Exact Formulae for the Fractional Partition Functions

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Abstract

The partition function $p(n)$ has been a testing ground for applications of analytic number theory to combinatorics. In particular, Hardy and Ramanujan invented the "circle method" to estimate the size of $p(n)$, which was later perfected by Rademacher who obtained an exact formula. Recently, Chan and Wang considered the fractional partition functions, defined for $\alpha \in \mathbb{Q}$ by $\sum_{n=0}^{\infty} p_\alpha(n)x^n := \prod_{k=1}^{\infty}(1-x^k)^{-\alpha}$. In this paper we use the Rademacher circle method to find an exact formula for $p_\alpha(n)$ and study its implications, including log-concavity and the higher-order generalizations (i.e., the Turán inequalities) that $p_\alpha(n)$ satisfies.

1 Introduction and Statement of Results

A *partition* of a nonnegative integer n is a non-increasing sequence of positive integers with sum n. We use $p(n)$ to denote the number of partitions of n. One powerful tool for analyzing the partition function is Euler's generating function:

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{1.1}$$

The study of the size of $p(n)$ spurred the development of the "circle method," which has had many applications, including the proof of the weak Goldbach conjecture [11]. In 1918, G. H. Hardy and S. Ramanujan [10] invented this method to obtain an infinite but divergent series expansion for $p(n)$ and the asymptotic formula:

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

This method was perfected by H. Rademacher [16], who determined the convergent exact formula

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right), \tag{1.2}$$

where

$$I_\nu(z) := \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)}$$

is the modified Bessel function of the first kind,

$$A_k(n) := \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k}$$

is a Kloosterman sum, and

$$s(h,k) := \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right) \tag{1.3}$$

is the usual Dedekind sum.

The partition function also satisfies certain congruences, which exhibit a great degree of structure. Ramanujan was the first to study these congruences, and he discovered examples including $p(5n+4) \equiv 0 \pmod 5$. In a recent paper, Chan and Wang [4] defined for $\alpha \in \mathbb{Q}$ the *fractional partition function* $p_\alpha(n)$ in terms of its generating function

$$P(x)^\alpha = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^\alpha} =: \sum_{n=0}^{\infty} p_\alpha(n) x^n, \tag{1.4}$$

and studied its congruences, showing, for instance, that $p_{1/2}(29n+26) \equiv 0 \pmod{29}$. A general theory of such congruences has recently been developed by Bevilacqua, Chandran, and Choi [2]. The discussion of congruences for $p_\alpha(n)$ is possible because $p_\alpha(n)$ is rational whenever α is rational.

When $\alpha \in \mathbb{Z}^+$, $p_\alpha(n)$ counts the number of partitions of n in which each term is labeled with one of α different colors, where the order of the colors does not matter [12]. Moreover, in such cases, the function

$$\eta(\tau)^{-\alpha} = q^{-\frac{\alpha}{24}} P(q)^\alpha \tag{1.5}$$

is a weakly holomorphic modular form of weight $-\alpha/2 \in (1/2)\mathbb{Z}$, where τ is in the upper half-plane, $\eta(\tau) := q^{1/24} \prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function, and $q := e^{2\pi i \tau}$. This makes it possible to compute the values of $p_\alpha(n)$ using Maass-Poincaré series, as described by Bringmann et al. [3, §6.3], which give a Rademacher-type infinite series expansion that reduces to (1.2) when $\alpha = 1$. To do this, one computes the principal part of $\eta(\tau)^{-\alpha}$, which correspond to the values $p_\alpha(n)$ for $0 \leq n \leq \lfloor \alpha/24 \rfloor$. Then, using the fact that a weakly holomorphic modular form is determined by its weight and principal part, one can write it as a finite sum of Maass-Poincaré series and apply a known formula for the coefficients of such series. While these observations shed light on the case where α is a positive integer, there is currently no known combinatorial or modular-form interpretation of $p_\alpha(n)$ for arbitrary rational α.

In this paper, we extend the definition of $p_\alpha(n)$ to arbitrary real α via (1.4) and give exact formulas for $p_\alpha(n)$ in the spirit of Rademacher. For real $\alpha > 0$, $n > \alpha/24$, and $m \leq \alpha/24$, we define the functions

$$\nu_\alpha(n) := \sqrt{n - \frac{\alpha}{24}}, \quad \mu_\alpha(m) := \sqrt{\frac{\alpha}{24} - m} \tag{1.6}$$

and the *α-Kloosterman sum*

$$A_k^{(\alpha)}(n,m) := \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{\alpha \pi i s(h,k) + \frac{2\pi i}{k}(mH - nh)}, \tag{1.7}$$

where H denotes an inverse of h modulo k and $s(h,k)$ is the Dedekind sum defined in (1.3). Our exact formulas for $p_\alpha(n)$ are the content of the following theorem.

Theorem 1.1. *For all $\alpha > 0$ and $n > \alpha/24$, we have*

$$p_\alpha(n) = \nu_\alpha(n)^{-\frac{\alpha}{2}-1} \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2}+1} p_\alpha(m) \sum_{k=1}^{\infty} \frac{2\pi}{k} A_k^{(\alpha)}(n,m) I_{\frac{\alpha}{2}+1}\left(\frac{4\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right), \tag{1.8}$$

where $q := \lfloor \frac{\alpha}{24} \rfloor$.

Theorem 1.1 also enables the calculation of explicit error bounds for approximations of $p_\alpha(n)$ obtained by truncating (1.8). These have several implications, including a simple description of the asymptotic behavior of $p_\alpha(n)$ for large n, given in Corollary 1.2.

Corollary 1.2. *For all $\alpha > 0$, as $n \to \infty$, we have*

$$p_\alpha(n) \sim 2\pi \frac{I_{\frac{\alpha}{2}+1}\left(\frac{\pi\alpha}{6}\lambda_\alpha(n)\right)}{\lambda_\alpha(n)^{\frac{\alpha}{2}+1}} \sim \sqrt{\frac{12}{\alpha}} \cdot \frac{e^{\frac{\alpha\pi}{6}\lambda_\alpha(n)}}{\lambda_\alpha(n)^{\frac{\alpha+3}{2}}},$$

where $\lambda_\alpha(n) := \sqrt{\frac{24n}{\alpha} - 1}$.

We remark that because $p_\alpha(n)$ is rational for any $\alpha \in \mathbb{Q}$, Theorem 1.1 implies that the series in (1.8) converges to a rational number when $n \in \mathbb{Z}$, $n > \alpha/24$. We make use of this fact later in the paper (Corollary 4.2) to provide a finite formula for $p_\alpha(n)$ in the case where $\alpha \in \mathbb{Q}$.

When considering a sequence of real numbers, one is often interested in more than just its asymptotic behavior. One property that is often studied is log-concavity. A sequence $\{a(n)\}$ is called *log-concave* if we have

$$a(n+1)^2 - a(n)a(n+2) \geq 0$$

for all n. Nicolas [14] and DeSalvo and Pak [7] independently proved that $p(n)$ is log-concave for $n \geq 25$. In fact, the condition of log-concavity is a special case of what are known as the *higher Turán inequalities* [5]. One can show that a sequence satisfies the higher Turán inequalities of degree d if and only if the Jensen polynomials

$$J_a^{d,n}(x) := \sum_{j=0}^{d} \binom{d}{j} a(n+j)x^j \tag{1.9}$$

have strictly real roots for all n—we say that such a polynomial is *hyperbolic* [6]. Chen, Jia, and Wang [5] conjectured that for any fixed degree d, $J_p^{d,n}(x)$ is eventually hyperbolic, and proved this for $d = 3$; Larson and Wagner [13] independently proved this conjecture for $d \in \{3, 4, 5\}$. Griffin, Ono, Rolen, and Zagier [9] established the conjecture of Chen et al. for all d by showing that, after suitable renormalization, the Jensen polynomials of $p(n)$ converge to the Hermite polynomials $H_d(x)$ as $n \to \infty$. We apply their methods to prove the analogue of Chen et al.'s conjecture for $p_\alpha(n)$.

Theorem 1.3. *For $\alpha > 0$ and $d \in \mathbb{N}$, there exists $N_d(\alpha)$ such that $J_{p_\alpha}^{d,n}(X)$ is hyperbolic for all $n > N_d(\alpha)$.*

Our paper is divided into five main sections. In Section 2, we establish some preliminary results, including a modified version of the Dedekind functional equation for $\eta(\tau)$. In Section 3, we use the circle method along with this identity to prove Theorem 1.1. In Section 4, we use Theorem 1.1 to prove more results about $p_\alpha(n)$, including the estimate given in Corollary 1.2. We also analyze the hyperbolicity of the Jensen polynomials associated with $p_\alpha(n)$. Finally, in Section 5 we provide numerical illustrations of our main theorems.

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2 Proof of the Functional Equation for $P(x)^\alpha$

In order to apply the circle method to $p_\alpha(n)$, we first require a precise statement of Dedekind's functional equation for the eta function. We derive this from Iseki's formula [1, §3.5]. For convenience, when $\mathrm{Re}(x) > 0$, we set

$$\lambda(x) := \sum_{m=1}^{\infty} \frac{e^{-2\pi m x}}{m} = -\log\bigl(1 - e^{-2\pi x}\bigr).$$

Remark. Throughout this section, we let $\log z$ denote the branch of the logarithm with a branch cut along the negative imaginary axis and $\log 1 = 0$, and we define $\arg z := \mathrm{Im}(\log z)$.

2.1 Derivation of the Logarithmic Functional Equation from Iseki's Formula

In order to derive the required modification of the functional equation for $\eta(\tau)$, we first prove a lemma which follows from Iseki's formula [1, §3.5].

Theorem 2.1 (Iseki's Formula). *For* $\mathrm{Re}\, z > 0$*,* $0 < \alpha < 1$*, and* $0 \le \beta \le 1$*, let*

$$\Lambda(\alpha, \beta, z) := \sum_{r=0}^{\infty} \left[\lambda((r+\alpha)z - i\beta) + \lambda((r+1-\alpha)z + i\beta)\right].$$

Then we have

$$\Lambda(\alpha, \beta, z) = \Lambda(1-\beta, \alpha, z^{-1}) - \pi z\left(\alpha^2 - \alpha + \frac{1}{6}\right) + \frac{\pi}{z}\left(\beta^2 - \beta + \frac{1}{6}\right) + 2\pi i\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right). \quad (2.1)$$

Lemma 2.2. *For* $\mathrm{Re}\, z > 0$*, we have*

$$\sum_{r=1}^{\infty} \lambda(rz) = \sum_{r=1}^{\infty} \lambda\left(\frac{r}{z}\right) + \frac{1}{2}\log z - \left(\frac{\pi z}{12} - \frac{\pi}{12 z}\right). \quad (2.2)$$

Proof. Letting $\beta = 0$ in Iseki's formula, we obtain

$$\Lambda(\alpha, 0, z) = \Lambda(1, \alpha, z^{-1}) - \pi z\left(\alpha^2 - \alpha + \frac{1}{6}\right) + \frac{\pi}{6z} - \pi i\left(\alpha - \frac{1}{2}\right). \quad (2.3)$$

From here, bringing $\Lambda(1, \alpha, z^{-1})$ to the left side, reordering the summations, and setting $a(\alpha) := \lambda(\alpha z) - \lambda(i\alpha)$ and

$$b_r(\alpha) := \lambda((r+\alpha)z) + \lambda((r-\alpha)z) - \lambda\left(\frac{r}{z} - i\alpha\right) - \lambda\left(\frac{r}{z} + i\alpha\right)$$

yields

$$a(\alpha) + \sum_{r=1}^{\infty} b_r(\alpha) = -\pi z\left(\alpha^2 - \alpha + \frac{1}{6}\right) + \frac{\pi}{6z} - \pi i\left(\alpha - \frac{1}{2}\right). \quad (2.4)$$

The reordering is valid because the sum over each of the four terms in $b_r(\alpha)$ converges absolutely, since $\lambda(\gamma z) \sim e^{-2\pi\gamma z}$ as $\gamma \to \infty$. We proceed by taking the limit as $\alpha \to 0^+$. We start by observing that

$$\begin{aligned}
\lim_{\alpha\to 0^+} a(\alpha) &= \lim_{\alpha\to 0^+} [\lambda(\alpha z) - \lambda(i\alpha)] \\
&= \lim_{\alpha\to 0^+} \left[\log\bigl(1 - e^{-2\pi i\alpha}\bigr) - \log\bigl(1 - e^{-2\alpha\pi z}\bigr)\right] \\
&= \lim_{\alpha\to 0^+} \log\left(\frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\alpha\pi z}}\right),
\end{aligned} \quad (2.5)$$

where the last step is justified because $\arg(1 - e^{-2\pi i\alpha}) - \arg(1 - e^{-2\alpha\pi z}) \in (-\pi, \pi)$ for $\alpha > 0$. By L'Hôpital's rule,

$$\lim_{\alpha \to 0^+} \frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\alpha\pi z}} = \lim_{\alpha \to 0^+} \frac{2\pi i e^{-2\pi i\alpha}}{2\pi z e^{-2\alpha\pi z}} = \frac{i}{z},$$

and so

$$\lim_{\alpha \to 0^+} a(\alpha) = \log\left(\frac{i}{z}\right) = \frac{\pi i}{2} - \log z,$$

using the fact that $\arg(i/z) \in (0, \pi)$ and $\log(1/z) = -\log z$ for our definition of the logarithm. We now show that

$$\lim_{\alpha \to 0^+} \sum_{r=1}^{\infty} b_r(\alpha) = \sum_{r=1}^{\infty} \lim_{\alpha \to 0^+} b_r(\alpha) = \sum_{r=1}^{\infty} (2\lambda(rz) - 2\lambda(r/z)).$$

For this purpose, start by noting that for $\operatorname{Re} x > 0$, we have $|\lambda(x)| \leq \lambda(\operatorname{Re} x)$ by the series expansion for λ, and that $\lambda(\operatorname{Re} x)$ is monotonically decreasing. In particular, we have

$$|\lambda((r \pm \alpha)z)| \leq \lambda\left(\left(r - \frac{1}{2}\right)\operatorname{Re} z\right) \leq \lambda\left(r \cdot \frac{\operatorname{Re} z}{2}\right),$$

and $|\lambda(rz \pm i\alpha)| \leq \lambda(r \operatorname{Re} z)$. For $x > 0$, we can verify that $\sum_{r=1}^{\infty} \lambda(rx)$ converges by the asymptotic behavior of $\lambda(rx)$ as $r \to \infty$. Consequently, by the discrete version of the dominated convergence theorem, we may exchange the order of the limit and the summation over b_r. Thus, in the limit, (2.4) becomes

$$\frac{\pi i}{2} - \log z + 2\sum_{r=1}^{\infty} \lambda(rz) - 2\sum_{r=1}^{\infty} \lambda\left(\frac{r}{z}\right) = -\frac{\pi z}{6} + \frac{\pi}{6z} + \frac{\pi i}{2}. \tag{2.6}$$

This is equivalent to (2.2). □

For the main theorem of this section, we begin by citing a fact proven in [1, §3.6].

Proposition 2.3. *Let* $\operatorname{Re} z > 0$*, let* $h, k \in \mathbb{Z}$ *be coprime with* $k > 0$*, and choose* H *such that* $hH \equiv -1 \pmod{k}$*. Then we have that*

$$\sum_{\substack{n=1 \\ n \not\equiv 0 \pmod{k}}}^{\infty} \lambda\left(\frac{n}{k}(z - ih)\right) = \sum_{\substack{n=1 \\ n \not\equiv 0 \pmod{k}}}^{\infty} \lambda\left(\frac{n}{k}(z^{-1} - iH)\right) + \left(\frac{\pi z}{12} - \frac{\pi}{12z}\right)\left(1 - \frac{1}{k}\right) + \pi i s(h, k). \tag{2.7}$$

With this fact and Lemma 2.2, we may finally provide the desired logarithmic version of the functional equation for $\eta(\tau)$.

Theorem 2.4. *For* $\operatorname{Re} z > 0$ *and* $h, k, H \in \mathbb{Z}$ *with* $k > 0$*,* $\gcd(h, k) = 1$*, and* $hH \equiv -1 \pmod{k}$*, we have*

$$\sum_{n=1}^{\infty} \lambda\left(\frac{n}{k}(z - ih)\right) = \sum_{n=1}^{\infty} \lambda\left(\frac{n}{k}(z^{-1} - iH)\right) + \frac{1}{k}\left(\frac{\pi}{12z} - \frac{\pi z}{12}\right) + \frac{1}{2}\log z + \pi i s(h, k). \tag{2.8}$$

Proof. Using the periodicity of λ, we note that

$$\sum_{r=1}^{\infty} \lambda(rz) = \sum_{\substack{n=1 \\ n \equiv 0 \pmod{k}}}^{\infty} \lambda\left(\frac{n}{k}(z - ih)\right) \quad \text{and} \quad \sum_{r=1}^{\infty} \lambda\left(\frac{r}{z}\right) = \sum_{\substack{n=1 \\ n \equiv 0 \pmod{k}}}^{\infty} \lambda\left(\frac{n}{k}(z^{-1} - iH)\right).$$

Substituting this into (2.2) and adding equation (2.7) yields the desired result. □

2.2 Application of the Logarithmic Functional Equation to $P(x)^\alpha$

We recall the generating function

$$P(x) := \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=0}^{\infty} p(n)x^n,$$

which is holomorphic for x in the open unit disk. In deriving the Hardy-Ramanujan-Rademacher series formula for the partition function, we rely on the fact that the equation above holds analytically as well as formally. We extend this observation to $p_\alpha(n)$ by showing that the generating function $P(x)^\alpha$ is well-defined.

Lemma 2.5. *For x in the open unit disk and $\alpha > 0$, we have*

$$\sum_{n=0}^{\infty} p_\alpha(n)x^n = \prod_{k=1}^{\infty} e^{-\alpha \log\left(1-x^k\right)} =: P(x)^\alpha. \tag{2.9}$$

Proof. Start by observing that our branch of the logarithm ensures that $\exp\left(-\alpha \log\left(1-x^k\right)\right)$ is formally equivalent to $(1-x^k)^{-\alpha}$. Thus, because the $p_\alpha(n)$ are defined in terms of the formal equivalence in (1.4), it suffices to show that $P(x)^\alpha$ as defined above is holomorphic for $|x| < 1$. For this purpose, let $0 < r < 1$, and observe that $-\sum_{k=1}^{\infty} \alpha \log\left(1-x^k\right)$ converges uniformly for $|x| \leq r$ by the ratio test, as

$$\lim_{k\to\infty} \left| \frac{\log\left(1-x^{k+1}\right)}{\log(1-x^k)} \right| = \lim_{k\to\infty} \left| \frac{-\log(x)x^{k+1}/(1-x^{k+1})}{-\log(x)x^k/(1-x^k)} \right| = \lim_{k\to\infty} \left| x \cdot \frac{1-x^k}{1-x^{k+1}} \right| = |x| \leq r.$$

Thus, $\prod_{k=1}^{\infty} e^{-\alpha \log\left(1-x^k\right)}$ converges uniformly for $|x| \leq r$, from which it follows that $P(x)^\alpha$ is holomorphic in every closed disk $|x| \leq r$ and hence in the open unit disk $|x| < 1$ as desired. □

We are finally ready for the main result of this section, which expresses the functional equation for $\eta(\tau)$ in terms of $P(x)^\alpha$.

Theorem 2.6 (Modified Functional Equation)**.** *For* $\operatorname{Re} z > 0$*,* $\alpha > 0$*,* $h, k, H \in \mathbb{Z}$ *with* $k > 0$*,* $\gcd(h,k) = 1$*, and* $hH \equiv -1 \pmod{k}$*, we have*

$$P(x)^\alpha = e^{\pi i \alpha s(h,k)} \left(\frac{z}{k}\right)^{\alpha/2} \exp\left(\frac{\alpha\pi}{12k}\left(\frac{k}{z} - \frac{z}{k}\right)\right) P(x')^\alpha, \tag{2.10}$$

where

$$x := \exp\left(\frac{2\pi}{k}\left(ih - \frac{z}{k}\right)\right), \qquad x' := \exp\left(\frac{2\pi}{k}\left(iH - \frac{k}{z}\right)\right), \tag{2.11}$$

and real powers are given for the precise branch of the logarithm described in Section 2.1.

Proof. Applying Theorem 2.4 with z/k in place of z and multiplying by α, we obtain

$$\alpha \sum_{n=1}^{\infty} \lambda\left(\frac{n}{k}\left(\frac{z}{k} - ih\right)\right) = \frac{\alpha\pi}{12k}\left(\frac{k}{z} - \frac{z}{k}\right) + \frac{\alpha}{2}\log\left(\frac{z}{k}\right) + \pi i \alpha s(h,k) + \alpha \sum_{n=1}^{\infty} \lambda\left(\frac{n}{k}\left(\frac{k}{z} - iH\right)\right).$$

Exponentiating both sides yields

$$\prod_{n=1}^{\infty} \exp\left(-\alpha \log(x)\right) = \exp\left(\frac{\alpha\pi}{12k}\left(\frac{k}{z} - \frac{z}{k}\right)\right) \exp\left(\frac{\alpha}{2}\log\left(\frac{z}{k}\right)\right) e^{\pi i \alpha s(h,k)} \prod_{n=1}^{\infty} \exp\left(-\alpha \log(x')\right) \tag{2.12}$$

for x and x' defined above, which is equivalent to (2.10). □

3 Proof of the Series Formula for $p_\alpha(n)$

In this section, we use Radamacher's circle method to prove the series formula for $p_\alpha(n)$. We closely follow Apostol's proof of the $\alpha = 1$ case [1, §5.7].

Proof of Theorem 1.1. Using Cauchy's residue theorem and Lemma 2.5, we can write

$$p_\alpha(n) = \frac{1}{2\pi i}\int_C \frac{P(x)^\alpha}{x^{n+1}}\,dx, \tag{3.1}$$

where C is any simple closed contour in the unit disk which encloses the origin. To evaluate this, we consider the change of variables $x = e^{2\pi i\tau}$, under which the closed unit disk $|x| \le 1$ is the image of the infinite vertical strip $\{\tau : 0 \le \operatorname{Re}\tau \le 1,\ 0 \le \operatorname{Im}\tau\}$. We start by recalling the Farey sequences F_N, defined by enumerating the rational numbers in $[0,1]$ with reduced denominators at most N. In addition, for $\gcd(h,k) = 1$, we let $C(h,k)$ denote the Ford circle associated with h/k, which has center $h/k + i/(2k^2)$ and radius $1/(2k^2)$ (details are given in [1, §5.6]). As in Rademacher's original work, we integrate along the Rademacher paths $R(N)$ in the τ-plane, consisting of the upper arcs of the Ford circles associated with F_N, with the intent to later take the limit as $N \to \infty$ (depicted in Figure 2). For $N \ge 1$, we write (3.1) as

$$p_\alpha(n) = \int_{R(N)} P(e^{2\pi i\tau})^\alpha e^{-2\pi i n\tau}\,d\tau. \tag{3.2}$$

Decomposing $R(N)$ into its component arcs, we may write the above integral as

$$\int_{R(N)} = \sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}}\int_{\gamma(h,k)} =: \sum_{h,k}\int_{\gamma(h,k)}, \tag{3.3}$$

where we define the right side as a shorthand for the double sum over h and k, and $\gamma(h,k)$ is the upper arc of the Ford circle $C(h,k)$ of radius $1/(2k^2)$ tangent to the real axis at h/k.

We now introduce a second change of variables given by

$$z = -ik^2\left(\tau - \frac{h}{k}\right), \tag{3.4}$$

which maps the circle $C(h,k)$ onto the circle K of radius $1/2$ centered at $1/2$. Let $z_1(h,k)$ and $z_2(h,k)$ be the respective endpoints of the image of $\gamma(h,k)$, and let x and x' be defined as in Theorem 2.6. Then

$$p_\alpha(n) = \sum_{h,k} ik^{-2}e^{-\frac{2\pi i nh}{k}}\int_{z_1(h,k)}^{z_2(h,k)} e^{\frac{2n\pi z}{k^2}}P(x)^\alpha\,dz,$$

from which the modified functional equation from Theorem 2.6 yields

$$p_\alpha(n) = \sum_{h,k} ik^{-\frac{\alpha}{2}-2}e^{-\frac{2\pi i nh}{k}}\omega^{(\alpha)}(h,k)\int_{z_1(h,k)}^{z_2(h,k)} e^{\frac{2\pi nz}{k^2}}\Psi_k^{(\alpha)}(z)P(x')^\alpha\,dz,$$

where

$$\omega^{(\alpha)}(h,k) := e^{\alpha\pi i s(h,k)}, \quad \text{and} \quad \Psi_k^{(\alpha)}(z) := z^{\frac{\alpha}{2}}\exp\left(\frac{\alpha\pi}{12z} - \frac{\alpha\pi z}{12k^2}\right).$$

Let $q = \lfloor \alpha/24 \rfloor$, and define

$$Q^{(\alpha)}(x) := \sum_{m=0}^{q} p_\alpha(m)x^m.$$

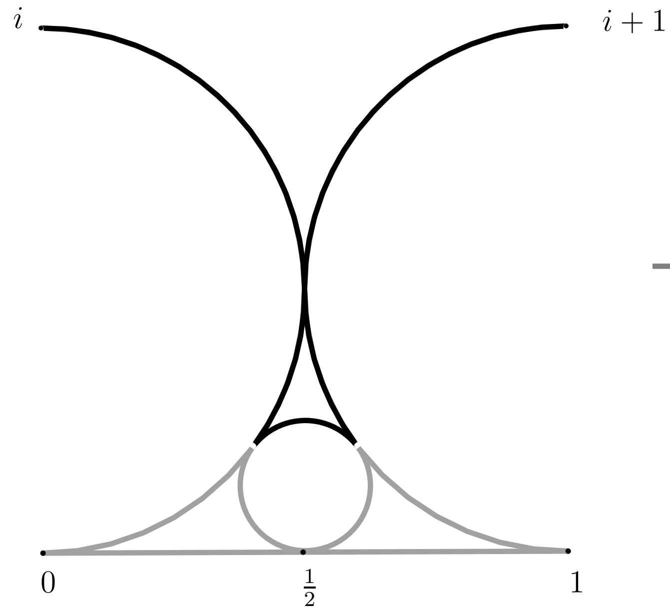

Figure 1: The Rademacher path $R(2)$

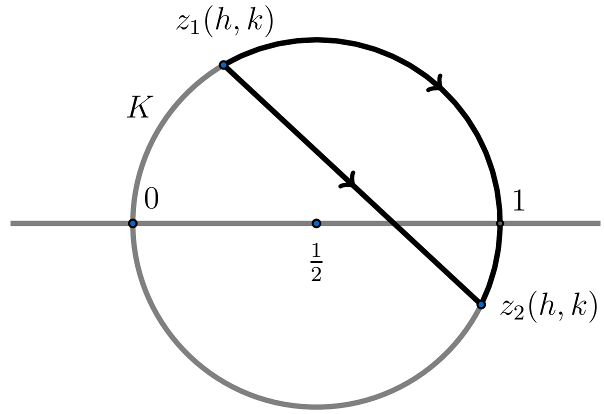

Figure 2: Path of integration in the z-plane

We proceed by separating out a part of the integral that corresponds to $Q^{(\alpha)}(x)$ and showing that the remaining part goes to zero as $N \to \infty$. In particular, we write

$$I_1(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k^{(\alpha)}(z) e^{\frac{2\pi n z}{k^2}} Q^{(\alpha)}(x') dz$$

and

$$I_2(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k^{(\alpha)}(z) e^{\frac{2\pi n z}{k^2}} \left(P(x')^\alpha - Q^{(\alpha)}(x')\right) dz$$

to obtain

$$p_\alpha(n) = \sum_{h,k} i k^{-2-\frac{\alpha}{2}} e^{-\frac{2\pi i n h}{k}} \omega^{(\alpha)}(h,k) \cdot (I_1(h,k) + I_2(h,k)). \tag{3.5}$$

We now show that $I_2(h,k)$ is "small" for large N by considering the integral along the chord in the z-plane joining $z_1(h,k)$ and $z_2(h,k)$. Because $0 < \operatorname{Re} z \leq 1$ and $\operatorname{Re}(z^{-1}) \geq 1$ for z on the path of integration, we can write

$$\left| \Psi_k^{(\alpha)}(z) \cdot e^{\frac{2n\pi z}{k^2}} \cdot \left\{ P(x')^\alpha - Q^{(\alpha)}(x') \right\} \right| \tag{3.6}$$

$$= |z|^{\frac{\alpha}{2}} \exp\left(\frac{\alpha\pi}{12} \operatorname{Re}(z^{-1}) - \frac{\alpha\pi}{12k^2} \operatorname{Re} z + \frac{2n\pi}{k^2} \operatorname{Re} z \right) \cdot \left| \sum_{m=q+1}^{\infty} p_\alpha(m) \exp\left(\frac{2\pi i H m}{k} - \frac{2\pi m}{z} \right) \right|$$

$$\leq |z|^{\frac{\alpha}{2}} \exp\left(\frac{\alpha\pi}{12} \operatorname{Re}(z^{-1}) + \frac{2n\pi}{k^2} \right) \sum_{m=q+1}^{\infty} p_\alpha(m) e^{-2\pi m \operatorname{Re}(z^{-1})}$$

$$\leq |z|^{\frac{\alpha}{2}} \sum_{m=q+1}^{\infty} p_\alpha(m) e^{-2\pi\left(m - \frac{\alpha}{24}\right) \operatorname{Re}(z^{-1})} \tag{3.7}$$

$$\leq |z|^{\frac{\alpha}{2}} \sum_{m=q+1}^{\infty} p_\alpha(m) e^{-2\pi\left(m - \frac{\alpha}{24}\right)} = |z|^{\frac{\alpha}{2}} e^{\frac{\alpha\pi}{12}} (P(e^{-2\pi})^\alpha - Q^{(\alpha)}(e^{-2\pi})). \tag{3.8}$$

Since $|z| < \sqrt{2}k/N$ for z on the chord from $z_1(h,k)$ to $z_2(h,k)$, the integrand is less than $C(k/N)^{\alpha/2}$ for some constant C not depending on N. Thus, because the length of the chord is at most $2\sqrt{2}k/N$, we have

$$|I_2(h,k)| < \frac{Ck^{\frac{\alpha}{2}+1}}{N^{\frac{\alpha}{2}+1}}. \tag{3.9}$$

Substituting this bound into the sum of the I_2 terms in (3.5) yields

$$\left|\sum_{h,k} ik^{-\frac{\alpha}{2}-2}e^{-\frac{2\pi inh}{k}}\omega^{(\alpha)}(h,k)I_2(h,k)\right| < \sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} Ck^{-1}N^{-\frac{\alpha}{2}-1} \le CN^{-\frac{\alpha}{2}-1}\sum_{k=1}^{N}1 = CN^{-\frac{\alpha}{2}}.$$

Thus, we have

$$p_\alpha(n) = \left(\sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-\frac{\alpha}{2}-2}e^{-\frac{2\pi inh}{k}}\omega^{(a)}(h,k)I_1(h,k)\right) + O(N^{-\frac{\alpha}{2}}). \tag{3.10}$$

Next we consider $I_1(h,k)$. We can write

$$I_1(h,k) = \int_{-K} - \int_0^{z_1(h,k)} - \int_{z_2(h,k)}^0 =: \int_{-K} - J_1 - J_2, \tag{3.11}$$

where we omit the integrands for brevity, and where $-K$ indicates that we integrate in the negative direction along K. Because $|z| \le \sqrt{2}k/N$ on the paths of integration, we can bound the integrands of J_1 and J_2 by

$$\begin{aligned}
&\left|\Psi_k^{(\alpha)}(z)e^{\frac{2\pi nz}{k^2}}Q^{(\alpha)}(x')\right| \\
&\le |z|^{\frac{\alpha}{2}}\exp\left(\frac{\alpha\pi}{12}\operatorname{Re}(z^{-1}) - \frac{\alpha\pi}{12k^2}\operatorname{Re}z + \frac{2n\pi}{k^2}\operatorname{Re}z\right)\left|\sum_{m=0}^{q}p_\alpha(m)\exp\left(\frac{2\pi iHm}{k} - \frac{2\pi m}{z}\right)\right| \\
&\le |z|^{\frac{\alpha}{2}}\exp\left(\frac{\alpha\pi}{12} + \frac{2\pi}{k^2}\left(n - \frac{\alpha}{24}\right)\operatorname{Re}z\right)\left|\sum_{m=0}^{q}p_\alpha(m)e^{-2\pi m}\right| &(3.12)\\
&\le \frac{e^{2n\pi}2^{\frac{\alpha}{4}}k^{\frac{\alpha}{2}}}{N^{\frac{\alpha}{2}}}\left|\sum_{m=0}^{q}p_\alpha(m)e^{-2\pi m}\right|. &(3.13)
\end{aligned}$$

The lengths of the arcs from 0 to $z_1(h,k)$ and $z_2(h,k)$ are less than $\pi|z_1(h,k)|$ and $\pi|z_2(h,k)|$, respectively, and both of these are bounded by $\pi\sqrt{2}k/N$, so we get that $|J_1|, |J_2| < C_1k^{\frac{\alpha}{2}+1}N^{-\frac{\alpha}{2}-1}$ for some constant C_1.

Combining (3.10), (3.11), and the bounds for J_1 and J_2 above, we find that

$$p_\alpha(n) = \sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-\frac{\alpha}{2}-2}e^{-\frac{2\pi inh}{k}}\omega^{(\alpha)}(h,k)\int_{-K}\Psi_k^{(\alpha)}(z)e^{\frac{2n\pi z}{k^2}}Q^{(\alpha)}(x')\,dz + O(N^{-\frac{\alpha}{2}}), \tag{3.14}$$

which in the limit as N goes to infinity becomes

$$\begin{aligned}
p_\alpha(n) = & \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} ik^{-\frac{\alpha}{2}-2} e^{-\frac{2\pi inh}{k}} \omega^{(\alpha)}(h,k) \\
& \cdot \int_{-K} z^{\frac{\alpha}{2}} \exp\left(\frac{2\pi nz}{k^2} + \frac{\alpha\pi}{12z} - \frac{\alpha\pi z}{12k^2} + \frac{2\pi imH}{k} - \frac{2\pi m}{z}\right) dz \\
= & \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} ik^{-\frac{\alpha}{2}-2} e^{\frac{2\pi i}{k}(mH-nh)} \omega^{(\alpha)}(h,k) \\
& \cdot \int_{-K} z^{\frac{\alpha}{2}} \exp\left(\frac{2\pi z}{k^2}\nu_\alpha(n)^2 + \frac{2\pi}{z}\mu_\alpha(m)^2\right) dz \\
= & \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} i\frac{A_k^{(\alpha)}(n,m)}{k^{\frac{\alpha}{2}+2}} \int_{-K} z^{\frac{\alpha}{2}} \exp\left(\frac{2\pi z}{k^2}\nu_\alpha(n)^2 + \frac{2\pi}{z}\mu_\alpha(m)^2\right) dz.
\end{aligned}$$

To evaluate the integral on the right, we make the change of variables $t = 2\pi(\alpha/24-m)/z$ to obtain

$$\begin{aligned}
p_\alpha(n) = 2\pi & \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \frac{A_k^{(\alpha)}(n,m)}{k^{\frac{\alpha}{2}+2}} \left[2\pi\mu_\alpha(n)^2\right]^{\frac{\alpha}{2}+1} \\
& \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\frac{\alpha}{2}-2} \exp\left(t + \left(\frac{2\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right)^2 \frac{1}{t}\right) dz,
\end{aligned}$$

where $c = \alpha\pi/12$. Now recall that the modified Bessel function of the first kind satisfies

$$I_\beta(z) = \frac{(z/2)^\beta}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-\beta-1} e^{t+\frac{z^2}{4t}}\, dt \tag{3.15}$$

for $c > 0, \mathrm{Re}(\nu) > 0$ [17, p. 181]. Consequently, for $n \ge \alpha/24$, we find that

$$p_\alpha(n) = 2\pi \sum_{m=0}^{q} p_\alpha(m) \sum_{k=1}^{\infty} \frac{A_k^{(\alpha)}(n,m)}{k^{\frac{\alpha}{2}+2}} \left(2\pi\mu_\alpha(m)^2\right)^{\frac{\alpha}{2}+1} \left(\frac{2\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right)^{-\frac{\alpha}{2}-1} I_{\frac{\alpha}{2}+1}\left(\frac{4\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right) \tag{3.16}$$

$$= \nu_\alpha(n)^{-\frac{\alpha}{2}-1} \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2}+1} p_\alpha(m) \sum_{k=1}^{\infty} \frac{2\pi}{k} A_k^{(\alpha)}(n,m) I_{\frac{\alpha}{2}+1}\left(\frac{4\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right). \tag{3.17}$$

□

4 Applications of the Series Formula for $p_\alpha(n)$

4.1 Estimates of $p_\alpha(n)$

In this section, we consider the error of the approximation

$$p_\alpha(n;\delta) := \nu_\alpha(n)^{-\frac{\alpha}{2}-1} \sum_{m=0}^{q} \mu_\alpha(m)^{\frac{\alpha}{2}+1} p_\alpha(m) \sum_{1\le k<\frac{2\pi}{\delta}\mu_\alpha(m)} \frac{2\pi}{k} A_k^{(\alpha)}(n,m) I_{\frac{\alpha}{2}+1}\left(\frac{4\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right). \tag{4.1}$$

for $p_\alpha(n)$. Note in particular that in the limit as $\delta \to 0^+$, we have $p_\alpha(n;\delta) \to p_\alpha(n)$.

Theorem 4.1. *For all $\alpha > 0$, $0 < \delta < 2\pi\mu_\alpha(0)$, and $n > \alpha/24$, we have*

$$|p_\alpha(n) - p_\alpha(n;\delta)| < \frac{C}{\delta}\frac{I_{\frac{\alpha}{2}+1}(2\delta\nu_\alpha(n))}{\nu_\alpha(n)^{\frac{\alpha}{2}+1}} < C\delta^{\frac{\alpha}{2}}\frac{I_{\frac{\alpha}{2}+1}(4\pi\mu_\alpha(0)\nu_\alpha(n))}{(2\pi\mu_\alpha(0)\nu_\alpha(n))^{\frac{\alpha}{2}+1}}, \tag{4.2}$$

where

$$C := 4\pi^2\left(1+\frac{2}{\alpha}\right)\mu_\alpha(0)\sum_{m=0}^{q}\mu_\alpha(m)^{\frac{\alpha}{2}+1}p_\alpha(m).$$

Proof. Start by noting that

$$\left|A_k^{(\alpha)}(n,m)\right| \le \sum_{\substack{0\le h<k\\(h,k)=1}}\left|e^{\alpha\pi i s(h,k)+\frac{2\pi i}{k}(mH-nh)}\right| = \sum_{\substack{0\le h<k\\(h,k)=1}} 1 \le k. \tag{4.3}$$

Moreover, using the fact from [15] that for $0 < x < y$ and $\nu > 1$, the modified Bessel function of the first kind satisfies

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^\nu, \tag{4.4}$$

we have that

$$\begin{aligned}\sum_{k\ge\frac{2\pi}{\delta}\mu_\alpha(m)}\frac{I_{\frac{\alpha}{2}+1}\left(\frac{4\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right)}{I_{\frac{\alpha}{2}+1}(2\delta\nu_\alpha(n))} &< \sum_{k\ge\frac{2\pi}{\delta}\mu_\alpha(m)}\left(\frac{2\pi}{k\delta}\mu_\alpha(m)\right)^{\frac{\alpha}{2}+1}\\ &< 1+\int_{\frac{2\pi}{\delta}\mu_\alpha(m)}^{\infty}\left(\frac{2\pi}{t\delta}\mu_\alpha(m)\right)^{\frac{\alpha}{2}+1}dt\\ &= 1+\frac{4\pi}{\alpha\delta}\mu_\alpha(m)\end{aligned}$$

for $0 \le m \le q$. Thus, we find that

$$\begin{aligned}\nu_\alpha(n)^{\frac{\alpha}{2}+1}|p_\alpha(n)-p_\alpha(n;\delta)| &\le 2\pi\sum_{m=0}^{q}\mu_\alpha(m)^{\frac{\alpha}{2}+1}p_\alpha(m)\sum_{k\ge\frac{2\pi}{\delta}\mu_\alpha(m)} I_{\frac{\alpha}{2}+1}\left(\frac{4\pi}{k}\nu_\alpha(n)\mu_\alpha(m)\right)\\ &< 2\pi I_{\frac{\alpha}{2}+1}(2\delta\nu_\alpha(n))\sum_{m=0}^{q}\mu_\alpha(m)^{\frac{\alpha}{2}+1}p_\alpha(m)\left[1+\frac{4\pi}{\alpha\delta}\mu_\alpha(m)\right].\end{aligned}$$

Since $1 < \frac{2\pi}{\delta}\mu_\alpha(0)$ and $\mu_\alpha(m) \le \mu_\alpha(0)$, it follows that

$$|p_\alpha(n)-p_\alpha(n;\delta)| < \frac{4\pi^2}{\delta}\frac{I_{\frac{\alpha}{2}+1}(2\delta\nu_\alpha(n))}{\nu_\alpha(n)^{\frac{\alpha}{2}+1}}\left(1+\frac{2}{\alpha}\right)\mu_\alpha(0)\sum_{m=0}^{q}\mu_\alpha(m)^{\frac{\alpha}{2}+1}p_\alpha(m),$$

or applying the Paris inequality a second time using $2\delta\nu_\alpha(n) < 4\pi\mu_\alpha(0)\nu_\alpha(n)$,

$$|p_\alpha(n)-p_\alpha(n;\delta)| < 4\pi^2\delta^{\frac{\alpha}{2}}\frac{I_{\frac{\alpha}{2}+1}(4\pi\mu_\alpha(0)\nu_\alpha(n))}{(2\pi\mu_\alpha(0)\nu_\alpha(n))^{\frac{\alpha}{2}+1}}\left(1+\frac{2}{\alpha}\right)\mu_\alpha(0)\sum_{m=0}^{q}\mu_\alpha(m)^{\frac{\alpha}{2}+1}p_\alpha(m).$$

□

We are now in a position to prove the simple asymptotic formula for $p_\alpha(n)$ stated in the introduction.

Proof of Corollary 1.2. Observe that since $\frac{4\pi}{\delta}\mu_\alpha(m)$ is strictly increasing in m, there exists a $0 < \delta < 2\pi\mu_\alpha(0)$ such that $\frac{2\pi}{\delta}\mu_\alpha(m) \leq 2$ for $0 < m \leq q$ and so

$$p_\alpha(n;\delta) = 2\pi \left(\frac{\mu_\alpha(0)}{\nu_\alpha(n)}\right)^{\frac{\alpha}{2}+1} I_{\frac{\alpha}{2}+1}\left(4\pi\nu_\alpha(n)\mu_\alpha(0)\right) = 2\pi \frac{I_{\frac{\alpha}{2}+1}\left(\frac{\pi\alpha}{6}\lambda_\alpha(n)\right)}{\lambda_\alpha(n)^{\frac{\alpha}{2}+1}}.$$

Moreover, by Theorem 4.1, we have

$$|p_\alpha(n) - p_\alpha(n;\delta)| \leq C\frac{I_{\frac{\alpha}{2}+1}(2\delta\nu_\alpha(n))}{\nu_\alpha(n)^{\frac{\alpha}{2}+1}}$$

for some constant C. Using the fact that $I_\nu(z) \sim e^z/\sqrt{2\pi z}$ from [8, 10.30.4], we easily verify that $C\nu_\alpha(n)^{-\frac{\alpha}{2}-1} I_{\frac{\alpha}{2}+1}(2\delta\nu_\alpha(n)) \ll p_\alpha(n)$, from which it follows that

$$p_\alpha(n) \sim p_\alpha(n;\delta) \sim \frac{e^{\frac{\alpha\pi}{6}\lambda_\alpha(n)}}{\lambda_\alpha(n)^{\frac{\alpha+3}{2}}}.$$

□

Theorem 4.1 also allows us to derive a finite exact formula for $p_\alpha(n)$ when α is rational. This is made possible by a formula for the denominator of $p_\alpha(n)$ from [4], which states that if $\alpha = a/b$ for coprime $a, b \in \mathbb{Z}$ with $b > 0$, then

$$\operatorname{denom}(p_\alpha(n)) := b^n \prod_{p|b} p^{\operatorname{ord}_p(n!)},$$

where $\operatorname{ord}_p(n)$ denotes the multiplicity of a prime p as a factor of n.

Corollary 4.2. *Let $\alpha, \varepsilon > 0$ and $n > \alpha/24$ with α rational. Then*

$$p_\alpha(n) = \frac{\lfloor Dp_\alpha(n;\delta)\rceil}{D}, \tag{4.5}$$

where $D = \operatorname{denom}(p_\alpha(n))$ and

$$\delta := \left(\frac{(2\pi\mu_\alpha(0)\nu_\alpha(n))^{\frac{\alpha}{2}+1}}{2DCI_{\frac{\alpha}{2}+1}(4\pi\mu_\alpha(0)\nu_\alpha(n))}\right)^{\frac{2}{\alpha}},$$

with C defined as in Theorem 4.1.

Proof. Observe that by Theorem 4.1, we have

$$|p_\alpha(n) - p_\alpha(n;\delta)| < C\delta^{\frac{\alpha}{2}} \frac{I_{\frac{\alpha}{2}+1}(4\pi\mu_\alpha(0)\nu_\alpha(n))}{(2\pi\mu_\alpha(0)\nu_\alpha(n))^{\frac{\alpha}{2}+1}} = \frac{1}{2D}. \tag{4.6}$$

Thus, $D|p_\alpha(n) - p_\alpha(n;\delta)| < 1/2$, implying that $Dp_\alpha(n)$ is the nearest integer to $Dp_\alpha(n;\delta)$. □

4.2 Hyperbolicity of the Jensen Polynomials of $p_\alpha(n)$

In this section, we demonstrate how the asymptotics of $p_\alpha(n)$ in this paper can be used to generalize a recent hyperbolicity result for the usual partition function.

Proof of Theorem 1.3. Set

$$m = \frac{\alpha}{24} \quad \text{and} \quad c_0 = \log\left(\sqrt{\frac{12}{\alpha}} \cdot \left(\frac{\alpha}{24}\right)^{\frac{\alpha+3}{4}}\right).$$

Then by Corollary 1.2,

$$p_\alpha(n) \sim e^{c_0 + 4\pi\sqrt{mn}} n^{-\frac{\alpha+3}{4}}.$$

Thus, as in [9, §3], we have

$$\log\left(\frac{p_\alpha(n+j)}{p_\alpha(n)}\right) \sim 4\pi\sqrt{m}\sum_{i=1}^{\infty}\binom{1/2}{i}\frac{j^i}{n^{i-1/2}} - \frac{\alpha+3}{4}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}j^i}{in^i},$$

from which it is clear that $p_\alpha(n)$ satisfies the conditions of Theorem 3 from [9] with $A(n) = 2\pi\sqrt{m/n} + O(1/n)$ and $\delta(n) = (\pi/2)^{1/2}m^{1/4}n^{-3/4} + O(n^{-5/4})$. It follows immediately that for all d the Jensen polynomials associated with $p_\alpha(n)$ are hyperbolic for sufficiently large n. □

Remark. The proof of Theorem 1.3 follows [9, §3]. In particular, we consider the renormalization of the Jensen polynomials given by

$$\widehat{J}_{p_\alpha}^{d,n}(X) = \frac{\delta(n)^{-d}}{p_\alpha(n)} \cdot J_{p_\alpha}^{d,n}\left(\frac{\delta(n)X - 1}{\exp(A(n))}\right). \tag{4.7}$$

Theorem 1.3 follows from the fact that for fixed d,

$$\lim_{n\to\infty} \widehat{J}_{p_\alpha}^{d,n}(X) = H_d(x), \tag{4.8}$$

where $H_d(x)$ is the degree d renormalized Hermite polynomial in [9].

5 Numerical Data[1]

In this section, we illustrate the theorems of the previous sections using numerical examples. For simplicity, we limit our examples to cases where $0 < \alpha < 24$. For such α, it will be convenient to define

$$r_\alpha(n;m) = \frac{\mathrm{Re}\left(p_\alpha\left(n; \frac{2\pi\mu_\alpha(0)}{m+1}\right)\right)}{p_\alpha(n)}, \tag{5.1}$$

the ratio between the real part of the m-term approximation to $p_\alpha(n)$ and the actual value. Note that a value of $r_\alpha(n;m)$ closer to 1 indicates that the m-term approximation to $r_\alpha(n)$ is more accurate.

By Corollary 1.2, we know that $p_\alpha(n)$ is asymptotically equivalent to the first term in the series expansion in Theorem 1.1 as n goes to infinity. Table 1 displays the accuracy of the first-term expansion for $\alpha = e$ and n varying from 1 to 10. Table 3 shows the ratio of both the first-term and the five-term approximation to $p_\alpha(n)$ where $\alpha = 1/\pi$ and $\alpha = 5$. Note that the sign of the error term $|p_\alpha(n) - p_\alpha(n;m)|$ is usually periodic with period $m+1$. This is a consequence of the periodicity of the Kloosterman sums.

Table 2 displays how $p_\alpha(n,m)$ converges to $p_\alpha(n)$ for $\alpha = 1/e, n = 50$, and $1 \le m \le 10$. Table 4 displays the ratio of the m-term approximation of $p_\alpha(n)$ to the actual value for $n = 100$ and various

n	$p_e(n)$	$\mathrm{Re}(p_e(n;1))$	$r_e(n;1)$
1	2.71	2.83	1.04253
2	7.77	7.65	0.98444
3	18.05	18.23	1.01014
4	40.26	39.96	0.99263
5	81.84	82.28	1.00543
6	161.99	161.41	0.9964
7	303.75	304.41	1.00217
8	556.32	555.61	0.99873
9	985.41	986.27	1.00086
10	1710.31	1709.07	0.99927

Table 1: Accuracy of first-term approximations to $p_e(n)$

m	$\mathrm{Re}(p_{1/e}(50;m))$	$r_{1/e}(50;m)$
1	356.2898	0.997668
2	357.2586	1.000381
3	357.1278	1.000014
4	357.053	0.999805
5	357.1236	1.000003
6	357.1195	0.999991
7	357.1169	0.999984
8	357.1208	0.999995
9	357.1201	0.999993
10	357.1296	1.00002

Table 2: Accuracy of m-term approximations to $p_{1/e}(50) = 357.1225$

n	$r_{1/\pi}(n;1)$	$r_{1/\pi}(n;5)$	$r_5(n;1)$	$r_5(n;5)$
1	1.294180591	0.953980957	1.015286846	1.000097277
2	0.970982400	0.982523054	0.994583967	1.000042848
3	1.083673986	1.018088216	1.002732222	1.000007177
4	0.923295102	1.02170408	0.998466124	0.999992664
5	1.124698668	1.016001474	1.000871244	0.999999382
6	0.897139773	1.004350338	0.999524823	1.000000088
7	1.108496000	0.978153497	1.000255655	1.000000217
8	0.943494666	1.002688299	0.999854031	1.000000092
9	1.034408356	1.003218418	1.000093623	0.999999982
10	0.961090657	1.005487344	0.999935881	0.999999997
11	1.076769973	0.993996646	1.000043109	0.999999968
12	0.923558631	1.005396386	0.999972215	1.000000007
13	1.058750442	0.996292489	1.000017874	1.000000008
14	0.980265489	0.993723758	0.999987986	1.000000000

Table 3: Accuracy of approximation to $p_\alpha(n)$ as n increases

m	$r_{0.01}(100;m)$	$r_{0.1}(100;m)$	$r_1(100;m)$	$r_{10}(100;m)$
1	0.846079580	0.988058877	0.999998178	1.000000000
2	0.969774117	0.999386989	1.000000009	1.000000000
3	0.920711483	0.997246602	0.999999995	1.000000000
4	0.973881495	0.999016179	0.999999999	1.000000000
5	1.040636574	1.000923931	1.000000000	1.000000000
6	1.028999226	1.000579623	1.000000000	1.000000000
7	1.020829553	1.000421683	1.000000000	1.000000000
8	0.995326778	0.999817677	1.000000000	1.000000000
9	0.995461037	0.999846688	1.000000000	1.000000000
10	1.011689149	1.000211135	1.000000000	1.000000000

Table 4: Accuracy of approximation to $p_\alpha(n)$ as number of terms in series increases

values of α and m. As we increase α, we see that the relative error of the approximation for $p_\alpha(n)$ decreases.

Table 5 depicts the convergence of $\widehat{J}^{2,n}_{p_\alpha}(X)$ to the Hermite polynomial $H_2(x) = x^2 - 2$, and the convergence of $\widehat{J}^{3,n}_{p_\alpha}(X)$ to the Hermite polynomial $H_3(x) = x^3 - 6x$. Here,

$$A(n) = 2\pi\sqrt{\frac{\alpha}{24n-\alpha} - \frac{24}{24n-\alpha}}, \quad \text{and} \quad \delta(n) = \sqrt{\frac{12\pi\alpha^{\frac{1}{2}}}{(24n-\alpha)^{\frac{3}{2}}} - \frac{288\alpha}{(24n-\alpha)^2}}$$

as in Theorem 1.3, for $\sqrt{3}$. To compute $p_\alpha(n)$ for large n, we used the 100-term approximation of our series formula; this is valid for our purposes because by Theorem 4.1, the relative error $|r_{\sqrt{3}}(n, 100) - 1|$ is bounded by 10^{-75} for the values of n we consider.

n	$\widehat{J}^{2,n}_{p_{\sqrt{3}}}(x)$	$\widehat{J}^{3,n}_{p_{\sqrt{3}}}(x)$
10000	$0.999598x^2 + 0.120905x - 2.03828$	$0.999942x^3 + 0.0939817x^2 - 6.03526x - 0.648632$
20000	$0.999804x^2 + 0.0966267x - 2.02711$	$0.999971x^3 + 0.0767061x^2 - 6.02522x - 0.543473$
30000	$0.999871x^2 + 0.0852795x - 2.02216$	$0.999981x^3 + 0.0683801x^2 - 6.0207x - 0.495049$
40000	$0.999904x^2 + 0.0782302x - 2.0192$	$0.999986x^3 + 0.0631174x^2 - 6.01799x - 0.435239$
50000	$0.999923x^2 + 0.0732538x - 2.01719$	$1.00252x^3 + 0.0595086x^2 - 6.03131x - 0.429626$
$\vdots$	$\vdots$	$\vdots$
∞	$x^2 - 2$	$x^3 - 6x$

Table 5: Convergence to the Hermite polynomial of degree 2, $x^2 - 2$, and of degree 3, $x^3 - 6x$

In Table 6, we provide the actual value of $p_{51/7}(n)$ alongside the minimum number $M_{51/7}(n)$ for which Corollary 4.2 guarantees that $p_{51/7}(n)$ is given by a suitable rounding of $p_\alpha\left(n; \frac{2\pi\mu_\alpha(0)}{M_{51/7}(n)+1}\right)$, which has $M_{51/7}(n)$ terms. We also provide $M^*_{51/7}(n)$, the minimum number of terms such that this is numerically true.

n	$p_{51/7}(n)$	$M_{51/7}(n; D)$	$M^*_{51/7}(n; D)$
1	51/7	2	1
2	1836/49	3	2
3	52751/343	5	3
4	1322226/2401	8	4
5	29852442/16807	14	7
6	623075585/117649	23	10
7	85346705106/5764801	67	26
8	1583888229297/40353607	114	43
9	28093059550223/282475249	194	63
10	479246612549889/1977326743	330	109

Table 6: Number of terms for exact solution for $p_{51/7}(n)$

[1] All computations in this section were done with Wolfram Mathematica.

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