

Floating and Illumination Bodies for Polytopes: Duality Results*

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Abstract: We consider the question how well a floating body can be approximated by the polar of the illumination body of the polar. We establish precise convergence results in the case of centrally symmetric polytopes. This leads to a new affine invariant which is related to the cone measure of the polytope.

Key words and phrases: floating body, illumination body

1 Introduction

Floating bodies are a fundamental notion in convex geometry. Early notions of floating bodies are motivated by the physical description of floating objects. The first systematic study of floating bodies appeared 1822 in a work by C. Dupin [7] on naval engineering. Blaschke, in dimensions 2 and 3, and later Leichtweiss in higher dimensions, used the floating body in the study of affine differential geometry, in particular affine surface area (see [3], [10]). Affine surface area is among the most powerful tools in convexity. It is widely used, for instance in approximation of convex bodies by polytopes, e.g., [4, 16, 21] and affine differential geometry, e.g., [1, 9, 22, 23].

Dupin's floating body may not exist and related to that, originally affine surface area was only defined for sufficiently smooth bodies. Schütt and Werner [20] and independently Bárány and Larman [2] introduced the convex floating body which, in contrast to Dupin's original definition, always exists and coincides with Dupin's floating body if the latter exists. This, in turn, allowed to define affine surface area for all convex bodies as carried out in [20].

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The illumination body was introduced much later in [24] as a further tool to study affine properties of convex bodies. It was pointed out in [26] that the convex floating body and the illumination body are dual notions in the sense that the polar of the floating body and the illumination body of the polar should be close both, in a conceptual and geometric sense. In [13] the duality relation is studied in detail for C_+^2 -bodies, i.e. bodies with twice differentiable boundary and everywhere positive curvature, and ℓ_p^n -balls. The paper provides asymptotic sharp estimates how well the polar of the floating body can be approximated by an illumination body of the polar. A limiting procedure leads to a new affine invariant that is different from the affine surface area. It is related to the cone measure of the convex body. These measures play a central role in many aspects of convex geometry, e.g., [5, 6, 14, 15].

The purpose of this paper is to make the duality relation between floating body and illumination body precise when the convex body is a polytope P . It was shown by Schütt [19] that the limit of the (appropriately normalized) volume difference of a polytope P and its floating body leads to a quantity related to the combinatorial structure of the polytope, namely the flags of P (see section 5). Likewise, as shown in [25], the limit of the (appropriately normalized) volume difference of a simplex and its illumination body is related to the combinatorics of the boundary. Now, as in the smooth case [13], a limit procedure leads to a new affine invariant that is not related to the combinatorial structure of the boundary of the polytope, but, as in the smooth case, to cone measures.

The techniques in [13] rely on comparing “extremal” directions, i.e., directions where the boundary of the convex body and its floating body, and the illumination body of its polar, differ the most and the least. The techniques used in [13] employ tools from differential geometry which is possible due to the C_+^2 smoothness assumptions. Such tools are no longer available in our present setting of polytopes and we have to use a completely different approach.

It would be interesting to have results for more general classes of convex bodies other than polytopes and the ones investigated in [13]. A major obstruction is that in general it is hard to compute the polar body and even harder to compute the floating body if we do not have smoothness assumptions or are in the case of polytopes. The smooth case [13] and the polytope case seem to be the extremal cases. Indeed, ellipsoids are C_+^2 -bodies where equality of the polar of the floating body and the illumination body of the polar can be achieved. And we show here that polytopes display the worst behavior one can expect. Furthermore, the limit is not continuous with respect to the polytope involved since it depends only on the local structure of the boundary (see section 6).

The paper is organized as follows. In the next section we present the main theorem and some consequences. In Section 3 we give the necessary background material and several lemmas needed for the proof of the main theorem. In section 5 we discuss properties of the new affine invariant. We show, with an example, that it is not continuous with respect to the Hausdorff distance. We also show that for this invariant the combinatorial structure of the polytope is less relevant. The relation to the cone measures is the dominant feature. In the final section we address questions of approximation of the floating body by the polar of the illumination of the polar. We show that our convergence results are of pointwise nature and we derive a uniform upper bound for general convex bodies.

2 Main theorem and consequences

We work in a similar framework as in [13]. We first recall the notions and definitions that we will need. Let K be a n -dimensional convex body and $\delta \geq 0$. The convex floating body K_δ of K was introduced by Schütt and Werner [20] and independently by Barany and Larman [2] as

$$K_\delta = \bigcap_{|K \cap H^-|_n \leq \delta |K|_n} H^+, \quad (2.1)$$

where H is a hyperplane and H^+, H^- are the corresponding closed half-spaces and $|K|_n$ is the volume of K .

We denote by $\text{conv}[A, B]$ the convex hulls of two sets $A, B \subseteq \mathbb{R}^n$. If $B = \{x\}$ we simply write $\text{conv}[A, x]$ for the convex hull of A and the vector x .

The illumination body K^δ of K was defined by E. Werner [24] as

$$K^\delta = \{x \in \mathbb{R}^n : |\text{conv}[K, x]|_n \leq (1 + \delta)|K|_n\}. \quad (2.2)$$

If 0 is in the interior of K , the polar K° of K is given by

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \text{ for all } x \in K\}. \quad (2.3)$$

It is a simple fact of convex geometry that for a hyperplane H with corresponding halfspaces H^+, H^- there is a corresponding point $x_H \in \mathbb{R}^n$ such that

$$(K \cap H^+)^\circ = \text{conv}[K^\circ, x_H],$$

whenever 0 is in the interior of $K \cap H^+$. As noted in [26], this polarity relation gives rise to the idea that cutting off with hyperplanes sets of a certain volume of a convex body and including points such that their convex hull with the convex body has a certain volume, should be dual operations,

$$(K_\delta)^\circ \approx (K^\circ)^{\delta'}. \quad (2.4)$$

In the same paper it is pointed out that equality cannot be achieved in general since the floating body is always strictly convex and the illumination body of a polytope is always a polytope.

As in [13] we like to measure how “close” these two bodies are in the polytope case. A further outcome of such a study shows how well the floating body of a polytope can be approximated by a polytope, namely the polar of an illumination of the polar. For $x \in \mathbb{R}^n \setminus \{0\}$ and K a convex body with $0 \in K$ we denote by $r_K(x) = \sup\{\lambda \geq 0 : \lambda x \in K\}$ the radial function of K . To measure how close two centrally symmetric convex bodies S_1, S_2 are, we use the distance

$$d(S_1, S_2) = \sup_{u \in \mathbb{S}^{n-1}} \max \left[\frac{r_{S_1}(u)}{r_{S_2}(u)}, \frac{r_{S_2}(u)}{r_{S_1}(u)} \right] = \inf \left\{ a \geq 1 : \frac{1}{a} S_1 \subseteq S_2 \subseteq a S_1 \right\}. \quad (2.5)$$

It is worthwhile to mention that $\log d(\cdot, \cdot)$ is a metric on the space of convex bodies in $\mathbb{R}^n$ which induces the same topology as the Hausdorff distance.

For a centrally symmetric convex body S and $0 < \delta < \frac{1}{2}$, we put $\langle S \rangle^\delta = \left((S^\circ)^\delta \right)^\circ$. We then define

$$d_S(\delta) = \inf_{\delta' > 0} d\left(S_\delta, \langle S \rangle^{\delta'}\right). \quad (2.6)$$

Please note that $d_{L(S)}(\delta) = d_S(\delta)$ for every linear invertible map L . Observe also that $d_{B_2^n}(\delta) = 1$.

One of the main theorems in [13] states that for origin symmetric convex bodies C in $\mathbb{R}^n$ that are C_+^2 , i.e. the Gauss curvature $\kappa(x)$ exists for every $x \in \partial C$ and is strictly positive, the relation (2.4) can be made precise in terms of the cone measures of C and C° .

For a Borel set $A \subset \partial C$, the cone measure M_C of A is defined as $M_C(A) = |\text{conv}[0, A]|_n$. The density function of M_C is $m_C(x) = \frac{1}{n} \langle x, N(x) \rangle$ (see [15]), and we write $n_C(x) = \frac{1}{n|C|_n} \langle x, N(x) \rangle$ for the density of the normalized cone measure $\mathbb{P}_C$ of C (again see e.g., [15]). This means that, e.g., [15],

$$dM_C(x) = m_C(x) d\mu_C(x) \quad \text{and} \quad d\mathbb{P}_C(x) = n_C(x) d\mu_C(x).$$

Denote by $N_C : \partial C \rightarrow S^{n-1}$, $x \mapsto N(x)$ the Gauss map of C , see e.g., [18]. Then, similarly, $m_{C^\circ}(x) = \frac{1}{n} \frac{\kappa_C(x)}{\langle x, N(x) \rangle^n}$ is the density function of the “cone measure” M_{C° of C° . For a Borel set $A \subset \partial C$, $M_{C^\circ}(A) = |\text{conv}[0, N_C^{-1}(N_C(A))]|_n$ and $n_{C^\circ}(x) = \frac{1}{n|C^\circ|_n} \frac{\kappa_C(x)}{\langle x, N(x) \rangle^n}$ is the density of the normalized cone measure $\mathbb{P}_{C^\circ}$ of C° , see, e.g., [15]. When C is C_+^2 , this formula holds for all $x \in \partial C$. Thus

$$dM_{C^\circ}(x) = m_{C^\circ}(x) d\mu_C(x) \quad \text{and} \quad d\mathbb{P}_{C^\circ}(x) = n_{C^\circ}(x) d\mu_C(x).$$

As observed in [13], we then have for a centrally symmetric C_+^2 convex body C that

$$\lim_{\delta \rightarrow \infty} \frac{d_C(\delta) - 1}{\delta^{\frac{2}{n+1}}} = c_n (|C|_n |C^\circ|_n)^{\frac{1}{n+1}} \left[\max_{x \in \partial C} \left(\frac{n_{C^\circ}(x)}{n_C(x)} \right)^{\frac{1}{n+1}} - \min_{x \in \partial C} \left(\frac{n_{C^\circ}(x)}{n_C(x)} \right)^{\frac{1}{n+1}} \right],$$

where $c_n = \frac{1}{2} \left(\frac{n+1}{|B_2^{n-1}|_{n-1}} \right)^{\frac{2}{n+1}}$.

In the case of a polytope the (discrete) densities n_P and n_{P° of the normalized cone measures can be expressed as follows. Let ξ be an extreme point of P . Let F_ξ be the facet of P° that has ξ as an outer normal. The (discrete) density of the normalized cone measure of P° is

$$n_{P^\circ}(\xi) = \frac{1}{n |P^\circ|_n} \frac{1}{\|\xi\|} |F_\xi|_{n-1}, \quad (2.7)$$

where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. Let $s(F_\xi)$ be the $(n-1)$ -dimensional Santaló point (see, e.g., [8, 18]) of F_ξ and $(F_\xi - s(F_\xi))^\circ$ be the polar of $(F_\xi - s(F_\xi))$ with respect to the $(n-1)$ -dimensional subspace in which $F_\xi - s(F_\xi)$ lies. We put

$$n_P(\xi) = \frac{1}{n |P|_n} \|\xi\| |(F_\xi - s(F_\xi))^\circ|_{n-1}. \quad (2.8)$$

Let C_ξ be the cone with base F_ξ and apex at the origin and let

$$C_\xi^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \text{ for all } x \in C_\xi\}$$

be the cone dual to C_ξ . Then $|(F_\xi - s(F_\xi))^\circ|_{n-1}$ is the $(n-1)$ -dimensional volume of the base of

$$Z = C_\xi^* \cap \left\{ x \in \mathbb{R}^n : \left\langle x, \frac{\xi}{\|\xi\|} \right\rangle \leq \xi \right\}$$

and thus $\frac{1}{n} \|\xi\| |(F_\xi - s(F_\xi))^\circ|_{n-1}$ is the n -dimensional volume of the finite cone Z . The expression $n_P(\xi)$ is the ratio of this volume and the volume of P . We see $n_P(\xi)$ as a cone measure associated to ξ since this volume is the cone measure of the set of all points of Z with outer normal $\frac{\xi}{\|\xi\|}$.

Our main theorem can be expressed in terms of $n_P(\xi)$ and $n_{P^\circ}(\xi)$ and reads as follows.

Theorem 2.1 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then*

$$\lim_{\delta \rightarrow 0} \frac{d_P(\delta) - 1}{\delta^{1/n}} = \min_{c \geq 0} \left[\max_{\xi \in \text{ext}(P)} \left(\frac{n_{P^\circ}(\xi) - c n_P(\xi)^{\frac{1}{n}}}{n_P(\xi)^{\frac{1}{n}} n_{P^\circ}(\xi)}, \frac{c}{\min_{\eta \in \text{ext}(P)} n_{P^\circ}(\eta)} \right) \right].$$

Recall that for $1 \leq p < \infty$, the ℓ_p^n -unit balls are defined as $B_p^n = \{x \in \mathbb{R}^n : (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \leq 1\}$. The subsequent corollary about the cube $B_\infty^n = \{x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i| \leq 1\}$ and the crosspolytope $B_1^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$, is an immediate consequence of Theorem 2.1.

Corollary 2.2

$$\lim_{\delta \rightarrow 0} \frac{d_{B_\infty^n}(\delta) - 1}{\delta^{1/n}} = \frac{\sqrt[n]{n!}}{n} \text{ and } \lim_{\delta \rightarrow 0} \frac{d_{B_1^n}(\delta) - 1}{\delta^{1/n}} = \frac{2^{1/n}}{2}.$$

3 Tools and Lemmas

Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. In [11] it was shown that for centrally symmetric convex bodies Dupin's floating body exists and coincides with the convex floating body. This means that every support hyperplane of P_δ cuts off the volume $\delta|P|_n$ from P . We use this fact throughout the remainder of the paper. We denote by $\text{ext}(P)$ the set of extreme points of P . Note that this set coincides with the set of vertices of P . For $\xi \in \text{ext}(P)$, let $F_1, \dots, F_k$ be the $(n-1)$ -dimensional facets of P such that $\xi \in F_i$. Then there are $y_1, \dots, y_k \in \mathbb{R}^n$ such that for $1 \leq i \leq k$,

$$F_i \subseteq H_i := \{x \in \mathbb{R}^n : \langle x, y_i \rangle = 1\}.$$

Observe that $y_1, \dots, y_k$ are vertices of P° and that $F_\xi := \text{conv}[y_1, \dots, y_k]$ is a facet of P° . Let $s(F_\xi)$ be the $(n-1)$ -dimensional Santaló point of F_ξ and $(F_\xi - s(F_\xi))^\circ$ be the polar of $(F_\xi - s(F_\xi))$ with respect to the $(n-1)$ -dimensional subspace in which $F_\xi - s(F_\xi)$ lies (see (2.7) and (2.8)).

For $\delta > 0$, let P_δ be the floating body of P . Let $\xi \in \text{ext}(P)$. We denote by ξ_δ the unique point in the intersection of ∂P_δ with the line segment $[0, \xi]$ and by $\langle x \rangle^\delta$ the unique point in the intersection of $\partial \langle P \rangle^\delta$

with $[0, \xi]$. We denote by ξ^δ the unique point such that ξ is the unique point in the intersection of ∂P^δ with $[0, \xi^\delta]$.

The next lemma provides a formula for $\frac{\|\xi^\delta\|}{\|\xi\|}$ if $\delta > 0$ is sufficiently small.

Lemma 3.1 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then there is $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$ and every vertex $\xi \in \partial P$ we have*

$$\frac{\|\xi^\delta\|}{\|\xi\|} = 1 - \left(\frac{n|P|_n}{|(F_\xi - s(F_\xi))^\circ|_{n-1}\|\xi\|} \right)^{1/n} \delta^{1/n}.$$

Proof. Let $e_i \in \mathbb{R}^n$ be the vector with i -th entry 1 and the other entries are 0. We first consider the case that $\xi = e_n$ and $s(F_\xi) = e_n$. For $v \in \mathbb{R}^n \setminus \{0\}$ we denote by $v^\perp = \{w \in \mathbb{R}^n : \langle v, w \rangle = 0\}$ the orthogonal complement of v . We show that

$$e_n^\perp \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n : \langle x, y_i \rangle \leq 1\}$$

is an $(n-1)$ -dimensional convex body with centroid in the origin. For self similarity reasons the $(n-1)$ -dimensional centroid of

$$(\alpha e_n + e_n^\perp) \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n : \langle x, y_i \rangle \leq 1\}$$

is αe_n for every $\alpha < 1$. Let $\bar{y}_i \in \mathbb{R}^{n-1}$ be such that $(\bar{y}_i, 1) = y_i$, $1 \leq i \leq k$. Put

$$F = \{\bar{y} \in \mathbb{R}^{n-1} : (\bar{y}, 1) \in F_{e_n}\} = \text{conv}[\bar{y}_1, \dots, \bar{y}_k] \subseteq \mathbb{R}^{n-1}$$

and

$$B = F^\circ = \bigcap_{i=1}^k \{\bar{x} \in \mathbb{R}^{n-1} : \langle \bar{x}, \bar{y}_i \rangle \leq 1\}$$

where the polar is taken in $\mathbb{R}^{n-1}$. Then $s(F) = 0$ and

$$\bigcap_{i=1}^k \{x \in \mathbb{R}^n : \langle x, y_i \rangle \leq 1\} = \{(\lambda \bar{x}, 1 - \lambda) : \lambda \geq 0, \bar{x} \in B\}.$$

It is a well-known fact (see [17]) that for a convex body C we have the identities

$$g((C - s(C))^\circ) = 0 = s((C - g(C))^\circ).$$

It follows immediately that the centroid of

$$e_n^\perp \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n : \langle x, y_i \rangle \leq 1\}$$

lies in the origin. The volume of the cone with base $B \times \{0\}$ and apex $\xi = e_n$ is given by $|B|_{n-1}/n$. Let $0 \leq \Delta \leq 1$. Then the volume of the cone with base

$$((1-\Delta)e_n + e_n^\perp) \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n : \langle x, y_i \rangle \leq 1\}$$

and apex e_n is given by $\frac{\Delta^n}{n}|B|_{n-1}$. There is $\Delta_0 > 0$ such that for every $0 \leq \Delta \leq \Delta_0$ the point e_n is the only vertex of P contained in the half-space

$$\{x \in \mathbb{R}^n : x_n \geq 1 - \Delta\}.$$

Hence, the above described cone is given by

$$\{x \in \mathbb{R}^n : x_n \geq 1 - \Delta\} \cap P.$$

Let $\delta > 0$ and choose Δ such that

$$\frac{1}{n}|B|_{n-1}\Delta^n = \delta|P|_n,$$

or, equivalently,

$$\Delta = \left(\frac{n|P|_n}{|B|_{n-1}} \right)^{\frac{1}{n}} \delta^{1/n}.$$

Choose $\delta_0 > 0$ sufficiently small such that for every $0 \leq \delta \leq \delta_0$ the value of Δ is smaller than or equal to Δ_0 . It was shown in [11] that for centrally symmetric convex bodies, the floating body coincides with the convex floating body. Thus, since P is centrally symmetric, the floating body of P coincides with the convex floating body, and therefore the hyperplane $\{x \in \mathbb{R}^n : x_n \geq 1 - \Delta\}$ touches P_δ at the centroid of

$$\{x \in \mathbb{R}^n : x_n \geq 1 - \Delta\} \cap P.$$

This centroid is given by

$$(1-\Delta)e_n = \left(1 - \left(\frac{n|P|_n}{|(B^\circ - s(B^\circ))^\circ|_{n-1}} \right)^{\frac{1}{n}} \delta^{1/n} \right) e_n = \left(1 - \left(\frac{n|P|_n}{|(F - s(F))^\circ|_{n-1} \|e_n\|} \right)^{\frac{1}{n}} \delta^{1/n} \right) e_n.$$

For a general vertex ξ and general $s(F_\xi)$ note first that $\langle \xi, s(F_\xi) \rangle = 1$ and thus, $\xi \notin s(F_\xi)^\perp$. Let $L \in \mathbb{R}^{n \times n}$ be a matrix with last row $s(F_{x_i})$ and the other rows are a basis of $\xi^\perp$. Let L^{-t} the inverse of the transpose. Since $\langle \xi, s(F_\xi) \rangle = 1$ it follows that L is a full rank matrix with $L(\xi) = e_n$ and $L^{-t}(s(F_\xi)) = e_n$. Then, LP is a centrally symmetric polytope with vertex $L(\xi) = e_n$ and $s(F_{L(\xi)}) = s(F_{e_n}) = e_n$. Note that $\frac{\|\xi_\delta\|}{\|\xi\|} = \frac{\|(L(\xi))_\delta\|}{\|L(\xi)\|}$. The lemma follows from

$$\begin{aligned} \frac{n|LP|_n}{|(F_{L\xi} - s(F_{L\xi}))^\circ|_{n-1} \|L\xi\|} &= \frac{n|\det(L)| \cdot |P|_n}{|(L^{-t}(F_\xi - s(F_\xi)))^\circ|_{n-1} \|L\xi\|} \\ &= \frac{n|\det(L)| \cdot |P|_n}{\left(|\det(L^{-t})| \cdot \|L_{\|\xi\|}^\xi\| \right)^{-1} |(F_\xi - s(F_\xi))^\circ|_{n-1} \|L\xi\|} \\ &= \frac{n|P|_n}{|(F_\xi - s(F_\xi))^\circ|_{n-1} \|\xi\|}. \end{aligned}$$

The second equality follows from the fact that for every $(n-1)$ -dimensional vector space V with normal u , every linear invertible map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and every Borel set $A \subseteq V$, we have $|S(A)|_{n-1} = |\det(S)| \cdot \|S^{-t}(u)\| \cdot |A|_{n-1}$. $\square$

For a vertex $\xi \in P$, $\langle \xi \rangle^\delta$ is the unique point in the intersection of $\partial \left(\langle P \rangle^\delta \right)$ and the line segment $[0, \xi]$.

Lemma 3.2 *Let P be a centrally symmetric polytope. Then there is a $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$ and every extreme point $\xi \in \text{ext}(P)$ we have*

$$\frac{\|\langle \xi \rangle^\delta\|}{\|\xi\|} = \left(1 + \frac{n|P^\circ|_n \|\xi\|}{|F_\xi|_{n-1}} \delta \right)^{-1}.$$

Proof. We show that

$$\left\{ y \in \mathbb{R}^n : \langle y, \xi \rangle = 1 + \frac{n|P^\circ|_n \|\xi\|}{|F_\xi|_{n-1}} \delta \right\}$$

is a support hyperplane of $(P^\circ)^\delta$. The lemma then follows immediately.

Let $z \in F_\xi$ and $\Delta \geq 0$. The volume of the cone with base F_ξ and apex $z + \Delta \frac{\xi}{\|\xi\|}$ is $\frac{1}{n} |F_\xi|_{n-1} \Delta$. There is a $\Delta_0 > 0$ and an $\eta > 0$ such that

$$F_\xi^\eta = \{z \in F_\xi : \text{dist}(z, \partial F_\xi) \geq \eta\}$$

has non-empty relative interior and such that for every $z \in F_\xi^\eta$ and every $0 \leq \Delta \leq \Delta_0$ we have

$$\text{conv} \left[P^\circ, z + \Delta \frac{\xi}{\|\xi\|} \right] = P^\circ \cup \text{conv} \left[F_\xi, z + \Delta \frac{\xi}{\|\xi\|} \right].$$

Let $\delta_0 > 0$ be such that $\frac{n|P^\circ|_n}{|F_\xi|_{n-1}} \delta \leq \Delta_0$ for every $0 \leq \delta \leq \delta_0$. It is obvious that for every $z \in F_\xi^\eta$ the vector

$$z + \frac{n|P^\circ|_n}{|F_\xi|_{n-1}} \delta \frac{\xi}{\|\xi\|}$$

lies on the boundary of $(P^\circ)^\delta$. Since F_ξ is contained in the hyperplane $\{y \in \mathbb{R}^n : \langle y, \xi \rangle = 1\}$, it follows that

$$\left\{ y \in \mathbb{R}^n : \langle y, \xi \rangle = 1 + \frac{n|P^\circ|_n \|\xi\|}{|F_\xi|_{n-1}} \delta \right\}$$

is a support hyperplane of $(P^\circ)^\delta$. $\square$

Lemma 3.3 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then there is a $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$*

$$\text{conv}[\{\langle \xi \rangle^\delta : \xi \in \text{ext}(P)\}] \subseteq \langle P \rangle^\delta \subseteq \text{conv} \left[\left\{ \langle \xi \rangle^\delta : \xi \in \text{ext}(P) \right\} \cup \left\{ \frac{1}{2}(\xi + \xi') : \xi, \xi' \in \text{ext}(P) : \xi \neq \xi' \right\} \right].$$

Proof. The first inclusion is obvious. Choose $\delta_0 > 0$ as in Lemma 3.2.

The second inclusion will follow from the fact that $(P^\circ)^\delta \supseteq C(\delta)$, where

$$C(\delta) = \bigcap_{\xi \in \text{ext}(P)} \left\{ y \in \mathbb{R}^n : \langle y, \xi \rangle = 1 + \frac{n|P^\circ|_n \|\xi\|}{|F_\xi|_{n-1}} \delta \right\} \cap \bigcap_{\xi, \xi' \in \text{ext}(P), \xi \neq \xi'} \{ y \in \mathbb{R}^n : \frac{1}{2} \langle \xi + \xi', y \rangle \leq 1 \}.$$

Let $y_0 \in C(\delta) \setminus P^\circ$. It follows that there is a $\xi \in \text{ext}(P)$ with $\langle y_0, \xi \rangle \geq 1$. Let $\xi' \in \text{ext}(P) \setminus \{\xi\}$. Since $\langle y_0, \frac{1}{2}(\xi + \xi') \rangle \leq 1$ it follows that

$$\langle y_0, \xi' \rangle = \langle y_0, \xi + \xi' \rangle - \langle y_0, \xi \rangle \leq 2 - 1 = 1.$$

We obtain

$$\text{conv}[P^\circ, y_0] = P^\circ \cup \text{conv}[F_\xi, y_0].$$

Since $y_0 \in \left\{ y \in \mathbb{R}^n : \langle y, \xi \rangle = 1 + \frac{n|P^\circ|_n \|\xi\|}{|F_\xi|_{n-1}} \delta \right\}$, we deduce

$$|\text{conv}[F_\xi, y_0]|_n = \frac{1}{n} \left(\left\langle y_0, \frac{\xi}{\|\xi\|} \right\rangle - \frac{1}{\|\xi\|} \right) |F_\xi|_{n-1} \leq \frac{1}{n} \cdot \frac{n|P^\circ|_n}{|F_\xi|_{n-1}} \delta \cdot |F_\xi|_{n-1} = \delta |P^\circ|,$$

which means that $y_0 \in (P^\circ)^\delta$. □

Corollary 3.4 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then there is a $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$*

$$\begin{aligned} & \text{conv}[\{\langle \xi \rangle^\delta : \xi \in \text{ext}(P)\}] \subseteq (P)^\delta \\ & \subseteq \text{conv} \left[\left\{ \langle \xi \rangle^\delta : \xi \in \text{ext}(P) \right\} \cup \left\{ \frac{1}{2}(\xi + \xi') : \xi, \xi' \in \text{ext}(P) : \xi \neq \xi' \text{ and } \frac{1}{2}(\xi + \xi') \in \partial P \right\} \right]. \end{aligned}$$

Proof. We only need to prove

$$(P)^\delta \subseteq \text{conv} \left[\left\{ \langle \xi \rangle^\delta : \xi \in \text{ext}(P) \right\} \cup \left\{ \frac{1}{2}(\xi + \xi') : \xi, \xi' \in \text{ext}(P) : \xi \neq \xi' \text{ and } \frac{1}{2}(\xi + \xi') \in \partial P \right\} \right].$$

Consider the set

$$\left\{ \frac{1}{2}(\xi + \xi') : \xi, \xi' \in \text{ext}(P) : \xi \neq \xi' \text{ and } \frac{1}{2}(\xi + \xi') \in \text{int}(P) \right\}.$$

This set is finite and $\lim_{\delta \rightarrow 0} \langle \xi \rangle^\delta = \xi$ for every $\xi \in \text{ext}(P)$. It follows that there is a $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$ the following holds

$$\left\{ \frac{1}{2}(\xi + \xi') : \xi, \xi' \in \text{ext}(P) : \xi \neq \xi' \text{ and } \frac{1}{2}(\xi + \xi') \in \text{int}(P) \right\} \subseteq \text{conv} \left[\left\{ \langle \xi \rangle^\delta : \xi \in \text{ext}(P) \right\} \right]$$

which yields the claim of the corollary. □

Lemma 3.5 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then there is a function $t : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ with $\lim_{\delta \rightarrow 0} t(\delta) = 0$ such that*

$$P_\delta \subseteq (1 - \Lambda \delta (1 - t(\delta)))P,$$

where $\Lambda = \min_{\zeta \in \text{ext}(P^\circ)} \frac{|P|_n \|\zeta\|}{|F_\zeta|_{n-1}}$.

Proof. Let $\delta > 0$ be given. Let $\zeta \in \text{ext}(P^\circ)$. We choose $\Delta = \Delta(\zeta, \delta)$ such that

$$\left| P \cap \left\{ x \in \mathbb{R}^n : \left\langle x, \frac{\zeta}{\|\zeta\|} \right\rangle \geq \frac{1}{\|\zeta\|} - \Delta \right\} \right|_n = \delta |P|_n.$$

For $\delta > 0$ and hence $\Delta = \Delta(\zeta, \delta) \geq 0$ sufficiently small, the volume of $P \cap \{x \in \mathbb{R}^n : \langle x, \frac{\zeta}{\|\zeta\|} \rangle \geq \frac{1}{\|\zeta\|} - \Delta\}$ is up to an error given by $\Delta |F_\zeta|_{n-1}$, i.e. there is a function T_ζ with $\lim_{\Delta \rightarrow 0} T_\zeta(\Delta) = 0$ such that

$$\left| \left\{ x \in \mathbb{R}^n : \left\langle x, \frac{\zeta}{\|\zeta\|} \right\rangle \geq \frac{1}{\|\zeta\|} - \Delta \right\} \cap P \right|_n = \Delta |F_\zeta|_{n-1} (1 + T_\zeta(\Delta)).$$

Hence, for every $\zeta \in \text{ext}(P^\circ)$, there is a function t_ζ with $\lim_{\delta \rightarrow 0} t_\zeta(\delta) = 0$ such that

$$P_\delta \subseteq \left\{ x \in \mathbb{R}^n : \langle \zeta, x \rangle \leq 1 - \frac{|P|_n \|\zeta\|}{|F_\zeta|_{n-1}} \delta (1 - t_\zeta(\delta)) \right\}.$$

Let $t(\delta) = \max_{\zeta \in \text{ext}(P^\circ)} t_\zeta(\delta)$ and $\Lambda = \min_{\zeta \in \text{ext}(P^\circ)} \frac{|P|_n \|\zeta\|}{|F_\zeta|_{n-1}}$. Then

$$P_\delta \subseteq \bigcap_{\zeta \in \text{ext}(P^\circ)} \{x \in \mathbb{R}^n : \langle \zeta, x \rangle \leq 1 - \Lambda \delta (1 - t(\delta))\} = (1 - \Lambda \delta (1 - t(\delta)))P.$$

□

Lemma 3.6 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope and $x \in \partial P \setminus \text{ext}(P)$. Then there exists $\delta_0 > 0$ and $k > 0$ such that for every $0 \leq \delta \leq \delta_0$ we have*

$$\frac{\|x_\delta\|}{\|x\|} \geq 1 - k \delta^{\frac{1}{n-1}}.$$

Proof. Since x is not an extreme point of P , there are points $x_1, x_2 \in \partial P$ with $x_1 \neq x \neq x_2$ and such that $x = \frac{1}{2}(x_1 + x_2)$. By a linear transformation of P we may assume without loss of generality that $x = e_2$, $x_1 = e_2 - e_1$ and $x_2 = e_2 + e_1$. There is an $0 < \varepsilon < 1$ such that $[-\varepsilon, \varepsilon] \times \{0\} \times [-\varepsilon, \varepsilon]^{n-2} \subseteq P$. It follows that the centrally symmetric convex body

$$S = \text{conv} [e_2 \pm \varepsilon e_1, -e_2 \pm \varepsilon e_1, [-\varepsilon, \varepsilon] \times \{0\} \times [-\varepsilon, \varepsilon]^{n-2}] = [-\varepsilon, \varepsilon] \times \text{conv} [\pm e_2, \{0\} \times \{0\} \times [-\varepsilon, \varepsilon]^{n-2}]$$

is contained in P . Put $\tilde{\delta} = \delta \frac{|P|_n}{|S|_n}$. We compute $(e_2)_{\tilde{\delta}}$ with respect to $S_{\tilde{\delta}}$. Let $0 \leq \Delta < 1$. A simple computation shows that

$$|S \cap \{x \in \mathbb{R}^n : x_2 \geq 1 - \Delta\}|_n = \frac{1}{n} (2\varepsilon \Delta)^{n-1}.$$

The $(n-1)$ -dimensional centroid of the $(n-1)$ -dimensional set $S \cap \{x \in \mathbb{R}^n : x_2 = 1 - \Delta\}$ lies on the line $\mathbb{R}e_2$. Since S is symmetric, the convex floating and the floating body of Dupin coincide [11] and it follows that for $\delta < \frac{1}{2}$,

$$\left(1 - \frac{(n|S|_n)^{\frac{1}{n-1}}}{2\varepsilon} \tilde{\delta}^{\frac{1}{n-1}}\right) e_2 \in \partial(S_{\tilde{\delta}}).$$

Since $S \subseteq P$, there exists $\delta_0 > 0$ and $k > 0$ such that for every $0 \leq \delta \leq \delta_0$,

$$\frac{\|x_\delta\|}{\|x\|} \geq 1 - k \delta^{\frac{1}{n-1}}$$

where x_δ is taken with respect to P_δ . □

4 Proof of Theorem 2.1 and Corollary 2.2

We recall the quantities that are relevant for our main theorem. For $\xi \in \text{ext}(P)$, we put

$$\alpha_\xi = \left(\frac{n|P|_n}{|(F_\xi - s(F_\xi))^\circ|_{n-1} \|\xi\|} \right)^{1/n}, \quad (4.1)$$

$$\beta_\xi = \frac{n|P^\circ|_n \|\xi\|}{|F_\xi|_{n-1}} \text{ and } \beta = \max_{\xi \in \text{ext}(P)} \beta_\xi. \quad (4.2)$$

For $c \geq 0$, we set

$$G_c(P) = \max_{\xi \in \text{ext}(P)} [a_\xi - c\beta_\xi, c\beta] \text{ and } G(P) = \min_{c \geq 0} G_c(P). \quad (4.3)$$

Then Theorem 2.1 reads.

Theorem 2.1 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then*

$$\lim_{\delta \rightarrow 0} \frac{d_P(\delta) - 1}{\delta^{1/n}} = G(P).$$

We split the proof of the theorem and show separately the upper and lower bound.

4.1 Upper bound

We prove the following proposition.

Proposition 4.1 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then*

$$\limsup_{\delta \rightarrow 0} \frac{d_P(\delta) - 1}{\delta^{1/n}} \leq G(P).$$

Proof. Let $c_0 \geq 0$ be such that $G(P) = G_{c_0}(P)$ and put $\delta' = c_0 \delta^{1/n}$. By Lemma 3.2, Lemma 3.4 and Lemma 3.5, a sufficient condition for $P_\delta \subseteq a \langle P \rangle^{\delta'}$ is that

$$1 - \Lambda\delta(1 - t(\delta)) \leq a(1 + \beta_\xi \delta')^{-1},$$

for every $\xi \in \text{ext}(P)$. Hence,

$$a \geq (1 - \Lambda\delta(1 - t(\delta))) \max_{\xi \in \text{ext}(P)} (1 + \beta_\xi \delta') = (1 - \Lambda\delta(1 - t(\delta)))(1 + \beta_{c_0} \delta^{1/n}).$$

By Lemma 3.1, Lemma 3.2 and Corollary 3.4, a sufficient condition for $\langle P \rangle^{\delta'} \subseteq aP_\delta$ is that

$$(1 + \beta_\xi \delta')^{-1} \leq a(1 - \alpha_\xi \delta^{1/n})$$

for every $\xi \in \text{ext}(P)$ and that

$$\left\| \frac{1}{2}(\xi + \xi') \right\| \leq a \left\| \left(\frac{1}{2}(\xi + \xi') \right)_\delta \right\|$$

for $\xi, \xi' \in \text{ext}(P)$, $\xi \neq \xi'$ such that $\frac{1}{2}(\xi + \xi') \in \partial P$. From the first condition we derive that

$$a \geq \frac{1}{(1 - \alpha_\xi \delta^{1/n})(1 + \beta_\xi \delta')}$$

for every $\xi \in \text{ext}(P)$. By Lemma 3.6 there is a constant $k > 0$ and $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$ we have

$$\left\| \left(\frac{1}{2}(\xi + \xi') \right)_\delta \right\| \geq \left(1 - k \delta^{\frac{1}{n-1}} \right) \left\| \frac{1}{2}(\xi + \xi') \right\|$$

and we may assume that k and δ_0 are taken uniformly with respect to all pairs (ξ, ξ') . Hence, for $\delta \leq \delta_0$ we have the condition that

$$a \geq \frac{1}{1 - k \delta^{\frac{1}{n-1}}}.$$

We check that all three conditions are met if one takes $a = 1 + G(P) \delta^{\frac{1}{n}} (1 + o(1))$. The condition

$$a \geq \frac{1}{1 - k \delta^{\frac{1}{n-1}}}$$

is obvious since $1 + G(P) \delta^{1/n} \geq (1 - k \delta^{1/(n-1)})^{-1}$ for sufficiently small $\delta > 0$. The condition

$$a \geq \frac{1}{(1 - \alpha_\xi \delta)(1 + \beta_\xi \delta')}$$

is true since

$$\begin{aligned} \frac{1}{(1 - \alpha_\xi \delta^{1/n})(1 + \beta_\xi \delta')} &= \frac{1}{(1 - \alpha_\xi \delta^{1/n})(1 + \beta_\xi c_0 \delta^{1/n})} = 1 + (\alpha_\xi - c_0 \beta) \delta^{1/n} + o(\delta^{1/n}) \\ &\leq 1 + G_{c_0}(P) \delta^{1/n} + o(\delta^{1/n}) \leq 1 + G(P) \delta^{1/n} + o(\delta^{1/n}). \end{aligned}$$

Finally, the condition

$$a \geq (1 - \Lambda\delta(1 - t(\delta)))(1 + \beta_{c_o}\delta^{1/n})$$

is true since

$$(1 - \Lambda\delta(1 - t(\delta)))(1 + \beta_{c_o}\delta^{1/n}) \leq 1 + c_0\beta\delta^{1/n} \leq 1 + G_{c_0}(P)\delta^{1/n} \leq 1 + G(P)\delta^{1/n} + o(\delta^{1/n}).$$

□

4.2 Lower Bound

We prove the following proposition.

Proposition 4.2 *Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope. Then*

$$\liminf_{\delta \rightarrow 0} \frac{d_P(\delta) - 1}{\delta^{1/n}} \geq G(P).$$

Proof. Let $c_0 \geq 0$ such that $G(P) = G_{c_0}(P)$ and let $\xi_1, \xi_2 \in \text{ext}(P)$ be such that

$$\beta_{\xi_1} = \max_{\zeta \in \text{ext}(P)} \beta_{\zeta} \quad \text{and} \quad \alpha_{\xi_2} - c_0\beta_{\xi_2} = \max_{\zeta \in \text{ext}(P)} [\alpha_{\zeta} - c_0\beta_{\zeta}].$$

We obtain that $c_0\beta_{\xi_1} = \alpha_{\xi_2} - c_0\beta_{\xi_2}$ and therefore that $c_0 = \frac{\alpha_{\xi_2}}{\beta_{\xi_1} + \beta_{\xi_2}}$ and $G(P) = \frac{\alpha_{\xi_2}\beta_{\xi_1}}{\beta_{\xi_1} + \beta_{\xi_2}}$. A necessary condition for $\langle P \rangle^{\delta'} \subseteq aP_{\delta}$ is that $\|\langle \xi_2 \rangle^{\delta'}\| \leq a\|(\xi_2)_{\delta}\|$, or, equivalently, also using Lemmas 3.1 and 3.2,

$$a \geq \frac{\|\langle \xi_2 \rangle^{\delta'}\|}{\|(\xi_2)_{\delta}\|} = (1 - \alpha_{\xi_2}\delta^{1/n})^{-1}(1 + \beta_{\xi_2}\delta')^{-1}.$$

By Lemma 3.4, there is δ_0 such that for every $0 \leq \delta \leq \delta_0$ we have

$$\langle P \rangle^{\delta'} \subseteq \text{conv} \left[\left\{ \langle \xi \rangle^{\delta'} : \xi \in \text{ext}(P) \right\} \cup \left\{ \frac{1}{2}(\xi + \xi') : \xi, \xi' \in \text{ext}(P) : \xi \neq \xi' \right\} \right] =: P(\delta').$$

If $\delta'_0 > 0$ is chosen sufficiently small, $\langle \xi_1 \rangle^{\delta'}$ is an extreme point of $P(\delta')$. Then there exists $\varepsilon > 0$ and a hyperplane $H^y = \{x \in \mathbb{R}^n : \langle x, y \rangle = 1\}$ such that $\langle \langle \xi_1 \rangle^{\delta'}, y \rangle > 1 + \varepsilon$ and such that all other extreme points of $P(\delta')$ lie in $\{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1\}$ for every $0 \leq \delta' \leq \delta'_0$. Hence,

$$\langle P \rangle^{\delta'} \cap \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 1\} \subseteq \text{conv} \left[P \cap H^y, \langle \xi_1 \rangle^{\delta'} \right].$$

Let $z \in (\partial P) \cap H^y$. Then $\lambda \langle \xi_1 \rangle^{\delta'} + (1 - \lambda)z \notin \text{int} \left[\langle P \rangle^{\delta'} \right]$, for every $\lambda \in [0, 1]$. Fix $\lambda \in [0, 1]$ and put $v = \lambda \xi_1 + (1 - \lambda)z \in \partial P$. Let $t, \mu \in (0, 1)$ be such that $tv = \mu \langle \xi_1 \rangle^{\delta'} + (1 - \mu)z$. Then

$$t\|v\| \geq \|\langle v \rangle^{\delta'}\|. \quad (4.4)$$

We determine t . By Lemma 3.2 and as ξ and $\langle \xi_1 \rangle^{\delta'}$, we know that

$$\langle \xi_1 \rangle^{\delta'} = (1 + \beta_{\xi_1} \delta')^{-1} \xi_1$$

if $\delta'_0 > 0$ is chosen sufficiently small. This means that t and μ satisfy the equation

$$t(\lambda \xi_1 + (1 - \lambda)z) = \mu(1 + \beta_{\xi_1} \delta')^{-1} \xi_1 + (1 - \mu)z.$$

Since ξ and z are linearly independent, t and μ satisfy the system of linear equations

$$\text{I. } t\lambda - \mu(1 + \beta_{\xi_1} \delta')^{-1} = 0 \quad \text{II. } t(1 - \lambda) + \mu = 1.$$

It follows that $t = (1 + \lambda \beta_{\xi_1} \delta')^{-1}$. Since v is not an extreme point of P , it follows from Lemma 3.6 that there is a $k_v \geq 0$ such that

$$\frac{\|v_\delta\|}{\|v\|} \geq 1 - k_v \delta^{\frac{1}{n-1}}.$$

By this and (4.4), a necessary condition for $a \langle P \rangle^{\delta'} \supseteq P_\delta$ is that $a(1 + \lambda \beta_{\xi_1} \delta')^{-1} \geq 1 - k_v \delta^{\frac{1}{n-1}}$, i.e., $a \geq (1 + \lambda \beta_{\xi_1} \delta')(1 - k_v \delta^{\frac{1}{n-1}})$. Assume that $\delta' \geq \frac{\alpha_{\xi_2}}{\lambda \beta_{\xi_1} + \beta_{\xi_2}} \delta^{1/n}$ then we get

$$a \geq (1 + \lambda \beta_{\xi_1} \delta')(1 - k_v \delta^{\frac{1}{n-1}}) \geq 1 + \frac{\alpha_{\xi_2} \beta_{\xi_1} \lambda}{\lambda \beta_{\xi_1} + \beta_{\xi_2}} \delta^{1/n} + o(\delta^{1/n}).$$

The assumption $\delta' \leq \frac{\alpha_{\xi_2}}{\lambda \beta_{\xi_1} + \beta_{\xi_2}} \delta^{1/n}$ together with the necessary condition $a \geq (1 - \alpha_{\xi_2} \delta^{1/n})^{-1}(1 + \beta_{\xi_2} \delta')^{-1}$ also yields

$$a \geq 1 + \frac{\alpha_{\xi_2} \beta_{\xi_1} \lambda}{\lambda \beta_{\xi_1} + \beta_{\xi_2}} \delta^{1/n} + o(\delta^{1/n}).$$

Thus,

$$\liminf_{\delta \rightarrow 0} \frac{d_P(\delta) - 1}{\delta^{1/n}} \geq \frac{\alpha_{\xi_2} \beta_{\xi_1} \lambda}{\lambda \beta_{\xi_1} + \beta_{\xi_2}}.$$

Letting $\lambda \rightarrow 1$, we get the desired result. □

4.3 Proof of Corollary 2.2

We first treat the case of the cube.

Corollary 4.3

$$\lim_{\delta \rightarrow 0} \frac{d_{B_\infty^n}(\delta) - 1}{\delta^{1/n}} = \frac{\sqrt[n]{n!}}{n}$$

Proof. By symmetry, α_ξ and β_ξ have the same value for all the extreme points of B_∞^n . Take $\xi = (1, \dots, 1)$. Then $\|\xi\| = \sqrt{n}$, $|B_\infty^n|_n = 2^n$, $|B_1^n|_n = \frac{2^n}{n!}$ and

$$|F_\xi|_{n-1} = |\text{conv}[e_1, \dots, e_n]|_{n-1} = \frac{\sqrt{n}}{(n-1)!}.$$

It is well known that the volume product $|S_{n-1}|_{n-1} |(S_{n-1})^\circ|_{n-1}$ of the $(n-1)$ -dimensional simplex is $\frac{n^n}{((n-1)!)^2}$. Hence, as F_ξ is an $(n-1)$ -dimensional regular simplex,

$$|(F_\xi - s(F_\xi))^\circ|_{n-1} = \frac{1}{|F_\xi|_{n-1}} \cdot \frac{n^n}{((n-1)!)^2} = \frac{n^n}{\sqrt{n}(n-1)!}.$$

Therefore,

$$\alpha_\xi = \left(\frac{n2^n}{\frac{n^n}{\sqrt{n}(n-1)!} \sqrt{n}} \right)^{1/n} = 2 \frac{\sqrt[n]{n!}}{n} \text{ and } \beta_\xi = 2^n.$$

The minimum over all $c \geq 0$ of $\max[\alpha_\xi - c\beta_\xi, c\beta_\xi]$ is attained for $c = \frac{\alpha_\xi}{2\beta_\xi}$. Thus

$$G(B_\infty^n) = \frac{\alpha_\xi}{2} = \frac{\sqrt[n]{n!}}{n},$$

which completes the proof. □

Now we show the statement of Corollary 2.2 in the case of the crosspolytope.

Corollary 4.4

$$\lim_{\delta \rightarrow 0} \frac{d_{B_1^n}(\delta) - 1}{\delta^{1/n}} = \frac{2^{1/n}}{2}.$$

Proof. As in the previous example, all α_ξ and all β_ξ are equal and $G(B_1^n) = \frac{\alpha_\xi}{2}$. Take $\xi = e_n$. Then $|B_1^n|_n = \frac{2^n}{n!}$, $\|\xi\| = 1$ and $F_\xi = \text{conv}[e_n + \sum_{i=1}^{n-1} \theta_i e_i : \theta \in \{-1, 1\}^{n-1}] = e_n + B_\infty^{n-1}$. It follows that

$$|(F_\xi - s(F_\xi))^\circ|_{n-1} = |B_1^{n-1}|_{n-1} = \frac{2^{n-1}}{(n-1)!}.$$

We obtain

$$\alpha_\xi = \left(\frac{n \frac{2^n}{n!}}{\frac{2^{n-1}}{(n-1)!} \cdot 1} \right)^{1/n} = 2^{1/n}.$$

□

5 The combinatorial structure of d_P

In [19], it was proved that the following relation holds for all polytopes $P \subseteq \mathbb{R}^n$,

$$\lim_{\delta \rightarrow 0} \frac{|P|_n - |P_\delta|_n}{\delta \ln(\delta)^{n-1}} = \frac{\text{fl}_n(P)}{n!n^{n-1}},$$

where $\text{fl}_n(P)$ denotes the number of flags of P . A flag of P is an $(n+1)$ -tuple $(F_0, \dots, F_n)$ such that F_i is an i -dimensional face of P and $F_0 \subset F_1 \subset \dots \subset F_n$.

This theorem suggests that also d_P , and hence $G(P)$, might only depend on the combinatorial structure of P . The fact that d_P is invariant under affine transformations of P supports this conjecture. However, this is not the case, as is illustrated by the following 2-dimensional example.

For $\varepsilon \in (0, 1)$, we consider the hexagon

$$P(\varepsilon) = \text{conv} \left[\pm e_2, \pm \sqrt{1 - \varepsilon^2} e_1 \pm \varepsilon e_2 \right].$$

We show that $d_{P(\varepsilon)}$ changes for different values of ε . We compute the 2-dimensional volume of $P(\varepsilon)$. The hexagon is, up to a nullset, the disjoint union of the two congruent trapezoids

$$T_1 = \text{conv}[-e_2, \sqrt{1 - \varepsilon^2} e_1 - \varepsilon e_2, \sqrt{1 - \varepsilon^2} e_1 + \varepsilon e_2, e_2]$$

and

$$T_2 = \text{conv}[e_2, -\sqrt{1 - \varepsilon^2} e_1 + \varepsilon e_2, -\sqrt{1 - \varepsilon^2} e_1 - \varepsilon e_2, -e_2].$$

The trapezoid T_1 has the two parallel sides $S_1 = \text{conv}[-e_2, e_2]$ and $S_2 = \text{conv}[\sqrt{1 - \varepsilon^2} e_1 - \varepsilon e_2, \sqrt{1 - \varepsilon^2} e_1 + \varepsilon e_2]$ and the height of T_1 with respect to S_1, S_2 is given by $\sqrt{1 - \varepsilon^2}$. Hence,

$$|T_1|_2 = \frac{|S_1|_1 + |S_2|_1}{2} \cdot \sqrt{1 - \varepsilon^2} = \frac{2 + 2\varepsilon}{2} \cdot \sqrt{1 - \varepsilon^2} = (1 + \varepsilon) \cdot \sqrt{1 - \varepsilon^2},$$

and we conclude that $|P(\varepsilon)|_2 = 2 \cdot |T_1|_2 = 2(1 + \varepsilon) \cdot \sqrt{1 - \varepsilon^2}$.

We compute the vertices of the polar of $P(\varepsilon)$. One vertex is given as the solution of the equations

$$y_2 = 1 \quad \text{and} \quad \sqrt{1 - \varepsilon^2} y_1 + \varepsilon y_2 = 1,$$

which yields $(y_1, y_2) = \left(\frac{1 - \varepsilon}{\sqrt{1 - \varepsilon^2}}, 1 \right)$. Another vertex is given as the solution of the equations

$$\sqrt{1 - \varepsilon^2} y_1 + \varepsilon y_2 = 1 \quad \text{and} \quad \sqrt{1 - \varepsilon^2} y_1 + \varepsilon y_2 = 1,$$

which yields $(y_1, y_2) = \left(\frac{1}{\sqrt{1 - \varepsilon^2}}, 0 \right)$. By symmetry, the six vertices of P° are given by

$$\left\{ \pm \frac{1}{\sqrt{1 - \varepsilon^2}} e_1, \pm \frac{1 - \varepsilon}{\sqrt{1 - \varepsilon^2}} e_1 \pm e_2 \right\}.$$

Since $P(\varepsilon)^\circ$ is the union of two trapezoids, computations similar to the case of $P(\varepsilon)$ yield that the 2-dimensional volume of $P(\varepsilon)^\circ$ is given by

$$|P(\varepsilon)^\circ|_2 = \frac{4-2\varepsilon}{\sqrt{1-\varepsilon^2}}.$$

If $\xi = \pm e_2$, we get that $|F_\xi|_1 = 2 \cdot \frac{1-\varepsilon}{\sqrt{1-\varepsilon^2}}$ and

$$|(F_\xi - s(F_\xi))^\circ|_1 = 2 \cdot \left(\frac{|F_\xi|_1}{2} \right)^{-1} = 2 \frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon}.$$

Hence,

$$\alpha_1 := \alpha_\xi = \left(\frac{2 \cdot 2(1+\varepsilon) \cdot \sqrt{1-\varepsilon^2}}{2 \cdot \frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon} \cdot 1} \right)^{1/2} = \sqrt{2} \cdot \sqrt{1-\varepsilon^2}$$

and

$$\beta_1 := \beta_\xi = \frac{2 \cdot \frac{4-2\varepsilon}{\sqrt{1-\varepsilon^2}} \cdot 1}{2 \cdot \frac{1-\varepsilon}{\sqrt{1-\varepsilon^2}}} = \frac{4-2\varepsilon}{1-\varepsilon}.$$

If $\xi = \pm\sqrt{1-\varepsilon^2}e_1 \pm \varepsilon e_2$ then

$$|F_\xi|_1 = \left\| \frac{1-\varepsilon}{\sqrt{1-\varepsilon^2}}e_1 + e_2 - \frac{1}{\sqrt{1-\varepsilon^2}}e_1 \right\| = \frac{1}{\sqrt{1-\varepsilon^2}}$$

and

$$|(F_\xi - s(F_\xi))^\circ|_1 = 2 \cdot \left(\frac{1}{2 \cdot \sqrt{1-\varepsilon^2}} \right)^{-1} = 4 \cdot \sqrt{1-\varepsilon^2}.$$

Hence,

$$\alpha_2 := \alpha_\xi = \left(\frac{2 \cdot 2(1+\varepsilon) \cdot \sqrt{1-\varepsilon^2}}{4 \cdot \sqrt{1-\varepsilon^2}} \right)^{1/2} = (1+\varepsilon)^{1/2}$$

and

$$\beta_2 := \beta_\xi = \frac{2 \cdot \frac{4-2\varepsilon}{\sqrt{1-\varepsilon^2}}}{\frac{1}{\sqrt{1-\varepsilon^2}}} = 8-4\varepsilon.$$

We compute $G(P(\varepsilon))$. If $0 < \varepsilon < \frac{1}{2}$, then $8-4\varepsilon > \frac{4-2\varepsilon}{1-\varepsilon}$ and therefore, $\beta = \max_{\xi \in \text{ext}(P_\varepsilon)} \beta_\xi = 8-4\varepsilon$. Moreover, for $0 < \varepsilon < \frac{1}{2}$, $\alpha_1 > \alpha_2$ and thus $\alpha_1 - c \cdot \beta_1 \geq \alpha_2 - c \cdot \beta_2$, for every $c \geq 0$. This yields

$$G_c(P(\varepsilon)) = \max \left[c(8-4\varepsilon), \sqrt{2} \cdot \sqrt{1-\varepsilon^2} - c \cdot \frac{4-2\varepsilon}{1-\varepsilon} \right]$$

and $G_c(P(\varepsilon))$ is minimized by

$$c_0 = \frac{(1+\varepsilon)^{1/2} (1-\varepsilon)^{3/2}}{\sqrt{2} (2-\varepsilon) (3-2\varepsilon)}.$$

It follows that

$$G(P(\varepsilon)) = G_{c_0}(P(\varepsilon)) = 2\sqrt{2} \cdot \frac{(1+\varepsilon)^{1/2}(1-\varepsilon)^{3/2}}{3-2\varepsilon}.$$

This means that, if $\varepsilon > 0$ is sufficiently small, $G(P(\varepsilon))$ and hence $d_{P(\varepsilon)}$, changes for different values of ε . Moreover, this example shows that the affine invariant $G(P(\varepsilon))$ is not continuous with respect to the Hausdorff distance, since $P(\varepsilon)$ converges to B_1^2 as ε goes to 0 but

$$\lim_{\varepsilon \rightarrow 0} 2\sqrt{2} \cdot \frac{(1+\varepsilon)^{1/2}(1-\varepsilon)^{3/2}}{3-2\varepsilon} = \frac{2\sqrt{2}}{3} \neq \frac{\sqrt{2}}{2} = G(B_1^2).$$

6 Approximation results for the floating body and open questions

The parameter d_S measures the best approximation of the floating body by the polar of an illumination body of the polar. We establish a uniform bound for this quantity, independent of the convex body.

Proposition 6.1 *Let $S \subseteq \mathbb{R}^n$ be a centrally symmetric convex body. Then there exist constants G_n, δ_n only depending on the dimension such that*

$$d(S_\delta, S) \leq 1 + G_n \delta^{1/n}$$

for $\delta \in [0, \delta_n]$.

In particular, the proposition yields

$$d_S(\delta) = \inf_{\delta' \geq 0} d(S_\delta, \langle S \rangle^{\delta'}) \leq d(S_\delta, \langle S \rangle^0) = d(S_\delta, S) \leq 1 + G_n \delta^{1/n}.$$

Thus, $\delta^{1/n}$ is already the worst order of convergence we can get in general. The polytopal case shows that we cannot hope for any better uniform rate of convergence. Proposition 6.1 does not involve the floating body any more. We address the question if we get a better uniform bound if we involve the illumination body. At best, how does the optimal G_n look like such that $d_S(\delta) \leq 1 + G_n \delta^{1/n} + o(\delta^{1/n})$ where the error term $o(\delta^{1/n})$ does only depend on the dimension? It would be interesting to know about the maximizers if they exist. Furthermore, it would be interesting to know something about the best uniform bound on subclasses like polytopes, C^1 -bodies or C^2 -bodies.

Proof of Proposition 6.1. The quantity $d(S_\delta, S)$ is invariant with respect to linear transformations and we may therefore assume that S is in John position, i.e., in particular,

$$B_2^n \subseteq S \subseteq \sqrt{n}B_2^n.$$

Let $\xi \in \partial S$. Then $1 \leq \|\xi\| \leq \sqrt{n}$. Put $B_2^{\xi^\perp} = B_2^n \cap \xi^\perp$. By central symmetry of S , the double cone $C_\xi = \text{conv}[\xi, B_2^{\xi^\perp}, -\xi]$ is contained in S ,

$$C_\xi = \text{conv}[\xi, B_2^{\xi^\perp}, -\xi] \subseteq S.$$

Put

$$\Delta = \left(\frac{n|S|_n}{\|\xi\| \|B_2^{n-1}\|_{n-1}} \right)^{1/n} \delta^{1/n}.$$

The halfspace

$$\{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq 1 - \Delta\}$$

cuts off exactly volume $\delta|S|_n$ from C_ξ . Similar to the proof of Lemma 3.6 we get that

$$\frac{\|\xi_\delta\|}{\|\xi\|} \geq 1 - \left(\frac{n|S|_n}{\|\xi\| \|B_2^{n-1}\|_{n-1}} \right)^{1/n} \delta^{1/n}.$$

Taking into account that $|S|_n \leq \sqrt{n^n} |B_2^n|$ and $\|\xi\| \geq 1$ we obtain

$$\frac{\|\xi_\delta\|}{\|\xi\|} \geq 1 - \sqrt{n} \left(\frac{n|B_2^n|}{|B_2^{n-1}|_{n-1}} \right)^{1/n} \delta^{1/n}.$$

The desired result follows. $\square$

One might ask if the convergence result in Theorem 2.1 is uniform, i.e., does

$$|\mathbf{d}_P(\delta) - 1 - G(P)\delta^{1/n}| \leq o(\delta^{1/n})$$

hold with an error term $o(\delta^{1/n})$ only depending on the dimension? This is not the case. Indeed, the floating and illumination bodies are stable with respect to the Hausdorff distance, [12] and hence, with respect to the distance $\mathbf{d}$. This means that if

$$\lim_{n \rightarrow \infty} \mathbf{d}(K_n, K) = 1,$$

then

$$\lim_{n \rightarrow \infty} \mathbf{d}((K_n)_\delta, K_\delta) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbf{d}((K_n)^\delta, K) = 1.$$

Consider now the polytopes $P(\varepsilon)$. By continuity of the floating and illumination body we get for fixed $\delta > 0$

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbf{d}_{P(\varepsilon)}(\delta) - 1}{\delta^{1/n}} = \frac{\mathbf{d}_{B_1^n}(\delta) - 1}{\delta^{1/n}}.$$

On the other hand, by Section 5,

$$\lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{\mathbf{d}_{P(\varepsilon)}(\delta) - 1}{\delta^{1/n}} = \frac{2\sqrt{2}}{3} > \frac{\sqrt{2}}{2} = \lim_{\delta \rightarrow 0} \frac{\mathbf{d}_{B_1^n}(\delta) - 1}{\delta^{1/n}}.$$

We also like to address the problem to compute the optimal constant $\tilde{G}(P)$ such that

$$\mathbf{d}_P(\delta) \leq 1 + \tilde{G}(P)\delta^{1/n} + o(\delta^{1/n})$$

for centrally symmetric polytopes such that $o(\delta^{1/n})$ is only a dimension dependent error. The problem of proving such a result is already illustrated by the example $P(\varepsilon)$ of Section 5. The facet and vertex structure of a polytope is not stable with respect to the distance d but the convergence result Theorem 2.1 depends highly on these quantities. On the other hand $P(\varepsilon)$ is close to B_1^n for small ε and therefore, $d_{P(\varepsilon)}(\delta)$ and $d_{B_1^n}(\delta)$ behave similarly for a wide range of δ not too close to 0. Deriving a uniform bound would demand techniques which take into account the global structure of the convex bodies.

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