, \ldots, 0)$ and $d = t^{r-1}$. Write $w(\vec{u}) = \sum_{i=0}^{r-2} t^i \vec{w}_i + t^{r-1}\vec{u}$ with $\vec{w}_i \in \Delta$. Thus, a computation shows that

$$||w(\vec{u}) - (\vec{a} + d\vec{u})|| = ||w(\vec{u}) - t^{r-1}\vec{u}|| = \left|\left|\sum_{i=0}^{r-2} t^i \vec{w}_i\right|\right| \leq \sum_{i=0}^{r-2} t^i ||\vec{w}_i||$$

for $\vec{w}_0, \ldots, \vec{w}_{r-2} \in \Delta$. Since $\mathrm{diam}(\Delta) = (k-1)\sqrt{m}$, it follows that

$$||w(\vec{u}) - (\vec{a} + d\vec{u})|| \leq (k-1)\sqrt{m}\left(\sum_{i=0}^{r-2} t^i\right) \leq kt^{r-2}\sqrt{m} < \varepsilon t^{r-1} = \varepsilon d,$$

by our choice of t. Since $\{w(\vec{u})\}_{\vec{u}\in\Delta} \subseteq X$, we conclude that X contains an $C_\varepsilon(m,k)$.

Case 2: There exists $\vec{u}_0 \in \Delta$ with $X \cap A_{r,\vec{u}_0} = \emptyset$.

Since $|X| \geq \alpha|A_r|$ and $|\Delta| = k^m$, by an average argument there exists $\vec{u}_1 \in \Delta$ such that

$$|X \cap A_{r,\vec{u}_1}| \geq \frac{\alpha|A_r|}{k^m - 1} = \frac{\alpha k^m |A_{r-1}|}{k^m - 1}.$$

Set $X' = X \cap A_{r,\vec{u}_1}$ and $\alpha' = \frac{\alpha k^m}{k^m - 1}$. Note that

$$\alpha' = \frac{\alpha k^m}{k^m - 1} > \left(\frac{k^m - 1}{k^m}\right)^r \cdot \frac{k^m}{k^m - 1} = \left(\frac{k^m - 1}{k^m}\right)^{r-1}.$$

Therefore, viewing $A_{r,\vec{u}}$ as a copy of A_{r-1} by the induction assumption we obtain that $X' \subseteq X$ contains an $C_\varepsilon(m,k)$. □

Let r be the smallest integer such that $\left(\frac{k^m-1}{k^m}\right)^r < \alpha$ and set $A = A_r$. A computation shows that

$$r = \left\lceil \frac{\log(1/\alpha)}{\log \frac{k^m}{k^m-1}} \right\rceil < 2k^m \log(1/\alpha).$$

Therefore by Proposition 3.5 we have that any set $X \subseteq A$ with $|X| \geq \alpha|A|$ contains an $C_\varepsilon(m,k)$. Finally, by the construction of A we have that $A \subseteq [N_0]^m$ for

$$N_0 \leq \mathrm{diam}(B_r) + 1 = (k-1)(1 + t + \ldots + t^{r-1}) + 1 \leq kt^{r-1} \leq \left(\frac{k\sqrt{m}}{\varepsilon}\right)^{2k^m \log(1/\alpha)}. \qquad \square$$

Lemma 3.4 gives us a set $A \subseteq [N]^m$ such that any α-proportion contains a $C_\varepsilon(m,k)$. However, this is still not good enough, since to obtain an upper bound we need a similar result for $[N]^m$. The next lemma shows by an average argument that the property of A can be extended to $[N]^m$ by losing a factor of a power of two in the proportion α.

Lemma 3.6. *Let $A \subseteq [N]^m$ be a configuration in the grid. For any $X \subseteq [N]^m$ with $|X| \geq \alpha N^m$, there exists a translation A' of A such that $|X \cap A'| \geq \frac{\alpha}{2^m}|A'|$.*

Proof. Consider a random translation $A' = A + \vec{u}$, where $\vec{u} = (u_1, \ldots, u_m)$ is an integer vector chosen uniformly inside $[-N+1, N]^m$. For every vector $\vec{x} \in X$, there exists exactly $|A|$

elements $\vec{v} \in [-N+1, N]^m$ such that $\vec{x} - \vec{v} \in A$. This means that $\mathbb{P}(\vec{x} \in A') = \mathbb{P}(\vec{x} - \vec{u} \in A) = \frac{|A|}{(2N)^m}$. Therefore

$$\mathbb{E}(|X \cap A'|) = \sum_{\vec{x} \in X} \mathbb{P}(\vec{x} \in A') = \frac{|X||A|}{(2N)^m} \geq \frac{\alpha}{2^m}|A| = \frac{\alpha}{2^m}|A'|$$

Consequently, by the first moment method, there is $\vec{u}$ and A' satisfying our conclusion. □

We finish the section putting everything together.

Proposition 3.7. *Let N, m and k be integers and* $\varepsilon > 0$*. Then there exists a positive constant* c_2 *depending only on k and m such that the following holds. If* $S \subseteq [N]^m$ *is such that*

$$|S| > N^{m - c_2(\log(1/\varepsilon))^{-1}},$$

then S contains an $C_\varepsilon(m, k)$*.*

Proof. Set $\alpha_0 = 2^m N^{-c'(\log(1/\varepsilon))^{-1}}$ where $c' = (4k^m \log(k\sqrt{m}))^{-1}$. Let $N_0 = N_0(\alpha_0/2^m, \varepsilon, m, k)$ be the integer obtained by Lemma 3.4 and $A \subseteq [N_0]$ be the set such that any $X \subseteq A$ with $|X| \geq \frac{\alpha_0}{2^m}|A|$ contains an $C_\varepsilon(m, k)$. Note that

$$N_0 \leq \left(\frac{k\sqrt{m}}{\varepsilon}\right)^{2k^m \log(2^m/\alpha_0)} = \exp\left(\frac{2c'k^m \log N \log(k\sqrt{m}/\varepsilon)}{\log(1/\varepsilon)}\right)$$
$$\leq \exp\left(4c'k^m \log N \log(k\sqrt{m})\right) = N,$$

which implies that $A \subseteq [N]$.

Let $S \subseteq [N]$ with $|S| \geq \alpha_0 N^m$. Then by Lemma 3.6, there exists a translation A' of A such that $|S \cap A'| \geq \frac{\alpha_0}{2^m}|A'|$. Hence, by Lemma 3.4, the set S contains a $C_\varepsilon(m, k)$. The result now follows since

$$|S| \geq \alpha_0 N^m = 2^m N^{m - c'(\log(1/\varepsilon))^{-1}} > N^{m - c_2(\log(1/\varepsilon))^{-1}}$$

for appropriate c_2. □

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