Affinely Representable Lattices, Stable Matchings, and Choice Functions

Yuri Faenza and Xuan Zhang(✉)

Columbia University, New York, NY 10027, USA
{yf2414,xz2569}@columbia.edu

Abstract. Birkhoff's representation theorem [11] defines a bijection between elements of a distributive lattice $\mathcal{L}$ and the family of upper sets of an associated poset $\mathcal{B}$. When elements of $\mathcal{L}$ are the stable matchings in an instance of Gale and Shapley's marriage model, Irving et al. [22] showed how to use $\mathcal{B}$ to devise a combinatorial algorithm for maximizing a linear function over the set of stable matchings. In this paper, we introduce a general property of distributive lattices, which we term as affine representability, and show its role in efficiently solving linear optimization problems over the elements of a distributive lattice, as well as describing the convex hull of the characteristic vectors of lattice elements. We apply this concept to the stable matching model with path-independent quota-filling choice functions, thus giving efficient algorithms and a compact polyhedral description for this model. To the best of our knowledge, this model generalizes all models from the literature for which similar results were known, and our paper is the first that proposes efficient algorithms for stable matchings with choice functions, beyond extension of the Deferred Acceptance algorithm [31].

Keywords: Stable matching · Choice function · Distributive lattice · Birkhoff's representation theorem

1 Introduction

Since Gale and Shapley's seminal publication [17], the concept of stability in matching markets has been widely studied by the optimization community. With minor modifications, the one-to-many version of Gale and Shapley's original stable *marriage* model is currently employed in the National Resident Matching Program [30], which assigns medical residents to hospitals in the US, and for matching eighth-graders to public high schools in many major cities in the US [1].

In this paper, matching markets have two sides, which we call firms F and workers W. In the marriage model, every agent from $F \cup W$ has a *strict preference list* that ranks agents from the opposite side of the market. The problem asks for a *stable matching*, which is a matching where no pair of agents prefer each other to their assigned partner. A stable matching can be found efficiently via the Deferred Acceptance algorithm [17].

M. Singh and D. P. Williamson (Eds.): IPCO 2021, LNCS 12707, pp. 89–103, 2021.
https://doi.org/10.1007/978-3-030-73879-2_7

Although successful, the marriage model does not capture features that have become of crucial importance both inside and outside academia. For instance, there is growing attention to models that can increase diversity in school cohorts [28,37]. Such constraints cannot be represented in the original model, or even its one-to-many or many-to-many generalizations, since admission decisions with diversity concerns cannot be captured by a strict preference list.

To model these and other markets, every agent $a \in F \cup W$ is endowed with a *choice function* $\mathcal{C}_a$ that picks a team she prefers the best from a given set of potential partners. See, e.g., [7,14,23] for more applications of models with choice functions, and the literature review section for more references. *Mutatis mutandis*, one can define a concept of stability in this model as well (for this and the other technical definitions mentioned below, see Sect. 2). Two classical assumptions on choices functions are *substitutability* and *consistency*, under which the existence of stable matchings is guaranteed [6,20]. Clearly, existence results are not enough for applications (and for optimizers). Interestingly, little is known about efficient algorithms in models with choice functions. Only extensions of the classical Deferred Acceptance algorithm for finding the one-side optimal matching have been studied for this model [13,31].

The goal of this paper is to study algorithms for optimizing a linear function w over the set of stable matchings in models with choice functions, where w is defined over firm-worker pairs. Such algorithms can be used to obtain a stable matching that is e.g., *egalitarian*, *profit-optimal*, and *minimum regret* [25]. We focus in particular on the model where choice functions are assumed to be substitutable, consistent, and *quota-filling*. This model (QF-Model) generalizes all classical models where agents have strict preference lists, on which results for the question above were known. For this model, Alkan [3] has shown that stable matchings form a distributive lattice. As we argue next, this is a fundamental property that allows us to solve our optimization problem efficiently. For missing proofs, extended discussions, and examples, see the full version of the paper [15].

Our contributions and techniques. We give a high-level description of our approach and results. For the standard notions of posets, distributive lattices, and related definitions, see [19]. All sets considered in this paper are finite.

Let $\mathcal{L} = (\mathcal{X}, \succeq)$ be a distributive lattice, where all elements of $\mathcal{X}$ are distinct subsets of a base set E and $\succeq$ is a partial order on $\mathcal{X}$. We refer to $S \in \mathcal{X}$ as an *element* (of the lattice). Birkhoff's theorem [11] implies that we can associate[1] to every distributive lattice $\mathcal{L}$ a poset $\mathcal{B} = (Y, \succeq^\star)$ such that there is a bijection $\psi : \mathcal{X} \to \mathcal{U}(\mathcal{B})$, where $\mathcal{U}(\mathcal{B})$ is the family of *upper sets* of $\mathcal{B}$. $U \subseteq Y$ is an upper set of $\mathcal{B}$ if $y \in U$ and $y' \succeq^\star y$ for some $y' \in Y$ implies $y' \in U$. We say therefore that $\mathcal{B}$ is a *representation poset* for $\mathcal{L}$ with *representation function* ψ. See Example 1 for a demonstration. $\mathcal{B}$ may contain much fewer elements than the lattice $\mathcal{L}$ it represents, thus giving a possibly "compact" description of $\mathcal{L}$.

The representation function ψ satisfies that for $S, S' \in \mathcal{X}$, $S \succeq S'$ if and only if $\psi(S) \subseteq \psi(S')$. Albeit $\mathcal{B}$ and ψ explain how elements of $\mathcal{X}$ are related to each

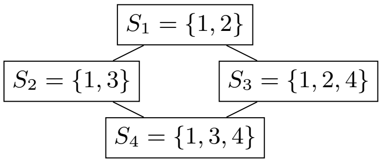
[1] The result proved by Birkhoff is actually a bijection between the families of lattices and posets, but in this paper we shall not need it in full generality.

other with respect to $\succeq$, they do not contain any information on which items from E are contained in each lattice element. We introduce therefore Definition 1. For $S \in \mathcal{X}$ and $U \in \mathcal{U}(\mathcal{B})$, we write $\chi^S \in \{0,1\}^E$ and $\chi^U \in \{0,1\}^Y$ to denote their characteristic vectors, respectively.

Definition 1. *Let $\mathcal{L} = (\mathcal{X}, \succeq)$ be a distributive lattice on a base set E and $\mathcal{B} = (Y, \succeq^\star)$ be a representation poset for $\mathcal{L}$ with representation function ψ. $\mathcal{B}$ is an* affine representation *of $\mathcal{L}$ if there exists an affine function $g : \mathbb{R}^Y \to \mathbb{R}^E$ such that $g(\chi^U) = \chi^{\psi^{-1}(U)}$, for all $U \in \mathcal{U}(B)$. In this case, we also say that $\mathcal{B}$* affinely represents *$\mathcal{L}$ via affine function g and that $\mathcal{L}$ is* affinely representable*.*

Note that in Definition 1, we can always assume $g(u) = Au + x_0$, where x_0 is the characteristic vector of the maximal element of $\mathcal{L}$ and $A \in \{0, \pm 1\}^{E \times Y}$.

Example 1. Consider the distributive lattice $\mathcal{L} = (\mathcal{X}, \succeq)$ with base set $E = \{1,2,3,4\}$ whose Hasse diagram is given below.



$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The representation poset $\mathcal{B} = (Y, \succeq^\star)$ of $\mathcal{L}$ is composed of two non-comparable elements, y_1 and y_2. The representation function ψ is defined as

$$\psi(S_1) = \emptyset =: U_1;\ \psi(S_2) = \{y_1\} =: U_2;\ \psi(S_3) = \{y_2\} =: U_3;\ \psi(S_4) = Y =: U_4.$$

That is, $\mathcal{U}(\mathcal{B}) = \{U_1, U_2, U_3, U_4\}$. One can think of y_1 as the operation of adding $\{3\}$ and removing $\{2\}$, and y_2 as the operation of adding $\{4\}$. $\mathcal{B}$ affinely represents $\mathcal{L}$ via the function $g(\chi^U) = A\chi^U + \chi^{S_1}$, with matrix A given above.

Now consider the distributive lattice $\mathcal{L}'$ obtained from $\mathcal{L}$ by switching S_3 and S_4. One can check that $\mathcal{L}'$ is not affinely representable [15]. △

As we show next, affine representability allows one to efficiently solve linear optimization problems over elements of a distributive lattice. In particular, it generalizes a property that is at the backbone of combinatorial algorithms for optimizing a linear function over the set of stable matchings in the marriage model and its one-to-many and many-to-many generalizations (see, e.g., [10,22]). In the marriage model, the base set E is the set of pairs of agents from two sides of the market, $\mathcal{X}$ is the set of stable matchings, and for $S, S' \in \mathcal{X}$, $S \succeq S'$ if every firm prefers its partner in S to its partner in S' or is indifferent between the two. Elements of its representation poset are certain (trading) cycles, called *rotations*.

Lemma 1. *Assume poset $\mathcal{B} = (Y, \succeq^\star)$ affinely represents lattice $\mathcal{L} = (\mathcal{X}, \succeq)$. Let $w : E \to \mathbb{R}$ be a linear function over the base set E of $\mathcal{L}$. Then the problem $\max\{w^\intercal \chi^S : S \in \mathcal{X}\}$ can be solved in time $\texttt{min-cut}(|Y|+2)$, where $\texttt{min-cut}(k)$ is the time complexity required to solve a minimum cut problem with nonnegative weights in a digraph with k nodes.*

Proof. Let $g(u) = Au + x_0$ be the affine function for the representation. Then,

$$\max_{S \in \mathcal{X}} w^\intercal \chi^S = \max_{U \in \mathcal{U}(\mathcal{B})} w^\intercal g(\chi^U) = \max_{U \in \mathcal{U}(\mathcal{B})} w^\intercal (A\chi^U + x_0) = w^\intercal x_0 + \max_{U \in \mathcal{U}(\mathcal{B})} (w^\intercal A)\chi^U.$$

Thus, our problem boils down to the optimization of a linear function over the upper sets of $\mathcal{B}$. It is well-known that the latter problem is equivalent to computing a minimum cut in a digraph with $|Y| + 2$ nodes [29].

We want to apply Lemma 1 to the QF-Model model. Observe that a choice function may be defined on all the (exponentially many) subsets of agents from the opposite side of the market. We avoid this computational concern by modeling choice functions via an oracle model. That is, choice functions can be thought of as agents' private information. The complexity of our algorithms will therefore be expressed in terms of $|F|$, $|W|$, and the time required to compute the choice function $\mathcal{C}_a(X)$ of an agent $a \in F \cup W$, where the set X is in the domain of $\mathcal{C}_a$. The latter running time is denoted by `oracle-call` and we assume it to be independent of a and X. Our first result is the following.

Theorem 1. *The distributive lattice* $(\mathcal{S}, \succeq)$ *of stable matchings in the* QF-Model *is affinely representable. Its representation poset* $(\Pi, \succeq^\star)$ *has* $O(|F||W|)$ *elements.* $(\Pi, \succeq^\star)$*, as well as its representation function* ψ *and affine function* $g(u) = Au + x_0$*, can be computed in time* $O(|F|^3|W|^3\texttt{oracle-call})$*. Moreover, matrix* A *has full column rank.*

In Theorem 1, we assumed that operations, such as checking if two sets coincide and obtaining an entry from the set difference of two sets, take constant time. If this is not the case, the running time needs to be scaled by a factor mildly polynomial in $|F| \cdot |W|$. Observe that Theorem 1 is the union of two statements. First, the distributive lattice of stable matchings in the QF-Model is affinely representable. Second, this representation and the corresponding functions ψ and g can be found efficiently. Those two results are proved in Sect. 3 and Sect. 4, respectively. Combining Theorem 1, Lemma 1 and algorithms for finding a minimum cut (see, e.g., [34]), we obtain the following.

Corollary 1. *The problem of optimizing a linear function over the set of stable matchings in the* QF-Model *can be solved in time* $O(|F|^3|W|^3\texttt{oracle-call})$*.*

As an interesting consequence of studying a distributive lattice via the poset that affinely represents it, one immediately obtains a linear description of the convex hull of the characteristic vectors of elements of the lattice (see Sect. 5). In contrast, most stable matching literature (see the literature review section) has focused on deducing linear descriptions for special cases of our model via ad-hoc proofs, independently of the lattice structure.

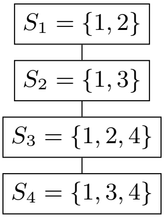
Theorem 2. *Let* $\mathcal{L} = (\mathcal{X}, \succeq)$ *be a distributive lattice and* $\mathcal{B} = (Y, \succeq^\star)$ *be a poset that affinely represents it via the affine function* $g(u) = Au + x_0$*. Then the extension complexity of* $\text{conv}(\mathcal{X}) := \text{conv}\{\chi^S : S \in \mathcal{X}\}$ *is* $O(|Y|^2)$*. If moreover* A *has full column rank, then* $\text{conv}(\mathcal{X})$ *has* $O(|Y|^2)$ *facets.*

Theorem 1 and Theorem 2 imply the following description for the stable matching polytope $\text{conv}(\mathcal{S})$, i.e., the convex hull of the characteristic vectors of stable matchings in the QF-Model.

Corollary 2. $\text{conv}(\mathcal{S})$ *has* $O(|F|^2|W|^2)$ *facets.*

We conclude with an example of a lattice represented via a non-full-column rank matrix A.

Example 2. Consider the distributive lattice given below.



$$A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

It can be represented via the poset $\mathcal{B} = (Y, \succeq^\star)$, that contains three elements y_1, y_2, and y_3 where $y_1 \succeq^\star y_2 \succeq^\star y_3$. Thus, $\mathcal{U}(\mathcal{B}) = \{\emptyset, \{y_1\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}\}$. In addition, $\mathcal{B}$ affinely represents $\mathcal{L}$ via the function $g(\chi^U) = A\chi^U + \chi^{S_1}$, where A is given below. It is clear that matrix A does not have full column rank. △

Relationship with the literature. Gale and Shapley [17] introduced the one-to-one stable marriage (SM-Model) and the one-to-many stable admission model (SA-Model), and presented an algorithm which finds a stable matching. McVitie and Wilson [27] proposed the break-marriage procedure that finds the full set of stable matchings. Irving et al. [22] presented an efficient algorithm for the maximum-weighted stable matching problem with weights over pairs of agents, using the fact that the set of stable matchings forms a distributive lattice [24] and that its representation poset – an affine representation following our terminology – can be constructed efficiently via the concept of rotations [21]. The above-mentioned structural and algorithm results have been also shown for its many-to-many generalization (MM-Model) in [8,10]. A complete survey of results on these models can be found, e.g., in [19,25].

For models with substitutable and consistent choice functions, Roth [31] proved that stable matchings always exist by generalizing the algorithm presented in [17]. Blair [12] proved that stable matchings form a lattice, although not necessarily distributive. Alkan [3] showed that if choice functions are further assumed to be quota-filling, the lattice is distributive. Results on (non-efficient) enumeration algorithms in certain models with choice functions appeared in [26].

It is then natural to investigate whether algorithms from [10,21] can be directly extended to construct the representation poset in the QF-Model. However, definition of rotations and techniques in [10,21] rely on the fact that there is a strict ordering of partners, which is not available with choice functions. This, for instance, leads to the fact that the symmetric difference of two stable matchings that are adjacent in the lattice is always a simple cycle, which is not always true in the QF-Model. We take then a more fundamental approach by showing a carefully defined ring of sets is isomorphic to the set of stable matchings, and thus we can construct the rotation poset following a maximal chain of the stable matching lattice. This approach conceptually follows the one in [19] for the SM-Model and leads to a generalization of the break-marriage procedure from [27]. Again, proofs in [19,27] heavily rely on having a strict ordering of partners, while we need to tackle the challenge of not having one.

Besides the combinatorial perspective, another line of research focuses on the polyhedral aspects. Linear descriptions of the convex hull of the characteristic vectors of stable matchings are provided for the SM-Model [32,33,38], the SA-Model [9], and the MM-Model [16]. In this paper, we provide a polyhedral description for the QF-Model, by drawing connection between the order polytope (i.e., the convex hull of the characteristic vectors of the upper sets of a poset) and Birkhoff's representation theorem of distributive lattices. A similar approach has been proposed in [5]: their result can be seen as a specialization of Theorem 2 to the SM-Model.

Aside from the stable matching problem, the feasible spaces of many other combinatorial optimization problems form a distributive lattice. Examples, as pointed out in [18], include feasible rooted trees for the shortest path problem, and market clearing prices for the assignment game [35].

2 The QF-Model

Let F and W denote two disjoint finite sets of agents, say firms and workers, respectively. Associated with each firm $f \in F$ is a *choice function* $\mathcal{C}_f : 2^{W(f)} \to 2^{W(f)}$ where $W(f) \subseteq W$ is the set of *acceptable* partners of f and $\mathcal{C}_f$ satisfies the property that for every $S \subseteq W(f)$, $\mathcal{C}_f(S) \subseteq S$. Similarly, a choice function $\mathcal{C}_w : 2^{F(w)} \to 2^{F(w)}$ is associated to each worker w. We assume that for every firm-worker pair (f, w), $f \in F(w)$ if and only if $w \in W(f)$. We let $\mathcal{C}_W$ and $\mathcal{C}_F$ denote the collection of firms' and workers' choice functions respectively. A *matching market* is a tuple $(F, W, \mathcal{C}_F, \mathcal{C}_W)$.

Following Alkan [3], we define the QF-Model by assuming that the choice function $\mathcal{C}_a$ of every agent $a \in F \cup W$ satisfies the three properties below.

Definition 2 (Substitutability). *$\mathcal{C}_a$ is substitutable if for any set of partners S, $b \in \mathcal{C}_a(S)$ implies that for all $T \subseteq S$, $b \in \mathcal{C}_a(T \cup \{b\})$.*

Definition 3 (Consistency). *$\mathcal{C}_a$ is consistent if for any sets of partners S and T, $\mathcal{C}_a(S) \subseteq T \subseteq S$ implies $\mathcal{C}_a(S) = \mathcal{C}_a(T)$.*

Definition 4 (Quota-filling). *$\mathcal{C}_a$ is quota-filling if there exists $q_a \in \mathbb{N}$ such that for any set of partners S, $|\mathcal{C}_a(S)| = \min(q_a, |S|)$. We call q_a the* quota *of a.*

Intuitively, substitutability implies that an agent selected from a set of candidates will also be selected from a smaller subset; consistency is also called "irrelevance of rejected contracts"; and quota-filling means that an agent has a number of positions and she tries to fill those as many as possible. A choice function is substitutable and consistent if and only if it is *path-independent* [2].

Definition 5 (Path-independence). *$\mathcal{C}_a$ is path-independent if for any sets of partners S and T, $\mathcal{C}_a(S \cup T) = \mathcal{C}_a\big(\mathcal{C}_a(S) \cup T\big)$.*

A *matching* μ is a mapping from $F \cup W$ to $2^{F \cup W}$ such that for all $w \in W$ and all $f \in F$, (1) $\mu(w) \subseteq F(w)$, (2) $\mu(f) \subseteq W(f)$, and (3) $w \in \mu(f)$ if and only if $f \in \mu(w)$. A matching can also be viewed as a collection of firm-worker pairs. That is, $\mu \equiv \{(f,w) : f \in F, w \in \mu(f)\}$. We say a matching μ is *individually rational* if for every agent a, $\mathcal{C}_a(\mu(a)) = \mu(a)$. An acceptable firm-worker pair $(f,w) \notin \mu$ is called a *blocking pair* if $w \in \mathcal{C}_f(\mu(f) \cup \{w\})$ and $f \in \mathcal{C}_w(\mu(w) \cup \{f\})$, and when such pair exists, we say μ is *blocked by* the pair or the pair *blocks* μ. A matching μ is *stable* if it is individually rational and it admits no blocking pairs. If f is matched to w in some stable matching, we say that f (resp. w) is a *stable partner* of w (resp. f). We denote by $\mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ the set of stable matchings in the market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$. Alkan [3] showed the following.

Theorem 3 ([3]). *Consider a matching market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$ in the* QF-Model. *Then, $\mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$ is a distributive lattice under the partial order $\succeq$ where $\mu_1 \succeq \mu_2$ if for all $f \in F$, $\mathcal{C}_f(\mu_1(f)) \cup \mu_2(f)) = \mu_1(f)$.*

We denote by μ_F and μ_W the firm- and worker-optimal stable matchings, respectively. For every $a \in F \cup W$, let $\Phi_a = \{\mu(a) : \mu \in \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)\}$. Alkan [3] showed that for all $S, T \in \Phi_a$ the following holds: $|S| = |T| =: \overline{q}_a$ (equal-quota); and $\overline{q}_a < q_a \implies |\Phi_a| = 1$ (full-quota).

3 Affine Representability of the Stable Matching Lattice

For the rest of the paper, we fix a matching market $(F, W, \mathcal{C}_F, \mathcal{C}_W)$ and often abbreviate $\mathcal{S} := \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)$. In this section, we show that the distributive lattice of stable matchings $(\mathcal{S}, \succeq)$ in the QF-Model is affinely representable. Our approach is as follows. First, we show that $(\mathcal{S}, \succeq)$ is isomorphic to a lattice $(\mathcal{P}, \subseteq)$ belonging to a special class, that is called *ring of sets*. We then show that rings of sets are always affinely representable. Next, we show a poset $(\Pi, \succeq^\star)$ representing $(\mathcal{S}, \succeq)$. We last show how to combine all those results and "translate" the affine representability of $(\mathcal{P}, \subseteq)$ to the affine representability of $(\mathcal{S}, \succeq)$. We note in passing that in this section we actually rely on weaker assumptions than those from the QF-Model, essentially matching those from [4]. That is, instead of quota-filling, we can assume a weaker condition called *cardinal monotonicity*: $\mathcal{C}_a$ is cardinal monotone if for all sets of partners $S \subseteq T$, $|\mathcal{C}_a(S)| \le |\mathcal{C}_a(T)|$.

Isomorphism between the stable matching lattice and a ring of sets. A family $\mathcal{H}$ of subsets of a *base set* B is a *ring of sets* over B if $\mathcal{H}$ is closed under set union and set intersection [11]. A ring of sets is a distributive lattice with the partial order relation $\subseteq$, and the join ($\vee$) and meet ($\wedge$) operations corresponding to set intersection and set union, respectively.

Let $\phi(a) \coloneqq \{b : b \in \mu(a) \text{ for some } \mu \in \mathcal{S}\}$ denote the set of stable partners of agent a. For a stable matching μ, let $P_f(\mu) \coloneqq \{w \in \phi(f) : w \in \mathcal{C}_f(\mu(f) \cup \{w\})\}$, and define the *P-set* of μ as $P(\mu) \coloneqq \{(f, w) : f \in F,\ w \in P_f(\mu)\}$.

The following theorem gives a "description" of the stable matching lattice as a ring of sets. Let $\mathcal{P}(\mathcal{C}_F, \mathcal{C}_W)$ denote the set $\{P(\mu) : \mu \in \mathcal{S}(\mathcal{C}_F, \mathcal{C}_W)\}$, and we often abbreviate $\mathcal{P} \coloneqq \mathcal{P}(\mathcal{C}_F, \mathcal{C}_W)$.

Theorem 4. *(1) the mapping $P : \mathcal{S} \to \mathcal{P}$ is a bijection;*
(2) $(\mathcal{P}, \subseteq)$ is isomorphic to $(\mathcal{S}, \succeq)$. Moreover, $P(\mu_1 \vee \mu_2) = P(\mu_1) \cap P(\mu_2)$ and $P(\mu_1 \wedge \mu_2) = P(\mu_1) \cup P(\mu_2)$. In particular, $(\mathcal{P}, \subseteq)$ is a ring of sets over the base set $E = \{(f, w) \in F \times W : w \in \phi(f)\}$.

Remark 1. An isomorphism between the lattice of stable matchings and a ring of set is proved in the SM-Model as well [19], where the authors define $P_f(\mu) := \{w : f \text{ prefers } w \text{ to } \mu(f)\}$, hence including firm-worker pairs that are not stable. Interestingly, there are examples showing that in the QF-Model, if we were to use the natural extension of the definition in [19], i.e., $P_f(\mu) := \{w \in W(f) : w \in \mathcal{C}_f(\mu(f) \cup \{w\})\}$, then $\mathcal{P}$ is not a ring of set, see [15].

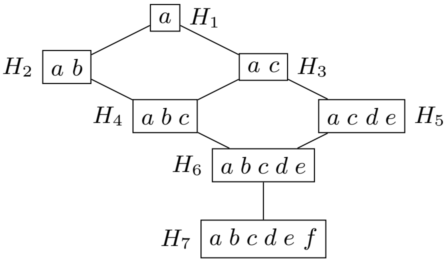
Affine representability of rings of sets. Consider a poset $(X, \geq)$. Let $a, a' \in X$. If $a' > a$ and there is no $b \in X$ such that $a' > b > a$, we say that a' is an *immediate predecessor* of a in $(X, \geq)$ and that a is an *immediate descendant* of a' in $(X, \geq)$. Fix a ring of set $(\mathcal{H}, \subseteq)$ over a base set B and define set $\mathcal{D}(\mathcal{H}) := \{H \backslash H' : H' \text{ is an immediate predecessor of } H \text{ in } (\mathcal{H}, \subseteq)\}$ of *minimal differences* among elements of $\mathcal{H}$. We note that minimal differences are disjoint [19]. We elucidate in Example 3 these definitions and the facts below.

Theorem 5 ([11]). *There is a partial order $\sqsupseteq$ over $\mathcal{D}(\mathcal{H})$ such that $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ is a representation poset for $(\mathcal{H}, \subseteq)$ where the representation function ψ is defined as follows: for any upper set $\overline{\mathcal{D}}$ of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$, $\psi^{-1}(\overline{\mathcal{D}}) = \bigcup\{K : K \in \overline{\mathcal{D}}\} \cup H_0$ where H_0 is the minimal element of $\mathcal{H}$. Moreover, $|\mathcal{D}(\mathcal{H})| = O(|B|)$.*

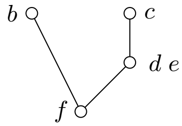
From Theorem 5, it is not hard to prove the following.

Theorem 6. *$(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ affinely represents $(\mathcal{H}, \subseteq)$ via the affine function $g(u) = Au + x_0$, where $x_0 = \chi^{H_0}$, and $A \in \{0, 1\}^{B \times \mathcal{D}(\mathcal{H})}$ has columns χ^K for each $K \in \mathcal{D}(\mathcal{H})$. Moreover, A has full column rank.*

Example 3. Consider the ring of sets and its representation poset given below.



(a) $(\mathcal{H}, \subseteq)$



$$A = \begin{pmatrix} 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}$$

(b) $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$

Representation function ψ maps $H_1, \cdots, H_7$ to $\emptyset$, $\{\{b\}\}$, $\{\{c\}\}$, $\{\{b\}, \{c\}\}$, $\{\{c\}, \{d, e\}\}$, $\{\{b\}, \{c\}, \{d, e\}\}$, and $\{\{b\}, \{c\}, \{d, e\}, \{f\}\}$, respectively. The affine function is $g(u) = Au + x_0$ with $x_0^\intercal = (1, 0, 0, 0, 0, 0)$ and matrix A given above. Note that columns of A correspond to $\{b\}, \{c\}, \{d, e\}, \{f\}$ in this order. △

Representation of $(\mathcal{S}, \succeq)$ via the poset of rotations. For $\mu' \succeq \mu \in \mathcal{S}$ with μ' being an immediate predecessor of μ in the stable matching lattice, let $\rho^+(\mu', \mu) = \{(f, w) : (f, w) \in \mu \setminus \mu'\}$ and $\rho^-(\mu', \mu) = \{(f, w) : (f, w) \in \mu' \setminus \mu\}$. We call $\rho(\mu', \mu) := (\rho^+(\mu', \mu), \rho^-(\mu', \mu))$ a *rotation* of $(\mathcal{S}, \succeq)$. Let $\Pi(\mathcal{S})$ denote the set of rotations of $(\mathcal{S}, \succeq)$. We abbreviate $\mathcal{D} := \mathcal{D}(\mathcal{P})$ and $\Pi := \Pi(\mathcal{S})$.

Theorem 7. *(1) the mapping $Q : \Pi \to \mathcal{D}$, with $Q(\rho) := \rho^+$, is a bijection;*
(2) $(\mathcal{D}, \sqsupseteq)$ is isomorphic to $(\Pi, \succeq^\star)$ where for two rotations $\rho_1, \rho_2 \in \Pi$, $\rho_1 \succeq^\star \rho_2$ if $Q(\rho_1) \sqsupseteq Q(\rho_2)$;
(3) $(\Pi, \succeq^\star)$ is a representation poset for $(\mathcal{S}, \succeq)$, where the representation function $\psi_\mathcal{S}$ is defined as follows: $\psi_\mathcal{S}^{-1}(\overline{\Pi}) = \mu_F \cup (\bigcup_{\rho \in \overline{\Pi}} \rho^+) \setminus (\bigcup_{\rho \in \overline{\Pi}} \rho^-)$, for any upper set $\overline{\Pi}$ of $(\Pi, \succeq^\star)$.

$(\Pi, \succeq^\star)$ is called the *rotation poset*. By Theorem 6 and Theorem 7, we deduce $|\Pi| = O(|F||W|)$ and the following, proving the structural statement from Theorem 1. The base set E of $(\mathcal{S}, \succeq)$ is the set of acceptable firm-worker pairs.

Theorem 8. *The rotation poset $(\Pi, \succeq^\star)$ affinely represents the stable matching lattice $(\mathcal{S}, \succeq)$ with affine function $g(u) = Au + \mu_F$, where $A \in \{0, \pm 1\}^{E \times \Pi}$ has columns $\chi^{\rho^+} - \chi^{\rho^-}$ for each $\rho \in \Pi$. Moreover, matrix A has full column rank.*

4 Algorithms

To conclude the proof of Theorem 1, we show how to efficiently find the elements of Π, and how they relate to each other via $\succeq^\star$. First, we employ Roth's extension of the Deferred Acceptance algorithm to find a firm- or worker-optimal stable matching. Second, we feed its output to an algorithm that produces a maximal chain $C_0, C_1, \ldots, C_k$ of $(\mathcal{S}, \succeq)$ and the set Π. We then provide an algorithm that, given a maximal chain of a ring of sets, constructs the partial order for the poset

Algorithm 1. `break-marriage(`μ', f', w'`)` with $(f', w') \in \mu' \in \mathcal{S}$

for each firm $f \neq f'$ **do** $X_f^{(0)} = \overline{X}_f(\mu')$ **end for**
let $X_{f'}^{(0)} = \overline{X}_{f'}(\mu') \setminus \{w'\}$; set the step count $s = 0$
repeat
  for each worker w **do**
    let $X_w^{(s)} = \{f \in F : w \in \mathcal{C}_f(X_f^{(s)})\}$
    if $w \neq w'$ **then** $Y_w^{(s)} = \mathcal{C}_w(X_w^{(s)})$ **else** $Y_w^{(s)} = \mathcal{C}_w(X_w^{(s)} \cup \{f'\}) \setminus \{f'\}$
  end for
  for each firm f **do** $X_f^{(s+1)} = X_f^{(s)} \setminus \{w \in W : f \in X_w^{(s)} \setminus Y_w^{(s)}\}$ **end for**
  update the step count $s = s + 1$
until $X_f^{(s-1)} = X_f^{(s)}$ for every firm f
Output: matching $\overline{\mu}$ with $\overline{\mu}(w) = Y_w^{(s-1)}$ for every worker w

of minimal differences. This and previous facts are then exploited to obtain the partial order $\succeq^\star$ on rotations of Π. Lastly, we argue on the overall running time.

For a matching μ and $f \in F$, let $\overline{X}_f(\mu) := \{w \in W(f) : \mathcal{C}_f(\mu(f) \cup \{w\}) = \mu(f)\}$. Define the *closure* of μ, denoted by $\overline{X}(\mu)$, as the collection of sets $\{\overline{X}_f(\mu) : f \in F\}$. If μ is individually rational, then $\mu(f) \subseteq \overline{X}_f(\mu)$ for every $f \in F$.

Lemma 2. *Let $\mu_1, \mu_2 \in \mathcal{S}$ such that $\mu_1 \succeq \mu_2$. Then, $\forall f \in F$, $\mu_2(f) \subseteq \overline{X}_f(\mu_1)$.*

Deferred Acceptance Algorithm. Roth [31] generalized to choice function models the algorithm proposed in [17]. There is one side that is proposing – for the following, we let it be F. Initially, for each $f \in F$, let $X_f := W(f)$, i.e., the set of acceptable workers of f. At every step, every $f \in F$ proposes to workers in $\mathcal{C}_f(X_f)$. Then, every $w \in W$ considers the set of firms X_w who made a proposal to w, *temporarily accepts* $Y_w := \mathcal{C}_w(X_w)$, and *rejects* the rest. Afterwards, each firm f removes from X_f all workers that rejected f. Hence, throughout the algorithm, X_f denotes the set of acceptable workers of f that have not rejected f yet. The *firm-proposing* algorithm iterates until there is no rejection.

Theorem 9. *Roth's algorithms outputs μ_F in time $O(|F||W|$`oracle-call`$)$.*

By symmetry, swapping the role of firms and workers, we have the *worker-proposing* deferred acceptance algorithm, which outputs μ_W.

Constructing Π via a maximal chain of $(\mathcal{S}, \succeq)$. A *maximal chain* $C_0, \cdots, C_k$ in $(\mathcal{S}, \succeq)$ is an ordered subset of $\mathcal{S}$ such that C_{i-1} is an immediate predecessor of C_i in $(\mathcal{S}, \succeq)$ for all $i \in [k]$, $C_0 = \mu_F$, and $C_k = \mu_W$.

We now extend to our setting the *break-marriage* idea proposed by McVitie and Wilson [27]. This algorithm produces a matching μ starting from $\mu' \in \mathcal{S}$. A formal description is given in Algorithm 1. Roughly speaking, the algorithm re-initiates the deferred acceptance algorithm from μ' after suitably breaking a matched pair. Repeated applications of Algorithm 1 allow us to obtain an immediate descendant of μ' in $(\mathcal{S}, \succeq)$.

Algorithm 2. Immediate descendant of $\mu' \in \mathcal{S}$

1: set $\mathcal{T} = \emptyset$
2: **for each** $(f', w') \in \mu' \setminus \mu_W$ **do**
3: run the `break-marriage(`μ', f', w'`)` procedure
4: **if** the procedure is successful **then** add the output matching $\overline{\mu}$ to $\mathcal{T}$
5: **end for**

Output: a maximal matching μ^* from $\mathcal{T}$ wrt $\succeq$, i.e. $\nexists\ \mu \in \mathcal{T}$ such that $\mu \succ \mu^*$

It is easy to see that `break-marriage(`μ', f', w'`)` always terminates. We let $s^\star$ be the value of step count s at the end of the algorithm. Note that by the termination condition, $\overline{\mu}(f) = \mathcal{C}_f(X_f^{(s^\star)})$ for every firm f. Let $(f, w) \in F \times W$. We say f is *rejected by* w *at step* s if $f \in X_w^{(s)} \setminus Y_w^{(s)}$, and we say f is *rejected by* w if f is rejected by w at some step during the break-marriage procedure. Note that a firm f is rejected by all and only the workers in $\overline{X}_f(\mu') \setminus X_f^{(s^\star)}$.

Theorem 10. *The running time of Algorithm 1 is* $O(|F||W|$`oracle-call`$)$.

Lemma 3. *The matching $\overline{\mu}$ output by Algorithm 1 is individually rational, and for every firm $f \in F$, we have $\mathcal{C}_f(\overline{\mu}(f) \cup \mu'(f)) = \mu'(f)$.*

We say `break-marriage(`μ', f', w'`)` is *successful* if $f' \notin \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\})$.

Lemma 4. *If* `break-marriage(`μ', f', w'`)` *is successful, then $\overline{\mu} \in \mathcal{S}$ and $\mu' \succ \overline{\mu}$.*

Next theorem shows a sufficient condition for the break-marriage procedure to output an immediate descendant in the stable matching lattice.

Theorem 11. *Let $\mu' \succeq \mu \in \mathcal{S}$ and assume μ' is an immediate predecessor of μ in the stable matching lattice. Pick $(f', w') \in \mu' \setminus \mu$ and let $\overline{\mu}$ be the output matching of* `break-marriage(`μ', f', w'`)`*. Then, $\overline{\mu} = \mu$.*

Proof. Note that by Lemma 2, $\mu(f) \subseteq \overline{X}_f(\mu')$ for every $f \in F$. We start by showing that during the algorithm, for every firm f, no worker in $\mu(f)$ rejects f. Assume by contradiction that this is not true. Let s' be the first step where such a rejection happens, with firm f_1 being rejected by worker $w_1 \in \mu(f_1)$.

Claim 1. There exists a firm $f_2 \in Y_{w_1}^{(s')} \setminus \mu(w_1)$ such that $f_2 \in \mathcal{C}_{w_1}(\mu(w_1) \cup \{f_2\})$.

Let f_2 be the firm whose existence is guaranteed by Claim 1. In particular, $f_2 \in Y_{w_1}^{(s')}$ implies $w_1 \in \mathcal{C}_{f_2}(X_{f_2}^{(s')})$. Note that by our choice of f_1, $\mu(f_2) \subseteq X_{f_2}^{(s')}$. Therefore, using substitutability and $w_1 \in X_{f_2}^{(s')}$, we have $w_1 \in \mathcal{C}_{f_2}(\mu(f_2) \cup \{w_1\})$. Thus, (f_2, w_1) is a blocking pair of μ, which contradicts the stability of μ.

Hence, for every firm f, no worker in $\mu(f)$ rejects f and thus, $\mu(f) \subseteq X_f^{(s^\star)}$. Because of path-independence and $\overline{\mu}(f) = \mathcal{C}_f(X_f^{(s^\star)})$, we have $\mathcal{C}_f(\overline{\mu}(f) \cup \mu(f)) = \mathcal{C}_f(\mathcal{C}_f(X_f^{(s^\star)}) \cup \mu(f)) = \mathcal{C}_f\big(\mathcal{C}_f(X_f^{(s^\star)} \cup \mu(f))\big) = \mathcal{C}_f(X_f^{(s^\star)}) = \overline{\mu}(f)$ (§). This in

Algorithm 3. A maximal chain of $(\mathcal{S}, \succeq)$ and the set of rotations Π

set counter $k = 0$; let $C_k = \mu_F$
while $C_k \neq \mu_W$ **do**
  run Algorithm 2 with $\mu' = C_k$, and let μ^* be its output
  update counter $k = k + 1$; let $C_k = \mu^*$
end while
Output: maximal chain $C_0, C_1, \cdots, C_k$; and $\Pi = \{\rho_i \coloneqq \rho(C_{i-1}, C_i)\}_{i \in [k]}$.

particular implies $|\overline{\mu}(f)| \geq |\mu(f)|$ due to individual rationality of μ and quota-filling. Note also that $|\mu(f)| = |\mu'(f)| = |\mathcal{C}_f(\overline{\mu}(f) \cup \mu'(f))| \geq |\mathcal{C}_f(\overline{\mu}(f))| = |\overline{\mu}(f)|$, where the first equality is due to the equal-quota property, the second and the last by Lemma 3, and the inequality by quota-filling. We deduce $|\mu(f)| = |\mu'(f)| = |\overline{\mu}(f)|$ ($\natural$). We now show that `break-marriage(`μ', f', w'`)` is successful.

Claim 2. $|\overline{\mu}(w)| = |\mu'(w)|$ for every worker $w \neq w'$.

Hence, $|\overline{\mu}(w')| = |\mu'(w')| = \overline{q}_{w'} = q_{w'}$, where the first equality holds from Claim 2 and ($\natural$), the second by the equal-quota property, and the last because $\mu(w') \neq \mu'(w')$ by choice of w' and the full-quota property. Therefore, $f' \notin \mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\})$ because otherwise $|\overline{\mu}(w')| = |\mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\}) \setminus \{f'\}| < |\mathcal{C}_{w'}(X_{w'}^{(s^\star-1)} \cup \{f'\})| \leq q_{w'}$, where the last inequality holds by quota-filling, a contradiction. Thus, `break-marriage(`μ', f', w'`)` is successful.

Finally, by Lemma 4, we have $\overline{\mu} \in \mathcal{S}$ and $\mu' \succ \overline{\mu}$. Because of (§), we also have $\overline{\mu} \succeq \mu$. Therefore, it must be that $\overline{\mu} = \mu$ by the choice of μ.

We now present in Algorithm 2 a procedure that finds an immediate descendant for any given stable matching, using the break-marriage procedure.

Lemma 5. *Let $\mu_1 \succ \mu_2 \succ \mu_3 \in \mathcal{S}$. If $(f, w) \in \mu_1 \setminus \mu_2$, then $(f, w) \notin \mu_3$.*

Theorem 12. *The output μ^* of Algorithm 2 is an immediate descendant of μ' in $(\mathcal{S}, \succeq)$. Its running time is $O(|F|^2|W|^2$`oracle-call`$)$.*

Proof. First note that due to Lemma 4, all matchings in the set $\mathcal{T}$ are stable matchings. Assume by contradiction that the output matching μ^* is not an immediate descendant of μ' in $(\mathcal{S}, \succeq)$. Then, there exists a stable matching μ such that $\mu' \succ \mu \succ \mu^*$. By Lemma 5, for every firm-worker pair $(f', w') \in \mu' \setminus \mu$, we also have $(f', w') \notin \mu_W$. Thus, $\mu \in \mathcal{T}$ due to Theorem 11. However, this means that μ^* is not a maximal matching from $\mathcal{T}$, which is a contradiction. The runtime follows from Theorem 10 and the fact that $|\mu'| = O(|F||W|)$.

Algorithm 3 employs Algorithm 2 to find a maximal chain of the stable matching lattice, as well as the set of rotations.

Theorem 13. *Algorithm 3 is correct and runs in time $O(|F|^3|W|^3$ `oracle-call`$)$.*

Algorithm 4. Construction of the rotation poset $(\Pi, \succeq^\star)$

1: Run Roth's algorithm [30], to obtain μ_F and μ_W.
2: Run Algorithm 3 to obtain a maximal chain $C_0, C_1, \cdots, C_k$ of the stable matching lattice $(\mathcal{S}, \succeq)$, and the set of rotations $\Pi \equiv \{\rho_1, \rho_2, \cdots, \rho_k\}$.
3: Run the algorithm from Theorem 14 to obtain the partial order $\succeq^\star$.

Proof. By Theorem 7 and Theorem 12, the maximal chain output is correct. Additionally, by Theorem 5, $|\Pi| = O(|F||W|)$ and the running time follows. By [19, Section 2.4.3], all elements of $\mathcal{D}(\mathcal{P})$ are found on a maximal chain of $\mathcal{P}$. Then, by Theorem 7, Π can also be found on a maximal chain of $\mathcal{S}$. Thus, output Π is correct.

Partial order $\succeq^\star$ over Π. Consider a ring of sets $(\mathcal{H}, \subseteq)$. We can produce an efficient algorithm that obtains the partial order $\sqsupseteq$ of $(\mathcal{D}(\mathcal{H}), \sqsupseteq)$ from a maximal chain of $(\mathcal{H}, \subseteq)$, based on the classical concept of *irreducible* elements: see [15] for details. Together with Theorem 4 and Theorem 7, we have the following.

Theorem 14. *There is an algorithm with runtime* $O(|F|^3|W|^3\texttt{oracle-call})$ *that constructs the partial order* $\succeq^\star$ *given as input the output of Algorithm 3.*

Summary and time complexity analysis. The complete procedure to build the rotation poset is summarized in Algorithm 4. Its correctness and runtime of $O(|F|^3|W|^3\texttt{oracle-call})$ follow from Theorem 9, Theorem 13, and Theorem 14. This concludes the proof of Theorem 1.

5 The Convex Hull of Lattice Elements: Proof of Theorem 2

The *order polytope* associated with poset $(Y, \succeq^\star)$ is defined as

$$\mathcal{O}(Y, \succeq^\star) \coloneqq \{y \in [0,1]^Y : y_i \geq y_j,\ \forall i, j \in Y \text{ s.t. } i \succeq^\star j\}.$$

Stanley [36] showed that the vertices of $\mathcal{O}(Y, \succeq^\star)$ are the characteristic vectors of upper sets of $(Y, \succeq^\star)$, and gave a complete characterization of the $O(|Y|^2)$ facets of $\mathcal{O}(Y, \succeq^\star)$. We claim that

$$\text{conv}(\mathcal{X}) = \{x_0\} \oplus A \cdot \mathcal{O}(Y, \succeq^\star) = \{x \in \mathbb{R}^E : x = x_0 + Ay,\ y \in \mathcal{O}(Y, \succeq^\star)\},$$

where $\oplus$ denotes the Minkowski sum operator. Indeed, g defines a bijection between vertices of $\mathcal{O}(Y, \succeq^\star)$ and vertices of $\text{conv}(\mathcal{X})$. The claim them follows by convexity. As $\mathcal{O}(Y, \succeq^\star)$ has $O(|Y|^2)$ facets, we conclude the first statement from Theorem 2.

Now suppose that A has full column rank. Then, since $\mathcal{O}(Y, \succeq^\star)$ is full-dimensional, $\text{conv}(\mathcal{X})$ is affinely isomorphic to $\mathcal{O}(Y, \succeq^\star)$. Hence, there is a one-to-one correspondence between facets of $\mathcal{O}(Y, \succeq^\star)$ and facets of $\text{conv}(\mathcal{X})$. The second statement then follows from the characterization given in [36]. Example 4 shows that statements above do not hold when A does not have full column-rank.

Example 4. Consider the lattice $(\mathcal{X}, \succeq)$ and its representation poset $(Y, \succeq^\star)$ from Example 2. Note that

$$\text{conv}(\mathcal{X}) = \{x \in [0,1]^4 : x_1 = 1,\ x_2 + x_3 = 1\}.$$

Thus, $\text{conv}(\mathcal{X})$ has dimension 2. On the other hand, $\mathcal{O}(Y, \succeq^\star)$ has dimension 3. So the two polytopes are not affinely isomorphic.

More generally, one can easily construct a distributive lattice $(\mathcal{X}, \succeq)$ such that the number of facets of $\mathcal{O}(Y, \succeq^\star)$ gives no useful information on the number of facets of $\text{conv}(\mathcal{X})$, where $(Y, \succeq^\star)$ is a poset that affinely represents $(\mathcal{X}, \succeq)$. In fact, the vertices of any 0/1 polytope can be arbitrarily arranged in a chain to form a distributive lattice $(\mathcal{X}, \succeq)$. A poset $\mathcal{O}(Y, \succeq^\star)$ that affinely represents $(\mathcal{X}, \succeq)$ is given by a chain with $|Y| = |\mathcal{X}| - 1$. It is easy to see that $\mathcal{O}(Y, \succeq^\star)$ is a simplex and has therefore $|Y| + 1 = |\mathcal{X}|$ facets. However, $\text{conv}(\mathcal{X})$ could have much more (or much less) facets than the number of its vertices. △

References

1. Abdulkadiroğlu, A., Sönmez, T.: School choice: a mechanism design approach. Am. Econ. Rev. **93**(3), 729–747 (2003)
2. Aizerman, M., Malishevski, A.: General theory of best variants choice: some aspects. IEEE Trans. Autom. Control **26**(5), 1030–1040 (1981)
3. Alkan, A.: On preferences over subsets and the lattice structure of stable matchings. Rev. Econ. Design **6**(1), 99–111 (2001)
4. Alkan, A.: A class of multipartner matching markets with a strong lattice structure. Econ. Theor. **19**(4), 737–746 (2002)
5. Aprile, M., Cevallos, A., Faenza, Y.: On 2-level polytopes arising in combinatorial settings. SIAM J. Discret. Mathe. **32**(3), 1857–1886 (2018)
6. Aygün, O., Sönmez, T.: Matching with contracts: comment. Am. Econ. Rev. **103**(5), 2050–51 (2013)
7. Aygün, o., Turhan, B.: Dynamic reserves in matching markets: Theory and applications. Available at SSRN 2743000 (2016)
8. Baïou, M., Balinski, M.: Many-to-many matching: stable polyandrous polygamy (or polygamous polyandry). Discret. Appl. Math. **101**(1–3), 1–12 (2000)
9. Baïou, M., Balinski, M.: The stable admissions polytope. Math. Program. **87**(3), 427–439 (2000). https://doi.org/10.1007/s101070050004
10. Bansal, v., Agrawal, A., Malhotra, V.S.: Polynomial time algorithm for an optimal stable assignment with multiple partners. Theoret. Comput. Sci. **379**(3), 317–328 (2007)
11. Birkhoff, G.: Rings of sets. Duke Math. J. **3**(3), 443–454 (1937)
12. Blair, C.: The lattice structure of the set of stable matchings with multiple partners. Math. Oper. Res. **13**(4), 619–628 (1988)
13. Chambers, C.P., Yenmez, M.B.: Choice and matching. Am. Econ. J. Microecon. **9**(3), 126–47 (2017)
14. Echenique, F., Yenmez, M.B.: How to control controlled school choice. Am. Econ. Rev. **105**(8), 2679–2694 (2015)
15. Faenza, Y., Zhang, X.: Affinely representable lattices, stable matchings, and choice functions (2020). Available on arXiv

16. Fleiner, T.: On the stable b-matching polytope. Math. Soc. Sci. **46**(2), 149–158 (2003)
17. Gale, D., Shapley, L.S.: College admissions and the stability of marriage. Am. Math. Month. **69**(1), 9–15 (1962)
18. Garg, V.K.: Predicate detection to solve combinatorial optimization problems. In: Proceedings of the 32nd ACM Symposium on Parallelism in Algorithms and Architectures, pp. 235–245 (2020)
19. Gusfield, D., Irving, R.W.: The stable marriage problem: structure and algorithms. MIT press (1989)
20. Hatfield, J.W., Milgrom, P.R.: Matching with contracts. Am. Econ. Rev. **95**(4), 913–935 (2005)
21. Irving, R.W., Leather, P.: The complexity of counting stable marriages. SIAM J. Comput. **15**(3), 655–667 (1986)
22. Irving, R.W., Leather, P., Gusfield, D.: An efficient algorithm for the "optimal" stable marriage. J. ACM (JACM) **34**(3), 532–543 (1987)
23. Kamada, Y., Kojima, F.: Efficient matching under distributional constraints: theory and applications. Am. Econ. Rev. **105**(1), 67–99 (2015)
24. Knuth, D.E.: Marriages stables. Technical report (1976)
25. Manlove, D.: Algorithmics of matching under preferences, vol. 2. World Scientific (2013)
26. Martínez, R., Massó, J., Neme, A., Oviedo, J.: An algorithm to compute the full set of many-to-many stable matchings. Math. Soc. Sci. **47**(2), 187–210 (2004)
27. McVitie, D.G., Wilson, L.B.: The stable marriage problem. Commun. ACM **14**(7), 486–490 (1971)
28. Nguyen, T., Vohra, R.: Stable matching with proportionality constraints. Oper. Res. **67**(6), 1503–1519 (2019)
29. Picard, J.-C.: Maximal closure of a graph and applications to combinatorial problems. Manage. Sci. **22**(11), 1268–1272 (1976)
30. Roth, A.E.: The evolution of the labor market for medical interns and residents: a case study in game theory. J. Polit. Econo. **92**(6), 991–1016 (1984)
31. Roth, A.E.: Stability and polarization of interests in job matching. Econom. J. Econ. Soc. **52**, 47–57 (1984)
32. Roth, A.E., Rothblum, U.G., Vate, J.H.V.: Stable matchings, optimal assignments, and linear programming. Math. Oper. Res. **18**(4), 803–828 (1993)
33. Rothblum, U.G.: Characterization of stable matchings as extreme points of a polytope. Math. Program. **54**(1–3), 57–67 (1992)
34. Schrijver, A.: Combinatorial Optimization: Polyhedra and Efficiency, vol. 24. Springer Science & Business Media, Heidelberg (2003)
35. Shapley, L.S., Shubik, M.: The assignment game I: the core. Int. J. Game Theory, **1**(1), 111–130 (1971). https://doi.org/10.1007/BF01753437
36. Stanley, R.P.: Two poset polytopes. Discret. Comput. Geom. **1**(1), 9–23 (1986). https://doi.org/10.1007/BF02187680
37. Tomoeda, K.: Finding a stable matching under type-specific minimum quotas. J. Econ. Theory **176**, 81–117 (2018)
38. Vate, J.H.V.: Linear programming brings marital bliss. Oper. Res. Lett. **8**(3), 147–153 (1989)