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A NOTE ON CONGRUENCES FOR WEAKLY HOLOMORPHIC MODULAR FORMS

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ABSTRACT. Let O_L be the ring of integers of a number field L. Write $q = e^{2\pi iz}$, and suppose that

$$f(z) = \sum_{n \gg -\infty} a_f(n) q^n \in M_k^!(\mathrm{SL}_2(\mathbb{Z})) \cap O_L[[q]]$$

is a weakly holomorphic modular form of even weight $k \leq 2$. We answer a question of Ono by showing that if $p \geq 5$ is prime and $2 - k = r(p-1) + 2p^t$ for some $r \geq 0$ and $t > 0$, then $a_f(p^t) \equiv 0 \pmod{p}$. For $p = 2, 3$, we show the same result, under the condition that $2 - k - 2p^t$ is even and at least 4. This represents the "missing case" of Theorem 2.5 from [Proc. Amer. Math. Soc. 144 (2016), pp. 4591–4597].

1. Introduction

For any even $k \geq 4$, let M_k denote the space of holomorphic modular forms of level 1 and weight k. A meromorphic modular form f is called *weakly holomorphic* if it is holomorphic on the upper half plane but may have a pole at infinity. For any even k, let $M_k^!$ denote the space of weakly holomorphic modular forms of level 1 and weight k. For any $f \in M_k^!$, we write its Fourier expansion in terms of $q = e^{2\pi iz}$ as

$$f(z) = \sum_{n \gg -\infty} a_f(n) q^n.$$

Suppose that $f \in M_k$ is a normalized cuspidal eigenform. We say that the prime p is *non-ordinary* for f if p divides $a_f(p)$, and otherwise we say p is *ordinary*. Non-ordinary primes are generally expected to be rare, but little is definitively known. Elkies [2] showed that for any weight 2 newform f, there are infinitely many primes p which are non-ordinary. However, aside from forms of weight 2, it is not known for any normalized eigenform without complex multiplication either that there are infinitely many ordinary primes, or that there are infinitely many non-ordinary primes.

In [1], Choie, Kohnen, and Ono show that for certain weights k and primes p, p is non-ordinary for all normalized eigenforms of level 1 and weight k. Specifically, they show that if $\delta(k) \in \{4, 6, 8, 10, 14\}$, if $k \equiv \delta(k) \pmod{12}$, and if p is a prime

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for which $k \equiv \delta(k) \pmod{p-1}$, then for all primes $\mathfrak{p} \subset O_L$ above p, we have

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

Applying this result appropriately, one recovers many known cases of non-ordinary primes. For instance, one can show that $2, 3, 5$, and 7 are non-ordinary for Δ, capturing all but one non-ordinary prime up to 10^6.

Jin, Ma, and Ono generalized this result in [3]. This allowed them to show that given any finite set of primes S, there are infinitely many level 1 normalized Hecke eigenforms f such that every $p \in S$ is non-ordinary for f.

Beyond Hecke eigenforms, the authors proved a congruence for the coefficients of weakly holomorphic modular forms. Let $p \geq 5$ be prime. Suppose that $f = \sum_{n \gg -\infty} a_f(n) q^n \in M_k^! \cap O_L[[q]]$, where k is even, that we have

$$2 - k = r(p-1) + sp^t$$

for some $s \neq 2$, and that $\operatorname{ord}_\infty(f) > -p^u$, where $u \leq t$. Then they show that for any integer v with $u \leq v \leq t$, we have

$$a_f(p^v) \equiv a_f(0) \equiv 0 \pmod{p}.$$

Ono [4] asked whether a similar result holds for $s = 2$ as well. In this note, we answer this question by proving the following.

Theorem 1.1. *Suppose that $f = \sum_{n \gg -\infty} a_f(n) q^n \in M_k^! \cap O_L[[q]]$, where $k \in 2\mathbb{Z}$ and O_L is the ring of algebraic integers of a number field L. Let p be a prime, and suppose that $\operatorname{ord}_\infty(f) > -p^t$. Then the following are true.*

(1) *Suppose that $p \geq 5$, and that $k \leq 2$, $r \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}_{>0}$ are integers for which*

$$2 - k = r(p-1) + 2p^t.$$

Then we have

$$a_f(p^t) \equiv 0 \pmod{p}.$$

(2) *Suppose that $p = 2, 3$, and that $k + 2p^t + m = 2$, where $m \geq 4$ is even. Then we have*

$$a_f(p^t) \equiv 0 \pmod{p}.$$

The proof is given in Section 2. The key idea is that the forms in question are related via congruences to weight 2 forms which arise as derivatives of modular functions.

Example 1.2. In the case $s = 2$, it is not true in general that $a_f(0) \equiv 0 \pmod{p}$. For instance, take $p = 5, t = 1$, and $k = -8$. The form

$$f = \frac{E_4}{\Delta} = q^{-1} + 264 + \cdots + 126745880q^5 + \cdots$$

is weakly holomorphic of weight k. We have $a_f(5) \equiv 0 \pmod{5}$, but $a_f(0) \not\equiv 0 \pmod{5}$.

2. The proof

In Subsection 2.1, we recall basic facts about the Theta operator and about congruences of modular forms. In Subsection 2.2, we prove Theorem 1.1.

2.1. **Preliminaries.** Let f in $M_k^!$ be a weakly holomorphic modular form. Ramanujan's Theta operator is defined by

$$\Theta(f) = q\frac{d}{dq}(f) = \frac{1}{2\pi i}\cdot\frac{df}{dz}.$$

Proposition 2.1. *If $f = \sum_{n\gg -\infty} a_f(n)q^n \in M_2^!$, then $f = \Theta(P(j))$ for some polynomial in j. In particular, we have $a_f(0) = 0$.*

Proof. For completeness, we present the proof, which is given in [3]. Every weakly holomorphic modular form f of weight 2 is of the form $P(j(z))E^{14}(z)\Delta(z)^{-1}$, for some polynomial $P(x)$. We have the following two key identities:

$$-\frac{1}{2\pi i}\frac{d}{dz}j = -\Theta(j) = \frac{E_{14}}{\Delta},$$

$$j^w\frac{d}{dz}j = \frac{1}{w+1}\frac{d}{dz}j^{w+1}.$$

If Q is a polynomial such that $Q'(x) = -P(x)$, it follows that we have

$$\Theta(Q(j)) = \frac{-1}{2\pi i}P(j)\frac{d}{dz}j = P(j)\frac{E_{14}}{\Delta} = f.$$

The derivative with respect to z of a q-series has vanishing constant term, so we have $a_f(0) = 0$. □

We will also need the following well-known congruences for Eisenstein series.

Proposition 2.2. *If $p \geq 5$ is prime, then as a q-series, $E_{p-1}(z) \equiv 1 \pmod p$. Additionally, for any $k \geq 2$, we have $E_k(z) \equiv 1 \pmod{24}$.*

Proof. See Lemma 1.22, parts (1) and (2), from [5]. □

2.2. **Proof of Theorem 1.1.** Let $g = \Theta(j) \in M_2^!$. We have

$$g = -q^{-1} + a_j(1)q + 2a_j(2)q^2 + \cdots.$$

It follows that we have

$$g^{p^t} \equiv \pm q^{-p^t} + O(q^{p^t}) \pmod p.$$

In the case where $p = 2, 3$, we can pick c_1, c_2 such that $4c_1 + 6c_2 = m$. We define

$$h = \begin{cases} g^{p^t}E_{p-1}^r f & p \geq 5, \\ g^{p^t}E_4^{c_1}E_6^{c_2} f & p = 2, 3. \end{cases}$$

In both cases, h is weakly holomorphic of weight 2. It follows by Proposition 2.1 that $a_h(0) = 0$. By Proposition 2.2, we have $E_{p-1}^r \equiv 1 \pmod p$ for $p \geq 5$, and $E_4^{c_1}E_6^{c_2} \equiv 1 \pmod p$ for $p = 2, 3$. Since additionally $\mathrm{ord}_\infty(f) > -p^t$, we have

$$a_h(0) \equiv \pm a_f(p^t) \equiv 0 \pmod p,$$

which proves the desired claim. □

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