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Brief paper

Finite-time integral control for a class of nonlinear planar systems with non-vanishing uncertainties ☆

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ABSTRACT

This paper considers the problem of globally regulating a class of planar nonlinear systems perturbed by various non-vanishing uncertainties including constant step disturbances, exogenous time-varying disturbances with unknown magnitudes, and modeling uncertainties with unknown system parameters. A new integral controller consisting of a nonlinear integral dynamic and a semi-linear control law is constructed to drive the states of the uncertain systems to the origin in a *finite time*. This is achieved by three major mechanisms: (i) for the purpose of finite-time convergence, a lower-order integral dynamic is first constructed; (ii) by revamping the technique of adding a power integrator, a semi-linear control law containing a linear corrective term is proposed to handle the various forms of uncertainties; and (iii) a new inequality is established to provide an effective estimating tool for the selection of a suitable control gain to guarantee finite-time stability.

1. Introduction

In this paper, we consider a class of planar nonlinear systems described by

$$\begin{cases} \dot{x}_1 &= x_2 + f_1(x_1) \\ \dot{x}_2 &= u + f_2(x_1, x_2) + d(t, x) \end{cases} \tag{1}$$

where $x = [x_1, x_2]^T \in \mathbb{R}^2$ are the system states, $u \in \mathbb{R}$ is control input, and $y = x_1$ is system output. In addition, $f_1(\cdot)$ and $f_2(\cdot)$ are unknown functions vanishing at the origin and $d(t, x)$ represents non-vanishing uncertainties. Planar systems are widely used to describe dynamics of practical systems such as a pendulum system considered in Example 5.1.

When $d(t, x) = 0$, system (1) with known differentiable nonlinearities f_i's can be globally stabilized by the backstepping approach (Sepulchre, Jankovic, & Kokotovic, 1997). In the case when f_i's are unknown, a domination approach was proposed in Tsinias (1991) to obtain an linear controller under the linear growth condition. In the case when f_i's only satisfy a Hölder growth condition (Muñoz-Vázquez, Parra-Vega, & Sánchez-Orta, 2016), a design methodology called adding a power integrator technique, which can be traced back to the technique of adding an integrator (Coron & Praly, 1991), was proposed in Polendo and Qian (2007) and Qian and Lin (2006) to globally stabilize system (1).

Recently, finite-time stabilization has been studied due to many interesting features such as faster convergence rate, higher accurateness, as well as better disturbance rejection property (Bhat & Bernstein, 2000; Du & Li, 2012; Li, Liu, & Ding, 2010). Many interesting results have been achieved for hyperbolic systems (Coron & Nguyen, 2020), switched nonlinear systems (Fu, Ma, & Chai, 2015; Song & Zhai, 2019), time-varying nonlinear systems (Sun, Yun, & Li, 2017), unknown nonlinear systems (Fu, Ma, & Chai, 2017; Hong, Wang, & Cheng, 2006; Wu, Chen, & Li, 2016), etc. Moreover, in order to meet the requirements on convergence time, fixed-time and prescribed-time controllers have been designed in Chen, Liu, and Zhang (2020) and Lopez-Ramirez, Polyakov, Efimov, and Perruquetti (2018). However, those methods are only applicable to systems with vanishing disturbances.

In the presence of non-vanishing uncertainties, i.e., $d(t, x) \neq 0$, the aforementioned results cannot guarantee that all states of the nonlinear system (1) converge to the origin. In the linear case when $f_1(\cdot) = 0$, $f_2(\cdot) = 0$, and $d(t, x) = \theta$ for an unknown constant θ, system (1) can be regulated by the commonly-used PID controller (O'Dwyer, 2009)

$$u(t) = -k_0 \int_0^t x_1(s)\, ds - k_1 x_1(t) - k_2 x_2(t) \tag{2}$$

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where k_0, k_1, k_2 are coefficients of the Hurwitz polynomial $s^3 + k_2 s^2 + k_1 s + k_0 = 0$ $(k_1 k_2 > k_0)$ (Dorf & Bishop, 2011). The corrective term $(\int_0^t x_1(s)\, ds)$ can counteract the effect of a constant disturbance but also will cause a trade-off between stability and convergence rate, i.e., a larger k_0 desired for a faster convergence rate may cause instability (when $k_0 \geq k_1 k_2$). Moreover, the controller (2) cannot handle exogenous time-varying disturbances, e.g., $d(t, x) = c(1 + 0.5\sin(2t))$ with an unknown magnitude c, which drive the states x_1 and x_2 away from the origin. The performance of the PID controller will be even worse when $d(t, x)$ is an internal modeling uncertainty such as $d(t, x) = \theta(1 + x_2^2)$ with an unknown system parameter θ. In fact, a finite-time escape phenomenon (Khalil, 1992) can be observed as θ increases. Therefore, it is of interest to design a controller to globally regulate the nonlinear systems in the presence of various uncertainties and achieve a faster convergence rate.

In this paper, we propose a novel integral controller consisting of a nonlinear integral dynamic and a semi-linear control law for the nonlinear system (1) with various non-vanishing uncertainties. First, similar to the linear case, we will introduce an extra integral state to be used in the control law to tackle $d(t, x)$. In order to achieve finite-time convergence rate, a lower-order integral dynamic will be used. Then, by revamping the technique of adding a power integrator (Polendo & Qian, 2007), we will construct a semi-linear control law with a linear corrective term to handle the various forms of uncertainties. To address the challenge in selecting the control coefficients of the semi-linear control law, we will establish an effective estimating tool in the form of a new inequality. The proposed integral controller will drive states of the planar system to the origin in a finite-time convergence rate, regardless of the presence of not-precisely-known nonlinearities and non-vanishing uncertainties.

2. Problem statement and a new inequality

The purpose of this paper is to solve the problem of **Global Finite-Time Regulation** for the nonlinear system (1) in the presence of non-vanishing uncertainties. More specifically, we aim to design a controller in the form

$$\dot{x}_0 = \eta(x_1), \quad u = \alpha(t, x_0, x_1, x_2) \tag{3}$$

with continuous nonlinear functions $\eta : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, such that (i) the trajectories of the closed-loop system (1) and (3) are uniformly globally stable in the sense of Lyapunov; and (ii) there is a finite time t^* such that $(x_1(t), x_2(t)) = (0, 0)$ for any $t \geq t^*$.

To solve the problem, what we need to do is to design a continuous controller in the form of (3) such that

- there is a constant x_0^* such that $(x_0^*, 0, 0)$ is the equilibrium point of the closed-loop system (1) and (3);
- under the new coordinates $z_1 = x_0 - x_0^*, z_2 = x_1, z_3 = x_2$, the closed-loop system has an uniformly globally finite-time stable equilibrium at the origin according to Definition A.1.

Next we introduce a new inequality which plays a crucial role in designing the proposed semi-linear controller.

Lemma 2.1. *For any constant $q \in \mathbb{R}_{\text{odd}}^+$[1] with $q < 1$ and positive constants ρ_1 and ρ_2, there exists a positive constant k^*, such that the following holds*

$$\rho_1 \lambda^2 + k(\lambda^q + (1 - \lambda)^q) \geq \rho_2 \tag{4}$$

for any real-valued functions λ and k satisfying $k \geq k^*$.

Proof. Denote $p = 1/q > 1, x = \lambda^q$ and $y = (\lambda - 1)^q$. It follows from Lemma B.1 that

$$\begin{aligned} 1 &= \left|(\lambda^q)^p - ((\lambda - 1)^q)^p\right| \\ &\leq p\left|\lambda^q + (1 - \lambda)^q\right| \left(\lambda^{1-q} + (\lambda - 1)^{1-q}\right) \\ &\leq p\left|\lambda^q + (1 - \lambda)^q\right| \left(|\lambda|^{1-q} + (1 + |\lambda|)^{1-q}\right). \end{aligned} \tag{5}$$

Note that $0 < (1 - q) < 1$ and $(\lambda^q + (1 - \lambda)^q) \geq 0$. Applying Lemma B.2. (2) to the last part of (5) yields

$$1 \leq p\left(\lambda^q + (1 - \lambda)^q\right)\left(2\,|\lambda|^{1-q} + 1\right). \tag{6}$$

With the help of (6), we have

$$\begin{aligned} &\rho_1 \lambda^2 + k(\lambda^q + (1 - \lambda)^q) \\ &\quad \geq \rho_1 \lambda^2 + \frac{k}{p(1 + 2|\lambda|^{1-q})} \\ &\quad = \frac{\rho_1}{2}(3 + 2\lambda^2) + \frac{k}{p(1 + 2|\lambda|^{1-q})} - \frac{3}{2}\rho_1 \\ &\quad \geq 2\sqrt{\frac{k\rho_1}{2p}\,\frac{3 + 2\lambda^2}{1 + 2|\lambda|^{1-q}}} - \frac{3}{2}\rho_1. \end{aligned} \tag{7}$$

Since $(1 - q) < 1$, it is clear that

$$3 + 2\lambda^2 = 1 + 2(1 + \lambda^2) \geq 1 + 2|\lambda|^{1-q}. \tag{8}$$

Substituting (8) into (7), we have

$$\rho_1 \lambda^2 + k(\lambda^q + (1 - \lambda)^q) \geq \sqrt{k}\sqrt{2\rho_1 q} - \frac{3}{2}\rho_1. \tag{9}$$

Hence, given positive constants $\rho_1,\ \rho_2,\ q$, the inequality (4) holds for any real function $k \geq k^* = \frac{(\rho_2 + 1.5\rho_1)^2}{2\rho_1 q}$. □

Remark 2.1. Lemma 2.1 is a previously missing counterpart of Lemma B.2. (1) in the case when the power is less than 1. Lemma B.2. (1) pertains to the case when $p \geq 1$ for $p \in \mathbb{R}_{\text{odd}}^+$, i.e.,

$$k(\lambda^p + (1 - \lambda)^p) \geq \rho_2, \quad \forall \lambda \in \mathbb{R}, \tag{10}$$

for $k \geq \rho_2 2^{p-1}$. However, when p is less than 1, (10) does not hold any more. Instead, when $q < 1$ and $q \in \mathbb{R}_{odd}^+$, we can only obtain

$$(\lambda^q + (1 - \lambda)^q) \leq 2^{1-q}, \quad \forall \lambda \in \mathbb{R}, \tag{11}$$

which is unusable for our particular purpose (see (30)) in this paper. Therefore, Lemma 2.1 is established to get the new relationship (4) by leveraging the term of $\rho_1 \lambda^2$.

3. Semi-linear controller design

In this section, for the following system

$$\dot{z}_1 = z_2^\gamma, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = u \tag{12}$$

with $\gamma \in \mathbb{R}_{\text{odd}}^+$ satisfying $\gamma < 1$, we design a novel finite-time stabilizer with a special semi-linear structure by revamping the technique of adding a power integrator.

Theorem 3.1. *There are positive constants $k^*,\ a_1$ and a_2, such that for any $K(t, z) \geq k^*$, the following semi-linear controller*

$$u = -K(t, z)\left(a_1 z_1 + \left(z_3^{\frac{3}{\gamma+2}} + a_2 z_2\right)^{\frac{2\gamma+1}{3}}\right) \tag{13}$$

globally stabilizes (12) in a finite time.

[1] $\mathbb{R}_{\text{odd}}^+ := \{m/n \mid m, n \text{ are positive odd integers}\}$.

Proof. Based on the adding a power integrator technique, we propose a constructive method to design a controller and a Lyapunov function in a recursive manner.

Step 1: First, construct $U_1 = \frac{2\gamma+1}{7-\gamma} z_1^{\frac{7-\gamma}{2\gamma+1}}$. The time derivative of U_1 along system (12) is

$$\dot{U}_1 = (z_1^{\frac{3}{2\gamma+1}})^{2-\gamma} z_2^{\gamma} = \xi_1^{2-\gamma} z_2^{*\gamma} + \xi_1^{2-\gamma}(z_2^{\gamma} - z_2^{*\gamma}) \tag{14}$$

where $\xi_1 = z_1^{3/(2\gamma+1)}$ and z_2^* is a virtual controller. For (14), selecting $z_2^{*\gamma} = -\beta_1 \xi_1^{\gamma} = -3\xi_1^{\gamma}$ yields

$$\dot{U}_1 = -3\xi_1^2 + \xi_1^{2-\gamma}(z_2^{\gamma} - z_2^{*\gamma}). \tag{15}$$

Step 2: Construct $W_2 = \frac{3}{7-\gamma}(z_2 - z_2^*)^{\frac{7-\gamma}{3}} = \frac{3}{7-\gamma}\xi_2^{\frac{7-\gamma}{3}}$ with $\xi_2 = z_2 - z_2^*$. Therefore, for $U_2 = U_1 + W_2$, the time derivative along system (12) is

$$\begin{aligned}\dot{U}_2 =& -3\xi_1^2 + \xi_1^{2-\gamma}(z_2^{\gamma} - z_2^{*\gamma}) + \xi_2^{\frac{4-\gamma}{3}} z_3^* \\ &+ \xi_2^{\frac{4-\gamma}{3}}(z_3 - z_3^*) + \xi_2^{\frac{4-\gamma}{3}} \frac{\partial(-z_2^*)}{\partial z_1}\dot{z}_1.\end{aligned} \tag{16}$$

By the fact that $\gamma < 1$ and Lemma B.2. (1), we have

$$\begin{aligned}\xi_1^{2-\gamma}(z_2^{\gamma} - z_2^{*\gamma}) \leq& 2^{1-\gamma}|\xi_1|^{2-\gamma}\left|z_2 - z_2^*\right|^{\gamma} \\ =& 2^{1-\gamma}|\xi_1|^{2-\gamma}|\xi_2|^{\gamma} \\ \leq& \frac{1}{2}\xi_1^2 + \eta_{21}\xi_2^2\end{aligned} \tag{17}$$

where $\eta_{21} \geq 0$ is a constant obtained by applying Lemma B.3. On the other hand, by the definitions of z_2^*, and Lemma B.2. (2), it is easy to have

$$\begin{aligned}\left|\xi_2^{\frac{4-\gamma}{3}} \frac{\partial(-z_2^*)}{\partial z_1}\dot{z}_1\right| =& |\xi_2|^{\frac{4-\gamma}{3}} \frac{3^{\frac{1}{\gamma}+1}}{2\gamma+1}|z_1|^{\frac{2-2\gamma}{2\gamma+1}}\left|\xi_2 - 3^{\frac{1}{\gamma}}\xi_1\right|^{\gamma} \\ \leq& \bar{\eta}_{21}|\xi_2|^{\frac{4-\gamma}{3}}|\xi_1|^{\frac{2-2\gamma}{3}}(|\xi_2|^{\gamma} + |\xi_1|^{\gamma})\end{aligned}$$

for a constant $\bar{\eta}_{21} > 0$. Applying Lemma B.3 to the above equation yields

$$\left|\xi_2^{\frac{4-\gamma}{3}} \frac{\partial(-z_2^*)}{\partial z_1}\dot{z}_1\right| \leq \frac{1}{2}\xi_1^2 + \eta_{22}\xi_2^2 \tag{18}$$

for a constant $\eta_{22} > 0$. Then, substituting (17) and (18) into (16) and constructing $z_3^* = -\beta_2 \xi_2^{(\gamma+2)/3}$ with $\beta_2 = \eta_{21} + \eta_{22} + 2$, we have the following

$$\dot{U}_2 \leq -2\xi_1^2 - 2\xi_2^2 + \xi_2^{\frac{4-\gamma}{3}}(z_3 - z_3^*). \tag{19}$$

Step 3: Construct a C^1 function $W_3 = \int_{z_3^*}^{z_3}(s^{\frac{3}{\gamma+2}} - z_3^{*\frac{3}{\gamma+2}})^{\frac{5-2\gamma}{3}}ds$. The time derivative of $U_3 = U_2 + W_3$ along (12) is

$$\begin{aligned}\dot{U}_3 \leq& -2\xi_1^2 - 2\xi_2^2 + \xi_2^{\frac{4-\gamma}{3}}(z_3 - z_3^*) + \xi_3^{\frac{5-2\gamma}{3}}u \\ &+ \frac{\partial W_3}{\partial z_2}\dot{z}_2 + \frac{\partial W_3}{\partial z_1}\dot{z}_1\end{aligned} \tag{20}$$

where $\xi_3 = z_3^{3/(\gamma+2)} - z_3^{*3/(\gamma+2)}$.

In what follows, we estimated cross-terms in (20). First, by Lemmas B.2–B.3 and the definition of ξ_3, we have

$$\begin{aligned}\xi_2^{\frac{4-\gamma}{3}}(z_3 - z_3^*) =& \xi_2^{\frac{4-\gamma}{3}}((z_3^{\frac{3}{\gamma+2}})^{\frac{\gamma+2}{3}} - (z_3^{*\frac{3}{\gamma+2}})^{\frac{\gamma+2}{3}}) \\ \leq& 2^{1-\frac{\gamma+2}{3}}|\xi_2|^{\frac{4-\gamma}{3}}|\xi_3|^{\frac{\gamma+2}{3}} \\ \leq& \frac{1}{3}\xi_2^2 + \eta_{31}\xi_3^2\end{aligned} \tag{21}$$

where η_{31} is a positive constant. Considering the definition of W_3 and the relation $\left|z_3 - z_3^*\right| \leq 2^{\frac{1-\gamma}{3}}|\xi_3|^{\frac{\gamma+2}{3}}$

$$\begin{aligned}\sum_{j=1}^{2}\frac{\partial W_3}{\partial z_j}\dot{z}_j \leq& \frac{5-2\gamma}{3}\left|z_3 - z_3^*\right||\xi_3|^{\frac{2-2\gamma}{3}}\left|\sum_{j=1}^{2}\frac{\partial z_3^{*\frac{3}{\gamma+2}}}{\partial z_j}\dot{z}_j\right| \\ \leq& \varpi|\xi_3|^{\frac{4-\gamma}{3}}\left|\sum_{j=1}^{2}\frac{\partial z_3^{*\frac{3}{\gamma+2}}}{\partial z_j}\dot{z}_j\right|,\end{aligned} \tag{22}$$

where $\varpi = \frac{(5-2\gamma)2^{\frac{1-\gamma}{3}}}{3}$. Based on $z_3^{*\frac{3}{\gamma+2}} = -\beta_2^{\frac{3}{\gamma+2}}(z_2 + \beta_1^{\frac{1}{\gamma}} z_1^{\frac{3}{2\gamma+1}})$ and by Lemma B.3, there is a constant $\bar{\eta}_{32} > 0$

$$\begin{aligned}\left|\sum_{j=1}^{2}\frac{\partial z_3^{*\frac{3}{\gamma+2}}}{\partial z_j}\dot{z}_j\right| =& |\beta_2^{\frac{3}{\gamma+2}}(z_3 + \beta_1 \frac{3}{2\gamma+1} z_1^{\frac{2-\gamma}{2\gamma+1}} z_2^{\gamma})| \\ \leq& \bar{\eta}_{32}(|z_1|^{\frac{\gamma+2}{2\gamma+1}} + |z_2|^{\frac{\gamma+2}{3}} + |z_3|) \\ \leq& \hat{\eta}_{32}(|\xi_1|^{\frac{\gamma+2}{3}} + |\xi_2|^{\frac{\gamma+2}{3}} + |\xi_3|^{\frac{\gamma+2}{3}})\end{aligned} \tag{23}$$

where $\hat{\eta}_{32}$ is an appropriate positive constant. Substituting (23) into (22) and applying Lemma B.3 to cross-terms $|\xi_3|^{\frac{4-\gamma}{3}}|\xi_1|^{\frac{\gamma+2}{3}}$ and $|\xi_3|^{\frac{4-\gamma}{3}}|\xi_2|^{\frac{\gamma+2}{3}}$, we can find a constant $\eta_{32} > 0$ such that

$$\sum_{j=1}^{2}\frac{\partial W_3}{\partial z_j}\dot{z}_j \leq \frac{3}{4}\xi_1^2 + \frac{2}{3}\xi_2^2 + \eta_{32}\xi_3^2. \tag{24}$$

Substituting (21) and (24) into (20) yields

$$\dot{U}_3 \leq -\frac{5}{4}\xi_1^2 - \xi_2^2 + \xi_3^{\frac{5-2\gamma}{3}}u + (\eta_{31} + \eta_{32})\xi_3^2. \tag{25}$$

By the traditional adding a power integrator technique (Polendo & Qian, 2007), we can choose the following conventional controller

$$\begin{aligned}u =& -\beta_3\left(z_3^{\frac{3}{\gamma+2}} + \beta_2^{\frac{3}{\gamma+2}} z_2 + \beta_2^{\frac{3}{\gamma+2}}\beta_1^{\frac{1}{\gamma}} z_1^{\frac{3}{2\gamma+1}}\right)^{\frac{2\gamma+1}{3}} \\ =& -\beta_3\left(z_3^{\frac{3}{\gamma+2}} + a_2 z_2 + (a_1 z_1)^{\frac{3}{2\gamma+1}}\right)^{\frac{2\gamma+1}{3}}\end{aligned} \tag{26}$$

with $\beta_3 = \eta_{31} + \eta_{32} + 1$, $a_1 = \beta_2^{\frac{2\gamma+1}{\gamma+2}}\beta_1^{\frac{2\gamma+1}{3\gamma}}$ and $a_2 = \beta_2^{\frac{3}{\gamma+2}}$. Under the conventional controller (26), (25) becomes

$$\dot{U}_3 \leq -\frac{5}{4}\xi_1^2 - \xi_2^2 - \xi_3^2. \tag{27}$$

Noting that the Lyapunov function U_3 is positive definite and proper with respect to (z_1, z_2, z_3), the closed-loop system (12) and (26) is globally asymptotically stable.

Semi-linear Controller Design: It is clear that the controller (26) does not have a linear z_1 term. Next, we design a new controller in the form of (13). To this end, we first rewrite (25) as

$$\dot{U}_3 \leq -\xi_1^2 - \xi_2^2 - \xi_3^2 - \frac{\xi_1^2}{4} + \beta_3\xi_3^2 + \xi_3^{\frac{5-2\gamma}{3}}u, \tag{28}$$

where β_3 is a constant same as the one in (26). Based on the conventional controller (26), we construct the following new controller

$$u = -K(t,z)\left(a_1 z_1 + (z_3^{\frac{3}{\gamma+2}} + a_2 z_2)^{\frac{2\gamma+1}{3}}\right), \tag{29}$$

with the same constants a_1 and a_2 as those in (26). When $\gamma < 1$ in the new controller, by (11), we only can have

$$\begin{aligned}\xi_3^{\frac{5-2\gamma}{3}}u|_{(29)} =& -K\xi_3^{\frac{5-2\gamma}{3}}(a_1 z_1 + (\xi_3 - (a_1 z_1)^{\frac{3}{2\gamma+1}})^{\frac{2\gamma+1}{3}}) \\ \geq& -K\xi_3^2\end{aligned}$$

which is not in the correct direction to dominate $\beta_3\xi_3^2$ in (28). Fortunately, the new Lemma 2.1 can help resolve this issue by leveraging the negative term $-\frac{1}{4}\xi_1^2$. Specifically, in what follows, we show that there is a large enough k^*, such that when $K(t,z) \geq k^*$ the following holds

$$-\frac{1}{4}\xi_1^2 + \beta_3\xi_3^2 + \xi_3^{\frac{5-2\gamma}{3}} u|_{(29)} \leq 0. \tag{30}$$

Clearly, when $\xi_3 = 0$, (30) holds. When $\xi_3 \neq 0$, let

$$\lambda = \frac{(a_1z_1)^{3/(2\gamma+1)}}{z_3^{3/(2\gamma+1)}+a_2z_2+(a_1z_1)^{3/(2\gamma+1)}} = \frac{a_1^{3/(2\gamma+1)}\xi_1}{\xi_3},$$
$$1-\lambda = \frac{z_3^{3/(\gamma+2)}+a_2z_2}{z_3^{3/(\gamma+2)}+a_2z_2+(a_1z_1)^{3/(2\gamma+1)}} = \frac{z_3^{3/(\gamma+2)}+a_2z_2}{\xi_3},$$

$\rho_1 = 1/(4a_1^{6/(2\gamma+1)})$, $\rho_2 = \beta_3$, and $q = (2\gamma+1)/3 < 1$. By Lemma 2.1, there is a $k^* > 0$ such that for any $K > k^*$ the following holds

$$\rho_2 \leq \rho_1\lambda^2 + K(\lambda^q + (1-\lambda)^q). \tag{31}$$

Multiplying both sides of (31) by ξ_3^2 yields,

$$\begin{aligned}\rho_2\xi_3^2 \leq& \rho_1\Big(a_1^{\frac{3}{2\gamma+1}}\xi_1\Big)^2 + K\xi_3^{2-q}\Big[\Big(a_1^{\frac{3}{2\gamma+1}}\xi_1\Big)^{\frac{2\gamma+1}{3}} \\ &+ \Big(z_3^{\frac{3}{\gamma+2}} + a_2z_2\Big)^{\frac{2\gamma+1}{3}}\Big] \\ =& \frac{1}{4}\xi_1^2 + K\xi_3^{\frac{5-2\gamma}{3}}\Big[a_1z_1 + (z_3^{\frac{3}{\gamma+2}} + a_2z_2)^{\frac{2\gamma+1}{3}}\Big]\end{aligned}$$

for any $K(t,z) \geq k^*$, which implies the inequality (30) holds. Substituting (30) into (28) leads to

$$\dot{U}_3|_{(12)\&(29)} \leq -\xi_1^2 - \xi_2^2 - \xi_3^2 \tag{32}$$

which implies that the closed-loop system (12)–(13) is globally asymptotically stable.

Finite-Time Stability Analysis: Next, we will show that the states of the closed-loop system (12)–(13) converge to the origin in a finite time (Bhat & Bernstein, 2000). From the definition of W_3 and Lemma B.2. (1), we arrive at

$$\begin{aligned}W_3 \leq& (z_3 - z_3^*)(z_3^{\frac{3}{\gamma+2}} - z_3^{*\frac{3}{\gamma+2}})^{\frac{5-2\gamma}{3}} \\ \leq& 2^{\frac{1-\gamma}{3}}|\xi_3|^{\frac{\gamma+2}{3}}|\xi_3|^{\frac{5-2\gamma}{3}} = 2^{\frac{1-\gamma}{3}}|\xi_3|^{\frac{7-\gamma}{3}}.\end{aligned} \tag{33}$$

This, together with $U_1 = \frac{2\gamma+1}{7-\gamma}\xi_1^{\frac{7-\gamma}{3}}$ and $W_2 = \frac{3}{7-\gamma}\xi_2^{\frac{7-\gamma}{3}}$, yields

$$\begin{aligned}U_3 \leq& \frac{2\gamma+1}{7-\gamma}\xi_1^{\frac{7-\gamma}{3}} + \frac{3}{7-\gamma}\xi_2^{\frac{7-\gamma}{3}} + 2^{\frac{1-\gamma}{3}}\xi_3^{\frac{7-\gamma}{3}} \\ \leq& d_1\Big((\xi_1^2)^{(7-\gamma)/6} + (\xi_2^2)^{(7-\gamma)/6} + (\xi_3^2)^{(7-\gamma)/6}\Big)\end{aligned} \tag{34}$$

where $d_1 = 2^{\frac{1-\gamma}{3}}$. By Lemma B.2. (3),

$$U_3 \leq d_1(\xi_1^2 + \xi_2^2 + \xi_3^2)^{(7-\gamma)/6}. \tag{35}$$

Therefore, by (32) and (35), there is a constant $\varrho = d_1^{\frac{-6}{7-\gamma}} = 4^{\frac{\gamma-1}{7-\gamma}}$, such that

$$\dot{U}_3 \leq -\varrho U_3^m \tag{36}$$

with $m = 6/(7-\gamma) < 1$. From (36), we can see that

$$0 \leq U_3^{1-m}(t) \leq U_3^{1-m}(0) - \varrho(1-m)t, \tag{37}$$

for $t \leq t^* = \frac{U_3^{1-m}(0)}{\varrho(1-m)}$, and $U_3(t) = 0$ for $t \geq t^*$. Therefore, the states of the closed-loop system (12)–(13) will converge to the origin in a finite time. □

Remark 3.1. By observation, the term z_1 in the conventional finite-time stabilizer (26) has a total power of 1, but it is not a line term due to the nonlinear composition. In Theorem 3.1, we move this term out of the nonlinear composition as a linear term in the semi-linear controller (29), which however destroys the negative definite property of (28). In this case, a larger control gain in (29) has been selected with the help of Lemma 2.1.

4. Global finite-time regulation of (1)

In this section, we will solve the global finite-time regulation problem of (1) by utilizing Theorem 3.1. First, we consider the relatively simple case of (1) when $f_i = 0$, $i = 1, 2$, i.e.,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u + d(t,x). \tag{38}$$

To solve the problem, we assume that the non-vanishing uncertainty $d(t,x)$ satisfies the following assumption.

Assumption 4.1. Assume there are an *unknown* constant θ and a *known* function $\alpha(t,x) \geq 1$ such that

$$d(t,x) = \theta\alpha(t,x).$$

Remark 4.1. The uncertain function $d(t,x)$ satisfying Assumption 4.1 encompasses several types of uncertainties in system (38). First, it includes constant disturbances as its special case when $\alpha(t,x) = 1$. For exogenous time-varying disturbances such as $d(t,x) = c(1+0.5\sin(2t))$ with unknown magnitude c, we can simply choose $\alpha(t,x) = 2(1+0.5\sin(2t)) \geq 1$ and $\theta = c/2$. Moreover, $d(t,x)$ can include internal modeling uncertainties such as $d(t,x) = \theta(1+x_2^2)$ with unknown system parameter θ.

Theorem 4.1. *Under Assumption 4.1, there are constants $\gamma \in \mathbb{R}^+_{odd}$ with $\gamma < 1$, k_1^*, a_1, and a_2, such that the following integral controller*

$$\begin{cases} u &= -k_1^*\alpha(t,x)\Big(a_1x_0 + (x_2^{\frac{3}{\gamma+2}} + a_2x_1)^{\frac{2\gamma+1}{3}}\Big) \\ \dot{x}_0 &= x_1^\gamma \end{cases} \tag{39}$$

solves the global finite-time regulation problem of (38).

Proof. Define $z_1 = x_0 - \frac{\theta}{k_1^*a_1}$, $z_2 = x_1$ and $z_3 = x_2$. Under the new coordinates, it is clear that the closed-loop system (38) and (39) can be rewritten as

$$\dot{z} = \begin{bmatrix} z_2^\gamma \\ z_3 \\ -k_1^*\alpha(\cdot)\Big(a_1z_1 + (z_3^{\frac{3}{\gamma+2}} + a_2z_2)^{\frac{2\gamma+1}{3}}\Big) \end{bmatrix} = F(t,z). \tag{40}$$

By Theorem 3.1, we can find constants a_1, a_2 and k^* such that for $K(t,z) = k_1^*\alpha(t,x) \geq k^*$ the system (40) is globally finite time stable. Therefore, there is a finite time t^*, such that $x_1(t) = x_2(t) = 0$ for $t \geq t^*$. By the definition of t^* and (34), we have

$$t^* \leq \frac{(\frac{2\gamma+1}{7-\gamma}\xi_1^{\frac{7-\gamma}{3}}(0) + \frac{3}{7-\gamma}\xi_2^{\frac{7-\gamma}{3}}(0) + 2^{\frac{1-\gamma}{3}}\xi_3^{\frac{7-\gamma}{3}}(0))^{1-m}}{\varrho(1-m)}$$

Therefore, the upper bound t_1 of t^* can be calculated by

$$\begin{aligned}t_1 =& \Bigg(\frac{(2\gamma+1)z_1^{\frac{7-\gamma}{2\gamma+1}}(0) + 3\Big(z_2(0) + \beta_1^{\frac{1}{\gamma}}z_1^{\frac{3}{2\gamma+1}}(0)\Big)^{\frac{7-\gamma}{3}}}{7-\gamma} \\ &+ 2^{\frac{1-\gamma}{3}}\Big(z_3^{\frac{3}{\gamma+2}}(0) + \beta_2^{\frac{3}{\gamma+2}}(z_2(0) + \beta_1^{\frac{1}{\gamma}}z_1^{\frac{3}{2\gamma+1}}(0))\Big)^{\frac{7-\gamma}{3}}\Bigg)^{\frac{1-\gamma}{7-\gamma}} \\ &\times 4^{\frac{1-\gamma}{7-\gamma}}\frac{7-\gamma}{1-\gamma}\end{aligned} \tag{41}$$

with the initial conditions $[z_1(0) = x_0(0) - \frac{\theta}{k_1^*a_1}, z_2(0) = x_1(0), z_3(0) = x_2(0)]$. □

In the case when $d(t,x) = \theta$ for an unknown constant, Theorem 4.1 holds for a controller with constant gains.

Corollary 4.1. *The system (38) with $d(t,x) = \theta$ for an unknown constant θ can be globally finite-time regulated by the following integral controller*

$$\begin{cases} u &= -k^*\Big(a_1x_0 + (x_2^{\frac{3}{\gamma+2}} + a_2x_1)^{\frac{2\gamma+1}{3}}\Big) \\ \dot{x}_0 &= x_1^{\gamma} \end{cases} \tag{42}$$

for appropriate positive constants k^*, a_1, a_2 and $\gamma \in \mathbb{R}^+_{odd}$ with $\gamma < 1$.

Remark 4.2. According to homogeneous system theory (Bhat & Bernstein, 2000; Hermes, 1991; Kawski, 1990), it is inevitable that a system needs a negative homogeneous degree in order to achieve finite-time convergence. To this end, a nonlinear integral auxiliary equation $\dot{x}_0 = y^{\gamma}$ with $\gamma < 1$ is proposed in this paper. However, when $\gamma < 1$, the traditional adding a power integrator approach (Polendo & Qian, 2007) will result in a controller with a nonlinear structure (see (26)) which cannot be used to handle the uncertain term $d(t,x)$. Therefore, in this paper a novel *"semi-linear"* controller structure is proposed in Theorem 4.1 to derive a linear corrective term of the integral state x_0 (see (39) and (42)) while still preserving homogeneity. This is archived by revamping the technique of adding a power integrator and introducing a new inequality described by Lemma 2.1.

Remark 4.3. The result can be extended to handle higher-dimensional and high-order systems such as those considered in Fu et al. (2017). To this end, a finite-time controller with a semi-linear term z_1 will be designed for the following system $\dot{z}_1 = z_2^{\gamma}$, $\dot{z}_i = z_{i+1}^{p_i}, i = 2, \ldots, n$, $\dot{z}_{n+1} = u$ where p_i's are odd positive integers. In this case, the power can be selected as $\gamma := \frac{\tau+1}{(\cdots((1-\tau)p_{n-1}-\tau)p_{n-2}\cdots)p_1-\tau}$ for an appropriate constant $\tau \in (-1, 0)$.

For the nonlinear system (1), we can now solve the global finite-time regulation problem if the nonlinear functions f_i's satisfy the following assumption.

Assumption 4.2. There are constants $\gamma \in \mathbb{R}^+_{odd}$ satisfying $\gamma < 1$, c_1 and c_2 such that

$$\begin{aligned} |f_1(x_1)| \leq& c_1|x_1|^{\frac{\gamma+2}{3}}, \\ |f_2(x_1,x_2)| \leq& c_2(|x_1|^{\frac{2\gamma+1}{3}} + |x_2|^{\frac{2\gamma+1}{\gamma+2}}). \end{aligned} \tag{43}$$

Remark 4.4. Assumption 4.2 is known as the Hölder condition which is commonly used for control of inherently nonlinear systems (Muñoz-Vázquez et al., 2016; Sun et al., 2017). When $\gamma = 1$, the condition becomes the well-known linear growth condition (Tsinias, 1991). There are many nonlinear functions satisfying Assumption 4.2 with a constant $\gamma \in (0, 1]$. For example, we know $|\sin(x_1)| \leq |x_1|^{\alpha}$ for any $\alpha \in (0, 1]$. For condition (43), we can choose $\alpha = \frac{11}{15}$ and $\gamma = 3/5$ such that $|\sin(x_1)| \leq |x_1|^{\frac{2\gamma+1}{3}}$. Similarly, for any $\alpha \in (0, 1]$, we can show that $|\ln(1 + x_2^2)| \leq (\frac{2}{\alpha} - 1)|x_2|^{\alpha}$ which will lead to Assumption 4.2 by appropriately selecting α and γ.

Theorem 4.2. *Under Assumptions 4.1 and 4.2, there are constants k_1^*, a_1, and a_2 such that for a large enough $L \geq 1$, the following integral controller*

$$\begin{cases} u &= -L^2k_1^*\alpha(\cdot)\Big(a_1x_0 + \big((\tfrac{x_2}{L})^{\frac{3}{\gamma+2}} + a_2x_1\big)^{\frac{2\gamma+1}{3}}\Big) \\ \dot{x}_0 &= Lx_1^{\gamma} \end{cases} \tag{44}$$

with $\alpha(t,x)$ from Assumption 4.1 and the constant γ from Assumption 4.2, solves the global finite-time regulation problem of system (1).

Proof. Define $z_1 = x_0 - \frac{\theta}{k_1^*a_1L^2}$, $z_2 = x_1$ and $z_3 = \frac{x_2}{L}$. By choosing the same constants a_1, a_2 and k_1^* as in Theorem 4.1, the closed-loop system (1) and (44) can be rewritten in the new coordinates as the following system

$$\dot{z} = LF(t,z) + \left[0, f_1(x_1), \frac{f_2(x_1,x_2)}{L}\right]^T \tag{45}$$

where $F(t,z)$ is the same as the one in (40). By using the same Lyapunov function U_3 constructed in Theorem 3.1, we can see that the derivative of U_3 along (45) is

$$\dot{U}_3 = L\frac{\partial U_3}{\partial z}F(t,z) + \frac{\partial U_3}{\partial z_2}f_1(\cdot) + \frac{\partial U_3}{\partial z_3}\frac{f_2(\cdot)}{L}. \tag{46}$$

By (32), it is straightforward to see that (46) becomes

$$\begin{aligned} \dot{U}_3 \leq& -L\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) + \frac{\partial U_3}{\partial z_2}f_1(\cdot) + \frac{\partial U_3}{\partial z_3}\frac{f_2(\cdot)}{L} \\ =& -L\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) \\ &+ (\frac{\partial W_2}{\partial z_2} + \frac{\partial W_3}{\partial z_2})f_1(z_2) + \frac{\partial W_3}{\partial z_3}\frac{f_2(z_2, Lz_3)}{L}. \end{aligned} \tag{47}$$

By Assumption 4.2 and the fact that $L \geq 1$ and $\frac{2\gamma+1}{\gamma+2} < 1$

$$\begin{aligned} \left|\frac{f_2(\cdot)}{L}\right| \leq& c_2\Big(\frac{|z_2|^{\frac{2\gamma+1}{3}}}{L} + \frac{|Lz_3|^{\frac{2\gamma+1}{\gamma+2}}}{L}\Big) \\ \leq& c_2(|\xi_2 - \beta_1^{\frac{1}{\gamma}}\xi_1|^{\frac{2\gamma+1}{3}} + |\xi_3 - \beta_2^{3/(\gamma+2)}\xi_2|^{\frac{2\gamma+1}{3}}) \\ \leq& \bar{c}_2(|\xi_1|^{\frac{2\gamma+1}{3}} + |\xi_2|^{\frac{2\gamma+1}{3}} + |\xi_3|^{\frac{2\gamma+1}{3}}) \end{aligned} \tag{48}$$

for a constant $\bar{c}_2 > 0$. Similarly, there is a constant $\bar{c}_1 > 0$ such that

$$|f_1(\cdot)| \leq c_1|z_2|^{\frac{\gamma+2}{3}} \leq \bar{c}_1(|\xi_1|^{\frac{\gamma+2}{3}} + |\xi_2|^{\frac{\gamma+2}{3}}). \tag{49}$$

In addition, from the definitions W_2 and W_3, we have

$$\frac{\partial W_2}{\partial z_2} = \xi_2^{\frac{4-\gamma}{3}}, \qquad \frac{\partial W_3}{\partial z_3} = \xi_3^{\frac{5-2\gamma}{3}}. \tag{50}$$

By inequality (22) and the definition of z_3^*, we can find a positive constant $\bar{c}_3$, such that

$$\begin{aligned} \left|\frac{\partial W_3}{\partial z_2}\right| \leq& \frac{(5-2\gamma)2^{\frac{1-\gamma}{3}}}{3}|\xi_3|^{\frac{4-\gamma}{3}}\left|\frac{\partial z_3^{*\frac{3}{\gamma+2}}}{\partial z_2}\right| \\ \leq& \bar{c}_3|\xi_3|^{\frac{4-\gamma}{3}}. \end{aligned} \tag{51}$$

Substituting (50) and (51) into (47), we have

$$\begin{aligned} \dot{U}_3 \leq& -L\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) \\ &+ \bar{c}_1(|\xi_2|^{\frac{4-\gamma}{3}} + \bar{c}_3|\xi_3|^{\frac{4-\gamma}{3}})(|\xi_1|^{\frac{\gamma+2}{3}} + |\xi_2|^{\frac{\gamma+2}{3}}) \\ &+ \bar{c}_2|\xi_3|^{\frac{5-2\gamma}{3}}(|\xi_1|^{\frac{2\gamma+1}{3}} + |\xi_2|^{\frac{2\gamma+1}{3}} + |\xi_3|^{\frac{2\gamma+1}{3}}). \end{aligned} \tag{52}$$

Note the summations of the powers of any two $|\xi_i|$ terms in (52) are 2. Therefore, by Lemma B.3, there is a positive constant M and $\Xi = L - M$ such that

$$\begin{aligned} \dot{U}_3 \leq& -L(\xi_1^2 + \xi_2^2 + \xi_3^2) + M(\xi_1^2 + \xi_2^2 + \xi_3^2) \\ \leq& -\Xi(\xi_1^2 + \xi_2^2 + \xi_3^2). \end{aligned} \tag{53}$$

Selecting a large enough L, we can get a relation same as (32). As a result, the integral controller (44)) can stabilize system (1) in a finite time under Assumptions 4.1–4.2. Considering (35) and (53), the upper bound t_2 of t^* is

$$t_2 =\Big(\frac{(2\gamma+1)z_1^{\frac{7-\gamma}{2\gamma+1}}(0)+3\big(z_2(0)+\beta_1^{\frac{1}{\gamma}}z_1^{\frac{3}{2\gamma+1}}(0)\big)^{\frac{7-\gamma}{3}}}{7-\gamma} + 2^{\frac{1-\gamma}{3}}\big(z_3^{\frac{3}{\gamma+2}}(0)+\beta_2^{\frac{3}{\gamma+2}}(z_2(0)+\beta_1^{\frac{1}{\gamma}}z_1^{\frac{3}{2\gamma+1}}(0))\big)^{\frac{7-\gamma}{3}}\Big)^{\frac{1-\gamma}{7-\gamma}} \times \frac{4^{\frac{1-\gamma}{7-\gamma}}}{\Xi}\frac{7-\gamma}{1-\gamma} \tag{54}$$

with the initial conditions $[z_1(0) = x_0(0) - \frac{\theta}{k_1^* a_1 L^2}, z_2(0) = x_1(0), z_3(0) = \frac{x_2(0)}{L}]$. □

Remark 4.5. Compared with the PID controller, our semi-linear controller has several advantages. First, the linear coefficients of the PID controller are fixed constants which contribute to the trade-off between stability and convergence rate. The nonlinear structure and coefficients of our controller however are chosen in the way such that both fast convergence rate such as finite-time convergence (by a negative homogeneous degree) and global stability (by Lyapunov function) can be guaranteed at the same time. Second, for exogenous time-varying disturbances, due to the domination nature of our design method, we can choose $K = k_1^*\alpha(\cdot)$ as a time-varying function to handle time-varying disturbances and use a scaling gain L to dominate nonlinearities f_1 and f_2 in system (1). Finally, the total power of each variable in controller (42) are not greater than one since $\gamma < 1$, and therefore the closed-loop system (1) and (42) will not have the finite-time escape phenomenon according to the Gronwall–Bellman inequality.

5. Examples

In this section, we show how those results in the previous section can be applied to some examples. First, we consider a pendulum system perturbed by a disturbance.

Example 5.1. The state model of a pendulum system (Moreno & Osorio, 2008) is described by

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{J}u - \frac{M_p g L_1}{2J}sin(x_1) - \frac{V_s}{J}x_2 + d(t,x) \end{cases} \tag{55}$$

where $y = x_1$ is the angle of oscillation, x_2 is the angular velocity, M_p is the pendulum mass, g is the gravitational force, L_1 is the pendulum length, $J = M_p L_1^2$ is the arm inertia, V_s is the pendulum viscous friction coefficient, and $d(t,x)$ is a non-vanishing unknown bounded disturbance. If J, $M_p g L_1$ and V_s are known constants, under $v = \frac{1}{J}u - \frac{M_p g L_1}{2J}sin(x_1) - \frac{V_s}{J}x_2$, system (55) can be rewritten as follows

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = v + d(t,x), \quad y = x_1. \tag{56}$$

We first consider the case when the disturbance is a unknown constant in (56), i.e., $d(t,x) = \theta$. By Corollary 4.1, a finite-time integral controller is designed as

$$v = -k^*(a_1 x_0 + (x_2^{15/13} + a_2 x_1)^{11/15}), \quad \dot{x}_0 = x_1^{3/5}. \tag{57}$$

The controller parameters are selected as $k^* = 3, a_1 = 0.8, a_2 = 3, \gamma = 3/5$ and the initial conditions are set as $[x_0(0), x_1(0), x_2(0)] = [2, -1.5, 3]$, i.e., $[z_1(0), z_2(0), z_3(0)] = [1.1667, -1.5, 3]$. Reviewing the definition of a_1 and a_2 in (26), we can obtain $\beta_1 = 0.431$ and $\beta_2 = 2.5912$. Thus, by (41), the settling time upper bound of $t_1 = 17.6723$ seconds can be obtained. The simulation result is shown in Fig. 1 where we choose $\theta = 2$, $M_p = 1$ (kg), $g = 9.8$ (m/s^2), $L_1 = 0.5$ (m), and $V_s = 0.18$ (kg m/s^2). It is clear that the states x_1 and x_2 will converge to zero and x_0 will converge to the constant 5/6 in a finite time bounded by t_1.

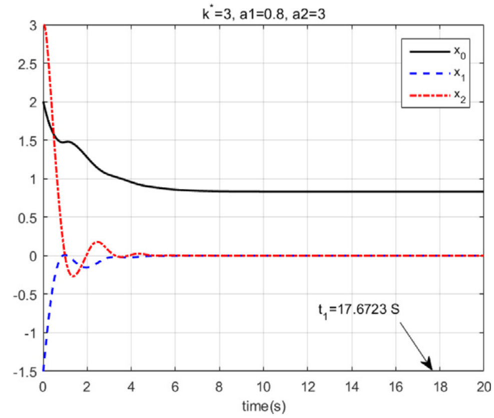

Fig. 1. Trajectories of (38) and (57) with $d(t,x) = 2$.

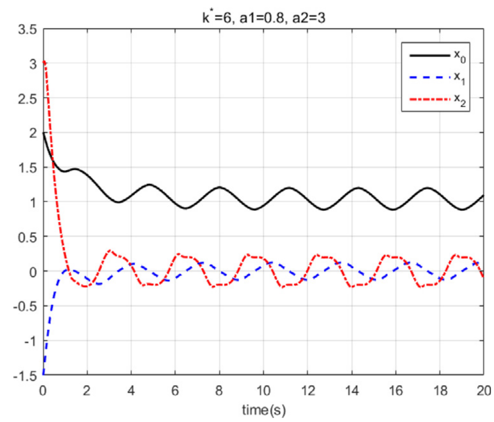

Fig. 2. Trajectories of (56) and (57) with time-varying $d(t,x)$.

When $d(t,x)$ is a time-varying function with an unknown magnitude, for example $d(t,x) = \theta(1+0.5\sin(2t))$, the controller (57) with a constant gain will not be sufficient to drive the output to zero in a finite time. As a matter of fact, as shown in the simulation in Fig. 2 under $\theta = 5, k^* = 6$ and other conditions the same as in Fig. 1, there are oscillations even for a large k^*.

Next, by Theorem 4.1, we design an integral controller with a time-varying gain $k(t,x) = 6(1+0.5\sin(2t)) \geq k^* = 3$ as

$$\begin{cases} v &= -k(t,x)(a_1 x_0 + (x_2^{15/13} + a_2 x_1)^{11/15}) \\ \dot{x}_0 &= x_1^{3/5} \end{cases} \tag{58}$$

In the simulation, we set $\theta = 5$ and choose $a_1 = 0.8, a_2 = 3, \gamma = 3/5$ and $[x_0(0), x_1(0), x_2(0)] = [2, -1.5, 3]$, i.e., $[z_1(0), z_2(0), z_3(0)] = [0.9583, -1.5, 3]$. In this case, we have $\beta_1 = 0.4310, \beta_2 = 2.5912$ and $t_1 = 17.5908$ seconds. Clearly, as shown in Fig. 3 the states x_1 and x_2 of the closed-loop system (56) and (58) converge to zero in a finite time.

Next, we consider a system with both a vanishing nonlinearity and a non-vanishing uncertainty.

Example 5.2. Consider the following system

$$\dot{x}_1 = x_2, \; \dot{x}_2 = u + \theta(1+x_2^2) + \sin(x_1)\delta(t), \tag{59}$$

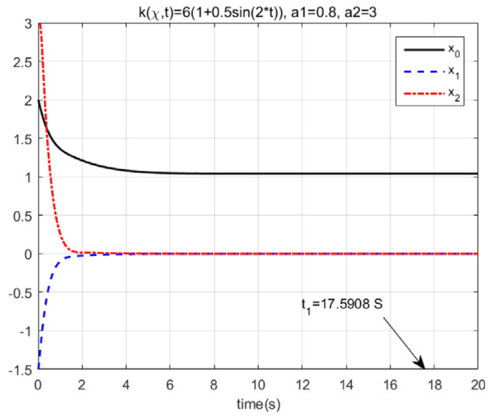

Fig. 3. Trajectories of (56) and (58) with time-varying $d(t, x)$.

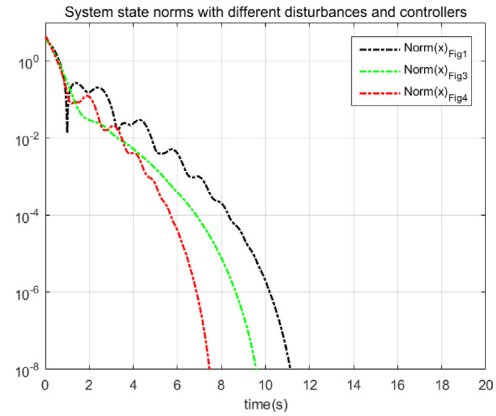

Fig. 5. Norms of the trajectories in logarithmic scale.

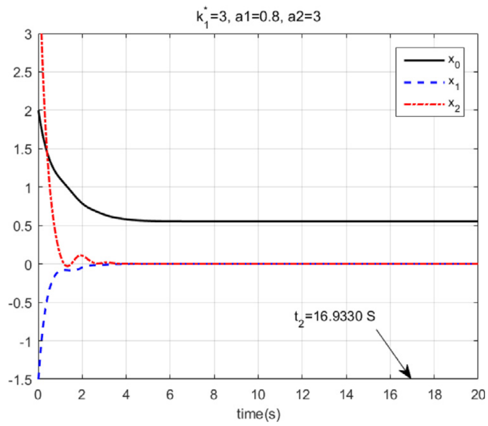

Fig. 4. Trajectories of (59) and (61).

where θ is an unknown constant and $\delta(t)$ is an unknown disturbance satisfying $|\delta(t)| \leq 1$. We can verify that

$$|\sin(x_1)\delta(t)| \leq |x_1|^{\frac{11}{15}} \leq |x_1|^{\frac{2\gamma+1}{3}} \tag{60}$$

which satisfies Assumption 4.2 with $\gamma = 3/5$. By Theorem 4.2, we can design the following integral controller

$$\begin{cases} u &= -L^2 k_1^*(1+x_2^2)\Big(a_1 x_0 + \big(\frac{x_2}{L}\big)^{\frac{15}{13}} + a_2 x_1\Big)^{\frac{11}{15}} \\ \dot{x}_0 &= L x_1^{3/5} \end{cases} \tag{61}$$

In the simulation shown in Fig. 4, we choose $\delta(t) = \sin(t)$, $\theta = 3$, $L = 1.5$, $k_1^* = 3$, $a_1 = 0.8$, $a_2 = 3$ and the initial conditions $[x_0(0), x_1(0), x_2(0)] = [2, -1.5, 3]$, i.e., $[z_1(0), z_2(0), z_3(0)] = [1.4444, -1.5, 2]$. Because a_1 and a_2 are the same as in Example 5.1, the values of β_1, β_2 are same too. By (54), we can get the settling time upper bound $t_2 = 16.9330$ seconds for $M = 0.401$ and $\Xi = L - M = 1.099$.

In order to better observe the convergence performance of the system with different disturbances, we draw the norms of states shown in Fig. 5.

Remark 5.1. As demonstrated in Fig. 1, 3, 4, and 5, the states x_1, x_2 convergence to zero in a finite time, regardless of various formats of the uncertainties. The unknown constant θ in the step disturbance, time-varying disturbance, or system uncertainty can also be recovered in a finite time from the final value of the integral state x_0, guaranteed by Theorem 4.1 or 4.2.

6. Conclusion

This paper has presented a new method to design a finite-time integral controller to regulate the states of a class of uncertain planar systems. Compared to the traditional PID controller, our proposed controller can handle more general uncertainties beyond constant step disturbances, such as external time-varying disturbances with unknown magnitudes and internal modeling uncertainties with unknown parameters. Moreover, owing to the use of a homogeneous integral element and a special semi-linear control law, the system states will converge to the origin and the unknown constant magnitude/parameter can be recovered from the integral state in a finite time.

Appendix A. Finite-Time Stability

Definition A.1 (*Finite-Time Stability* (Bhat & Bernstein, 2000; Fu et al., 2015)). Considering the following system

$$\dot{z}(t) = f(t, z(t)),\ \ t \geq t_0,\ \ z(t_0) = z_0, \tag{A.1}$$

where $z \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear continuous vector field satisfying $f(t, 0_n) = 0_n$.[2]

The origin of system (A.1) is said to be globally uniformly finite-time stable if it is uniformly Lyapunov stable and finite-time attractive, i.e. there exists a locally bounded function $T : \mathbb{R}^n \longrightarrow \mathbb{R}_+$, such that $z(t, t_0, z_0) = 0_n$ for all $t \geq t_0 + T(z_0)$, where $z(t, t_0, z_0)$ is a solution of (A.1) with $z_0 \in \mathbb{R}^n$. The function T is called the settling time function of system (A.1).

Appendix B. Useful lemmas

In this section, we list three lemmas whose proofs can be found in the literature (e.g. Hardy, Littlewood, and Polya (1952), Polendo and Qian (2007), Qian and Lin (2001)).

Lemma B.1. *Let $p \in \mathbb{R}^+_{odd}$ with $p \geq 1$, and x, y be real-valued functions. Then the following holds:*

$$\left|x^p - y^p\right| \leq p\,|x - y|\,(x^{p-1} + y^{p-1}).$$

Lemma B.2. *For $x, y, z \in \mathbb{R}$, and $p \in \mathbb{R}^+_{odd}$ with $p \geq 1$, the following inequalities holds:*

(1) $|x + y|^p \leq 2^{p-1}\,|x^p + y^p|$,

[2] 0_n is the n-dimensional zero vector $[0, \ldots, 0]^T$.

(2) $|x+y|^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}}$,
(3) $(|x|^p + |y|^p + |z|^p) \leq (|x| + |y| + |z|)^p$.

Lemma B.3. *Let c, d be positive constants. Given any positive number $\varepsilon \geq 0$, the following inequality holds:*

$$|x|^c\,|y|^d \leq \frac{c}{c+d}\varepsilon\,|x|^{c+d} + \frac{d}{c+d}\varepsilon^{-\frac{c}{d}}\,|y|^{c+d}.$$

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