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Three-Dimensional Mirror Symmetry and Elliptic Stable Envelopes

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$(X; X')$

1 Introduction

1.1 Mirror symmetries

Mirror symmetry is one of the most important physics structures that enter the world of mathematics and arouse lots of attention in the past several decades. Its general

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philosophy is that a space X should come with a dual X, which, though usually different from and unrelated to X in the appearance, admits some deep connections with X in geometry. Mirror symmetry in two dimensions turns out to be extremely enligntening in the study of algebraic geometry, symplectic geometry, and representation theory. In particular, originated from the 2D topological string theory, the Gromov–Witten theory has an intimate connection with 2D mirror symmetry; for an introduction, see [8, 27].

Similar types of duality also exist in three dimensions. More precisely, as introduced in [6, 7, 9, 10, 17, 18, 26, 28], the 3D mirror symmetry is constructed between certain pairs of 3D N = 4 supersymmetric gauge theories, under which they exchanged their *Higgs branches* and *Coulomb branches*, as well as their Fayet-Iliopoulos parameters and mass parameters. In mathematics, the N = 4 supersymmetries imply that the corresponding geometric object of our interest should admit a hyperKähler structure, or if one prefers to stay in the algebraic context, a holomorphic symplectic structure. In particular, for theories of the class as mentioned above, the Higgs branch, which is a certain branch of its moduli of vacua, can be interpreted as a holomorphic symplectic quotient in mathematics, where the prequotient and group actions are determined by the data defining the physics theory. The FI parameters and mass parameters of the theory are interpreted as *Kähler parameters* and *equivariant parameters*, respectively.

The Coulomb branch, however, did not have such a clear mathematical construction until recently [5, 43, 46]. In this general setting, it is not a holomorphic symplectic quotient, and it is difficult to study its geometry. Nevertheless, in many special cases for example, already appearing in the physics literature [6, 16], the Coulomb branch might also be taken as some holomorphic symplectic quotient. Those special cases include hypertoric varieties, Hilbert schemes of points on C^2, the moduli space of instantons on the resolved A_N surfaces, and so on. For a mathematical exposition, see [3, 4], where 3D mirror symmetry is refered to as *symplectic duality*.

A typical mirror symmetry statement for a space X and its mirror X is to relate certain geometrically defined invariants on both sides. For example, in the application of 2D mirror symmetry to genus-zero Gromow–Witten theory, the J-function counting rational curves in X is related to the I-function, which arises from the mirror theory.

In the 3D case, instead of cohomological counting, one should consider counting in the K-theory. One of the K-theoretic enumerative theories in this setting, which we are particularly interested in, is developed by Okounkov and his collaborators [1, 37, 48, 51]. The 3D mirror symmetry statement in this theory looks like

$(X) \cong V(X')$

(X')

On both sides, the vertex functions, which depend on Kähler parameters z_i and equivariant parameters a_i, can be realized as solutions of certain geometrically defined q-difference equations. We call those solutions that are holomorphic in Kähler parameters and meromorphic in equivariant parameters the *z-solutions* and those in the other way the *a-solutions*. In particular, vertex functions are by definition z-solutions.

Under the correspondence (1), the Kähler and equivariant parameters on X and X are exchanged with each other, and hence z-solutions of one side should be mapped to a-solutions of the other side and vice versa. In particular, for the correspondence to make sense, (1) should involve a transition between a basis of z-solutions and a basis of a-solutions. In [1], this transition matrix is introduced geometrically as the *elliptic stable envelope*.

1.2 Elliptic stable envelopes

The notion of stable envelopes first appear in [38] to generate a basis for Nakajima quiver varieties, which admits many good properties. Their definition depends on a choice of cocharacter or, equivalently, a chamber in the Lie algebra of the torus that acting on the space X. The transition matrices between stable envelopes defined for different chambers turn out to be certain R-matrices, satisfying the Yang–Baxter equation and hence defining quantum group structures. The stable envelopes are generalized to K-theory [37, 48, 51], where they not only depend on the choice of cocharacter σ but also depend piecewise linearly on the choice of slope s, which lives in the space of Kähler parameters.

In [1], stable envelopes are further generalized to the equivariant elliptic cohomology, where the piecewise linear dependence on the slope s is replaced by the meromorphic dependence to a Kähler parameter z. In particular, the elliptic version of the stable envelope is the most general structure, K-theoretic, and cohomological stable envelopes can be considered as limits of elliptic. The elliptic stable envelopes depend on both the equivariant and Kähler parameters, which makes it a natural object for the study of 3D mirror symmetry.

In this paper, we will concentrate on a special case where $X = T^*Gr(k,n)$, the cotangent bundle of the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$. This variety is a simplest example of Nakajima quiver variety associated to the A_1-quiver, with dimension vector $\mathsf{v} = k$ and framing vector $\mathsf{w} = n$. We will always assume that $n \geq 2k$. (Only in the case $n \geq 2k$ the dual variety X can also be realized as quiver variety.) Its mirror, which we denote by X, can also be constructed as a Nakajima quiver variety, associated to the A_{n-1}-quiver. It has dimension vector

$$v = (1,2,...,k - \underbrace{1,k,...,k}_{(n-2k+1)\text{-times}},k-1,...,2,1)$$

and framing vector

$$w_i = \delta_{i,k} + \delta_{i,n-k}.$$

For Nakajima quiver varieties, there is always a torus action induced by that on the framing spaces. Let T and T be the tori on X and X, respectively. They both have $n!/(k!(n-k)!)$ fixed points, which admit nice combinatorial descriptions. Elements in X^T can be interpreted as k-subsets $\mathbf{p} \subset \mathbf{n} := \{1,2,\cdots,n\}$, while $(X')^T$ is the set of Young diagrams λ that fit into a $k\times(n-k)$ rectangle. There is a natural bijection (54) between those fixed points

$$(X')^{T'} \xrightarrow{\sim} X^T$$

(X')

$$E_T(X) \to E_T \times E_{\mathrm{Pic}_T(X)}, \qquad E_T(X') \to E_T \times E_{\mathrm{Pic}_T(X')},$$

(X')

$$\kappa : K \to T, \qquad T \to K$$

(X')

By localization theorems, the equivariant elliptic cohomology of X has the form

$$\mathsf{E}_{\mathsf{T}}(X) = \coprod \mathcal{O}_p / \Delta,$$
$$p \in X^{\mathsf{T}}$$

where each $\mathcal{O}_{\mathbf{p}}$ is isomorphic to the base $E_{\mathsf{T}} \times E_{\mathrm{Pic}_{\mathsf{T}}(X)}$. The T-action on X is good enough, in the sense that it is of the *GKM type*, which means that it admits finitely many isolated fixed points and finitely many one-dimensional orbits. Due to this GKM property, the

(X')

form as above; however, the gluing data are more complicated.

By definition, the elliptic stable envelope $\mathrm{Stab}_{\sigma}(\mathbf{p})$ for a given fixed point $\mathbf{p} \in X^{\mathsf{T}}$ is the section of a certain line bundle T $(\mathbf{p})$. We will describe this section in terms of its components

$$T_{\mathbf{p},\mathbf{q}} := \mathrm{Stab}_{\sigma}(\mathbf{p})_{\mathcal{O}_{\mathbf{q}}},$$

which are written explicitly in terms of theta functions and satisfy prescribed quasiperiodics and compatibility conditions. Similar for X, we will describe the components $T'_{\lambda,\mu} := \mathrm{Stab}'_{\sigma'}(\lambda)_{\mu}$.

1.3 Coincidence of stable envelopes for dual variates

Our main result is that the restriction matrices for elliptic stable envelopes on the dual varieties coincide (up to transposition and normalization by the diagonal elements):

Corollary 1. Restriction matrices of elliptic stable envelopes for X and X are related by

$$'_{\lambda,\mu} = T'_{\mu,\mu}$$

where $\mathbf{p} = \mathrm{bj}(\lambda)$, $\mathbf{q} = \mathrm{bj}(\mu)$ and parameters are identified by (55).

In (2), the prefactors $T_{\mathbf{p},\mathbf{p}}$ and $T_{\mu,\mu}$ have very simple expressions as product of theta functions. The explicit formula for matrix elements $T_{\lambda,\mu}$ and $T_{\mathbf{q},\mathbf{p}}$, however, involves complicated summations.

Explicit formulas (see Theorems 3 and 4) for elliptic stable envelopes are obtained by the *abelianzation technique* [1, 60–62]. In the spirit of abelianization, the formula for $T_{\mathbf{q},\mathbf{p}}$ involves a symmetrization sum over the symmetric group S_k, the Weyl group of the gauge group $GL(k)$. However, the formula $T_{\lambda,\mu}$ involves not only a symmetrization over $S_{n,k}$, the Weyl group of the corresponding gauge group, but also a *sum over trees*. Similar phenomenon already appear in the abelianization formula for the elliptic stable envelopes of $\mathrm{Hilb}(\mathbb{C}^2)$ [62]. The reason for this sum over trees to occur is that in the abelianization for X, the preimage of a point is no longer a point, as in the case of X.

As a result, the correspondence (57) we obtained here actually generates an infinite family of nontrivial identities among product of theta functions. See Sections 7 and 8 for examples in the simplest cases $k = 1$ and $n = 4, k = 2$. In particular, in the $n = 4, k = 2$ case, we obtain the well-known 4-term theta identity.

Motivated by the correspondence (57) and the Fourier–Mukai philosophy, a natural guess is that the identity might actually come from a universal *"mother function"* m, living on the product $X \times X$. Consider the following diagram of embeddings

$$X = X \times \{\lambda\} \overset{i\lambda}{\dashrightarrow} X \times X \overset{i\mathbf{p}}{\dashleftarrow} \{\mathbf{p}\} \times X = X.$$

Corollary 1 then follows directly from our main theorem:

Theorem 1. There exists a holomorphic section m (the mother function) of a line bundle M on the $\mathsf{T} \times \mathsf{T}$equivariant elliptic cohomology of $X \times X$ such that

$${}^{*}_{\mathbf{p}}(\quad) = \qquad {}'(\lambda)$$

where $\mathbf{p} = \mathsf{bj}(\lambda)$.

The existence of the mother function was already predicted by Aganagic and Okounkov in the original paper [1]. This paper originated from our attempt to check their conjecture and construct the mother function for the simplest examples of dual quiver varieties.

1.4 Relation to ($\mathfrak{gl}_n$, $\mathfrak{gl}_m$)-duality

The 3D-mirror symmetry for A-type quiver varieties is closely related with the so-called ($\mathfrak{gl}_n$, $\mathfrak{gl}_m$)-duality in representation theory. For the case of X, which is A_1-quiver variety and X which is A_n-1 quiver variety, we are dealing with a particular example of ($\mathfrak{gl}_n$, $\mathfrak{gl}_2$)-duality (i.e., $m = 2$).

Let $\mathbb{C}^2(u)$ be the fundamental evaluation module with evaluation parameter u of the quantum affine algebra $U_{\hbar}(\widehat{\mathfrak{gl}}_2)$. Similarly, let $\wedge^k_{\hbar} \mathbb{C}^n(a)$ be the k-th fundamentalU evaluation module with the evaluation parameter a of quantum affine algebra $_{\hbar}(\mathfrak{gl}_n)$. Recall that the equivariant K-theory of quiver varieties is naturally equipped with an action of quantum affine algebras [45]. In particular, for $X = T^*Gr(k,n)$, we have isomorphism of weight subspaces in $U_{\hbar}(\widehat{\mathfrak{gl}}_2)$-modules:

$$K_{\mathsf{T}}(X) \cong \text{weight } k \text{ subspace in } \mathbb{C}^2(u_1) \otimes \cdots \otimes \mathbb{C}^2(u_n). \tag{3}$$

In geometry, the evaluation parameters u_i are identified with equivariant parameters of torus T. Similarly, the dual variety X is related to representation theory of $U_{\hbar}(\mathfrak{gl}_n)$:

$$(X') \subset \quad k\mathbb{C}^n(a_1) \otimes$$

the corresponding weight subspace is spanned by the following vectors:

$$(X') =$$

where e_i is the canonical basis in $\mathbb{C}^n$. So that both spaces have dimension $n!/(k!(n-k)!)$.

Let us recall that the elliptic stable envelopes feature in the representation theory as a building block for solutions of quantum Knizhnik–Zamilodchikov equations and quantum dynamical equations associated to affine quantum groups [14]. The integral solutions of these equations have the form [2, 29, 33, 52]:

$$\Psi_{p,q} \sim \quad \Phi_p(x_1,\ldots,x_n)\mathbf{Stab}_q(x_1,\ldots,x_n)$$

C i

$\Phi_p(x_1,...,x_n)$

$\mathbf{Stab}_q(x_1,...,x_n)$ denotes the elliptic stable envelope of the fixed point (elliptic weight function). The variables of integration x_i correspond to the Chern roots of tautological bundles.

(X, X')

1.5 Further progress

In this final section, we would like to overview recent progress in the study of 3D-mirror symmetry and elliptic stable envelopes made since the 1st release of this paper.

In his last two papers [49, 50], Okounkov proves that the elliptic stable envelopes exist for very general examples of symplectic varieties, improving the results of the original paper [1] dealing only with quiver varieties. Applications of the elliptic stable envelopes to problems in enumerative geometry, such us constructing integral solutions of the quantum differential and difference equations, description of monodromies of these equations, etc., are the central topics of these papers.

In particular, an interesting class of varieties for which the stable envelopes exists (by [49]) is given by the Cherkis–Nakajima–Takayama bow varieties [47]. Unlike quiver varieties, the bow varieties are closed under 3D-mirror symmetry, i.e., 3D-mirror of a bow variety is a bow variety again. For instance, the mirror X for $X = T{*}Gr(k,n)$ is a bow variety for every value $0 \leq k \leq n$ (note that X is a bow variety but not a quiver variety if $n < 2k$, which is why we consider only the "quiver" case $n \geq 2k$ in this paper). It is thus very natural to study the elliptic stable envelope classes and the corresponding mother functions for the bow varieties. This investigation is currently pursued in [53].

The results obtained in our paper were further generalized to the case of X given by the cotangent bundles over complete flag varieties of type A_n in [54]. This result is further generalized to flag varieties of arbitrary type in [56]. In [63], Theorem 1 was proved for the hypertoric varieties, see also [58] for the toric case. In particular, the mother function for the hypertoric varieties can be written very explicitly, see [63, Theorem 6.4]. The categorical generalization of Theorem 1 for hypertoric varieties is recently proposed in [39]. In this case, the elliptic cohomology of X is substituted by the category of coherent sheaves on the spaces of loops in X and m is substituted by the kernel of a Fourier–Mukai transform describing the mirror symmetry. This leads to a possible categorification of the elliptic stable envelopes.

Alternative proofs of our results, based on analysis of the vertex functions and q-difference equations, were given by Dinkins [11, 12]. Applications of 3D-mirror symmetry in enumerative geometry of threefolds were also considered in [35]. An approach to 3D-mirror symmetry based on the theory of quantum opers is investigated by Koroteev–Zeitlin, see [34] for the current progress.

The 3D-mirror symmetry for the K-theoretic stable envelope (which are limits of the elliptic ones) is investigated in the ongoing project [31, 32]. We expect that this work results in new geometric theory of the quantum differential and quantum difference equations associated with symplectic varieties.

2 Overview of Equivariant Elliptic Cohomology

We start with a pedestrian exposition of the equivariant elliptic cohomology. For more detailed discussions, we refer to [19, 20, 23, 25, 36, 57].

2.1 Elliptic cohomology functor

Let X be a smooth variety endowed with an action of torus $\mathsf{T} \cong (\mathbb{C}^\times)^r$. We say X is a T-variety. Recall that taking spectrums of the equivariant cohomolory and Ktheory, $\mathrm{Spec} H^*_{\mathsf{T}}(X)$ can be viewed as an affine scheme over the Lie algebra of the torus $\mathrm{Spec} H^*_{\mathsf{T}}(\mathrm{pt}) \cong \mathbb{C}^r$ and $\mathrm{Spec} K_{\mathsf{T}}(X)$ is an affine scheme over the algebraic torus $\mathrm{Spec} K_{\mathsf{T}}(\mathrm{pt}) \cong (\mathbb{C}^\times)^r$. Equivariant elliptic cohomology is an elliptic analogue of this viewpoint.

Let us fix an elliptic curve

$$E = \mathbb{C}^\times/q^{\mathbb{Z}},$$

that is, fix the modular parameter q. The equivariant elliptic cohomology is a covariant functor:

$$\mathrm{Ell}_{\mathsf{T}} : \{\mathsf{T}\text{-varieties}\} \to \{\text{schemes}\},$$

which assigns to a T-variety X certain scheme $\mathrm{Ell}_{\mathsf{T}}(X)$. For example, the equivariant elliptic cohomology of a point is

$$\mathrm{Ell}_{\mathsf{T}}(\mathrm{pt}) = \mathsf{T}/q^{\mathrm{cochar}(\mathsf{T})} \cong E^{\dim(\mathsf{T})}.$$

We denote this abelian variety by $E_{\mathsf{T}} := \mathrm{Ell}_{\mathsf{T}}(\mathrm{pt})$. We will refer to the coordinates on E_{T} (same as coordinates on T) as *equivariant parameters*.

Let $\pi : X \to \mathrm{pt}$ be the canonical projection to a point. The functoriality of the elliptic cohomology provides the map π_*: $\mathrm{Ell}_{\mathsf{T}}(X) \to E_{\mathsf{T}}$. For each point $t \in E_{\mathsf{T}}$, we take a small anallytic neighborhoods U_t, which is isomorphic via the exponential map to a small analytic neighborhood in $\mathbb{C}^r$. Consider the sheaf of algebras

$$\mathcal{H}_{U_t} := H^\bullet_{\mathsf{T}}(X^{\mathsf{T}_t}) \otimes_{H^\bullet_{\mathsf{T}}(\mathrm{pt})} \mathcal{O}_{U^{an}_t},$$

where

$$T_t := \bigcap_{\chi \in \mathrm{Char}(\mathsf{T}),\, \chi(t)=0} \ker\chi \subset \mathsf{T}.$$

Those algebras glue to a sheaf $\mathcal{H}$ over $\mathcal{E}_{\mathsf{T}}$, and we define $\mathrm{Ell}_{\mathsf{T}}(X) := \mathrm{Spec}_{\mathcal{E}_{\mathsf{T}}} \mathcal{H}$. The fiber of $\mathrm{Ell}_{\mathsf{T}}(X)$ over t is obtained by setting local coordinates to 0, as described in the following diagram [1]:

$$\begin{tikzcd}
\mathrm{Spec}\, H^\bullet(X^{\mathsf{T}_t}) \arrow[r, hook] \arrow[d, "\pi^*"] & \mathrm{Spec}\, H^\bullet_{\mathsf{T}}(X^{\mathsf{T}_t}) \arrow[d] & (\pi^*)^{-1}(U_t) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Ell}_{\mathsf{T}}(X) \arrow[d, "\pi^*"] \\
\{t\} \arrow[r, hook] & \mathbb{C}^r & U_t \arrow[l] \arrow[r] & \mathcal{E}_{\mathsf{T}}.
\end{tikzcd} \tag{5}$$

This diagram describes a structure of the scheme $\mathrm{Ell}_{\mathsf{T}}(X)$ and gives one of several definitions of elliptic cohomology.

Example 1. Let us consider a 2D vector space $V = \mathbb{C}^2$ with coordinates (z_1, z_2) and a torus $\mathsf{T} = (\mathbb{C}^\times)^2$ acting on it by scaling the coordinates: $(z_1, z_2) \to (u_1 z_1, u_2 z_2)$. Let us set $X = \mathbb{P}(V)$. The action of T on V induces a structure of T-space on X. We have $\mathcal{E}_{\mathsf{T}} = E \times E$ and the equivariant parameters u_1 and u_2 represent the coordinates on the 1st and the 2nd factor. Note that for a generic point $t = (u_1, u_2) \in \mathcal{E}_{\mathsf{T}}$ the fixed set X^{T_t} consists of two points, which in homogeneous coordinates of $\mathbb{P}(V)$ are

$$p = [1 : 0], \qquad q = [0 : 1].$$

The stalk of $\mathcal{H}$ at t is $H^\bullet_{\mathsf{T}}(p \cup q) \otimes_{H^\bullet_{\mathsf{T}}} \mathcal{O}_{\mathcal{E}_{\mathsf{T}},t}$, and the fiber is $H^\bullet(p \cup q)$. We conclude that over a general point $t \in \mathcal{E}_{\mathsf{T}}$ the fiber of π^* in (5) consists of two points.

At the points $t = (u_1,u_2)$ with $u_1 = u_2$ the torus T_t acts trivially on X, thus locally the sheaf H looks like

$$H_{\mathsf{T}}^{\bullet}(X^{\mathsf{T}_t}) = H_{\mathsf{T}}^{\bullet}(\mathsf{P}^1) = \mathsf{C}[\delta u_1,\delta u_2,z]/(z - \delta u_1)(z - \delta u_2),$$

where δu_1 and δu_2 are local coordinates centered at x. Taking Spec, this is the gluing of two copies of C^2 along the diagonal. Overall, we obtain that

$$\mathrm{Ell}_{\mathsf{T}}(X) = \mathsf{O}_p \cup_{\Delta} \mathsf{O}_q,$$

$$\Delta = \{(u_1,u_2) \mid u_1 = u_2\} \subset \mathscr{E}_{\mathsf{T}}.$$

We assume further that the set of fixed points X^{T} is a finite set of isolated points. We will only encounter varieties of this type in our paper. In this case, for a generic oneparametric subgroup $\mathsf{T}_t \subset \mathsf{T}$, we have

$$X^{\mathsf{T}_t} = X^{\mathsf{T}}.$$

By the localization theorem, we know that the irreducible components of $\mathrm{Ell}_{\mathsf{T}}(X)$ are parameterized by fixed points $p \in X^{\mathsf{T}}$ and each isomorphic to the base E_{T}. Therefore, similarly to Example 1, we conclude that set theoretically, $\mathrm{Ell}_{\mathsf{T}}(X)$ is union of $|X^{\mathsf{T}}|$ copies of E_{T}:

$$\mathrm{Ell}_{\mathsf{T}}(X) = \Big(\coprod_{p\in X^{\mathsf{T}}} \mathsf{O}_p\Big)\Big/, \tag{6}$$

where $\mathsf{O}_p \cong E_{\mathsf{T}}$ and / denotes the gluing of these abelian varieties along the subschemes $\mathrm{Spec}\,H^{\bullet}(X^{\mathsf{T}_t})$ corresponding to substori T_t for which the fixed sets X^{T_t} are larger than X^{T}. We call O_p the T-orbit associated to the fixed point p in $\mathrm{Ell}_{\mathsf{T}}(X)$.

In general, to describe the scheme structure of $\mathrm{Ell}_{\mathsf{T}}(X)$ in terms of the gluing data (6) can be quite involved. There is, however, a special case when it is relatively simple.

Definition 1. We say that T-variety X is a GKM variety if it satisfies the following conditions:

- X^{T} is finite,
- for every two fixed points $p,q \in X^{\mathsf{T}}$ there is no more than one T-equivariant curve connecting them.

Note that by definition, a GKM variety contains finitely many T-equivariant compact curves (i.e., curves starting and ending at fixed points). We note also that all these curves are rational $C \cong \mathbb{P}^1$ because T-action on C exists only in this case.

For a compact curve C connecting fixed points p and q, let $\chi_C \in \mathrm{Char}(\mathsf{T}) = \mathrm{Hom}(E_{\mathsf{T}},E)$ be the character of the tangent space T_pC. For all points t on the hyperplane $\chi_C^{\perp} \subset E_{\mathsf{T}}$, we thus have $p,q, \in C \subset X^{\mathsf{T}_t}$. As in Example 1, this means that in (6) the T-orbits

$\mathcal{O}_p$ and $\mathcal{O}_q$ in the scheme $\mathrm{Ell}_{\mathsf{T}}(X)$ are glued along the common hyperplane

$$\mathcal{O}_p \supset \chi_C^{\perp} \subset \mathcal{O}_q.$$

Proposition 1. If X is a GKM variety, then

$$\mathrm{Ell}_{\mathsf{T}}(X) = \Big(\coprod_{p \in X^{\mathsf{T}}} \mathcal{O}_p \Big) \Big/ \Delta,$$

where Δ denotes the intersections of T-orbits $\mathcal{O}_p$ and $\mathcal{O}_q$ along the hyperplanes

$$\mathcal{O}_p \supset \chi_C^{\perp} \subset \mathcal{O}_q,$$

for all p and q connected by an equivariant curve C where χ_C is the T-character of the tangent space T_pC. The intersections of orbits O_p and O_q are transversal and hence the scheme $Ell_{\mathsf{T}}(X)$ is a variety with simple normal crossing singularities.

Proof. Locally around $t \in E_{\mathsf{T}}$, the stalk of H is given by $H_{\mathsf{T}}^\bullet(X^{\mathsf{T}_t}) \otimes \mathcal{O}_{E_{\mathsf{T}},t}$. Let $s \in E_{\mathsf{T}}$ be another point, such that $T_s \supset T_t$. We have by T_s-equivariant localization,

$$H_{\mathsf{T}}^\bullet(X^{\mathsf{T}_t}) \otimes \mathrm{Frac}(H_{\mathsf{T}_s}^\bullet(\mathrm{pt})) \cong H_{\mathsf{T}}^\bullet(X^{\mathsf{T}_s}) \otimes \mathrm{Frac}(H_{\mathsf{T}_s}^\bullet(\mathrm{pt})). \tag{7}$$

In other words, if U_t and U_s are small analytic neighborhoods around t and s, such that $U_s \subset U_t$, then by definition of the elliptic cohomlogy, the restriction map of H from U_t to U_s is equivalent to the isomorphism given by the T_s-equivariant localization.

By the property of equivariant cohomology of GKM varieties [24], the variety $\mathrm{Spec} H_{\mathsf{T}}^\bullet(X^{\mathsf{T}_t})$ is the union of $\mathfrak{t}_p$'s along hyperplanes $\chi_C{}^\perp$, where $\mathfrak{t}_p \cong \mathbb{C}^r$ are Lie algebras of the torus, associated to fixed points. Moreover, the intersection of $\mathfrak{t}_p$'s for $p \in X^{\mathsf{T}}$ is transversal. More precisely, we have the exact sequence

$$0 \longrightarrow H_{\mathsf{T}}^\bullet(X^{\mathsf{T}_t}) \longrightarrow H_{\mathsf{T}}^\bullet(X^{\mathsf{T}}) \longrightarrow H_{\mathsf{T}}^\bullet(X_1^{\mathsf{T}_t}, X^{\mathsf{T}}),$$

where $X_1{}^{\mathsf{T}_t}$ is the 1-skeleton of X^{T_t} under the T-action and the last map is given by χ_C for all one-dimensional orbits in X^{T_t}.

We see that the exact sequence is compatible with the localization isomorphism (7), which means that analytically, the local descriptions of $\mathrm{Ell}_{\mathsf{T}}(X)$ glue over E_{T}, and globally $\mathrm{Ell}_{\mathsf{T}}(X)$ can be described exactly as in the proposition.

Here by "gluing", we mean the pushout in the category of schemes, in the sense of [59].

The classical examples of GKM varieties include Grassmannians or more generally, partial flag varieties. For non-GKM varieties, the structure of subschemes $\mathrm{Spec} H^\bullet(X^{\mathsf{T}_t})$ and intersection of orbits in (6) can be more complicated. In particular, more than two orbits can intersect along the same hyperplane.

2.3 Extended elliptic cohomology

We define

$$\mathcal{E}_{\mathrm{Pic}(X)} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} E \cong E^{\dim(\mathrm{Pic}(X))}. \tag{8}$$

For Nakajima quiver varieties $\mathrm{Pic}(X) \cong \mathbb{Z}^{|Q|}$ and thus $\mathcal{E}_{\mathrm{Pic}(X)} \cong E^{|Q|}$, where $|Q|$ denotes the number of vertices in the quiver. We will refer to the coordinates in this abelian variety as *Kähler parameters*. We will usually denote the Kähler parameters by the symbol z_i, $i = 1,...,|Q|$.

The extended T-orbits are defined by

$$\widehat{\mathsf{O}}_p := \mathsf{O}_p \times \mathcal{E}_{\mathrm{Pic}(X)},$$

and the extended elliptic cohomology by

$$\mathsf{E}_\mathsf{T}(X) := \mathrm{Ell}_\mathsf{T}(X) \times \mathcal{E}_{\mathrm{Pic}(X)}.$$

In particular, if X is GKM, $\mathsf{E}_\mathsf{T}(X)$ is a bouquet of extended orbits:

$$\mathsf{E}_\mathsf{T}(X) = \Big(\coprod_{p \in X^\mathsf{T}} \widehat{\mathsf{O}}_p\Big)/\Delta,$$

where Δ denotes the same gluing of orbits as in (6), that is, the extended orbits are glued only along the equivariant directions.

2.4 Line bundles on elliptic cohomology

We have the following description of a line bundle on the variety $\mathsf{E}_\mathsf{T}(X)$.

Proposition 2. Let X be a GKM variety.

- A line bundle $\mathcal{T}$ on the scheme $E_T(X)$ is a collection of line bundles $\mathcal{T}_p$ on extended orbits $\mathcal{O}_p$, $p \in X^T$, which coincide on the intersections:

$$\mathcal{T}_p|_{\mathcal{O}_p\cap\mathcal{O}_q} = \mathcal{T}_q|_{\mathcal{O}_p\cap\mathcal{O}_q},$$

- A meromorphic (holomorphic) section s of a line bundle $\mathcal{T}$ is the collection of meromorphic (holomorphic) sections s_p of $\mathcal{T}_p$ which agree on intersections:

$$s_p|_{\mathcal{O}_p\cap\mathcal{O}_q} = s_q|_{\mathcal{O}_p\cap\mathcal{O}_q}. \tag{9}$$

Since each orbit $\mathcal{O}_p$ is isomorphic to the base $E_T \times E_{\mathrm{Pic}(X)}$, each $\mathcal{T}_p$ is isomorphic via the pull back along π^* to a line bundle on the base. In practice, we often use the coordinates on the base to describe $\mathcal{T}_p$s.

Example 2. Characterization of line bundles and sections is more complicated for non-GKM varieties. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, with homogeneous coordinates $([x : y],[z : w])$, and let $\mathbb{C}^*$ acts on it by

$$t \cdot ([x : y],[z : w]) = ([x : ty],[z : tw]).$$

There are four fixed points, but infinitely many $\mathbb{C}^*$-invariant curves: the closure of $\{([x : y],[x : \lambda y])\}$ for any $\lambda \in \mathbb{C}^*$ is a $\mathbb{C}^*$-invariant curve, connecting the points $([1 : 0],[1 : 0])$ and $([0 : 1],[0 : 1])$. Locally near the identity $1 \in E_{\mathbb{C}^*}$, the elliptic cohomology $\mathrm{Ell}_{\mathbb{C}^*}(X)$ looks like

$$\mathrm{Spec} H_{\mathsf{T}}^{\bullet}(X) = \mathrm{Spec}\, \mathbb{C}[H_1, H_2, u]/(H_1^2 - u^2, H_2^2 - u^2) \quad \to \quad \mathrm{Spec}\mathbb{C}[u],$$

The gluing of 4 affine lines along the origin, as abstract schemes, would no longer be a subscheme in $\mathbb{C}^3$ and hence not isomorphic to $\mathrm{Spec} H_{\mathsf{T}}^{\bullet}(X)$. To express $\mathrm{Spec} H_{\mathsf{T}}^{\bullet}(X)$ still as a gluing, one has to allow each orbit O_p to have certain embedded non-reduced point at the origin. For an example of this type, see [59].

2.5 Theta functions

By Proposition 2, to specify a line bundle T on $\mathsf{E}_{\mathsf{T}}(X)$, one needs to define line bundles T_p on each orbit O_p. As O_p is an abelian variety, to fix T_p, it suffices to describe the transformation properties of sections as we go around periods of O_p. In other words, to define T_p, one needs to fix quasiperiods w_i of sections

$$s(x_i q) = w_i s(x_i),$$

for all coordinates x_i on O_p, that is, for all equivariant and Kähler parameters.

The abelian variates O_p are all some powers of E, which implies that sections of a line bundle on $\mathsf{E}_{\mathsf{T}}(X)$ can be expressed explicitly through the Jacobi theta function associated with E:

$$\theta(x) := (qx)_\infty (x^{1/2} - x^{-1/2})(q/x)_\infty, \qquad (x)_\infty = \prod_{i=0}^{\infty}(1 - xq^i).$$

$$\theta(xq) = -\frac{1}{x\sqrt{q}}\theta(x), \qquad \theta(1/x) = -\theta(x). \tag{10}$$

We also extend it by linearity and define

$$\theta\Big(\sum_i a_i - \sum_j b_j\Big) := \frac{\prod_i \theta(a_i)}{\prod_j \theta(b_j)}. \qquad (11)$$

By definition, the elliptic stable envelope associated with a T-variety X is a section of certain line bundle on $\mathsf{E}_{\mathsf{T}}(X)$ [1]. Thus, one can use theta-functions to give explicit formulas for stable envelopes, see Theorem 3 for an example of such formulas.

It will also be convenient to introduce the following combination:

$$\phi(x,y) = \frac{\theta(xy)}{\theta(x)\theta(y)}$$

This function has the following quasiperiods:

$$\varphi(xq,y) = y^{-1}\varphi(xq,y), \qquad \varphi(x,yq) = x^{-1}\varphi(x,y).$$

These transformation properties define the so-called Poincaré line bundle on the product of dual elliptic curves $E \times E^{\vee}$ with coordinates x and y and $\varphi(x,y)$ is a meromorphic section of this bundle.

3 Elliptic Stable Envelope for X

In this section, we discuss algebraic variety $X = T^*Gr(k,n)$—the cotangent bundle over the Grassmannian of k-dimensional subspaces in an n-dimensional complex space.

3.1 X as a Nakajima quiver variety

We consider a Nakajima quiver variety X defined by the A_1-quiver, with dimension $\mathbf{v} = k$ and framing $\mathbf{w} = n$. Explicitly, this variety has the following construction. Let $R = \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$ be a vector space of complex $k \times n$ matrices. There is an obvious action of $GL(k)$ on this space, which extends to a Hamiltonian action on its cotangent bundle:

$$T^*R = R \oplus R^* \cong \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^k),$$

with the Hamiltonian moment map

$$\mu : T^*R \to \mathrm{gl}(k)^*, \qquad \mu(\mathbf{j},\mathbf{i}) = \mathbf{ij}.$$

Then X is defined as

$$X := \mu^{-1}(0) \cap \{\theta\text{-semistable points}\}/GL(k),$$

where $\mathbf{j} \in R$ and $\mathbf{i} \in R^*$ are $n \times k$ and $k \times n$ matrices, respectively. There are two choices of stability conditions $\theta < 0$ and $\theta > 0$. In the 1st case, the semistable points are those pairs $(\mathbf{j},\mathbf{i})$ with injective $\mathbf{j}$:

$$\{\theta\text{-semistable points}\} = \{(\mathbf{j},\mathbf{i}) \in T^*R \mid \mathrm{rank}(\mathbf{j}) = k\}, \tag{12}$$

In the case $\theta > 0$, the semistable points are $(\mathbf{j},\mathbf{i})$ with $\mathbf{i}$ surjective [22]:

$$\{\theta\text{-semistable points}\} = \{(\mathbf{j},\mathbf{i}) \in T^*R \mid \mathrm{rank}(\mathbf{i}) = k\}.$$

By construction, X is a smooth holomorphic symplectic variety. In this paper, we choose

$$\theta = (-1) \in \mathrm{Lie}_{\mathbb{R}}(\mathsf{K}),$$

where $\mathsf{K} := U(1)$, as the stability condition defining X, in which case it is isomorphic to the cotangent bundle of the Grassmannian of complex k-dimensional vector subspaces in an n-dimensional space.

3.2 Torus action on X

Let $\mathsf{A} = (\mathbb{C}^\times)^n$ be a torus acting on $\mathbb{C}^n$ by scaling the coordinates:

$$(z_1,\ldots,z_n) \quad \to \quad (z_1u_1^{-1},\ldots,z_nu_n^{-1}), \tag{13}$$

which induces an action of A on T^*R. We denote by $\mathbb{C}^\times_{\hbar}$ the torus acting on T^*R by scaling the 2nd component:

$$(\mathbf{j},\mathbf{i}) \to (\mathbf{j},\mathbf{i}\hbar^{-1}).$$

We denote the whole torus[1](0) and thus descends to action on $\mathsf{T} = \mathsf{A} \times \mathbb{C}^\times_{\hbar}$. The action of X. Simple check shows that the action of T preserves semistable locus A

of μ- preserves the symplectic form on X, while $\mathbb{C}^\times_{\hbar}$ scales it by $\hbar$.

Note that the action (13) leaves invariant k-dimensional subspaces spanned by arbitrary k coordinate vectors. Thus, the set of T-fixed points X^{T} consists of $n!/((n-k)!k!)$ points corresponding to k-dimensional coordinate subspaces in $\mathbb{C}^n$. In other words, a fixed point $\mathbf{p} \in X^{\mathsf{T}}$ is described by a k-subset in the set $\{1,2,...,n\}$.

3.3 T-equivariant K-theory of X

Let us denote the tautological bundles on X associated to $\mathbb{C}^k$ and $\mathbb{C}^n$ by V and W, respectively. The bundle W is a topologically trivial rank-n vector bundle because $\mathbb{C}^n$ is a trivial representation of $GL(k)$. In contrast, V is a nontrivial rank-k subbundle of W. One can easily see that V is the standard tautological bundle of k-subspaces on the Grassmannian. We assume that the tautological bundle splits in K-theory into a sum of virtual line bundles,

$$\mathsf{V} = y_1 + \cdots + y_k. \tag{14}$$

In other words, y_i denote the Chern roots of V. The T-equivariant K-theory of X has the form:

$$K_{\mathsf{T}}(X) = \mathbb{C}[y_1^{\pm 1},\dots,y_k^{\pm 1}]^{S_k} \otimes \mathbb{C}[u_i^{\pm 1},\hbar^{\pm 1}]/I,$$

where S_k is the symmetric group of k elements and I denotes the ideal of Laurent polynomials vanishing at the fixed points, that is, at (15). For our choice of stability condition, the matrix $\mathbf{j}$ representing a fixed point is of rank k; thus, if $\mathbf{p}$ is a fixed point corresponding to the k-subset $\{\mathbf{p}_1,\dots,\mathbf{p}_k\} \subset \{1,2,\dots,n\}$, then

$$= u_{\mathbf{p}_i}^{-1}, \quad i = 1,\dots,k$$

This means that if a K-theory class is represented by a Laurent polynomial $f(y_i)$ then its restriction to a fixed point is given by the substitution $f(y_i)\big|_{\mathbf{p}} = f(u_{\mathbf{p}_i}^{-1})$.

$$(y_i)\big|_{\mathbf{p}} = f(u_{\mathbf{p}_i}^{-1}\hbar^{-1})$$

3.4 Tangent and polarization bundles

The definition of the elliptic stable envelope requires the choice of a polarization and a chamber [1]. The polarization $T^{1/2}X$, as a virtual bundle, is a choice of the half of the tangent space. In other words,

$$TX = T^{1/2}X + \hbar^{-1}T^{1/2}X^{*}.$$

We choose the polarization dual to the canonical polarization (which is defined for all Nakajima varieties, see [38, Example 3.3.3]):

$$T^{1/2}X = \hbar^{-1}\mathsf{W}^{*}\otimes\mathsf{V} - \hbar^{-1}\mathsf{V}^{*}\otimes\mathsf{V}. \tag{16}$$

Expressing TX through the Chern roots by (14) and restricting it to a fixed point $\mathbf{p}$ by (15), we find the T-character of the tangent space at $\mathbf{p}$ equals:

$$T_{\mathbf{p}}X = \sum_{i\in\mathbf{p},\ j\in\mathbf{n}\setminus\mathbf{p}} \frac{u_j}{u_i} + \hbar^{-1}\frac{u_i}{u_j}, \tag{17}$$

where $\mathbf{p}$ denotes the k-subset in $\mathbf{n} = \{1,\ldots,n\}$.

The definition of the stable envelope also requires the choice of a chamber, or equivalently, a cocharacter of the torus A. We choose σ explicitly as

$$\sigma = (1,2,\ldots,n) \in \mathrm{Lie}_{\mathbb{R}}(\mathsf{A}). \tag{18}$$

The choice of σ fixes the decomposition $T_{\mathbf{p}}X = N_{\mathbf{p}+} \oplus N_{\mathbf{p}-}$, where $N_{\mathbf{p}\pm}$ are the subspaces whose A-characters take positive or negative values on σ. From (17), we obtain

$$N_{\mathbf{p}-} = \bigoplus_{\substack{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p}\\ i<j}} \frac{u_i}{u_j} + \hbar^{-1}\bigoplus_{\substack{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p}\\ i>j}} \frac{u_j}{u_i}, \quad N_{\mathbf{p}+} = \bigoplus_{\substack{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p}\\ i>j}} \frac{u_i}{u_j} + \hbar^{-1}\bigoplus_{\substack{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p}\\ i<j}} \frac{u_j}{u_i}. \tag{19}$$

3.5 Elliptic cohomology of X

Let us first note that X is a GKM variety. Two fixed points $\mathbf{p}$, $\mathbf{q}$ are connected by an equivariant curve C if and only if the corresponding k-subsets differ by one index $\mathbf{p} = \mathbf{q} \setminus \{i\} \cup \{j\}$. In this case, the T-character of the tangent space equals

$$T_{\mathbf{p}}C = u_j/u_i.$$

By Proposition 1, we conclude that the extended elliptic cohomology scheme equals

$$\mathrm{E}_{\mathrm{T}}(X) = \Big(\coprod_{\mathbf{p}\in X^{\mathrm{T}}} \widehat{\mathrm{O}}_{\mathbf{p}}\Big)\Big/\Delta \tag{20}$$

$\widehat{\mathrm{O}}_{\mathbf{p}} \cong E_{\mathrm{T}} \times E_{\mathrm{Pic}(X)}$ with and $/\Delta$ denotes gluing of abelian varieties $\mathrm{O}_{\mathbf{p}}$ and $\mathrm{O}_{\mathbf{q}}$ with $\mathbf{p} = \mathbf{q} \setminus \{i\} \cup \{j\}$ along the hyperplanes $u_i = u_j$.

By definition, the elliptic stable envelope of a fixed point $\mathbf{p}$ is a section of the twisted Thom class of the polarization:

$$\mathcal{T}(\mathbf{p}) = \Theta(T^{1/2}X) \otimes \dots$$

$$\Theta \,:\, K\ (X)$$

We refer to [1, Sections 2.5-2.8] for the details of this construction. Sections of T $(\mathbf{p})|\mathcal{O}_{\mathbf{q}}$ transform as the following explicit function:

$$\frac{\phi(u_{\mathbf{p}_i}^{-1}, z^{-1})}{\phi(u_{\mathbf{q}_i}^{-1}, z^{-1})} \prod_{i=1}^{n} \frac{\phi(u_i, z_{u_i}^{-1}\hbar^{D_i})}{\phi(u_i, z_{u_i}^{-1})} \quad \mathrm{U}_{\mathbf{p},\mathbf{q}}(X) = \quad T^{1/2}X_{\mathbf{q}} \quad k \quad \mathbf{p} \quad i=1 \quad . \tag{22}$$

(The variables z_{u_i} correspond to Kähler variables of the T-*equivariant* Picard group. One checks that all quasiperiods of line bundles in these directions are trivial and the elliptic stable envelopes are actually independent on these variables, see discussion in [1, Section 3.3.7]. It is, however, convenient to keep these directions to describe shifts of stable envelopes by the index.) Here $\Theta \quad T^{1/2}X_{\mathbf{q}}$ for the Laurent polynomial $T^{1/2}X_{\mathbf{q}}$ is given by a product of theta functions via (11). It has the same transformation properties as the elliptic Thom class $\Theta \quad T^{1/2}X\mathcal{O}_{\mathbf{p}}$. Similarly, other terms given by products in (22) describe the transformation properties of the term denoted by ... in (21).

The powers $D^{\mathbf{p}}_i$ come from the *index* of the polarization bundle. They are computed as follows: for our choice of polarization (16) and chamber (18) the index of a fixed point $\mathbf{p}$ equals

$$\mathrm{ind}_{\mathbf{p}} = T^{1/2}X\Big|_{\mathbf{p},>} = \sum_{\substack{i\in \mathbf{p}\\ j\in /\mathbf{p}\\ j>i}} \frac{u_j}{u_i} \ \sum \ u_i h^{-} ,$$

and the integers $D^{\mathbf{p}}_i$ are the degrees of the index bundle, that is, the degree in variable u_i of the monomial:

$$\det \mathrm{ind}_{\mathbf{p}}. \quad \prod_{\substack{j\notin \mathbf{p}\\ j>i}} \frac{u_j}{u_i\hbar} = \prod_{i\in \mathbf{p}}$$

Note that $\mathrm{U}_{\mathbf{p},\mathbf{q}}$ are certain explicit products of the theta functions and their quasiperiods in all variables are easily determined from (10). In particular (22) conveniently packages the information about quasiperiods of the elliptic stable envelopes: the matrices of restrictions (23) transform in all variables, under shifts by q, as $\mathrm{U}_{\mathbf{p},\mathbf{q}}$.

The elliptic stable envelope $\mathrm{Stab}_{\sigma}(\mathbf{p})$ of a fixed point $\mathbf{p}$ (corresponding to the choice of chamber σ and polarization $T^{1/2}X$) is a section of $\mathrm{T}(\mathbf{p})$ fixed uniquely by a list of properties [1]. Alternative version of the elliptic stable envelope for cotangent bundles to partial flag variates was defined in [15, 55]. Comparing explicit formulas for elliptic stable envelopes in the case of the variety X from [1] and from [15, 55] one observes that they differ by a multiple. The definition of [15, 55] is based on the fact that X is a GKM variety, while definition of [1] is more general and is not restricted to GKM varieties. In fact, the Nakajima varieties are almost never GKM varieties. In this paper, we choose the approach of [15, 55] because GKM structure of X will simplify the computations. As we mentioned already, in the case of variety X, both approaches lead to the same explicit formulas; thus, there is no ambiguity in this choice.

Definition 2. The elliptic stable envelope of a fixed point $\mathrm{Stab}_{\sigma}(\mathbf{p})$ is the unique section of $\mathrm{T}(\mathbf{p})$, such that its components

$$\mathrm{Stab}_{\sigma}(\mathbf{p})\circ \qquad T_{\mathbf{p},\mathbf{q}} := \quad \mathbf{q} \tag{23}$$

satisfy the following properties

(1) $T_{\mathbf{p},\mathbf{p}} = \prod_{\substack{i\in\mathbf{p},\\ j\in\mathbf{n}\setminus\mathbf{p},\\ i<j}} \theta \prod_{\substack{i\in\mathbf{p}\\ \in\mathbf{n}\setminus\\ i>j}} \quad j \quad \mathbf{p},,\ \theta\ \text{—} .\ u_j u_j$ $\quad u_i u_i h$

(2) $T_{\mathbf{p},\mathbf{q}} = f_{\mathbf{p},\mathbf{q}} \prod_{\substack{i\in\\ j\in\mathbf{n}\setminus\\ i>j}} \ \theta\ \ j\ \ \mathbf{q},\ u_i\ \_h^{-1}$, where $f_{\mathbf{p},\mathbf{q}}$ is holomorphic in parameters u_i. u $\quad \mathbf{q},$

Let us note that the fact that $\mathrm{Stab}_\sigma(\mathbf{p})$ is a section of T $(\mathbf{p})$ implies that its restrictions $T_{\mathbf{p},\mathbf{q}}$ are sections of line bundles on abelian varietiesU O $\mathbf{q}$, which have the same transformation properties in all variables as $_{\mathbf{p},\mathbf{q}}(X)$.

3.6 Uniqueness of stable envelope for X

To justify the last definition, we need the following uniqueness theorem.

Theorem 2. [15,Appendix A] The matrix $T_{\mathbf{p},\mathbf{q}}$ satisfying the following:

(1) For a given fixed $\mathbf{p}$, the collection $\{T_{\mathbf{p},\mathbf{q}} \mid \mathbf{q} \in X^T\}$ form a section of the line bundle T $(\mathbf{p})$ (as defined by (22)).

(2) $T_{\mathbf{p},\mathbf{p}} = \prod_{\substack{\mathbf{p}, \\ j\in \mathbf{n}\setminus\mathbf{p}, \\ i<j}} \theta \prod_{\substack{i\in\mathbf{p} \\ \in\mathbf{n}\setminus \\ i>j}} \theta\left(\frac{u_j}{u_i}\right) \quad i \in \frac{\quad}{u_i h}\, j \quad \mathbf{p},, \quad \frac{\quad}{}.\ u_j u$

(3) $T_{\mathbf{p},\mathbf{q}} = \prod_{\substack{i\in \\ j\in\mathbf{n}\setminus \\ i>j}} f_{\mathbf{p},\mathbf{q}} \quad \mathbf{q}, \quad \theta \frac{u_j}{u_i} - h^{-1}$, where $f_{\mathbf{p},\mathbf{q}}$ is holomorphic in parameters $u_i.\ u$ $\mathbf{q},$

is unique.

Proof. Assume that we have two matrices that satisfy (1), (2), and (3) and let $\kappa_{\mathbf{p},\mathbf{q}}$ be their difference. Assume that $\kappa_{\mathbf{p},\mathbf{q}} = 0$ for some $\mathbf{p}$. Let $\mathbf{q}$ be a maximal (in the partial order defined by the chamber) fixed point such that $\kappa_{\mathbf{p},\mathbf{q}} = 0$. (The partial order defined by a chamber σ is

$$\mathbf{p} \quad \mathbf{q}, \qquad \Leftarrow\Rightarrow \mathbf{q} \in \mathrm{Attr}^f_\sigma(\mathbf{p})$$

where $\mathrm{Attr}^f_\sigma(\mathbf{p})$ is the full attracting set of a fixed point $\mathbf{p}$, see Section 3.1 in [1]. For $X = T{*}Gr(k,n)$ and the chamber (18) this is the standard Bruhat order on $S_n/(S_k \times S_{n-k})$. For fixed points $\mathbf{p} = \{\mathbf{p}_1,\dots,\mathbf{p}_k\}$ and $\mathbf{q} = \{\mathbf{q}_1,\dots,\mathbf{q}_k\}$ with $\mathbf{p}_1 < \cdots < \mathbf{p}_k$, $\mathbf{q}_1 < \cdots < \mathbf{q}_k$ we have

$$\mathbf{p} \quad \mathbf{q} \qquad \Leftarrow\Rightarrow \qquad \mathbf{p}_i \geq \mathbf{q}_i,\ i = 1,\dots,k.)$$

By (3), we know that

$$\kappa_{\mathbf{p},\mathbf{q}} = f_{\mathbf{p},\mathbf{q}} \prod_{i\in\mathbf{q},\, j\in\mathbf{n}\setminus\mathbf{q},\, j>i} \theta\left(\frac{u_j}{u_i}\hbar^{-1}\right), \quad (24)$$

where $f_{\mathbf{p},\mathbf{q}}$ is a holomorphic function of u_i.

For $i \in \mathbf{q}$ and $j \in \mathbf{n}\setminus\mathbf{q}$ with $i < j$, consider the point $\mathbf{q}' = \mathbf{q}\setminus\{i\} \cup \{j\}$. By construction, $\mathbf{q}$ and $\mathbf{q}'$ are connected by an equivariant curve with character u_i/u_j. The condition (1) means

$$\kappa_{\mathbf{p},\mathbf{q}} - \kappa_{\mathbf{p},\mathbf{q}'}\big|_{u_i=u_j} = 0.$$

$$\kappa_{\mathbf{p},\mathbf{q}'} =$$

$$\mathsf{I} =$$

$$i\in \quad \in\mathbf{n} \quad j\; i<j \quad \in\setminus\mathbf{q},$$

where $f_{\mathbf{p},\mathbf{q}}$ is holomorphic in u_i. As a holomorphic function in $u_i \in \mathbb{C}^{\times}$, it can be expanded as the Laurent series $f_{\mathbf{p},\mathbf{q}} = \sum_{k\in\mathbb{Z}} c_k u_i^k$ with nonzero radius of convergence.

The quasiperiods of functions $T_{\mathbf{p},\mathbf{q}}$ are the same as those of the functions $\mathrm{U}_{\mathbf{p},\mathbf{q}}(X)$. In particular, for all $i \notin \mathbf{p} \cap \mathbf{q}$ from (22), we find

$$f_{\mathbf{p},\mathbf{q}}(u_i q) = f'_{\mathbf{p},\mathbf{q}}(u_i) z^{\pm 1}\hbar^{m}$$

for some integer m. We obtain

$$\sum_{k\in\mathbb{Z}} c_k (z^{\pm 1}\hbar^m - q^k) u_i^k = 0$$

and thus $c_k = 0$ for all k, that is, $f_{\mathbf{p},\mathbf{q}} = 0$.

3.7 Existence of elliptic stable envelope for X

The following result is proven in [1, 15, 30]:

Theorem 3. For canonical polarization (16) and chamber (18), the elliptic stable envelope of a fixed point $\mathbf{p} \in X^{\mathsf{T}}$ has the following explicit form:

Stab$_\sigma(\mathbf{p})$ = Sym $l=1$ $i_l=1$ $\theta(y_l u_i$ $<\theta(z-{}_l1 \hbar y^{-}$ ki $-n+\mathbf{p}y_l-j2_l l)$ $i=\mathbf{p}_l+1$ $l u_i)$
(25) k $\mathbf{p}$ -1 $-1)\theta(y_l u_\mathbf{p}$ $z-1\hbar^{-}$ $k-n+\mathbf{p}$ $-2l)$ n $\theta(y\ \hbar^{-}$
θ θ

Note that the components $T_{\mathbf{p},\mathbf{q}}$ are defined by this explicit formula as restriction $T_{\mathbf{p},\mathbf{q}} = \mathrm{Stab}_{\theta,\sigma}(\mathbf{p})_\mathbf{q}$ $= \mathrm{Stab}_{\theta,\sigma}(\mathbf{p})\ {}_{y_i=u_{-\mathbf{q}i}1}$. The proof of this theorem is by checking the properties 1)-3) from Theorem 2 explicitly, details can be found in [15].

3.8 Holomorphic normalization

Note that the stable envelope (25) has poles in the Kähler parameter z. It will be more convenient to work with a different normalization of the stable envelope in which it is holomorphic in z:

$$\mathbf{Stab}(\mathbf{p}) := \Theta_\mathbf{p}\, \mathrm{Stab}_\sigma(\mathbf{p}), \tag{26}$$

where $_\mathbf{p}$ is the section of a line bundle on the Kähler part $E_{\mathrm{Pic}(X')}$ defined explicitly by

$$= \prod_{m=1} \theta(z^{-1-} \quad {}^{k} \quad {}_\mathbf{p}\hbar^{k-n+\mathbf{p}_m}). \tag{27}$$

(For X and T, see Section 4.) Similarly to Theorem 2, the stable envelope $\mathbf{Stab}(\mathbf{p})$ can be defined as a unique section of the twisted line bundle on $\mathsf{E}_{\mathsf{T}}(X)$:

$$\mathrm{M}(\mathbf{p}) = \mathcal{T}(\mathbf{p}) \otimes \Theta_{\mathbf{p}}, \tag{28}$$

with diagonal restrictions (Property 2 in Theorem 2) given by $T_{\mathbf{p},\mathbf{pp}}$. Note that the function $_{\mathbf{p}}$ only depends on Kähler variables. Thus, the twist of line bundle (28) does not affect quasiperiods of stable envelopes in the equivariant parameters.

We will see that the section $_{\mathbf{p}}$ has the following geometric meaning: it represents the elliptic Thom class of the repelling normal bundle on the dual variety X (see (35)):

$$\Theta_{\mathbf{p}} = \Theta(N'^{-}_{\lambda}),$$

where λ is related to $\mathbf{p}$ by (54), with parameter a_1/a_2 related to Kähler parameter z by (55).

4 Elliptic Stable Envelope for X

4.1 X as a Nakajima quiver variety

From now on, we always assume that $n \geq 2k$. In this section, we consider the variety X which is a Nakajima quiver variety associated to the A_{n-1} quiver. This variety is defined by the framing dimension vector:

$$\mathsf{w}_i = \delta_{k,i} + \delta_{n-k,i},$$

that is, all framing spaces are trivial except those at position k and $n-k$. Both nontrivial framing spaces are one dimensional. The dimension vector has the form

$$\mathsf{v} = (1,2,\ldots,k-1,\underbrace{k,\ldots,k}_{(n-2k+1)\text{-times}},k-1,\ldots,2,1).$$

By definition, this variety is given by the following symplectic reduction. Let us consider the vector space:

$$R = {}^{n-2}\,\mathrm{Hom}(\mathbb{C}^{v_i},\mathbb{C}^{v_{i+1}})\mathrm{Hom}(\mathbb{C},\mathbb{C}^{v_k})\mathrm{Hom}(\mathbb{C}^{v_{n-k}},\mathbb{C}), \tag{29}$$

$i=1$

and denote the representatives by $(\mathbf{a}_l,\mathbf{i}_k,\mathbf{j}_{n-k})$, $l = 1,\ldots,n-2$. Similarly, the dual vector space

$$R* = {}^{n-2}\,\mathrm{Hom}(\mathbb{C}^{v_{i+1}},\mathbb{C}^{v_i})\mathrm{Hom}(\mathbb{C}^{v_k},\mathbb{C})\mathrm{Hom}(\mathbb{C},\mathbb{C}^{v_{n-k}})$$

$i=1$

with representatives by $(\mathbf{b}_l,\mathbf{j}_k,\mathbf{i}_{n-k})$. We consider the symplectic space $T*^R = {}^R \oplus R*$ and the moment map

$$n-1\ \mu : T*R$$

$$\to \mathfrak{gl}(v_i)*$$

$i=1$

Denote $\mathbf{a} = \oplus \mathbf{a}_i$, $\mathbf{b} = \oplus \mathbf{b}_i$, $\mathbf{i} = \oplus \mathbf{i}_i$ and $\mathbf{j} = \oplus \mathbf{j}_i$, then the moment takes the explicit form $\mu((\mathbf{a},\mathbf{b},\mathbf{i},\mathbf{j}) = [\mathbf{b},\mathbf{a}] + \mathbf{i}\circ\mathbf{j}$. With this notation, X is defined as the quotient:

$n-1$

$$= \mu^{-1}(0) \cap \{\theta'$$

$i=1$

We will use the canonical choice of the stability parameter

$$\theta' = (1,1,\ldots,1) \in \mathrm{Lie}_{\mathbb{R}}(\mathsf{K}'), \tag{30}$$

where $\mathsf{K} := U(1)^{n-1}$. (We use the same notations for stability condition as in the MaulikOkounkov [38].

In particular, for us the stability parameter $\theta = (\theta_i)$ corresponds to a character $\chi : {}_i\, GL(v_i) \to \mathbb{C}^\times$ given by

$$\chi : (g_i) \to (\det g_i)^{\theta_i}.$$

i

The set of the θ-semistable points in $T*R$ is described as follows: a point $((\mathbf{a},\mathbf{i}_k,\mathbf{j}_{n-k}),(\mathbf{b},\mathbf{j}_k,\mathbf{i}_{n-k})) \in \mu^{-1}(0)$ is θ'-semistable, if and only if the image of $\mathbf{i}_k \oplus \mathbf{i}_{n-k}$ under the actions of $\{\mathbf{a}_l,\mathbf{b}_l, 1 \leq l \leq n-2\}$ generate the entire space $\overset{n-1}{\underset{i=1}{}}\mathbb{C}^{\mathsf{v}_i}$.

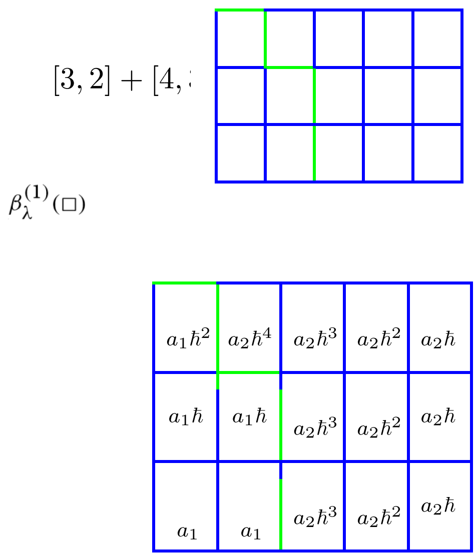


Fig. 4. The point $\lambda = [4,4,4,3,3,2] \subset R_{10,4}$ corresponds to $\mathbf{p} = \{4,7,9,10\} \subset \{,2,...,10\}$.

4.2 Tautological bundles over X

We denote by V_i the rank v_i tautological vector bundle on X associated to $\mathbb{C}^{\mathsf{v}_i}$. It will be convenient to represent the dimension vector and associated tautological bundles using the following combinatorial description. Let us consider a rectangle $\mathsf{R}_{n,k}$ with dimensions $k \times (n-k)$. We turn $\mathsf{R}_{n,k}$ by 45° as in the Figure 3. We will denote by $= (i,j) \in \mathsf{R}_{n,k}$ a box in $\mathsf{R}_{n,k}$ with coordinates (i,j), $i = 1,...,n-k$ and $j = 1,...,k$. We define a function of *diagonal number* on boxes:

$${}_{\square} = i - j + k$$

$\leq c_{\Box} \leq n -$

$$\mathsf{V}_m = x. \; c = m$$

The tautological bundles V_i generate the equivariant K-theory of X. The K-theory classes are represented by Laurent polynomials in x:

$$\prime(X') = \mathbb{C}[x_{ij}^{\pm 1}]^{\mathfrak{S}_{n,k}} \otimes \mathbb{C}[a_1^{\pm}, a_2^{\pm}, \hbar^{\pm}]/$$

where T is the torus described in the next subsection. These are the Laurent polynomials symmetric with respect to each group of Chern roots, that is, invariant under the group:

$$\mathsf{S}_{n,k} = \prod_{i=1}^{n-1} \mathsf{S}_{\mathsf{v}_i}, \tag{31}$$

$$= \{f(x_{i,j}) : f(x_{i,j}) \; {}_{x_{ij}=\varphi^{\lambda}_{ij}}, \forall \lambda \in (X')^{\mathsf{T}'}\}$$

see (32) below.

4.3 Torus action on X

Let $\mathsf{A} = (\mathbb{C}^{\times})^2$ be a 2D torus acting on the framing space $\mathbb{C} \oplus \mathbb{C}$ by

$$(z_1,z_2) \to (z_1a_1,z_2a_2).$$

Let $\mathbb{C}^\times_{\hbar}$ be the one-dimensional torus acting on T^*R by scaling the cotangent fiber

$$_{-2k})) \mapsto (($$

Denote their product by $\mathsf{T} = \mathsf{A} \times \mathbb{C}^\times_{\hbar}$. The fixed loci in X under the A-action admit a *tensor product* decomposition:

$$(X')^{\mathsf{A}'} = \coprod_{\mathsf{v}^{(1)}+\mathsf{v}^{(2)}=\mathsf{v}} \mathcal{M}(\mathsf{v}^{(1)},\delta_k) \times \mathcal{M}(\mathsf{v}^{(2)},\delta_{n-2k}),$$

where $\mathcal{M}(\mathsf{v}^{(1)},\delta_k)$ is the quiver variety associated with the A_{n-1} quiver with dimension vector $\mathsf{v}^{(1)}$, framing vector δ_k and the same stability condition θ; similar with $\mathcal{M}(\mathsf{v}^{(2)},\delta_{n-2k})$.

We now give a combinatorial description of the quiver variety $\mathcal{M}(\mathsf{v}^{(1)},\delta_k)$. By definition, a representative of a point in $\mathcal{M}(\mathsf{v}^{(1)},\delta_k)$ takes the form $(\mathbf{a},\mathbf{i},\mathbf{b},\mathbf{j})$. It is θsemistable, if and only if the image of $\mathbf{i}$ under the actions of all $\mathbf{a}$s and $\mathbf{b}$s generate the space

$$\mathsf{V}^{(1)} := \bigoplus_{i=1}^{n-1} \mathbb{C}^{\mathsf{v}^{(1)}_i}.$$

$$\mathbb{C}^{\mathsf{v}^{(1)}_{i-j+}}$$

$$\{(i,j) \in \mathbb{Z}^2_{>0} \quad {}^{-1}\mathbf{b}^{j-1}\mathbf{i}_k(1) \neq 0\}$$

In summary, the quiver variety $\mathcal{M}(\mathsf{v}^{(1)},\delta_k)$ is either empty or a single point, where the latter case only happens when there exists a partition λ, whose number of boxes in the m-th diagonal is $\mathbf{v}^{(1)}_{m+k}$. The quiver variety $\mathcal{M}(\mathbf{v}^{(2)},\delta_{n-2k})$ can be described in exactly the same way.

The restriction of Chern roots to the fixed point can be determined as follows. Consider

$$\mathbf{a}^{i-1}\mathbf{b}^{j-1}\mathbf{i}_k : \mathbb{C} \to V_{i-j+k}.$$

The action of the group $GL(\mathsf{v}^{(1)})$ on $\mathbf{a}^{i-1}\mathbf{b}^{j-1}\mathbf{i}_k$ is

$$\mathbf{a} \to g\mathbf{a}g^{-1}, \qquad \mathbf{b} \to g\mathbf{b}g^{-1}, \qquad \mathbf{i}_k \to g_k\mathbf{i}_k,$$

where $g = (g_1,\cdots,g_{n-1}) \in \prod_i GL(\mathsf{v}_i^{(1)})$. So

$$\mathbf{a}^{i-1}\mathbf{b}^{j-1}\mathbf{i}_k \to g\mathbf{a}^{i-1}\mathbf{b}^{j-1}\mathbf{i}_k,$$

and the action of A on the framing space $\mathbb{C}$, $z \to a_1 z$, induces the action

$$\mathbf{a}^{i-1}\mathbf{b}^{j-1}\mathbf{i}_k \to a_1^{-1}\mathbf{a}^{i-1}\mathbf{b}^{j-1}\mathbf{i}_k.$$

Here a_1 becomes a_1^{-1} because the framing $\mathbb{C}$ is the domain space of $\mathbf{i}_k$. To determine the restriction of the Chern root ϕ_{ij}, we need g to compensate the action of T, that is,

$$g_i = a_1, \qquad \forall i.$$

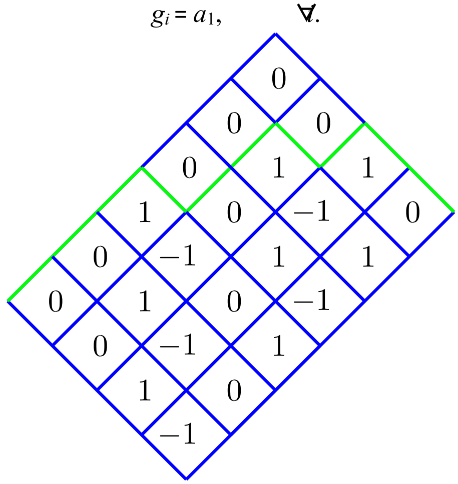


Fig. 1. An example of a fixed point represented by $[3,2] \in R_{8,3}$.

So the (A-equivariant) restriction is $\phi_{ij} = a_1$. For the $\bar{h}$-weight, $C_{\times \bar{h}}$ acts on **b** directly by

$_{j-1}\ \bar{h}$. So the T-equivariant restriction is $\phi_{ij} =$

$a_1\bar{h}$. Exactly same consideration applies to the 2nd part $M(\mathbf{v}^{(2)},\delta_{n-2k})$.

(X')

$\lambda \in (X')$

tautological bundles are given by the following formula:

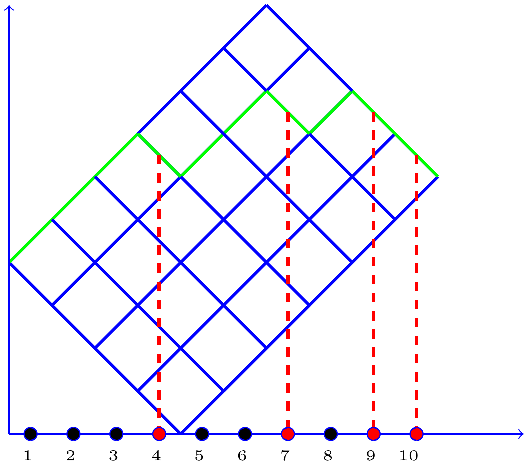
$$= \varphi^{\lambda}_{\square} := \quad a^1 \bar{h}^{j-1}, \text{ if } (i,j) \in \lambda, \; x^{\lambda} a_2^{\,-n-k-i+1}, \quad \text{if } (i,j) \in \bar{\lambda}.$$

(32) h

Our notations should be clear from the following example:

Example 3. Let us fix $n = 8, k = 3$ and consider a Young diagram $\lambda = [3,2]$, then $\bar{\lambda} = [4,3,3]$. The union of λ and $\bar{\lambda}$ is the rectangular of dimensions 5×3:

The values of Chern roots (which correspond to boxes of $R_{8,3}$) are given in Figure 2:



$\varphi^{\lambda}_{\square}$ for $\lambda =$

4.4 Tangent and polarization bundles for X

To define the elliptic stable envelope, we need to specify a polarization and a chamber. We choose the canonical polarization:

n_1

$$T^{1/2}X \;=\; {}_1 \quad + \;{}_2 \;{}_{n-k} + \sum_{i=}^{n-2} \mathcal{V}_{i+1}\mathcal{V}_i^* - \quad a\text{-}^1\mathrm{V}_k \qquad a$$

$\mathrm{V}*$ $\mathrm{V}_i*\mathrm{V}_i,$ (33)

1 i=1

such that the virtual tangent space takes the form:

$$= T^{1/2}X' + (T^{1/2}X')^* \otimes \hbar^{-1}$$

We choose a chamber in the following form:

$$\sigma' : (0,1) \in \mathrm{Lie}_{\mathbb{R}}(\mathsf{A}')$$

$$\lambda \;\in\; (X')$$

$$T_\lambda X' = TX' \quad \lambda$$

The tangent space at a fixed point decomposes into attracting and repelling parts:

$$\lambda X' = N_\lambda'^{+} \oplus N_\lambda'^{-}$$

where $N_{\lambda\pm}$ are the subspaces with A-characters, which take positive and negative values on the cocharacter (34), respectively. Explicitly these characters equal:

$$N_{\lambda-} = \sum_{m=1}^{k} a_2^{1-} a \_\_ h2k-n+\mathbf{p}_m-2m-1, \qquad N_{\lambda+} = \sum_{m=1}^{k} a_1^{2-} a \_\_ h-2k+n-\mathbf{p}_m+2m \tag{35}$$

where $\mathbf{p} = \{\mathbf{p}_1,...,\mathbf{p}_k\}$ = bj(λ), for bj described in (54).

4.5 Elliptic cohomology of X

The extended elliptic cohomology scheme of X is a bouquet of T orbits (as a set)

$$'(X') := \qquad {}'_\lambda/\Delta'$$

$$\lambda\in(X')$$

(X')

whereidentified with the coordinates in the 1st and 2nd factor ofO λ = ET × EPic. The equivariant parameters and Kähler parameters ofO λ, respectively. X are

By definition, the elliptic stable envelope classes are sections of the twisted elliptic Thom class of the polarization (see discussion in Section 3.5):

$$\mathcal{T}'(\lambda) = \Theta(T^{1/2}X') \otimes \ldots$$

which is a line bundle over the scheme (36) ($\mathcal{T}'(\lambda)$depends on λ via twist terms denoted by ...).

Sections of the line bundles $\mathcal{T}'(\lambda)$O $_\mu$ over abelian varieties O $_\mu$ have the same transformation

$$\mathcal{U}_{\lambda,\mu}(X') \qquad \prod_{\square\in \mathsf{R}_{n,k}} \frac{\phi(\varphi^\lambda_\square, z_{c_\square}^{-1})}{\phi(\varphi^\mu_\square, z_{c_\square}^{-1})} \prod_{i=1}^{2} \frac{\phi(a_i, z_{a_i}^{-1}\hbar^{D_i^\lambda})}{\phi(a_i, z_{a_i}^{-1})} = \Theta \quad T^{1/2}X' \quad \mu$$

properties as the following function: . (37)

)

The powers D_i^λ are determined as follows: let us consider the index of the fixed point

$$\mathrm{ind}_\lambda = T^{1/2}X'_{\lambda,>} \quad ./$$

The symbol > means that we choose only the T-weights of polarization $T^{1/2}X_\lambda$, which are positive at σ.

Let det(ind$_\lambda$) denote the product of all these weights, then D_i^λ is a degree of this monomial in variable

a_i.

$$'_{\sigma'}(\lambda) \qquad \mathcal{T}'(\lambda)$$

4.6 Holomorphic normalization

It will be convenient to work with stable envelopes, which differ from one defined in [1] by normalization

$$(\lambda) = \Theta'_\lambda \quad '_\sigma(\lambda)$$

Θ'_λ

λ = i — ——

p, ui uih

$j<\backslash j$ $>\backslash j$

where $\mathbf{p}$ = bj(λ) (see (54) below) and variables u_i are related to Kähler parameters z_i through (55). The stable envelope **Stab**(λ) is a section of the twisted line bundle on

(X')

$$'(\lambda) = \mathcal{T}'(\lambda) \otimes \Theta'_{\lambda}$$

Θ'_{λ}

Θ'_{λ}

$\Theta(N^-_{\mathbf{p}})$

5 Abelianization Formula for Elliptic Stable Envelope for X

5.1 Non-Kähler part of stable envelope

Define a function in the boxes of the rectangle $\mathsf{R}_{n,k}$ by

$$_{\square} = \quad - \quad \square \in \lambda \quad i \quad j, \quad \text{if} \quad \square \notin \lambda$$

ρ i j, if.

The following function describes the part of elliptic stable envelope of a fixed point λ, which is independent on Kähler parameters:

$$\mathbf{S}^{n,k}_{\lambda} = (-1)^{k(n-k)} \frac{\underset{\substack{c_I=k\\ I?\lambda}}{}\theta\left(\frac{x_I}{a_1}\right) \underset{\substack{c_I=k\\ I?\lambda}}{}\theta\left(\frac{a_1}{x_I\hbar}\right) \underset{c_I=n-k}{}\theta\left(\frac{a_2\hbar}{x_I}\right) \underset{\substack{c_I+1=c_J\\ \rho^{\lambda}_I>\rho^{\lambda}_J}}{}\theta\left(\frac{x_J\hbar}{x_I}\right) \underset{\substack{c_I+1=c_J\\ \rho^{\lambda}_I<\rho^{\lambda}_J}}{}\theta\left(\frac{x_I}{x_J}\right)}{\underset{\substack{c_I=c_J\\ \rho^{\lambda}_I>\rho^{\lambda}_J}}{}\theta\left(\frac{x_I}{x_J}\right)\theta\left(\frac{x_I}{x_J\hbar}\right)} \tag{41}$$

$$\prod_{\substack{I=k\\ I\notin\lambda}}$$

$$\prod_{=c}$$

$$\rho^{\lambda}_I > \rho^{\lambda}_J$$

ρλ>ρλ IJ

Example 4.

$$\mathbf{S}_{[1]3,1} = \theta\,\frac{x_{1,1}}{a_1}\quad \theta\,\frac{a_2\hbar}{x_{2,1}}\quad \theta\,\frac{x_{2,1}}{x_{1,1}}\quad \hbar^{-},$$

θx1,1 θa12,$\hbar$1 θ a21,2 θa22,$\hbar$2 θx21,,11 θx11,,12 θ$\hbar x$ 12,2,2 θx22,,12

$\mathbf{S}$4[1,2,1] = a1 x $\hbar x$ θxx12,,12 θ xx1,1 x x x ,

x $\hbar x$ 2,2

x 2,2 2,2 1,1 θ $a_2\hbar$ $\hbar x$12,2,2 x22,,21 θ θ $a_2\hbar$ θ $x_{2,1}\hbar$ θ $x_{1,1}\hbar$ θ

4,2 θθ

[2] = x1,1 $\hbar x$ x x x x x θθ

$\mathbf{S}$.

x1,1 x1,1

x2,2 $\hbar x$ 2,2

5.2 Trees in Young diagrams

Let us consider a Young diagram λ. We will say that two boxes $\square_1 = (i_1, j_1), \square_2 = (i_2, j_2) \in \lambda$ are *adjacent* if

$$i_1 = i_2, \qquad |j_1 - j_2| = 1 \qquad \text{or} \qquad j_1 = j_2, \qquad |i_1 - i_2| = 1.$$

Definition 3. A λ-tree is a rooted tree with

(⋆, ⋆

(⋆, ⋆, ⋆) edges connecting only the adjacent boxes.

Note that the number of λ-trees depends on the shape of λ. In particular, there is exactly one tree for “hooks”.

We assume that each $\lambda = (\lambda_1, 1, \cdots, 1)$ edge of a -tree is oriented in a certain way. In particular, on a set of edges, we λ have two well-defined functions

$$h,t : \{\text{edges of a tree}\} \dashrightarrow \{\text{boxes of } \lambda\},$$

which for an edge e return its head $h(e) \in \lambda$ and tail $t(e) \in \lambda$ boxes, respectively. In this paper, we will work with a distinguished *canonical orientation* on λ-trees.

Definition 4. We say that a λ-tree has canonical orientation if all edges are oriented from the root to the end points of the tree.

For a box $\square \in \lambda$ and a canonically oriented λ-tree t, we have a well-defined canonically oriented subtree $[\square, \mathrm{t}] \subset \mathrm{t}$ with root at . In particular, $[r,\mathrm{t}] = \mathrm{t}$ for a root r of t.

We rotate the rectangle $\mathsf{R}_{n,k}$ by 45◦ as in the Figure 3, such that the horizontal coordinate of the box is equal to c. The boundary of a Young diagram $\lambda \subset \mathsf{R}_{n,k}$ is a graph of a piecewise linear function. We define a function on boxes in $\mathsf{R}_{n,k}$ by

$$\beta^{(1)}_\lambda(\square) = \begin{cases} +1 & \text{if } \square \in \lambda \text{ and } \Gamma \text{ has maximum above} \\ -1 & \text{if } \square \in \lambda \text{ and } \Gamma \text{ has minimum above} \\ 0 & \text{else.} \end{cases} \tag{42}$$

$\beta^{(1)}_\lambda(\square) =$ $\square \in \bar{\lambda}$ $\beta^{(1)}_\lambda(\square)$

We also define $+1$ if $c < k$

$$\beta^{(2)}_\lambda(\square) = \begin{cases} +1 & \text{if } c < k \\ -1 & \text{if } c > n-k \\ 0 & \text{else} \end{cases}$$

and we set

$$(\square) = \beta_\lambda^{(1)}(\square) + \beta_\lambda^{(2)}(\square)$$

5.3 Kähler part of the stable envelope

Let $\lambda \subset R_{n,k}$ be a Young diagram and $\bar{\lambda} = R_{n,k} \setminus \lambda$ is the complement Young diagram as above. Let $\mathrm{t} \cup \bar{\mathrm{t}}$ be the (disjoint) union of λ-tree t and $\bar{\lambda}$-tree $\bar{\mathrm{t}}$. We define a function:

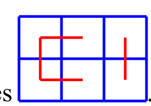
$$\mathbf{W}^{Ell}(\mathrm{t} \cup \bar{\mathrm{t}}; x_i, z_i) := \mathbf{W}^{Ell}(\mathrm{t}; x_i, z_i)\mathbf{W}^{Ell}(\bar{\mathrm{t}}, x_i, z_i),$$

for the elliptic weight of a tree, where

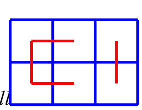
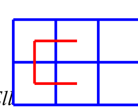
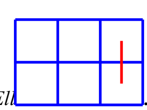
$$\mathbf{W}^{Ell}(\mathrm{t}; x_i, z_i) := (-1)^{\kappa(\mathrm{t})} \phi\left(\frac{\varphi_r^{\lambda}}{x_r}, z_{c_\square}^{-1} \prod_{\square \in [r,\mathrm{t}]} \hbar^{-v(\square)}\right) \prod_{e \in \mathrm{t}} \phi\left(\frac{\varphi_{t(e)}^{\lambda} x_{h(e)}}{x_{t(e)} \varphi_{h(e)}^{\lambda}}, z_{c_\square}^{-1} \prod_{\square \in [h(e),\mathrm{t}]} \hbar^{-v(\square)}\right), \quad (44)$$

and similarly for $\mathbf{W}^{Ell}(\bar{\mathrm{t}}, x_i, z_i)$.

Here $\square \in \mathrm{t}$ or $e \in \mathrm{t}$ means the box or edge belongs to the tree. The sign of a tree depends on the number $\kappa(\mathrm{t})$, which is equal to the number of edges in the tree with wrong orientation. In other words, $\kappa(\mathrm{t})$ is the number of edges in t directed down or to the left, while $\kappa(\bar{\mathrm{t}})$ is the number of edges in $\bar{\mathrm{t}}$ directed up or to the right. To avoid ambiguity, we also define $\mathbf{W}^{Ell}(\mathrm{t}; x_i, z_i) := 1$ for a tree in the empty Young diagram.

Example 5. Let us consider a Young diagram $[2,2] \subset R_{5,2}$ with trees .

By definition, we have

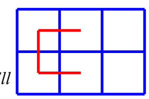
$\mathbf{W}^{Ell}$  $= \mathbf{W}^{Ell}$  $\mathbf{W}^{Ell}$ .

In this case, we have six boxes with the following characters:

$$\varphi^{\lambda}_{11} = a_1, \ \ \varphi^{\lambda}_{21} = a_1, \ \ \varphi^{\lambda}_{31} = a_2\hbar, \ \ \varphi^{\lambda}_{12} = a_1\hbar, \ \ \varphi^{\lambda}_{22} = a_1\hbar, \ \ \varphi^{\lambda}_{32} = a_2\hbar$$

Similarly for the h^{-}-weights of boxes (43) we obtain

$$\beta(1,1) = \beta^{(1)}(1,1) + \beta^{(2)}(1,1) = 1 + 0 = \quad 1,$$
$$\beta(1,2) = \beta^{(1)}(1,2) + \beta^{(2)}(1,2) = 0 + 1 = \quad 1,$$
$$\beta(2,1) = \beta^{(1)}(2,1) + \beta^{(2)}(2,1) = 0 + 0 = \quad 0,$$
$$\beta(2,2) = \beta^{(1)}(2,2) + \beta^{(2)}(2,2) = 1 + 0 = \quad 1,$$
$$\beta(3,1) = \beta^{(1)}(3,1) + \beta^{(2)}(3,1) = 0 - 1 = - \quad 1,$$
$$\beta(3,2) = \beta^{(1)}(3,2) + \beta^{(2)}(3,2) = 0 + 0 = 0.$$

First, let us consider $\mathbf{W}^{Ell}$ . In this case, we have a tree with the root at

r = (1,1) and three edgeswith the following heard and tails: $t(e_1)$ = (1,1), $h(e_1)$ = (1,2), $t(e_2)$ = (1,1),

$h(e_2)$ = (2,1), $t(e_3)$ = (1,2), $h(e_3)$ = (2,2).

For the 1st factor in (44), we obtain

$$\prod_{\square \in r,t} z_{c_\square}^{-1} \hbar^{-\mathsf{v}(\square)} = \phi\left(\frac{a_1}{x_{1,1}}, z_1^{-1} z_2^{-2} z_3^{-1} \ldots \phi_{r}^{\lambda}\right) \qquad 3$$

φ,h-

.

xr [ ]

For the edges in the product (44), we obtain:

$$\phi\left(\frac{x_{t(e_1)}}{\varphi^{\lambda}_{t(e_1)} x_{h(e_1)}} \frac{h(e_1)}{}\ \phi\lambda, z_{c_\square}^{-1}\right) \prod_{\square \in [h(e_1), t]} \hbar^{-\mathsf{v}(\square)} = \phi\left(\frac{x_{11}}{x_{12}}\hbar, z_1^{-1} z_2^{-1} \ldots h^{-2}\right),$$

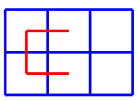
$t(e_2)$ $h(e_2)$ $\square\in[h(e_2),\mathfrak{t}]$ ϕ $x_{t(e_2)}$ $h(e_2)$ φ^{λ} x x $c_{\square}^{-1}\hbar^{-\mathrm{v}(\square)} = \phi$ $\frac{11}{x_{21}}, z_3^{-1}$ $\phi\lambda$

, z,

ϕ $x_{t(e_3)}$ $h(e_3)$ λ $\phi\lambda$ $c_{\square}^{-1}\hbar^{-\mathrm{v}(\square)} = \phi$ $\frac{x_{12}}{x}, z_2^{-1}$ - $\varphi_{t(e_3)}x_{h(e_3)}$ $\square\in[h(e_3),\mathfrak{t}]$ 1

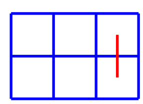
, zh- 22 .

Thus, overall we obtain

$\mathbf{W}_{Ell}$ 

$$= \phi \left(\frac{a_1}{x_{1,1}}, z_1 z_2^2 z_3 \hbar^3 \right) \phi \left(\frac{x_{11}\hbar}{x_{12}}, z_1 z_2 \hbar^2 \right) \left(\phi \frac{x_{11}}{x_{21}}, z_3^{1} \right) \left(\phi \frac{x_{12}}{x_{22}}, z_2^{1} \right).$$

2 - - h-

Similarly, for the 2nd multiple we obtain $\mathbf{W}_{Ell}$ 

$$= \phi \left(\frac{a_2\hbar}{x_{32}}, \frac{\hbar}{z_3 z_4} \right) \phi \left(\frac{x_{32}}{x_{31}}, \frac{\hbar}{z_4} \right).$$

5.4 Formula for elliptic stable envelope

Definition 5. The skeleton $_{\lambda}$ of a partition λ is the graph, whose vertices are given by the set of boxes of λ and whose edges connect all adjacent boxes in λ.

Definition 6. A ˥-shaped subgraph in λ is a subgraph $\gamma \subset \Gamma_{\lambda}$ consisting of two edges $\gamma = \{\delta_1,\delta_2\}$ with the following end boxes:

$$\delta_{1,1} = (i,j), \qquad \delta_{2,1} = \delta_{1,2} = (i+1,j), \qquad \delta_{2,2} = (i+1,j+1). \tag{45}$$

It is easy to see that the total number of L -shaped subgraphs in λ is equal to

$$= \sum_{l\in\mathbb{Z}} (\mathsf{d}_l(\lambda) - 1)$$

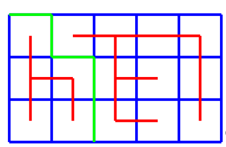
where $\mathsf{d}_l(\lambda)$ is the number of boxes in the l-diagonal of λ

$$(\lambda) = \#\{\square \in \lambda \mid c_{\square} = l\}$$

There is a special set of λ-trees, constructed as follows. For each ˥-shaped subgraph γ_i in λ we choose one of its two edges. We have 2^m of such choices. For each such choice, the set of edges $\Gamma_\lambda \setminus \{\delta_i\}$ is a λ -tree. We denote the set of 2^m λ-trees, which appear this way by Υ_λ.

Now let us define $\Upsilon_{n,k} = \Upsilon_\lambda \times \Upsilon_{\bar\lambda}$, whose elements of are pairs of trees $(\mathrm{t},\bar{\mathrm{t}})$, where t is a λ-tree with root (1,1) and $\bar{\mathrm{t}}$ is a $\bar\lambda$-tree with root $(n-k,k)$. Both trees are constructed in the way described as above, and they are disjoint, that is, do not have common vertices.

Example 6. Let us consider λ = [3,2] $\in \mathsf{R}_{8,3}$ and $\bar\lambda$ = [4,3,3]. A typical element of $\Upsilon_{8,3}$ looks like



$\in \Upsilon_{8,3}$

The following theorem can be proved using the same arguments as in [62].

Theorem 4. The elliptic stable envelope of a fixed point λ for the chamber σ defined by (34) and polarization (33) has the following form:

$$\boxed{\mathrm{Stab}_\sigma(\lambda) = \mathrm{Sym}_{\mathfrak{S}_{n,k}}\, \mathbf{S}_\lambda^{n,k} {}_{(\mathrm{t},\bar{\mathrm{t}})\square\Upsilon_{n,k}} \mathbf{W}^{Ell}(\mathrm{t} \,\square\, \bar{\mathrm{t}})} \tag{48}$$

where the symbol $\mathrm{Sym}_{\mathfrak{S}_{n,k}}$ denotes a sum over all permutations in the group (31).

Proof. The proof of this theorem is based on the abelianization of elliptic stable envelope developed in [1, Section 4.3], which, in turn, is inspired by the abelianization of stable envelopes in cohomology [60]. The proof follows closely the proof of the main result of [62]. To keep the presentation short, we will refer to the corresponding results in these papers when possible. We also refer to [1, 62] for definitions of all maps and objects appearing here.

Let us denote by **AX** the abelianization of the Nakajima variety X. This is a hypertoric variety defined by the following symplectic reduction:

$$\mathbf{AX} := T^*R////S,$$

where R is given by (29) and S is the maximal torus of $GL(\mathsf{v}_i)$. $\prod_i^n$ $\lambda, \bar{\lambda}$ The stability condition for this symplectic reduction is defined by (30). Let ( ) be a T fixed points in X. By definition, $(\lambda,\bar{\lambda})$ is a zero-dimensional $(\lambda,\bar{\lambda})$ Nakajima quiver variety of type A_{n-1}. We will denote by **AX** the $'_{(\lambda,\bar{\lambda})}$ abelianization of this Nakajima variety. It is a hypertoric subvariety **AX** ⊂ **AX** fixed by the action of torus A. We denote by $\mathrm{Stab}_{\mathsf{C}}$ the elliptic stable envelope map for these hypertoric varieties. The chamber C here is the chamber of A defined by cocharacter (34).

The abelianization diagram for Nakajima varieties (see [1, (74)]) expresses the elliptic stable envelope of the fixed point $(\lambda,\bar{\lambda})$ in X as the following composition:

$$(\lambda, \bar{\lambda}) = \pi_* \circ \mathsf{j}_+^* \circ (\mathsf{j}_{-*})^{-1} \qquad {}'_{-*} \circ ({}'^*_+)^{-1} \circ \pi'^{-1}_*$$

For the definition of all maps here, we refer to [1, Section 4.3].

Lemma 1. *The Nakajima quiver variety* $(\lambda,\bar{\lambda})$ *is a direct product of two zerodimensional Nakajima varieties of A -type corresponding to dimension vectors given*

$$\infty$$

$$'_{(\lambda,\bar{\lambda})} = \mathbf{AH}_\lambda \times \mathbf{AH}_{\bar{\lambda}}$$

∞

Proof. The fixed point set of a Nakajima quiver variety with respect to action of the framing torus is isomorphic to the direct product of Nakajima varieties for the same quiver and one-dimensional framings (this property of quiver varieties is known as tensor product structure). Non-empty A_{n-1} quiver varieties with one-dimensional framing are all zero-dimensional A quiver varieties and have dimension vectors ∞

$'$ $, \mathrm{j}'^{*}_{+}, \pi$

$$\mathrm{j}'_{-*} = \mathrm{j}'_{1,-*} \times \mathrm{j}'_{2,-*}, \quad \mathrm{j}'^{*}_{+} = \mathrm{j}'^{*}_{1,+} \times \mathrm{j}'^{*}_{2,+}, \quad \pi'_{*} = \pi'_{1,*} \times \pi'_{2,*}$$

$(\mathrm{j}'_{1,-*}, \mathrm{j}'^{*}_{1,+}, \pi'_{1,*})$ whereare maps for zero-dimensional Nakajima quiver variety λ (i.e., *A*

∞

$(\mathrm{j}'_{2,-*}, \mathrm{j}'^{*}_{2,+}, \pi'_{2,*})$

$\mathbf{AH}_{\bar\lambda}$.

The hypertoric varieties $\mathbf{AH}_{\lambda}$ were considered in [62, Section 6]. In particular, it was shown that $\mathbf{AH}_{\lambda}$ contains fixed points (of a maximal torus acting on $\mathbf{AH}_{\lambda}$ by automorphisms) labeled by λ-trees. For trees t, $\bar{t}$ in λ, and $\bar\lambda$, we denote by the same symbols the corresponding fixed points in $\mathbf{AH}_{\lambda}$ and $\mathbf{AH}_{\bar\lambda}$.

$''_{1}, \bar{\mathfrak{C}}''_{2}$

$''_{1}, \bar{\mathfrak{C}}''_{2}$

that C and C are faces of the chamber C).

The following is a version of [62,Proposition 6] for the case of *X*:

Proposition 3. Up to a shift of Kähler parameters $z_i \to z_i h^{-m_i}$, $m_i \in \mathbb{Z}$ the elliptic stable of $\mathbb{C}^{\times}_{t} \times \mathbb{C}^{\times}_{\bar t}$ -fixed point $(t,\bar t)$ in $\mathbf{AX}_{(\lambda,\bar\lambda)}$ corresponding to the chamber C equals

$$\mathrm{Stab}_{C}(t,\bar t) = \mathrm{Stab}_{C''_1}(t)\, \mathrm{Stab}_{C''_2}(\bar t), \tag{50}$$

where

$\mathrm{Stab}_{\mathfrak{C}_1}{}''(\mathrm{t}) =$

$$\prod_{=k} \theta\left(\frac{x_I}{a}\right) \prod_{c\ +1=c} \theta\left(\frac{x_J\hbar}{x}\right) \prod_{c\ +1=} {}_J\ \theta\ x \_\_ x_J \left(\quad\right) \qquad \theta\left(x_J\hbar\right)$$

$I \in \lambda$ 1 I J $\rho^\lambda > \rho^\lambda$ I I c $\rho^\lambda < \rho^\lambda$ I J $\in\lambda$ I J $J\in\lambda$

$$W_{\mathrm{t}}(z_i),\ \mathrm{Stab}_{\mathfrak{C}_2}{}''(\mathrm{t}) = \prod_{=n-k} \theta\ \frac{a_2\hbar}{x_I} \quad c_I{+}1{=}c_J \quad \frac{}{x_I} \quad I\in\bar\lambda \quad \prod_{\rho_I > \rho}$$

c_I λ λ J

$$\prod_{c\ +1=} \quad \rho_I < \rho_I \quad c\lambda J\ \theta \_\_ x x_J\ W_{\bar{\mathrm{t}}}(z_i),$$

I,J $I,$

λ

$I,J \notin \bar\lambda$ $I,J \notin \bar\lambda J$

the elliptic stable envelope of $\mathsf{A} \times \mathbb{C}^\times_{\mathrm{t}} \times \mathbb{C}^\times_{\bar{\mathrm{t}}}$ -fixed point $(\mathrm{t}, \bar{\mathrm{t}})$ in $\mathbf{AX}$ is given by

$$\mathrm{Stab}_{\mathfrak{C}}(\mathrm{t},\bar{\mathrm{t}}) = \prod_{\substack{I=k\\ c\ \in\lambda}} \theta\left(\frac{x_I}{a_1}\right) \prod_{\substack{c_I=k\\ \notin\lambda}} \theta\left(\frac{a_1}{x_I\hbar}\right) \prod_{c_I=n-k} \theta\left(\frac{a_2\hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J\\ \lambda\quad\lambda}} \theta\left(\frac{x_J\hbar}{x_I}\right) \prod_{\substack{c_I+1=c\\ \lambda\quad\lambda}} {}_J\,\theta\ x \frac{x_J}{J}\ W_{\mathrm{t}}(z_i) W_{\bar{\mathrm{t}}}(z_i) \tag{51}$$

I I $\rho I > \rho J$ $\rho I < \rho J$

with

$$W_{\mathrm{t}}(z_i) = (-1)^{\kappa_{\mathrm{t}}} \varphi \qquad x_r,\ i1\ z_i e\ \mathbf{t}\ \varphi \_\_\_\_ x_t{}^h(e) \phi h\lambda\left(\,^e{}_e\right),\ i\ [h(e),\mathrm{t}]\ z_i. \tag{52}$$

n $x\ (e)\ \varphi^\lambda_t$ $=$ $\in$ $\in$

Proof. By Lemma 1, elliptic stable envelope of a fixed point factors to a product of elliptic stable envelopes (50). The explicit formulas for elliptic stable envelopes of t in $\mathbf{AH}_\lambda$ are given by [62, Proposition 6], which gives the above explicit formulas.

We note that the variables z_i, $i \in R_{k,n}$ in (52) denote the Kähler parameters associated to the line bundles x_i on the abelianization **AX**.

Proposition 4.

$$\pi'_{*} \quad j'^{*} \quad (' \quad)^{-1} \qquad \sum_{(t,\bar{t})\in\Upsilon_{n,k}}$$

$$'_{-*}, j'^{*}_{+}, \pi'_{*}$$

Proposition 3, the stable envelope of the fixed point $(t,\bar{t})$ also factorizes into a product of stable envelopes. This gives

$$\pi'_{*} \circ j'^{*}_{+} \circ (j'_{-*})^{-1} \sum_{(t,\bar{t}) \in \Upsilon_{n,k}} \mathrm{Stab}_C(t,\bar{t}) =$$

$$\pi'_{1,*} \circ j'^{*}_{1,+} \circ (j'_{1,-*})^{-1} \quad {}''_{1}(t) \qquad \pi'_{2,*} \circ j'^{*}_{2,+} \circ (j'_{2,} \quad {}''(\bar{t})$$

Each factor here is equal to 1 by [62, Theorem 5].

The last proposition implies that the abelianization formula (49) can be written in the form:

$$\mathrm{Stab}(\lambda,\bar{\lambda}) = \pi_{*} \circ j^{*}_{+} \circ (j_{-*})^{-1} \circ \mathrm{Stab}_C \, \mathrm{Stab}_C \sum_{(t,\bar{t})\in\Upsilon_{n,k}} (t,\bar{t}) \Big) = \pi_{*} \circ j^{*}_{+} \circ (j_{-*})^{-1} \Big(\sum_{(t,\bar{t})\in\Upsilon_{n,k}} \mathrm{Stab}_C(t,\bar{t}) \Big),$$

where the 2nd identity $\mathrm{Stab}_C = \mathrm{Stab}_C \circ \mathrm{Stab}_C$ is the triangle lemma for elliptic stable envelope, see [1, Section 3.6]. We now see that the last expression coincides with (48).

$\prod_{\substack{c_I=c_J\\ \rho_I^\lambda>\rho_J^\lambda}} \theta$ x Indeed, the numerator of

(48) is given by (51), the product __^{I} in the denominator $\prod_{\substack{c_I=c_J\\ \rho_I^\lambda>\rho_J^\lambda}} \theta$

x_J

of (41) comes from pushforward π computed by localization, similarly ___^{J} x h $*x$

comes from the pushforward $(j_-)^{-1}_*$. We refer to [2, Section 4.3] for computations of the corresponding normal bundles to maps π and j_-.

By definition, the Kähler parameters z_l, $l = 1,\ldots,n-1$ of the Nakajima variety X are parameters associated to tautological line bundles $\mathcal{L}_l = \det\mathcal{V}_l$. Expressed in the corresponding Chern roots these line bundles have the form $\mathcal{L}_m = \sum_{i\in \mathsf{R}_{n,k},\, c_i=m} x_i$. This means that the Kähler parameters z_i, $i \in \mathsf{R}_{k,n}$ corresponding to the line bundles x_i on the abelianization $\mathbf{AX}$ restrict to the Kähler parameters of X by $z_i \to z_{c_i}$. This substitution gives desired dependence of stable envelope on Kähler variables.

The last step is to find correct shifts of the Kähler variables by powers of $\hbar$. Indeed, the proposition (3) provides the explicit formulas for elliptic stable envelopes up to shifts $z_i \to z_i\hbar^{m_i}$ for some integers m_i. The values of m_i are uniquely determined by the condition that quasi-periods $x_i \to x_i q$ of (48) coincide with the quasi-periods of the section (37). A calculation repeating the last part of [62, Section 8.3] gives exactly the combinatorial formula for the $\hbar$-powers (43).

5.5 Refined formula

In this subsection, we prove a refined version of formula (48), in the sense that when restricted to another fixed point μ, the summation will be rewritten as depending on the trees $\bar{\mathsf{t}}$ only, but not on the trees t. The refined formula will be of crucial use to us in the proof of the main theorem.

Given a fixed point λ, the original formula (48) has the following structure (for simplicity we omit the chamber subscript σ):

$$\mathrm{Stab}(\lambda) = \sum_{\sigma \in S_{n,k},\, \mathfrak{t}, \bar{\mathfrak{t}}} \frac{\mathcal{N}^{\sigma}}{\mathcal{D}^{\sigma}} \mathcal{R}^{\sigma}(\mathfrak{t}, \bar{\mathfrak{t}}) \mathcal{W}^{\sigma}(\mathfrak{t}, \bar{\mathfrak{t}})$$

where we denote

$$\mathcal{N} := (-1)^{k(n-k)-1} \prod_{\substack{c_I = k \\ I \in \lambda}} \theta\begin{pmatrix} x_I \\ a_1 \end{pmatrix} \prod_{\substack{c_I = k \\ I \notin \lambda}} \theta\begin{pmatrix} a_1 \\ x_I \end{pmatrix} \prod_{\substack{c_I = c_J,\ \rho_I > \rho_J + 1 \\ (I \leftrightarrow J) \notin \Gamma_\lambda \cup \Gamma_{\bar{\lambda}}}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_I = c_J,\ \rho_I < \rho_J + 1 \\ (I \leftrightarrow J) \notin \Gamma_\lambda \cup \Gamma_{\bar{\lambda}}}} \theta\left(\frac{x_I}{x_J}\right),$$

$$\mathcal{D} := \frac{\prod\limits_{\substack{c_I + 1 = c_J,\ \rho_I^\lambda = \rho_J^\lambda + 1 \\ (I \leftrightarrow J) \in \Gamma_\lambda \setminus \mathfrak{t} \cup \Gamma_{\bar{\lambda}} \setminus \bar{\mathfrak{t}}}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod\limits_{\substack{c_I + 1 = c_J,\ \rho_I^\lambda + 1 = \rho_J^\lambda \\ (I \leftrightarrow J) \in \Gamma_\lambda \setminus \mathfrak{t} \cup \Gamma_{\bar{\lambda}} \setminus \bar{\mathfrak{t}}}} \theta\left(\frac{x_I}{x_J}\right)}{\prod\limits_{c_I = c_J,\ \rho_i^\lambda = \rho_j^\lambda + 2} \theta\left(\frac{x_I}{x_J \hbar}\right)}$$

$$\mathcal{R}(\mathfrak{t}, \bar{\mathfrak{t}}) := \ldots \tag{52}$$

$$\mathcal{W}(\mathfrak{t}, \bar{\mathfrak{t}}) := \frac{\theta\left(\frac{a_1}{x_r} \hbar^{-v(I)} \prod_{I \in [r, \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)}{\theta\left(\prod_{I \in [r, \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)} \prod_{e \in \mathfrak{t}} \frac{\theta\left(\frac{x_{h(e)} \varphi^\lambda_{t(e)}}{x_{t(e)} \varphi^\lambda_{h(e)}} \hbar^{-v(I)} \prod_{I \in [h(e), \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)}{\theta\left(\prod_{I \in [h(e), \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)}$$

$$\cdot \frac{\theta\left(\frac{a_2 \hbar}{x_{\bar{r}}} \prod_{I \in [\bar{r}, \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)}{\theta\left(\prod_{I \in [\bar{r}, \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)} \prod_{e \in \bar{\mathfrak{t}}} \frac{\theta\left(\frac{x_{t(e)} \phi^\lambda_{h(e)}}{x_{h(e)} \phi^\lambda_{t(e)}} \prod_{I \in [h(e), \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)}{\theta\left(\prod_{I \in [h(e), \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)},$$

and N^σ, D^σ, $R^\sigma(t,\bar{t})$, $W^\sigma(t,\bar{t})$ are the functions obtained by permuting x_i's via $\sigma \in S_{n,k}$

in N, D, R, W.

We would like to consider its restriction to a fixed point $\nu \supset \lambda$; in other words, to evaluate $x_I = \phi_I^\nu$. The symmetrization ensures that Stab(λ) does not have poles for those values of x_I's, and hence Stab$(\lambda)_\nu$ is well defined.

For an individual term such as

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}\mathcal{R}^\sigma(\mathrm{t},\bar{\mathrm{t}})\mathcal{W}^\sigma(\mathrm{t},\bar{\mathrm{t}})$$

however, its restriction to ν is not well defined; in other words, it may depend on the order we approach the limit $x_I = \phi_I^\nu$. We discuss these properties in more details here. **Lemma 2.** The restriction to ν of

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}$$

is well defined, that is, does not depend on the ordering of evaluation.

Proof. The proof is the same as [62, Proposition 9]. **Lemma 3.** If

$$\frac{N^\sigma}{D^\sigma}\Big|_\nu = 0,$$

then σ fixes every box in $\bar{\nu}$.

Proof.when restricted toSuppose thatν. First note that$\frac{N^\sigma}{D^\sigma}\Big|_\nu = 0$. Then by LemmaN contains 2, N^σ contains no factors that vanish

σ

$\theta\left(\frac{a_2 h}{x}\right)$,

$=n-k, i\neq(n-k,k)$

$\sigma(i) \neq (n-k,k)$

We proceed by induction on the ρ-values of boxes in ν^{-}. Assume that σ fixes every box with $\rho \leq \rho_0$. Consider a box (a,b) with $\rho(a,b) = \rho_0 + 2$. Then either $(a+1,b)$ or $(a,b+1)$ lies in ν^{-}, and both of them have $\rho = \rho_0$. Suppose $\sigma^{-1}(a,b) = (a,b)$, then it is adjacent to neither $(a+1,b)$ nor $(a,b+1)$, and by induction hypothesis, $\rho_{\sigma^{-1}(a,b)} > \rho_{a+1,b}, \rho_{a,b+1}$. We see that N^{σ} contains the factor

$$\theta\left(\frac{x_{\sigma(a+1,b)}}{x_{\sigma(\sigma^{-1}(a,b))}}\right) = \theta \quad \pm 1,b \quad \text{or} \quad {}^{-}(a,b)) \quad \sigma(a,b+1)\hbar \quad = \quad \theta\, x$$

$$\sigma(\sigma\ 1 \quad \theta \quad x_{ab} \quad , x_a$$

$$x_{ab} \quad x x_{a,b+1}\hbar^{-}$$

which vanishes at ν. Hence, σ must fix (a,b) and the lemma holds.

Lemma 4. If

$$\cancel{ND}_{\sigma\sigma} = 0, \nu$$

then σ preserves the set of boxes of λ.

Proof. We proceed by induction on the diagonals. For the initial step, we need to show that the box with least content in λ, denoted by $(1,b)$, is fixed by σ. If $(1,b+1) \in \bar{\nu}$, then $(2,b+1) \in \bar{\nu}$, and σ fixes $(1,b)$ by Lemma 3. Now assume that $(1,b+1) \in \nu\backslash\lambda$. Let $X_1 = (1,b+1), X_2, \cdots$ be the boxes in the diagonal of $\nu\backslash\lambda$ with one less content than $(1,b)$. Since $\rho_{X_i} < 0 < \rho_{1,b}$, by Lemma 3 we always have in N_σ the factor

$$m\sigma(1,b) \quad \prod_{\geq 1} \theta\left(\frac{x_{\sigma(X_m)}}{x_{\sigma(1,b)}}\right) = \prod_{m\geq 1} \underline{\qquad x_{X_m}} \ \theta, x$$

which vanishes at ν unless $\sigma(1,b)$ has no box to the left of it. This implies $\sigma(1,b) = (1,b)$.

Now assume that σ preserves the l-th diagonal of λ. Consider the $(l + 1)$-th diagonal. There are several cases.

- Both the l-th and $(l + 1)$-th diagonals of $\nu\backslash\lambda$ are empty. The lemma holds trivially for $l +$ 1.
- $\nu\backslash\lambda$ is empty in the l-th diagonal but has one box X_1^{l+1} in the $(l + 1)$-th diagonal.
- In this case, let $Y_1^l, Y_2^l, \cdots$ be boxes in the l-th diagonal of λ. In N^σ, there is the theta factor

$$1 \left(\frac{}{x} \right) \quad m \; 1 \geq \quad \theta \quad \theta \quad \sigma(X_1^+) \; - \quad \prod_{\geq \; l \; 1} x_{\sigma(Y_m^l)} h =$$

$$\sigma(X_1^{l+1})^- \; xY_{ml} \qquad , xh \qquad m$$

$$\sigma(X_1^{l+1}) = X_1^{l+1}$$

- The l-th diagonal of $\nu\backslash\lambda$ is nonempty.
- In this case, let $X_1^l, X_2^l, \cdots$ be the boxes in the l-th diagonal of $\nu\backslash\lambda$, and consider a general box Y in the $(l + 1)$-th diagonal of λ. We have in N^σ the factor

$$\theta \left(\frac{x_{\sigma(X_m^l)}}{x_{\sigma(Y)}} \right)_{\geq 1} = \prod_{m \geq 1} \left. \frac{}{} \right) \prod \theta \frac{xX_{ml}}{} .$$

$$x\sigma(Y)\; m$$

If $\sigma(Y) \notin \lambda$, then it must be in $\nu\backslash\lambda$. Let Z be the box to the left of $\sigma(Y)$, which must either also lie in $\nu\backslash\lambda$ and has to be one of those X_i^l's, or lie in λ. In the former case, the product vanishes at ν; in the latter case, we have another factor $\theta \frac{x^Z}{}$, which also vanishes at ν.

$x_{\sigma(Y)}$

The lemma holds by induction.

Consider the subgroups in $S_{n,k}$ defined as

$S_\nu\backslash\lambda$:= {σ | σ fixes each box in$\lambda \cup \bar{\nu}$}, S_{λ^-} := {σ | σ fixes each box in λ}.

Lemma 5. If

$\in$ $\overline{N_\sigma}^\sigma = 0,$

D ν

then σ $S_\nu\backslash\lambda$.

Proof. The proof is exactly the same as Lemma 3, by induction on the ρ-values of boxes.

Now we would like to restrict the formula to the fixed point ν, in a specific choice of limit. We call the following the *row limit* for λ: first take

$x_I = x_J$

φ_I^ν

By previous lemmas, we see that only $\sigma \in S_\nu\backslash\lambda$ survives. Moreover, under the row limit, one can see that only one tree t (which contains all rows of λ) survives, and one can write all terms independent of trees in λ:

$$R^\sigma(t,t^-) = (-1)^{m(\lambda)} R^\sigma(t^-), \qquad W^\sigma(t,t^-) = W^\sigma(t^-),$$

where $m(\lambda) = \sum_{l\in\mathbb{Z}} (\mathsf{d}_l(\lambda) - 1)$, and

$$R(\bar{t}) := \frac{\displaystyle\theta_{\substack{c_I+1=c_J,\ \rho_I=\rho_J+1 \\ (I\leftrightarrow J)\ \bar{\lambda}\backslash t}} \frac{x_J\hbar}{x_I} \quad \theta_{\substack{c_I+1=c_J,\ \rho_I+1=\rho_J \\ (I\leftrightarrow J)\ \bar{\lambda}\backslash t}} \frac{x_I}{x_J}}{\displaystyle\theta_{\substack{c_I=c_J,\ \rho_I=\rho_J+2 \\ I,J\ \lambda}} \frac{x_I}{x_J\hbar}},$$

$$W(\bar{t}) := \frac{\theta\left(\hbar \prod_{I \in [\bar{r},\bar{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)}{\theta\left(\prod_{I \in [\bar{r},\bar{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)} \prod_{e \in \bar{t}} \frac{\theta\left(\frac{x_{t(e)}\phi^{\lambda}_{h(e)}}{x_{h(e)}\phi^{\lambda}_{t(e)}} \prod_{I \in [h(e),\bar{t}]} z_{c_I}^{-1}\right)}{\theta\left(\prod_{I \in [h(e),\bar{t}]} z_{c_I}^{-1}\right)}.$$

a_2 $h^{-v(I)}$ x

θ $h^{-v(I)}$

For N^{σ}, D^{σ}, and $\sigma \in S_{\bar{\lambda}}$, we have the factorization

$$\frac{\mathcal{N}^{\sigma}}{\mathcal{D}^{\sigma}} = \frac{\mathcal{N}_{\lambda}}{\mathcal{D}_{\lambda}} \cdot N'^{-}_{\lambda} \cdot \frac{\mathcal{N}_{\bar{\lambda}}}{\mathcal{D}^{\sigma}_{\bar{\lambda}}},$$

where

$$\Theta(N'^{-}_{\lambda}) = (-1)^{k\,n\;k\,-1}$$

$$(-)\frac{\prod_{\substack{c_I=k\\ I\in\lambda}} \theta\left(\frac{a_1}{x_I\hbar}\right) \prod_{\substack{c_I=n-k\\ I\in\bar{\lambda}}} \theta\left(\frac{a_2\hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J\\ I\in\lambda,\, J\in\bar{\lambda}}} \theta\left(\frac{x_J\hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J\\ I\in\bar{\lambda},\, J\in\lambda}} \theta\left(\frac{x_I}{x_J}\right)}{\prod_{\substack{c_I=c_J\\ I\in\lambda,\, J\in\bar{\lambda}}} \theta\left(\frac{x_I}{x_J}\right) \theta\left(\frac{x_I}{x_J\hbar}\right)},$$

$$\mathcal{N}^{\sigma}_{\bar{\lambda}} = \prod_{\substack{=n-k,\, i\in\bar{\lambda}\\ \neq(n-k,k)}} \theta\left(\frac{a_2\hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J,\, \rho_I>\rho_J\\ (I\leftrightarrow J)\notin\Gamma_{\bar{\lambda}},\, I,J\in\bar{\lambda}}} \theta\left(\frac{x_{\sigma(J)}\hbar}{x_{\sigma(I)}}\right) \prod_{\substack{c_I+1=c_J,\, \rho_I<\rho_J\\ (I\leftrightarrow J)\notin\Gamma_{\bar{\lambda}},\, I,J\in\bar{\lambda}}} \theta\left(\frac{x}{x_{\sigma(I)}}\right)$$

i

$$N_{\lambda} = \prod_{\substack{=k,\, I\in\lambda\\ \neq(1,1)}} \theta\left(\frac{x_I}{a_1}\right) \prod_{\substack{c_I+1=c_J,\, \rho_I>\rho_J\\ (I\leftrightarrow J)\notin\Gamma_{\lambda},\, I,J\in\lambda}} \theta\left(\frac{x_J\hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J,\, \rho_I<\rho_J\\ (I\leftrightarrow J)\notin\Gamma_{\lambda},\, I,J\in\lambda}} \theta\left(\frac{x_I}{x_J}\right),$$

c_I

$$\mathcal{D}^{\sigma}_{\bar\lambda} = \prod_{\substack{=c_J,\,\rho_I>\rho_J\\ I,J\in\bar\lambda}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right) \prod_{\substack{c_I=c_J,\,\rho_I>\rho_J+2\\ I,J\in\bar\lambda}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}\hbar}\right), \quad \mathcal{D}_{\lambda} = \prod_{\substack{c_I=c_J,\,\rho_I>\rho_J\\ I,J\in\lambda}} \theta$$

xI xI $=c_J,\,\rho_I>\rho_J+$ c_I $I,J\in\lambda$ $2\theta\, J$. $c_I x_J x\, h^{-}$

In summary, we have the following refined formula:

Proposition 5. For any choice of limit $x_i \to \phi_i^{\nu}$ for $i \in \lambda^{-}$, we have Stab$(\lambda)_\nu$

$$= \epsilon(\lambda)\ \Theta(N'^{-}_{\lambda})\ {}_{\nu} \cdot \%\ \ S_\nu\ N^{\sigma}\ \frac{\bar\lambda}{\mathcal{D}^{\sigma}_{-}} \mathcal{R}^{\sigma}(\bar{t})\mathcal{W}^{\sigma}(\bar{t})$$

,

$\sigma\in\ \backslash\lambda,t\ \ \lambda\ \ \ \nu$

$$\epsilon(\lambda) := (-1)^{m(\lambda)} \prod_{\substack{c_I+1=c_J\\ (I\ \ J)\notin\Gamma_\lambda,\,I,J\in\lambda}} (-1).$$

$\leftrightarrow$

As a corollary, we have the following identity in elliptic cohomology:

$$\boxed{\text{Stab}\,(\lambda) = (\lambda)(\ \ \%'^{-}_{\lambda})\ {}_{\sigma\,S_{\bar\lambda},\bar t}\ \frac{N^{\sigma}_{\bar\lambda}}{D^{\sigma}_{\bar\lambda}} R^{\sigma}(\bar t) W^{\sigma}(\bar t).} \tag{53}$$

Proof. Computations above show that

$$\text{Stab}(\lambda)_\nu = (-1)^{m(\lambda)} \frac{\lambda}{\mathcal{D}} \cdot \Theta(N'^{-}_{\lambda})\ \ N_{\lambda\nu}\ \%\ {}_{\nu\,\sigma\,S_\nu\,\lambda,t}\ ND_{\underline{\lambda}\lambda\sigma}^{-\,-\sigma} R_\sigma(\bar t)W_\sigma(\bar t)_\nu.$$

$\in\ \backslash\ ^{-}$

The refined formula is proved by the following lemma.

Lemma 6.

$$\mathrm{ND}\lambda\underline{\lambda}v = . \prod_{\substack{c_I+1=c_J \\ (I \quad J)\notin\Gamma_\lambda, I,J\in\lambda}} (-1)$$

Proof. Let $t_1 = h^{-}{}^{-1}$, $t_2 = 1$, $x_I \to x_I/a_1$ in [62, Proposition 10]. We have

$$\mathrm{ND}\lambda\underline{\lambda}v = \mathrm{ND}\lambda\underline{\lambda}\lambda = . \prod_{\substack{c_I+1=c_J,\, \rho_I<\rho_J \\ (I\leftrightarrow J)\notin\Gamma_\lambda, I,J\in\lambda}} (-1) \cdot \prod_{\substack{c_I+1=c_J,\, \rho_I>\rho_J \\ (I\leftrightarrow J)\notin\Gamma_\lambda, I,J\in\lambda}} (-1)$$

6 The mother function

6.1 Bijection on fixed points

Recall that the set$\mathbf{p} = \{\mathbf{p}_1,...,\mathbf{p}_k\}$ in the setX^{T} consists of$\mathbf{n} = \{1,2,n!.../((,nn-\}$. On the dual side, the set$k)!k!)$ fixed points corresponding to$\mathsf{T}$ consists of the$k$-subsets

(X')

same number of fixed points, labeled by Young diagrams λ, which fit into the rectangle $\mathsf{R}_{n,k}$ with dimensions $(n - k) \times k$. There is a natural bijection

$$(X')^{\mathsf{T}'} \xrightarrow{\sim}$$

defined in the following way.

$\lambda \in (X')$

We note that this bijection preserves the standard dominant ordering on the set of fixed points. For instance in the case $n = 4, k = 2$ the fixed points on X are labeled by 2-subsets in $\{1,2,3,4\}$, which are ordered as follows:

$$X^{\mathsf{T}} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

The fixed points on X correspond to Young diagrams, which fit into 2×2 rectangle. The bijection above

gives the following ordered list of fixed points in X:

$$(X')^{\mathsf{T}'} = \{\emptyset$$

parameters $u_i/u_{i+1},\hbar$ and the Kähler parameter z. The coordinates on $= \times \mathsf{O}$ $_\lambda = E_{\mathsf{T}} \times E_{\mathrm{Pic}}(X')$Recall that the coordinates on the abelian variety $\mathsf{O}_{\mathbf{p}}\, E_{\mathsf{T}}\, E_{\mathrm{Pic}(X)}$ are the equivariant

are the equivariant parameters $a_1/a_2,\hbar$ and Kähler parameters $z_1,\ldots,z_{n-1}$. Let us consider an isomorphism

identifying the equivariant and Kähler tori on the dual sides

$$\kappa : \mathsf{T} \to \mathsf{K}, \mathsf{K} \to \mathsf{T},$$

defined explicitly by

Recall that the stability and chamber parameters for X are defined by the following vectors:

$a_1 \to z\hbar^{k-1}, a_2$ $\hbar \to \hbar$

$$\sigma = (1,2,\ldots,n) \in \mathrm{Lie}_{\mathbb{R}}(\mathsf{A}), \qquad \theta = (-1) \in \mathrm{Lie}_{\mathbb{R}}(\mathsf{K}).$$

Using the map (55), we find that

$$\kappa^{-1}(\sigma) = (-1,\ldots,-1) = -\theta' \quad d. \qquad \kappa^{-1}(\theta) = (1) = -\sigma' \quad d,$$

We see that the isomorphisms κ is chosen such that the stability parameters are matched to chamber parameters on the dual side.

6.3 Mother function and 3D mirror symmetry

For the $(\mathsf{T} \times \mathsf{T}')$-variety $X \times X$, we consider equivariant embeddings defined by fixed points:

$$X = X \times \{\lambda\} \xrightarrow{i_\lambda} X \times X \xleftarrow{i_{\mathbf{p}}} \{\mathbf{p}\} \times X = X. \tag{56}$$

We consider $X \times \{\lambda\}$ as a $\mathsf{T} \times \mathsf{T}$ variety with trivial action on the 2nd component. This gives

$$\mathrm{Ell}_{\mathsf{T}\times\mathsf{T}}(X \times \{\lambda\}) = \mathrm{Ell}_{\mathsf{T}}(X) \times E_{\mathsf{T}} = \mathsf{E}_{\mathsf{T}}(X),$$

where in the last equality we used the isomorphism κ to identify $E_{\mathrm{Pic}(X)} = E_{\mathsf{T}}$. Similarly,

$$\mathrm{Ell}_{\mathsf{T}}(\{\mathbf{p}\} \times X') = \mathsf{E}_{\mathsf{T}'}(X').$$

We conclude that $\mathsf{T} \times \mathsf{T}$-equivariant embeddings (56) induce the following maps of extended elliptic cohomologies:

$$\mathsf{E}_{\mathsf{T}}(X) \xrightarrow{i^*_\lambda} \mathrm{Ell}_{\mathsf{T}\times\mathsf{T}'}(X \times X') \xleftarrow{i^*_{\mathbf{p}}} \mathsf{E}_{\mathsf{T}'}(X').$$

Here is our main result.

Theorem 5.

- There exists a line bundle $\mathfrak{M}$ on $\mathrm{Ell}_{\mathsf{T}\times\mathsf{T}}(X \times X)$ such that

$$(i^*_\lambda)^*(\mathfrak{M}) = \mathfrak{M}(\mathbf{p}), \qquad (i^*_{\mathbf{p}})^*(\mathfrak{M}) = \mathfrak{M}'(\lambda).$$

  where $\mathbf{p} = \mathrm{bj}(\lambda)$.

- There exists a holomorphic section $\mathfrak{m}$ (the mother function) of $\mathfrak{M}$, such that

$$(i^*_\lambda)^*(\mathfrak{m}) = \mathbf{Stab}(\mathbf{p}), \qquad (i^*_{\mathbf{p}})^*(\mathfrak{m}) = \mathbf{Stab}'(\lambda),$$

We will prove this theorem in Section 9. This theorem implies that (up to normalization by diagonal elements) the restriction matrices of elliptic stable envelopes of X and X are related by transposition:

(Similarly with notations in Definition 2, we denote $T'_{\lambda,\mu} := \mathrm{Stab}'(\lambda)\big|_{\mathcal{O}'_\mu}$; we also use the simplified notation $(-)|_{\mathbf{p}}$ for $(-)|_{\mathcal{O}_{\mathbf{p}}}$.)

Corollary 3. The restriction matrices of the elliptic stable envelopes for X and X in the basis of fixed points are related by

$$T_{\mathbf{p},\mathbf{p}} T'_{\lambda,\mu} = T'_{\mu,\mu} T_{\mathbf{q},\mathbf{p}}, \tag{57}$$

where $\mathbf{p} = \mathsf{bj}(\lambda)$, $\mathbf{q} = \mathsf{bj}(\mu)$ and parameters are identified by (55).

Proof. For fixed points $\lambda, \mu \in (X')^{\mathsf{T}}$, let $\mathbf{p} = \mathsf{bj}(\lambda)$, $\mathbf{q} = \mathsf{bj}(\mu)$ denote the corresponding fixed points in X^{T}. Note that $(X \times X)^{\mathsf{T}\times\mathsf{T}'} = X^{\mathsf{T}} \times (X')^{\mathsf{T}}$. Let us consider the point $(\mathbf{p},\mu)$ from this set. By Theorem 5, we have

$$\mathbf{Stab}(\mathbf{q})_{\mathbf{p}} = \mathfrak{m}|_{(\mathbf{p},\mu)} = \mathbf{Stab}(\lambda)_{\mu}.$$

By definition (26, 38), we have $\mathbf{Stab}^{(\lambda)}{}_{\mu} = \Theta'_{\lambda} T'_{\lambda,\mu}$, $\mathbf{Stab}^{(\mathbf{q})}{}_{\mathbf{p}} = \Theta_{\mathbf{q}} T_{\mathbf{q},\mathbf{p}}$. In the standard normalization of elliptic stable envelope, the diagonal elements of the restriction matrix are given by normal bundles of repelling part of the normal bundles:

$$T_{\mathbf{p},\mathbf{p}} = \Theta(N^{-}_{\mathbf{p}}), \qquad T'_{\lambda,\lambda} = \Theta(N'^{-}_{\lambda}),$$

with $N_{\mathbf{p}^-}$ and N'^{-}_{λ} as in (19), (35). We see that $\Theta'_{\lambda} = T_{\mathbf{p},\mathbf{p}}$, $\Theta_{\mathbf{q}} = T'_{\mu,\mu}$.

As we will see in Section 8, the equality (57) encodes certain infinite family of highly nontrivial identities for theta function.

7 The mother function in case $k = 1$

Before we prove the Theorem 5 in general, it might be very instructive to check its prediction in the case $k = 1$. In this case, the formulas for stable envelopes for X and X are simple enough to compute the mother function explicitly.

7.1 Explicit formula for the mother function

In the case $k = 1$, both X and X are hypertoric, $X = T^*\mathbb{P}^{n-1}$, and X is isomorphic to the A_{n-1} surface (resolution of singularity $\mathbb{C}^2/\mathbb{Z}_n$). The map κ has the following form:

$$\kappa : \quad \hbar \mapsto \frac{1}{\hbar}, \quad \frac{a_1}{a_2} \to z, \quad z_1 \to \frac{u_1}{u_2}, \quad \cdots, \quad z_{n-1} \to \frac{u_{n-1}}{u_n}. \tag{58}$$

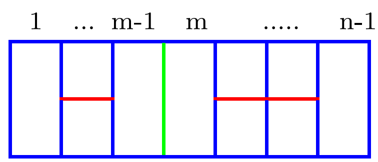


Fig. 5. The tree for the fixed point representing $[1^{m-1}] \subset R_{n,1}$.

We denote by $y = y_1$ the Chern root of the tautological bundle on X and by $x_i = x_{i,1}$, $i = 1,\cdots,n-1$ the Chern roots of tautological bundles on X. For symmetry, we also denote by $x_0 = a_1$ and $x_n = a_2$. In these notations, we have

Theorem 6. In the case $k = 1$, the mother function equals:

$$\mathrm{m} = (-1)^n \prod_{i=0}^{n} \theta\left(\frac{x_i \hbar}{x_{i-1}}\right) \ldots u_i y. \tag{59}$$

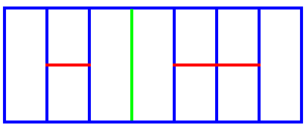
7.2 Stable envelope for X

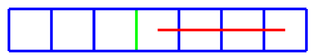
First, let us consider the elliptic stable envelopes of the fixed points in X. In the case $k = 1$, the fixed points on the variety X are labeled by Young diagrams inside the $1 \times (n-1)$ rectangle. There are exactly n such Young diagrams $\lambda_m = [\underbrace{1,1,...,1}_{m-1}]$ with $m = 0,\cdots,n-1$. To compute the stable envelope of λ_m, we need to consider trees in λ_m m and $\bar{\lambda}_m$. Obviously, there is only one possible tree in this case, see Figure 5:

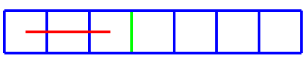
For (41), we obtain

$$\mathbf{S}^{Ell}_{\lambda_m,n,1} = (-1)^{n-1}\theta\left(\frac{x_1}{a_1}\right) \times \theta\left(\frac{a_2\hbar}{x_{n-1}}\right) \prod_{i=1}^{m-2}\theta\left(\frac{x_{i+1}}{x_i}\right) \times \theta\left(\frac{x_m}{x_{m-1}\hbar}\right) \times \prod_{i=m}^{n-2}\theta\left(\frac{x_{i+1}\hbar}{x_i}\right).$$

To compute the Kähler part of the stable envelope (44), we note that $\beta^{(2)}_{\lambda_m} = 0$ for all boxes of $R_{n,1}$ and $\beta^{(1)}_{\lambda}$ is equal to zero for all boxes except the box $(m-1,1)$ where it is equal to 1. Thus, $\beta_\lambda((i,1)) = \delta_{i,m-1}$.

$\mathbf{W}^{Ell}$ 



$= \mathbf{W}^{Ell}$  $\times\, \mathbf{W}^{Ell}$

$$= \varphi\left(\frac{a_1}{x_1}, \hbar^{-1}\prod_{i=1}^{m-1} z_i^{-1}\right)\prod_{i=1}^{m-2}\varphi\left(\frac{x_{i+1}}{x_i}, \hbar^{-1}\prod_{j=i+1}^{m-1} z_j^{-1}\right) \times \varphi\left(\frac{a_2\hbar}{x_{n-1}}, \prod_{i=m}^{n-1} z_i^{-1}\right)\prod_{i=m}^{n-2}\varphi\left(\frac{x_{i+1}\hbar}{x_i}, \prod_{j=i}^{m} z_j^{-1}\right).$$

We conclude that

$$\mathrm{Stab}'(\lambda_m) = \mathbf{S}^{Ell}_{\lambda_m,n,1}\mathbf{W}^{Ell}_{\lambda_m,n,1} = (-1)^{m} \frac{\prod_{i=1}^{m-1}\theta\left(\frac{x_i}{x_{i-1}}\hbar^{-1}\prod_{j=i}^{m-1} z_j\right)\times\theta\left(\frac{x_m}{x_{m-1}\hbar}\right)\times\prod_{i=m+1}^{n}\theta\left(\frac{x_i}{x_{i-1}}\hbar\prod_{j=m}^{i-1} z_j\right)}{\prod_{i=1}^{m-1}\theta\left(\hbar^{-1}\prod_{j=i}^{m-1} z_j\right)\times\prod_{i=m+1}^{n}\theta\left(\prod_{j=m}^{i-1} z_j\right)}, \tag{60}$$

where we denote $x_0 = a_1$ and $x_n = a_2$. The restriction of stable envelope to fixed points is given by evaluation of Chern roots (32). In this case, the restriction to m-th fixed point is given by

$$\{x_1 = a_1, \quad \cdots, \quad x_{m-1} = a_1, \quad x_m = a_2\hbar^{n-m}, \quad \cdots, \quad x_{n-1} = a_2\hbar\}. \tag{61}$$

Thus, for the diagonal matrix elements of restriction matrix, we obtain

$$T'_{\lambda_m,\lambda_m} = \mathrm{Stab}'(\lambda_m)\big|_{\lambda_m} = (-1)^n\theta\left(\frac{a_2}{a_1}\hbar^{n-m+1}\right).$$

Finally, the stable envelope written in terms of parameters of X, that is, all with the parameters substituted by (58), equals:

$$\mathrm{Stab}(\lambda_m) = (-1)^n \frac{\prod_{i=1}^{n}\theta\left(\frac{x_i}{x_{i-1}\hbar}\frac{u_i}{u_m}\right)}{\prod_{i=1}^{m-1}\theta\left(\frac{u_i}{u_m\hbar}\right)\prod_{i=m+1}^{n}\theta\left(\frac{u_i}{u_m}\right)}, \tag{62}$$

with diagonal elements of the restriction matrix:

$$T'_{\lambda_m,\lambda_m} = (-1)^n\theta(z^{-1}\hbar^{-n+m-1}). \tag{63}$$

7.3 Stable envelope for X

Under the bijection of fixed points, we have $\mathrm{bj}(\lambda_m) = \{m\} \subset \mathbf{n}$. From (25) for the stable envelope of X in the case $k = 1$, we obtain

$$\mathrm{Stab}(m) = \prod_{i=1}^{m-1}\theta(yu_i\hbar) \times \frac{\theta(yu_m z^{-1}\hbar^{-n+m-1})}{\theta(z^{-1}\hbar^{-n+m-1})} \times \prod_{i=m+1}^{n}\theta(yu_i). \tag{64}$$

The restriction to the m-th fixed point is given by substitution $y = u_m^{-1}$. Thus, for diagonal of restriction matrix, we obtain

$$= \mathtt{Stab}(m)|_{m} = \prod_{i=1}^{m} \theta\left(\frac{u_i}{u_m}\right.$$ —— -1 $\frac{n\ ui}{}$

$T_{m,m}\theta$. (65)

h $i{=}m{+}1$ um

7.4 Stable envelopes are restrictions of the mother functions

We are now ready to check Theorem 6 in the k = 1 case. Note that (65) gives exactly the denominator of (62) and we obtain

Stab(λ_m) = $T_{m,m}$ Stab(λ_m) $$= (-1)^n \prod_{i=}^{n} \theta\left(\frac{x_i}{x_{i-1}}\right.$$ ______ ___u_i =

m| m, h um

1

where m is defined by (59) by m| m we denotes the restriction of this class to the m-th fixed point on X, that is, the evaluation $y = u_{-m}{}^{1}$. Similarly, we note that (63) is exactly the denominator of (64) and we obtain

$$=(-1)^n \prod_{i=1}^{m} \theta\begin{pmatrix} yu_i \\ \hbar \end{pmatrix} \times \theta(yu_m z^{-1}\hbar^{-n+m-1})$$ -1 n

Stab$(m) = T'_{\lambda_m,\lambda}$ ${}_m$Stab(m) ——

$\times\ \theta(yu_i)$= m|λ_m,

$i{=}m{+}1$

where m|λ_m denoted the restriction to λ_m on X, that is, the substitution (61) (one should not forget to substitute $h^{-} \to h^{-\,-1}$ in (61), as all formulas written in terms of the parameters of X). Theorem 6 for k = 1 is proven.

8 Simplest Non-abelian Case n = 4,k = 2

8.1 Identification of parameters and fixed points

In the case $k = 1$ considered in the previous section, the matrix elements of restriction matrices $T_{\lambda,\mu}$ and $T_{\mathbf{p},\mathbf{q}}$ factorize into a product of theta functions and Theorem 5 can be proved by explicit computation. In contrast, when $k \geq 2$, the matrix elements are much more complicated. In particular, Theorem 5 (and Corollary 3) gives a set of very non-trivial identities satisfied by the theta functions. In this section, we consider the simplest example with $n = 4$ and $k = 2$. In this case, the fixed points on X are labeled by 2-subsets in {1,2,3,4}. We consider the basis ordered as

$$X^{\mathsf{T}} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

The fixed points on X' correspond to Young diagrams, which fit into a 2×2 square. The bijection on the fixed points described in the Section 6.1 gives the corresponding points on X (in the same order):

$$(X')^{\mathsf{T}'} = \{\emptyset, [1], [1,1], [2], [2,1], [2,2]\}.$$

The identification of Kähler and equivariant parameters (55) in this case reads

$$\frac{a_1}{a_2} \mapsto z\hbar, \quad \hbar \mapsto \hbar^{-1}, \quad z_1 \mapsto \frac{u_1 \hbar}{u_2}, \quad z_2 \mapsto \frac{u_2}{u_3}, \quad z_3 \mapsto \frac{u_3}{u_4 \hbar}. \tag{66}$$

We will denote a fixed point simply by its number $m = 1, \cdots, 6$. For example, $T_{2,3}$ will denote the coefficient of the restriction matrix for X given by $T_{\{1,3\},\{1,4\}}$. Similarly, $T'_{1,4}$ $_{\emptyset,[2]}$ on the dual side X'.

8.2 Explicit expressions for stable envelopes

Using (25), (26), (48), and (38), one can compute explicit expressions for stable envelopes. We list two of them here for example (after applying κ (66)):

$$\frac{\theta\left(\frac{y_1 u_1}{\hbar}\right)\theta\left(\frac{y_1 u_2}{\hbar}\right)\theta\left(\frac{y_1 u_3}{z\hbar}\right)\theta(y_1 u_4)\theta\left(\frac{y_2 u_1}{\hbar}\right)\theta\left(\frac{y_2 u_2}{\hbar}\right)\theta\left(\frac{y_2 u_3}{\hbar}\right)\theta\left(\frac{y_2 u_4}{z\hbar^2}\right)}{\theta\left(\frac{y_1}{y_2}\right)\theta\left(\frac{y_2}{y_1 \hbar}\right)} + (y_1 \leftrightarrow y_2)$$

$$= \frac{\theta\left(\frac{u_4}{u_1}\right)\theta\left(\frac{u_4}{u_2}\right)\theta\left(\frac{a_2 u_3 u_4}{\hbar x_{2,2} u_2 u_1}\right)\theta\left(\frac{x_{2,2} u_3}{x_{1,2} \hbar u_1}\right)\theta\left(\frac{x_{2,2} u_4}{x_{2,1} u_3}\right)\theta\left(\frac{x_{1,2} u_3}{x_{1,1} u_2}\right)\theta\left(\frac{x_{2,1}}{x_{1,1} \hbar}\right)\theta\left(\frac{\hbar a_1}{x_{1,1}}\right)\theta\left(\frac{\hbar a_1}{x_{2,2}}\right)\theta\left(\frac{a_2}{x_{1,1} \hbar}\right)}{\theta\left(\frac{u_3 u_4}{u_1 u_2}\right)\theta\left(\frac{u_4}{u_3}\right)\theta\left(\frac{x_{1,1}}{x_{2,2}}\right)\theta\left(\frac{\hbar x_{1,1}}{x_{2,2}}\right)}$$

$$+ \frac{\theta\left(\frac{u_3}{u_1}\right)\theta\left(\frac{u_3}{u_2}\right)\theta\left(\frac{a_2 u_3 u_4}{\hbar x_{2,2} u_1 u_2}\right)\theta\left(\frac{x_{2,2} u_4}{x_{1,2} \hbar u_1}\right)\theta\left(\frac{x_{1,2} u_4}{x_{1,1} u_2}\right)\theta\left(\frac{x_{1,1} u_4 \hbar}{x_{2,1} u_3}\right)\theta\left(\frac{x_{2,2}}{x_{2,1}}\right)\theta\left(\frac{\hbar a_1}{x_{1,1}}\right)\theta\left(\frac{\hbar a_1}{x_{2,2}}\right)\theta\left(\frac{a_2}{\hbar x_{1,1}}\right)}{\theta\left(\frac{u_3 u_4}{u_1 u_2}\right)\theta\left(\frac{u_4}{u_3}\right)\theta\left(\frac{x_{1,1}}{x_{2,2}}\right)\theta\left(\frac{\hbar x_{1,1}}{x_{2,2}}\right)}$$

$$+ \left(x_{1,1} \leftrightarrow x_{2,2}\right),$$

$0 \quad 1 \quad m = a_2.$

κ : denotes T

8.2

Stab(6) =

Stab(1)

where we denote $x = a$,x

8.3 Theorem 5 in case $n = 4, k = 2$

Corollary 1 means that the functions above are related by the following identities:

$$\mathbf{Stab}(a)\ _b = \mathbf{Stab}'(b)\ _a,$$

where the restriction to the fixed points on X is given by substitution of variables y_i (15). The restrictions to the fixed points on X are defined by (32) (together with identification of parameters (66)!). We only compute non-zero restrictions and only those $\mathbf{Stab}(a)_b$ with $a = b$ (the case $a = b$ is trivial).

For example:

Case $a = 2, b = 1$:

$$\mathbf{Stab}'(1)\big|_2 = \theta\left(z\hbar^3\right)\theta(\hbar)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{zu_2\hbar^3}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right),$$

$$\mathbf{Stab}(2)\big|_1 = \theta\left(z\hbar^3\right)\theta(\hbar)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{zu_2\hbar^3}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right).$$

We see that for $(a,b) = (2,1)$ the two are trivially equal as product of theta functions, which also happens in cases (a,b)=(3,2),(4,2),(5,2),(5,3),(4,3),(6,3),(5,4),(6,5). However, the identity is nontrivial for the remaining cases (a,b) = (3,1),(4,1),(5,1), (6,1),(6,2),(6,4). Case $a = 3, b = 1$:

$$\mathbf{Stab}(1)_3 = \frac{\theta(h)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{zu_3h^2}{u_4}\right)\theta\left(\frac{zu_2h^3}{u_3}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(h) - \theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{hu_4}{u_3}\right)\theta\left(zh^2\right)\theta\left(\frac{zu_2h^3}{u_4}\right)}{\theta\left(\frac{u_3}{u_4}\right)},$$

$$\mathbf{Stab}(3)\big|_1 = \theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta(\hbar)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{z\hbar^2u_2}{u_4}\right)\theta\left(zh^3\right).$$

Case $a = 4, b = 1$:

$$\mathbf{Stab}^{(1)}\big|_4 = \theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar)\,\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(z\hbar^3\right)\theta\left(\frac{z\hbar^2}{u_3}\right.$$

$$u_1,$$

$$\frac{\theta(\hbar)\theta\ \frac{u_1}{u_4}\ \theta\ \frac{u_2}{u_4}\quad \theta\ \frac{u_1}{u_3}\ \theta\ \frac{z\hbar\ u_1}{u_2}\ \theta(\hbar)\theta\ \frac{zu_2\hbar}{u_3}\ -\theta\ \frac{u_2}{u_3}\ \theta\ \frac{\hbar u_2}{u_1}\ \theta\ \frac{zu_1\hbar}{u_3}\ \theta\ z^{-}}{u \atop u_2} \quad {2 \atop 3}$$

$3\ h2$

$$\mathbf{Stab}(4)\ 1 = \qquad \theta\ \frac{1}{\qquad}.$$

Case $a = 5, b = 1$:

$$'(1) \quad \frac{\hbar)\ -\theta\ \frac{z^2\hbar\ u_1}{u_4}\ \theta\ \frac{\hbar u_2}{u_4}\ \theta\ \frac{\hbar u_4}{u_3}\ \theta\ z\hbar^2\ \theta\ \frac{u_1}{u_3}\ \theta\ \frac{^2u_2}{u_3}\ +\theta\ \frac{z\hbar\ u_1}{u_3}\ \theta\ \frac{\hbar u_2}{u_3}\ \theta\ \frac{zu_3\hbar}{u_4}\ \theta\ \frac{u_1^2}{u_4}\ \theta\ \frac{u_2}{u_4}\ \theta(^{-}\ \hbar)}{\frac{3}{u_4}\ \theta(}$$

$$\mathbf{Stab}\ 5 = \qquad \theta$$

u ,

$$\frac{\theta(\hbar)\ \theta\ \frac{u_1}{u_3}\ \theta\ \frac{u_1}{u_4}\ \theta\ \frac{z\hbar\ u_1}{u_2}\ \theta\ \frac{\hbar u_2}{u_3}\ \theta\ \frac{z\hbar\ u_2}{u_4}\ \theta(\hbar)-\theta\ \frac{u_2}{u_3}\ \theta\ \frac{u_2}{u_4}\ \theta\ \frac{\hbar u_1}{u_3}\ \theta\ \frac{z\hbar\ u_1}{u_4}\ \theta\ \frac{\hbar u_2}{u_1}\ \theta\ z^{-2}}{u_1}$$

$2 \qquad 2 \qquad 2 \qquad h$

$$\mathbf{Stab}(5)\ 1 = \qquad \theta\ u2 \qquad .$$

Case $a = 6, b = 1$:

$$\mathbf{Stab}'(1)\big|_6 = \frac{1}{\theta\left(\frac{u_1u_2}{u_3u_4}\right)\theta\left(\frac{u_3}{u_4}\right)}\left(\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{u_3\hbar}{u_4}\right)\theta\left(z\hbar^2\right)\theta\left(\frac{zu_1u_2}{u_3u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar)\right.$$

$$-\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta \quad z\hbar^2 \quad \theta(\hbar)\theta \quad \frac{z\hbar u_1u_2}{u_3u_4} \quad \theta \quad \frac{u}{u_3}$$

$$\left.-\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{z\hbar^2u_2u_1}{u_3u_4}\right)\theta\left(\hbar^2\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_3}{u_4}\right)\theta(z\hbar)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\right),$$

$$\frac{\theta(\hbar)^2\left(\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_2}\right)\theta\left(\frac{z\hbar u_1}{u_3}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{z\hbar^2u_2}{u_4}\right)-\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{z\hbar^2u_1}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\right)}{\theta\left(\frac{u_1}{u_2}\right)}$$

$\underline{1}\theta\ u_{\underline{2}}$

$u3$

Stab(6) 1 = .

Case $a = 6, b = 2$:

$$\mathbf{Stab}^{(2)}{}_6 = \theta \quad \frac{u_3\hbar}{u_2} \quad \theta(\hbar)\theta \quad \frac{\hbar u_1}{u_2}\theta \quad \frac{z\hbar u_1}{u_4}\theta \quad z\hbar^2 \quad \theta \quad \frac{u_3}{u_4}$$

h,

2 2 h $u\underline{3}$

$$\mathbf{Stab}(6)\ 2 = \theta(\ \frac{\hbar)\theta \ \frac{\hbar u_1}{u_2}\ \theta\ \frac{u_3\hbar}{u_2}\ \theta\ \frac{u_1}{u_4}\ \theta\ \frac{z\hbar u_1}{u_3}\ \theta\ \frac{zu_3\hbar}{u_4}\ \theta(\hbar)-\theta(z\hbar)\theta\ \frac{z\hbar\ u_1}{u_4}\ \theta\ \frac{u_3}{u_1}}{\theta\ uu\underline{1}3\ . \qquad -\theta\ u4}$$

Case $a = 6, b = 4$:

$$\mathbf{Stab}'(4)_6 = \theta\left(\frac{u_3\hbar}{u_1}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta\left(\frac{z\hbar u_2}{u_4}\right)\theta\left(\frac{u_3\hbar}{u_4}\right)\theta(\hbar)\theta\left(z^{-}\ \hbar^2\right),$$

(6) 4 $\theta(\hbar)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta\left(\frac{u_3\hbar}{u_1}\right)\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{zu_3\hbar}{u_4}\right)^2\theta(\hbar)-\theta(z\hbar)\theta\left(\frac{z\hbar\ u_2}{u_4}\right)^2\theta\ \hbar^{-1}\theta\ \frac{u_3}{u_2}\ 4$

Stab = $\theta\ \frac{u_2}{u_3}$.

8.4 Identities for theta functions

In all these cases the identity follows from the well-known 3-term identity

$\theta ayx1\ \theta hyx2\ \theta hyy2$ ^{1} $\theta(a) = \theta$ $x^2\ \theta\ x^2\ \theta\ y_2^1\ \theta\ \hbar$ $+\theta$ $x^2\ \theta\ x^2\ \theta(\hbar)\theta$ y_2 $ahy\ y\ y\ a\ hy$
ay ay 1 , (67)

and 4-term identity for theta functions:

$\theta(h)\theta yy12\ \theta hy\ x11\ \theta a2xhy1\ 2\ \theta a1ax22hy1\ \theta xy22\ \theta ax1x12$

$$-\theta\left(\frac{a_1 a_2 h y_1}{x_1}\right)\theta\left(\frac{x_1}{y_2}\right)\theta\left(\frac{h y_1}{x_2}\right)\theta\left(\frac{a_2 h y_2}{x_2}\right)\theta\left(\frac{h x_2}{x_1}\right)\theta\left(\frac{y}{y_{21}}\ \theta\ \&a1'\right.$$

(68)

$$= -\theta(h)\,\theta\Big(\frac{x_1}{x_2}\Big)\theta\Big(\frac{a_1a_2hy_2}{x_1}\Big)\theta\Big(\frac{a_2hy_1}{x_2}\Big)\theta\Big(\frac{a_1y_1}{y_2}\Big)\theta\Big(\frac{hy}{x_1}$$

$$\_\_ \quad \_\_\_\_ \quad \_\_\_ \quad \_\_ \quad \_\_1 \quad \theta^{x}y22$$

$$+\theta\Big(\frac{hy_2}{x_1}\Big)\theta\Big(\frac{x_2}{y_1}\Big)\theta\Big(\frac{x_1}{x_2}\Big)\theta\Big(\frac{hy_1}{y_2}\Big)\theta\Big(\frac{a_1a_2hy_1}{x_1}\Big)\theta\Big(\frac{a_2hy_2}{x_2}\Big)\theta\,(a_1.$$

Let us check the identity for the most complicated case $a = 6, b = 1$. The other cases are analyzed in the same manner. First, we specialize the parameters in the 4-term relation (68) to the following values:

$$(a_1 = \bar{h}^{-1}, \qquad a_2 = z\bar{h}, \qquad x_1 = u_3, \qquad x_2 = u_4, \qquad y_1 = u_2, \qquad y_2 = u_1, \qquad h = \bar{h}).$$

After this substitution, the above 4-term (up to a common multiple $\theta(\bar{h})$) takes the form:

$$-\theta\,\frac{u_1}{u_4}\,\theta\,\frac{\hbar u_3}{u_4}\,\theta\,\frac{u_1}{u_2}\,\theta\,\frac{zu_1\hbar^2}{u_3}\,\theta\,\frac{z\hbar u_2}{u_4}\,\theta\,\frac{\hbar u}{u_3}\quad 2$$

$$+\theta\,\frac{u_1}{u_2}\,\theta\,\frac{\hbar u_2}{u_4}\,\theta\,\frac{\hbar u_4}{u_3}\,\theta\,\frac{u_1}{u_3}\,\theta\,\frac{z\hbar u_2}{u_3}\,\theta\,\frac{zu_1\bar{}}{u_4}\quad h2 \tag{69}$$

$$= -\theta\,\frac{u_1}{u_4}\,\theta\,\frac{u_3}{u_4}\,\theta\,\frac{\hbar u_1}{u_2}\,\theta\,\frac{z\hbar u_1}{u_3}\,\theta\,\frac{z\hbar^2 u_2}{u_4}\,\theta\,\frac{\hbar u}{u_3}\quad 2$$

$$+\theta\,\frac{u_3}{u_4}\,\theta\,\frac{u_2}{u_4}\,\theta\,\frac{\hbar u_1}{u_3}\,\theta\,\frac{\hbar u_2}{u_1}\,\theta\,\frac{z\hbar u_2}{u_3}\,\theta\,\frac{zu_1\hbar^2}{u_4}\quad .$$

Now, the identity for $a = 6, b = 1$ has the form:

$$A_1 + A_2 + A_3 = B_1 + B_2,$$

where the terms have the following explicit form (after clearing the denominators):

$$A_1 = \theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{\hbar u_3}{u_4}\right)\theta\left(z\hbar^2\right)\theta\left(\frac{z\hbar u_1u_2}{u_3u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar)$$

$$A_2 = -\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta\left(z\hbar^2\right)\theta(\hbar)\theta\left(\frac{z\hbar u_1u_2}{u_3u_4}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2}{u_3}\right)$$

$$A_3 = -\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{z\hbar^2u_2u_1}{u_3u_4}\right)\theta\left(\hbar^2\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_3}{u_4}\right)\theta(z\hbar)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)$$

$$B_1 = \theta(\hbar)^2\theta\left(\frac{u_1u_2}{u_3u_4}\right)\theta\left(\frac{u_3}{u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_2}\right)\theta\left(\frac{z\hbar u_1}{u_3}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{z\hbar^2}{u_4}\right)$$

$$B_2 = -\theta(\hbar)^2\theta\left(\frac{u_1u_2}{u_3u_4}\right)\theta\left(\frac{u_3}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{zu_1\hbar^2}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)$$

For some values of the parameters, the three-term relation (67) can be written in the form

$$\theta\left(z\hbar^2\right)\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{z\hbar u_1u_2}{u_3u_4}\right) = -\theta\left(\frac{\hbar u_4}{u_2}\right)\theta\left(\frac{z\hbar^2u_2u_1}{u_3u_4}\right)\theta\left(\frac{u_1}{u_3}\right)\theta(z\hbar)$$
$$+\theta\left(\frac{u_1u_2}{u_3u_4}\right)\theta(\hbar)\theta\left(\frac{z\hbar u_2}{u_4}\right)\theta\left(\frac{zu_1\hbar^2}{u_3}\right)$$

and thus for A_1 we can write

$$A_1 = -\theta\left(\frac{\hbar u_4}{u_2}\right)\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_3}{u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta(\hbar)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{z\hbar^2u_2u_1}{u_3u_4}\right)\theta(z\hbar)$$
$$+\theta\left(\frac{zu_1\hbar^2}{u_3}\right)\theta\left(\frac{z\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta(\hbar)^2\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1u_2}{u_3u_4}\right)\theta\left(\frac{\hbar u_3}{u_4}\right).$$

Similarly we can write the 3-term relation as

$$\theta\left(z\hbar^2\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_4}\right)\theta\left(\frac{z\hbar u_1 u_2}{u_3 u_4}\right) = -\,\theta\left(\frac{\hbar u_3}{u_2}\right)\theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(z\hbar\right)$$

$$+\,\theta\left(\frac{u_1 u_2}{u_3 u_4}\right)\theta\left(\hbar\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{z u_1}{u_4}\right)$$

h_2

and thus

$$A_2 = \theta\left(\frac{\hbar u_3}{u_2}\right)\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\hbar\right)\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right)\theta\left(z\hbar\right)$$

$$-\theta\left(\frac{u_1}{u_2}\right)\theta\left(\hbar\right)^2\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_1 u_2}{u_3 u_4}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{z u_1}{u_4}\right)$$

h_2.

Finally,

$$\theta\left(\hbar^2\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{u_3}{u_4}\right) = \theta\left(\hbar\right)\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta\left(\frac{\hbar u_3}{u_2}\right) - \theta\left(\frac{\hbar u_4}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\hbar\right)\theta\left(\frac{\hbar u}{u_4}\right)$$

$_3$,

which gives

$$A_3 = \theta\left(\frac{\hbar u_4}{u_2}\right)\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_3}{u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\hbar\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right)\theta\left(z\hbar\right)$$

$$-\theta\left(\frac{\hbar u_3}{u_2}\right)\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\hbar\right)\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right)\theta\left(z\hbar\right)$$

.

Several terms in the sum $A_1 + A_2 + A_3$ cancels and we obtain

$$A_1+A_2+A_3 = \theta(\hbar)^2\,\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1u_2}{u_3u_4}\right)\left(\theta\left(\frac{zu_1\hbar^2}{u_3}\right)\theta\left(\frac{z\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{hu}{u_4}\right)\right.$$

$\_\_\_\_$ 3 θ $uu\underline{1}4$

$$-\theta\,\frac{\hbar u_2}{u_4}\;\theta\,\frac{\hbar u_4}{u_3}\;\theta\,\frac{zhu_2}{u_3}\;\theta\,\frac{zu_1{}^-}{u_4}\;h2\,\theta\,uu\underline{1}3\,.$$

Now, modulo a common multiple $\theta(\hbar)^2\theta$ u u_3 $\frac{1u}{u}\frac{2}{}4$ the relation $A_1 + A_2 + A_3 = B_1 + B_2$ is exactly the 4-term relation (69).

9 Proof of Theorem 5

Let us first discuss the idea of the proof. We denote the restriction matrices for the elliptic stable envelopes in (holomorphic normalization) by

$$\mathbf{T}_{\mathbf{q},\mathbf{p}} = \mathbf{Stab}(\mathbf{q})\mathrm{O}_{\mathbf{p}},\qquad \mathbf{T}'_{\lambda,\mu} = \mathbf{Stab}'(\lambda)\,\mathrm{O}'_{\mu}\,.$$

Recall that the isomorphism κ induces an isomorphism of extended orbits $\mathrm{O}'_{\mu} \cong \mathrm{O}_{\mathbf{p}} =$ ⍰

$E_{T\times T}$. First, we show that under this isomorphism we have the following identity

$$\mathbf{T}'_{\lambda,\mu} = \mathbf{T}_{\mathbf{q},\mathbf{p}},\quad \text{for}\quad \mathbf{p} = \mathrm{bj}(\lambda),\ \mathbf{q} = \mathrm{bj}(\mu). \tag{70}$$

By Theorem 2, to prove this identity, it is enough to check that the matrix elements $\mathbf{T}_{\lambda,\mu}$ satisfies the conditions (1), (2) and (3).

The condition (1) says that for fixed μ the set of functions $\mathbf{T}_{\lambda,\mu}$ is a section of the line bundle $\mathrm{M}(\mathbf{q})$ see (28). By Proposition 1, to check this property, it is enough to show that $\mathbf{T}_{\lambda,\mu}$ has the same quasiperiods in equivariant and Kähler variables as sections of $\mathrm{M}(\mathbf{q})|_{\mathrm{O}_{\mathbf{p}}}$ and that it satisfies the GKM conditions:

$$\mathbf{T}'_{\lambda,\mu}\big|_{u_i=u_j} = \mathbf{T}'_{\nu,\mu}\big|_{u_i=u_j}, \tag{71}$$

if the fixed points $\mathbf{p} = \mathsf{bj}(\lambda)$ and $\mathbf{s} = \mathsf{bj}(\nu)$ are connected by equivariant curve, that is, if $\mathbf{p} = \mathbf{s} \setminus \{i\} \cup \{j\}$ as k-sets. We recall that the quasiperiods of $\mathbf{T}_{\lambda,\mu}$ are the same as of function (37) multiplied by (39). The quesiperiods of sections of $\mathrm{M}(\mathbf{q})|_{\mathrm{O_p}}$ are the same as of function (22) multiplied by (27). Both resulting functions are explicit product of theta functions and the quasiperiods are determined immediately from (10). A long but straightforward calculation then shows that the quasiperiods coincide. To check (71) is however less trivial, we prove it in the next Subsection 9.2.

The condition (2) is trivial and follows from our choice of holomorphic normalization.

The condition (3) says that $\mathbf{T}_{\lambda,\mu}$ must be divisible by some explicit product of theta functions and the result of division is a holomorphic function in variables u_i. We will refer to these properties as *divisibility* and *holomorphicity*. These properties of the matrix $\mathbf{T}_{\lambda,\mu}$ will be proven in Subsections 9.3 and 9.4, respectively.

Let us consider the following scheme:

$$\mathsf{S}(X, X') := E_{\mathrm{T}\times\mathrm{T}} \times S^k(E) \times \prod_{i=1}^{n-1} S^{\mathrm{v}_i}(E). \tag{72}$$

Here $S^k(E)$ denotes k-th symmetric power of the elliptic curve E. We assume that coordinates on $S^k(E)$ are given by symmetric functions on Chern roots of tautological bundle on X. Similarly, $S^{\mathrm{v}_i}(E)$ denotes the scheme with coordinates given by Chern roots of i-th tautological bundle on X, that is, symmetric functions in x with $c = i$, see Section 4.2 for the notations.

Recall that the stable envelopes $\mathbf{Stab}(\mathbf{q})$ and $\mathbf{Stab}(\lambda)$ are defined explicitly by (25) and (48). In particular, they are symmetric functions in the Chern roots of tautological bundles. This means that the function defined by

$$\tilde{m} := \sum_{\mathbf{q}\in X} (\mathbf{T})^{-1}_{\mathbf{q},\mathbf{p}}\, \mathbf{Stab}(\mathbf{p})\, \mathbf{Stab}'(\mathsf{bj}^{-1}(\mathbf{q})) \quad (73)$$
$\mathbf{p},^{\mathsf{T}}$

can be considered as a meromorphic section of certain line bundle on $\mathsf{S}(X,X')$. We denote this line bundle by $\tilde{M}$. (Note that $\mathbf{T}_{\mathbf{q},\mathbf{p}}$ is triangular matrix with non-vanishing diagonal; thus, it is invertible and the sum in (73) is well defined. Note also that in (73) we assume that $\mathbf{q} = \mathsf{bj}(\mu)$ and the 2nd sum over μ is the same as sum over $\mathbf{q} \in X^{\mathsf{T}}$.) Let us consider the map

$$\tilde{\mathbf{c}} : \mathrm{Ell}_{\mathsf{T}\times\mathsf{T}}(X \times X' \qquad S(X,X')) \to \quad ,$$

which is defined as follows: the component of $\tilde{\mathbf{c}}$ mapping to the 1st factor of (72) is the projection to the base. The components of the map $\tilde{\mathbf{c}}$ to $S^k(E)$ and to $S^{\vee l}(E)$ are given by the elliptic Chern classes of the corresponding tautological classes. For the definition of elliptic Chern classes, see [21, Section 1.8] or [19, Section 5]. It is known that $\tilde{\mathbf{c}}$ is an embedding [40], see also [1, Section 2.4] for discussion.

Finally, the line bundle and the section of the Theorem 5 can be defined as $M = \tilde{\mathbf{c}}^*\tilde{M}$ and $m = \tilde{\mathbf{c}}^*(\tilde{m})$. Indeed, from the very definition (73) and (70), it is obvious that

$$(i^*_\lambda)^*(\mathfrak{m}) = \tilde{\mathfrak{m}}_{\ \lambda} = \mathbf{Stab}(\mathbf{p}), \qquad (i^*_{\mathbf{p}})^*(\mathfrak{m}) = \tilde{\mathfrak{m}}_{\ \mathbf{p}} = \mathbf{Stab}(\lambda).$$

that is, the section m is the mother function.

9.1 Cancellation of trees

Before checking conditions (1)–(3), we need a key lemma that describes that under specialization of some u_i parameters, the contributions from trees cancel out with each other and the summation simplifies dramatically.

Define the *boundary* of $\bar{\lambda}$ to be the set

$$\{(i,j) \in \bar{\lambda} \mid (i-1,j-1) \notin \bar{\lambda}\}.$$

Define the *upper boundary* of $\bar{\lambda}$ to be the set

$$U := \{(i,j) \in \bar{\lambda} \mid j = k\}.$$

Consider a 2×2 square in $\bar{\lambda}$, consisting of $(c,d),(c+1,d),(c,d+1),(c+1,d+1)$, where $(c+1,d)$ is in the a-th diagonal. Let $\bar{t}$ be a tree, which contains the edge $(c+1,d+1) \to (c+1,d)$.

The *involution* of $\bar{t}$ at the box $(c+1,d)$ is defined to be the tree $\mathrm{inv}(\bar{t},(c+1,d))$ obtained by removing $(c+1,d+1) \to (c+1,d)$ from $\bar{t}$ and adding the edge $(c,d) \to$

$(c+1,d)$. (The involution sends a tree to another tree. In fact, by definition, $\bar{t}$ contains the edge $\delta_1 : (c+1,d+1) \to (c+1,d)$, but does not contain the edge $\delta_2 : (c,d) \to (c+1,d)$. It also follows from definition that the subtree $\bar{s} := [(c+1,d),\bar{t}]$ does not contain the other 3 boxes in the 2×2 square (otherwise $\bar{t}$ would contain either a loop or a ⅂-shape). It is then easy to see that $\mathrm{inv}(\bar{t})$ is still a tree.) We abbreviate the notation as $\mathrm{inv}(\bar{t})$ if there is no confusion. Define $\mathrm{inv}(\mathrm{inv}(\bar{t})) = \bar{t}$. Involution is a well-defined operation on all trees at all boxes that are not in U or the boundary of $\bar{\lambda}$. Let $\bar{s}$ be the subtree

$$\bar{s} := [(c+1,d),\bar{t}] = [(c+1,d),\mathrm{inv}(\bar{t})].$$

The u-parameter contributed from $\bar{s}$ is

$$u(\mathrm{s}) := \prod_{I \in \bar{s}} \frac{u_{c_I+1}}{u_{c_I}}.$$

Lemma 7 (Cancellation lemma). SPiText

$$\frac{\mathcal{R}(\bar{t})\mathcal{W}(\bar{t})}{\mathcal{R}(\mathrm{inv}(\bar{t}))\mathcal{W}(\mathrm{inv}(\bar{t}))}\bigg|_{u(\bar{\mathfrak{s}})=1} = -1.$$

As a corollary,

$$\sum_{\sigma \in \mathfrak{S}_{\bar{\lambda}}} \frac{\mathcal{N}^\sigma_{\bar{\lambda}}}{\mathcal{D}^\sigma_{\bar{\lambda}}} \mathcal{R}^\sigma(\bar{t})\mathcal{W}^\sigma(\bar{t}) = -\sum_{\sigma \in \mathfrak{S}_{\bar{\lambda}}} \frac{\mathcal{N}^\sigma_{\bar{\lambda}}}{\mathcal{D}^\sigma_{\bar{\lambda}}} \mathcal{R}^\sigma(\mathrm{inv}(\bar{t}))\mathcal{W}^\sigma(\mathrm{inv}(\bar{t})).$$

Proof. Direct computation shows that

θ $x_{c+1,d}$ x

h

$$\frac{\mathcal{R}(\bar{\mathrm{t}})^{c,d}}{\mathcal{R}(\mathrm{inv}(\bar{\mathrm{t}}))} = \qquad . \tag{74}$$

$\theta x_{c\pm1,d\pm1}$

$x_{c+1,d}$

The quotient W(¯t)/W(inv(¯t)) has contribution from an edge e if the subtree $[h(e),\bar{t}]$ or $[h(e),\mathrm{inv}(\bar{t})]$ contains $(c+1,d+1)$ or (c,d). Those contributions are all of the form

$\theta\, xx_{th((ee))}\phi\phi h^{\lambda}{}_{t\lambda((ee))}\ I\notin h(e),\mathrm{inv}(\bar{t})]\ u^{u}{}_{c^{l}c+l\,1}\cdot u(\bar{s})$ or $\theta\, xx\ (e)\varphi_{t(e)}\ I\in[h(e),\bar{t}]$ λ

$t_{h(e)}\phi h^{\lambda}{}_{\lambda(e)}\ uu_{c^{l}c+l\,1}$,

$$\theta\ \frac{x_{t(e)}\phi^{\lambda}_{h(e)}}{x_{h(e)}\phi^{\lambda}{}_{e}}\ {}_{t(\ )\ I\notin h(e),\mathrm{inv}(\bar{t})]}\ u^{u}{}_{c^{l}c+_{l}1} \qquad \theta\, x^{x}\ {}_{h(\ )}\varphi^{\lambda}{}_{(\ )}\ (e)\ h(e)\ \prod_{(\ ),\bar{t}]\,t\,e\,\phi\,t\,e\,I\notin h\,e}$$

$u^{u}{}_{c^{l}c+l\,1}\,u(\bar{s})$

which are both 1 under $u(\bar{s})=1$, and the only remaining factor comes from the edges $(c+1,d+1)\to(c+1,d)$ and $(c,d)\to(c+1,d)$:

$$\theta x x_{c\pm c+1,1 d,\pm d 1} \,.\, \frac{\theta}{{}_{+1,d}\ \varphi^{\lambda}_{c}} \quad \frac{}{+1} \ \frac{\lambda}{c+1,d} x x_{c+1,}{}^{d} \qquad \phi + \qquad u(\bar{s})\,(s)$$

$$\frac{x_{c,d}}{x_{c+\ d}} \, \frac{\varphi^{\lambda}_{c+\ d}}{\varphi^{\lambda}_{c,d}} \qquad\qquad \frac{c,d}{x_{c+\ d}} \hbar^{-1}$$

$$c11_{,,d+1}\ u^{-} = 1 = \qquad x$$

$$\theta u(\bar{s})_{1,} \qquad \theta \quad 1,$$

The lemma follows.

9.2 GKM conditions

The goal of this section is to prove that the elliptic stable envelopes **Stab**(λ) satisfy the GKM condition (71). For simplicity, we assume that $(1,k) \in \bar{\lambda}$; in other words, $\bar{\lambda}$ starts with diagonal 1. The general case can be easily reduced to this.

A subtree of $\bar{t}$ is called a *strip* if it contains at most one box in each diagonal. We will also abuse the name *strip* for a connected subset in a partition that contains at most one box in each diagonal. We call a strip that starts from diagonal i to $j - 1$ an

(i,j)-*strip*.

Let λ and μ be two partitions, and $\mathbf{p} = \mathsf{bj}(\lambda)$, $\mathbf{q} = \mathsf{bj}(\mu)$. Suppose that as fixed points in X, $\mathbf{p}$ and $\mathbf{q}$ are connected by a torus-invariant curve, which means that

$$\mathbf{q} = \mathbf{p} \setminus \{i\} \cup \{j\},$$

for some $1 \le i,j \le n$ (assume $i < j$). On the dual side, that means $\mu \supset \lambda$, and $\mu \backslash \lambda$ is an

(i,j)-strip, lying the boundary of $\bar{\lambda}$.

Recall the GKM condition:

Proposition 6. For partitions λ and μ as above,

$$\mathbf{Stab}(\lambda)\big|_{u_i=u_j} = \mathbf{Stab}'(\mu)\big|_{u_i=u_j}.$$

By localization and the triangular property of stable envelopes, it suffices to show that for any partition $\nu \supseteq \lambda$,

$$\mathbf{Stab}(\lambda)\big|_{\nu,u_i=u_j} = \mathbf{Stab}'(\mu)\big|_{\nu,u_i=u_j}.$$

Before proving the GKM condition, we need some analysis on the specialization of the stable envelopes under $u_i = u_j$.

9.2.1 *Specialization of* $\mathbf{Stab}(\lambda)$ *under* $u_i = u_j$

Recall that $\mathbf{p} \subset \mathbf{n}$ and $i \in \mathbf{p}$, $j \notin \mathbf{p}$, $i < j$. We would like to study the specialization $u_i = u_j$.

By definition

$$\mathbf{Stab}(\lambda) = T_{\mathbf{p},\mathbf{p}} \cdot \mathrm{Stab}'(\lambda),$$

where

$$T_{\mathbf{p},\mathbf{p}} = (-1)^{(n-k)k} \prod_{i\in\mathbf{p},\, j\in\mathbf{n}\backslash\mathbf{p},\, i<j} \theta\left(\frac{u_j}{u_i}\right) \prod_{i\in\mathbf{p},\, j\in\mathbf{n}\backslash\mathbf{p},\, i>j} \theta\left(\frac{u_i}{u_j \hbar}\right).$$

In particular, $T_{\mathbf{p},\mathbf{p}}$ contains a factor $\theta\left(\frac{u_i}{u_j}\right)$.

For any tree $\bar{t}$ in $\bar{\lambda}$, consider all subtrees of $\bar{t}$ that are (i,j)-strips

$$\mathsf{B} = \{B_i, B_{i+1}, \cdots, B_{j-1}\},$$

where B_l is the box in the l-th diagonal. We define $B(\bar{t},i,j)$ to be one whose B_i has the *smallest height*. If $\bar{t}$ does not contain any (i,j)-stripes as subtrees, define $B(\bar{t},i,j) = \varnothing$ A tree $\bar{t}$ in $\bar{\lambda}$ is called *distinguished*, if its strip $B(\bar{t},i,j) \neq \emptyset$, and lies in the boundary of $\bar{\lambda}$.

A simple observation is that, for the contribution from $\bar{t}$ to **Stab**(λ) to be nonzero under $u_i = u_j$, $B(\bar{t},i,j)$ has to be nonempty.

Lemma 8. Let B be an (i,j)-strip in $\bar{t}$, which is a subtree. Let B_U be the box in $B \cap U$ with largest content. We have

- if $B_i \in U$, then B_i is the root of B;
- if $B_i \in U$, then B_U is the root of B.

Proof. If $B_i \in U$, and the root of B is some box other than B_i. Then the unique path from B_i to U has a box in its interior with local maximal content. must be connected to both the boxes to the left and above it, which is not allowed.

If $B_i \in U$, then every box in B from B_i to B_U is in U. It is clear that the root of B is B_U.

Lemma 9. Let B be an (i,j)-strip in $\bar{t}$ which is a subtree. If B_i lies in the boundary of $\bar{\lambda}$, then B lies entirely in the boundary of $\bar{\lambda}$; in other words, $B(\bar{t},i,j) = B$.

Proof. Suppose B_i lies in the boundary, but B does not. Then there exists a box in the boundary of $\bar{\lambda}$, not in B, but in a diagonal less than $j - 1$. Since $\bar{t}$ is a tree, there is a unique path from that box to some box in U. This path would contain a box with local maximal content in its interior. Contradiction.

Lemma 10. Under the specialization $u_i = u_j$,

Stab$(\lambda)_{u^i}$.

$$=u_j = T_{\mathbf{p},\mathbf{p}} \cdot \quad N_\lambda \;) \quad \sigma\in\ \lambda \quad \mathcal{D}^{\sigma}_{\bar\lambda} \quad (\mathfrak{t}) \quad (\mathfrak{t})\Big|_{u_a=}$$

$^-$ disinguished

$$S^- \quad u_b\,\mathfrak{t} \quad \epsilon(\lambda)\,\Theta(\tilde{\ }',- \quad \sum \quad \frac{\mathcal{N}^{\sigma}_{\bar\lambda}}{}\mathcal{R}^{\sigma}\ ^-\ \mathcal{W}^{\sigma}\ ^-$$

Proof. Let $B = B(\bar{\mathfrak{t}},i,j)$. Since $T_{\mathbf{p},p}$ contains a zero u_i/u_j, if $B = \varnothing$ it is clear that the stable envelope will vanish. Now assume $B = \varnothing$

If $i = 1$, then $B_i = (1,k)$. By Lemma 9 B lies in the boundary and $\bar{\mathfrak{t}}$ is distinguished.

If $i = 1$, it is easy to see that $B_i \in U$ (otherwise as a subtree B must contain $(1,k)$). If moreover B_i is not in the boundary, then one can construct its involution inv$(\bar{\mathfrak{t}})$. By Lemma 7, the contributions from $\bar{\mathfrak{t}}$ and inv$(\bar{\mathfrak{t}})$ cancel with each other. Therefore, in the summation over trees, we are left with those $\bar{\mathfrak{t}}$ whose B_i lies in the boundary of $\bar\lambda$, which by Lemma 9 are distinguished.

Fix a distinguished tree $\bar{\mathfrak{t}}$, and $B = B(\bar{\mathfrak{t}},i,j)$. Let's consider the restriction of **Stab**(λ) to a certain fixed point $\nu \supset \lambda$. For an individual contribution from given $\bar{\mathfrak{t}}$ and σ, we take the following limit, called *B-column limit* for $\nu\backslash\lambda$: first, for each pair of $I,J \in \nu\backslash\lambda$ such that I is above J and $I,J \in B$, take

$$x_I = x_J \bar{h};$$

for any $I \in B$, take $x_I = \phi_I^{\nu}$; finally take any well-defined evaluation of the remaining variables. Note that this limit only depends on the partition λ and the pair i,j, and does not depend on $\bar{\mathfrak{t}}$.

Lemma 11. The restriction

$$\frac{\mathcal{N}^{\sigma}_{\bar\lambda}}{}\mathcal{R}^{\sigma}(\bar{\ })\mathcal{W}^{\sigma}(\bar{\ })\mathcal{D}^{\sigma}_{\bar\lambda} \quad \mathfrak{t} \quad \mathfrak{t}\Big|_{\nu}$$

under the B-column limit vanishes unless σ fixes B.

Proof. Suppose that the restriction does not vanish under the chosen limit. Recall that B_l, $i \le l \le j-1$ is the box in the l-th diagonal of B. We use induction on l, from $j-1$ to i.

Recall that by the refined formula, σ lies in $S_{\nu\backslash\lambda}$.

First, we show that B_{j-1} is fixed by σ. Let $Y_1, Y_2, \cdots$ be the boxes in the j-th diagonal of ν\λ, such that the heights of Y_m's are increasing. Since $j \in \mathbf{p}$, Y_1 is the box to the right of B_{j-1}. Hence, we have the

theta factors θ x h ,

$x_{\sigma(B_{j-1}}$

$\frac{)}{Y_m} -$

$m \geq 1$

as $\rho_{B_{j-1}} > \rho_{Y_m}$ and B_{j-1} is not connected to Y_1. Under the B-column limit for ν\λ, this product vanishes unless $\sigma(B_{j-1})$ has no box below it, which implies $\sigma(B_{j-1}) = B_{j-1}$.

Next, suppose that B_{l+1} is fixed by σ, consider B_l. Let e be the edge connecting B_l and B_{l+1}. Let $X_1 = B_l, X_2, \cdots$ and $Y_1 = B_{l+1}, Y_2, \cdots$ be, respectively, the boxes in the l-th and $(l + 1)$-th diagonals of ν\λ.

If e is *horizontal*, then we have factors ,

$$\prod_{\geq 2} \theta \left(\frac{x_{\sigma(X_m)}}{x_{B_{l+1}}} \frac{m}{} \right.$$

since we know $\rho_X < {}_m$ $_{l+1}$ θ $x_{\sigma(X_m)}$ θ ρ_B and X_2 is not connected to x_{B^l} B_{l+1}. If $\sigma(X_m) = X_1 = B_l$

for some $m = 1$, then the factor $_{l\,1} = {}_l$ vanishes under the B-column ordering.

xB xB

$+$ $+^1$

Hence, $\sigma(B\ )\quad B$.

$l =\quad l$

If e is *vertical*, then we have factors

$$\left. \frac{x_{\sigma(B_l)}}{x_{\sigma(Y_m)}} - \right) = \prod_{m \geq 2} \theta \left(\frac{x_{\sigma(B_l)}}{Y_m \hbar} \right. \qquad \theta ,$$

h x

$m \geq 2$

since we know $\rho_{B_l} > \rho_{Y_m}$ and B_l is not connected to Y_2. If $\sigma(B_l) = X_m$ for some $m = 1$, then the factor $\theta\left(\frac{x_{\sigma(B_l)}}{x_Y h}\right) = \theta\left(\frac{x_{\bar{m}}}{x_{\bar{X}_m} h}\right)$ vanishes under the B-column ordering, since X_m is x x the box above Y_m and they are not in B for $m \geq 2$. Hence, σ fixes B_l.

In summary, after restriction to ν in the B-column limit for $\nu\backslash\lambda$, only contributions from distinguished $\bar{t}$ and permutations σ that fix B survive. We are now ready to prove Proposition 6.

9.2.2 *Proof of Proposition 6: μ is not contained in ν*

In this case, the strip $\mu\backslash\lambda$ is not entirely contained in $\nu\backslash\lambda$. Clearly, we have $\mathbf{Stab}(\mu)_\nu = 0$.

Lemma 12.

$$\mathbf{Stab}(\lambda)_{\nu, u_i = u_j} = 0.$$

Proof. By Lemma 10, only distinguished trees $\bar{t}$, with strip $B = B(\bar{t}, i, j)$ contributes. Let $B_i, \cdots, B_{j-1}$ be boxes in B, and X be the first box in B that does not lie in $\nu\backslash\lambda$. For restriction to ν of an individual contribution by given $\bar{t}$ and σ, we take the *column limit* for $\bar{\nu}$, that is, first let $x_I = x_J$ for any $I, J \in \bar{\nu}$ in the same column, and then take any limit for the remaining variables.

If $X = B_i$, then there is a box Y above it, which also lies in $\bar{\nu}$. Since $Y \in B$, the edge connecting X and Y is not in $\bar{t}$. The contribution from $\bar{t}$ then contains a factor $\theta(X/Y)$, which vanishes under the column limit.

If $X = B_i$, then either $i = 1$, or $i = 1$, and the entire $B \cap U$, and in particular B_U, lie in $\bar{\nu}$. By Lemma 8, we know the root $r_B = B_i$ or B_U, respectively. Denote the box not in B and connected to r by C. The factor in $W^{\sigma=1}(\bar{t})$ that contributes the pole u_i/u_j is

$$\frac{\varphi_r^{\lambda}}{} \; \frac{x_{C r_B C \lambda B}}{} \quad u_{ji}$$

$$\theta_\nu x \phi\, u\, u_j = 1.$$

$$\theta$$

$$u_i$$

Stab$(\lambda)_\nu = 0$ under $u_i = u_j$ because of the zero u_i/u_j in $T_{\mathbf{p},\mathbf{p}}$.

9.2.3 *Proof of Proposition 6:* $\mu \subset \nu$

In this case, B is contained entirely in $\nu \backslash \lambda$; in other words, $\lambda \subsetneq \mu \subset \nu$. Let r_B be the root of B, which if $i =$ 1, is B_U; and if $i = 1$, is B_a.

If $(n-k,k) \notin B$, let $C \in \mathfrak{t} \backslash B$ be the box connected to r_B. C could be in or not in $\nu \backslash \lambda$. If $(n-k,k) \in B$, we denote by convention that $x_C/\varphi_C^{\lambda} =$ 1. Then

$$u$$

$$\theta u_{ji} \mathrm{Stab}(\lambda)\%_{\nu, u_i = u_j} \; \mathrm{s}$$

$$\mathfrak{t} \quad \sum_{\nu \; \mu, \; \cap \bar{\mu}} \frac{\mathcal{N}_{\bar{\mu}}^{\sigma}}{\mathcal{D}_{\bar{\mu}}^{\sigma}} \mathcal{R}^{\sigma}(\bar{\mathfrak{t}} \cap \bar{\mu}) \mathcal{W}^{\sigma}(\bar{\mathfrak{t}} \cap -$$

$$= \theta \frac{u_j}{u} \; \epsilon(\lambda) \; \Theta(N'_{\lambda}{}'^{-})_{\nu} \cdot \_\_ \; \mu)_i \quad \sigma \in \quad \backslash \quad -$$

$$\nu$$

$$\cdot \; ck,k) \; \theta \; ax2lh^{-}\nu \prod_{\substack{= n-k, I \neq (n- \\ I \in \mu \;\; \lambda}} \quad I \quad \backslash$$

c_I $+1=c_J,\ \rho_I>\rho_J$ $J)\notin t, I$ or $J\in\mu\ \lambda$ $\theta\ x\sigma(\sigma(JI))h^{-}\ \nu\ I\ cIJ$

$$\prod_{\substack{=c_J,\ \rho_I>\rho_J\\ \text{or } J\in\mu\ \lambda}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right)^{1} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right)^{-}$$

1t,cIJor,$\rho IJ<\rho\mu J$ θ $x\sigma(\sigma(I)\ \nu\ xx$

$+ =$ $J)$

$(\ \leftrightarrow\ )\notin^{-}$

$(_{I\leftrightarrow}{}^{-}$ $\backslash$ $\in\backslash\lambda\cdot c_I$

$h\ \nu-1$

I $\backslash$

$$\cdot\frac{\theta\ \frac{x_C\phi^\lambda_{r_B}\ u_i}{x_{r_B}\phi^\lambda_C\ u_j}}{\theta\ \ \in\ \in\ u} \ \nu\ \frac{x\ \phi^\lambda}{\theta\ \frac{u_j}{u_{h(e)}}} \ \frac{\theta\ \frac{x_{t(e)}\phi^\lambda_{h(e)}}{x_{h(e)}\phi^\lambda_{t(e)}}\frac{u_{t(e)}}{u_i}}{\theta\ \frac{u_{t(e)}}{u_i}}^{\nu},$$

which can be compared with $\mathrm{Stab}'(\mu)$. Direct computation shows that

(μ) $_{\nu,u}$ u λ $=$ -1

$$\theta\left(\frac{u_j}{u_i}\right)\frac{\mathrm{Stab}'(\lambda)}{\mathrm{Stab}'}\Big|_{i=\ j} \quad (\quad)^{j-i}\theta(\hbar^{-1})\prod_{e\in B\setminus U}\left(\frac{\quad {}_e \quad {}_{t(\ )} \quad {}^{j}_{h(e)}}{\theta \qquad}\right)\Big|_{\nu,u_i=u_j}\prod_{e\in B\cap U}\frac{\theta\left(\frac{x_{t(e)}\varphi^{\lambda}_{h(e)}}{x_{h(e)}\varphi^{\lambda}_{t(e)}}\frac{u_{t(e)}}{u_i}\right)\Big|_{\nu,u_i=u_j}}{\theta\left(\frac{u_{t(e)}}{u_i}\right)}$$

$$=(-1)^{j-i}\theta(\hbar^{-1})\prod_{\substack{i<\quad<j\\ m\in\mathbf{p}}}\theta$$

where the last equality is because for $e\in B\setminus U$,

$$\frac{\theta\left(\frac{x_{t(e)}\varphi^{\lambda}_{h(e)}}{x_{h(e)}\varphi^{\lambda}_{t(e)}}\frac{u_j}{u_{h(e)}}\right)\Big|_{\nu}}{\theta\left(\frac{u_j}{u_{h(e)}}\right)} = \begin{cases} \theta\left(\frac{u_j}{u_{h(e)}}\right) & (e)\in p,\ e\notin U \\ u \quad (e) \quad \Big) & (e)\notin p,\ e\notin U \end{cases}$$

and for $e\in B\cap U$,

$$\frac{\theta\left(\frac{x_{t(e)}\quad{}_{h(e)}}{x_{h(\ )}\varphi^{\lambda}_{t(e)}}\frac{u_{t(e)}}{u_i}\right)}{\theta\left(\frac{u_{t(e)}}{u_i}\right)}$$

The proposition is proved by making the change of variable $\hbar\mapsto\hbar^{-1}$ in the above result, and compare with the following lemma.

Lemma 13.

$$\theta\left(\frac{u_i}{u_j}\right)^{-1}\frac{T_{\mathbf{p},\mathbf{p}}}{T_{\mathbf{q},q}}\Big|_{u_i=u_j}=\theta(\hbar^{-1})^{-1}\prod_{\substack{i<m<j\\ \mathbf{n}\ \ \mathbf{p}}}\theta\left(\frac{u_j}{u_m}\right)\prod_{\substack{i<m<j\\ m\in\mathbf{p}}}\theta\left(\frac{u_m}{u_j}\right)\prod_{\substack{i<m<j\\ m\in\mathbf{n}\ \ \mathbf{p}}}\theta\left(\frac{u_m}{u_j\hbar}\right)^{-1}\prod_{\substack{i<m<j\\ m\in\mathbf{p}}}\theta\left(\frac{u_j}{u_m\hbar}\right)^{-1}$$

Proof. Straightforward computation. ■

$$\theta \quad \frac{\frac{x_{t(e)}\phi_{h(e)}}{x_{h(\ )}\phi^{\lambda}_{e}} \frac{u}{u}}{\frac{u_j}{u_{h(e)}}}$$

$\theta\ h u \bar{u}\ {}_{mj}$ $\theta\ u_{mj} h$

$u u^{j}\ {-m}\mathbf{n}{<}j\ \theta\ u u_j^{-}$-,

m m $mi{<}\in\backslash\mathbf{p}$ u_m

$\theta\ h_{-}\ u u_{h(j e)},\ h$

$\theta\ h_{--1}\ u_{hj},\ h$

u_j

$\theta\ u_{h(e)}$

$e\ \phi\ {}_{v,ui=uj}\ \theta\theta h^{-}\ u_{t(j}\ \lambda$ $u\ {}_{e)}$

$= u_{t(e)}$.

u_j

$m \in \setminus \qquad \setminus$

9.3 Divisibility

In this subsection, we aim to prove the following divisibility result. Let $\mathbf{p} = \mathsf{bj}(\lambda)$, $\mathbf{q} = \mathsf{bj}(\mu) \in X^T$ be two fixed points.

Proposition 7. The function $\frac{T_{\mathbf{p},\mathbf{p}}}{T_{\mu,\mu}} \cdot T_{\lambda,\mu}$ is of the form

$$f_{\mu,\lambda} \cdot \theta \prod_{\substack{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p}\\ i>j}} \left(\frac{u_j}{u_i}\hbar\right)^{-1},$$

where $f_{\mu,\lambda}$ is holomorphic in parameters u_i.

Proof. Recall that

$$T_{\mathbf{p},\mathbf{p}} = (-1)^{k(n-k)} \prod_{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p},\, i>j} \theta\left(\frac{u_i}{u_j}\right) \prod_{i\in\mathbf{p},\, j\in\mathbf{n}\setminus\mathbf{p},\, i<j} \theta\left(\frac{u_j}{u_i}\hbar\right)^{-},$$

and $T_{\mu,\mu}$ does not depend on u_i's. By formula (53), we can see that all possible poles of $T_{\lambda,\mu}$ take the form u_i/u_j. Therefore, all possible poles of the function $f_{\mu,\lambda}$ in the proposition are of the form u_i/u_j. Moreover, by the proof of holomorphicity (Theorem 9 below), they have no poles at u_i/u_j. We conclude that $f_{\mu,\lambda}$ is holomorphic in u_i.

9.4 Holomorphicity

In this subsection, we will prove the holomorphicity, that is, the normalized restriction matrices of stable envelopes on X are holomorphic in u_is. The idea is to apply general results for q-difference equations associated to Nakajima quiver varieties.

9.4.1 *Quantum differential equations*

Let X be a Nakajima variety. For the cone of effective curves in $H_2(X,\mathbb{Z})$, we consider the semigroup algebra, which is spanned by monomials z^d with $d \in H_2(X,\mathbb{Z})_{\mathrm{eff}}$. It has a natural completion, which we denote by $\mathbb{C}[[z^d]]$. The cup product in the equivariant cohomology $H_T^\bullet(X)$ has a natural commutative deformation, parametrized by z:

$$\alpha \star \beta = \alpha \cup \beta + O(z) \tag{75}$$

known as the quantum product.

The quantum multiplication defines a remarkable flat connection on the trivial $H_T^\bullet(X)$-bundle over Spec($\mathbb{C}[[z^d]]$). Flat sections $\Psi(z)$ of this connection, considered as $H_T^\bullet(X)$-valued functions, are defined by the following system of differential equations (known as the quantum differential equation or Dubrovin connection):

$$\varepsilon \frac{d}{d\lambda}\Psi(z) = \lambda \star \Psi(z), \qquad \Psi(z) \in H_T^\bullet(X)[[z]],$$

where $\lambda \in H^2(X,\mathbb{C})$ and the differential operator is defined by

$$\frac{d}{d\lambda} z^d = (\lambda,d) z^d. \tag{76}$$

9.4.2 *Quantum multiplication by divisor*

The equivariant cohomology of Nakajima varieties is equipped with a natural action of certain Yangian $Y_\hbar(\mathfrak{g}_X)$ [65]. In the case of Nakajima varieties associated to quivers of ADE type this algebra coincides with the Yangian of the corresponding Lie algebra (but in general can be substantially larger).

The Lie algebra $\mathfrak{g}_X$ has a root decomposition:

$$\mathfrak{g}_X = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

in which $\mathfrak{h} = H^2(X,\mathbb{C}) \oplus$ center, and $\alpha \in H_2(X,\mathbb{Z})_{\text{eff}}$. All root subspaces $\mathfrak{g}_\alpha$ are finite dimensional and $\mathfrak{g}_{-\alpha} = \mathfrak{g}^*_\alpha$ with respect to the symmetric nondegenerate invariant form.

The quantum multiplication (75) for Nakajima varieties can be universally described in therms of the corresponding Yangians:

Theorem 7 ([38,Theorem 10.2.1]). The quantum multiplication by a class $\lambda \in H^2(X)$ is given by

$$\lambda\star = \lambda \cup +\hbar \sum_{(\theta,\alpha)>0} \alpha(\lambda)\frac{z^\alpha}{1-z^\alpha} e_\alpha e_{-\alpha} + \cdots \tag{77}$$

where $\theta \in H^2(X,\mathbb{R})$ is a vector in the ample cone (i.e., in the summation, θ selects the effective representative from each $\pm\alpha$ pair) and $\cdots$ denotes a diagonal term, which can be fixed by the condition $\lambda \star 1 = \lambda$.

Let z_i with $i = 1,\cdots,n-1$ denote the Kähler parameters of the Nakajima variety X from Section 4.

Corollary 4. The quantum connection associated with the Nakajima variety X is a connection with regular singularities supported on the hyperplanes

$$z_i z_{i+1} \ldots z_j = 1, \qquad 1 \le i < j \le n-1.$$

Proof. The variety X is a Nakajima quiver variety associated with the A_{n-1}-quiver. Thus, the corresponding Lie algebra $\mathfrak{g}_X = \mathfrak{sl}_n$. The Kähler parameters z_i associated to the tautological line bundles on X correspond to the simple roots of this algebra. In other words, in the notation of (76) they correspond to $z_i = z^{\alpha_i}$, where α_i, $i = 1,\cdots,n-1$ are the simple roots of $\mathfrak{sl}_n$ (more precisely, simple roots with respect to positive Weyl chamber $(\theta', \alpha_i) > 0$ where θ is the choice stability parameters for X).

By (77), the singularities of quantum differential equation of X are located at

$$z^{\alpha} = 1$$

for positive roots α. All positive roots of sl_n are of the form $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ with $1 \le i < j \le n - 1$. Thus, the singularities are at

$$z^{\alpha} = z_i z_{i+1} \cdots z_j = 1.$$

9.4.3 *Quantum difference equation*

In the equivariant *K*-theory, the differential equation is substituted by its *q*-difference version:

$$\Psi(zq^{\mathcal{L}})\mathcal{L} = \mathbf{M}_{\mathcal{L}}(z)\Psi(z) \tag{78}$$

where $\mathcal{L} \in \mathrm{Pic}(X)$ is a line bundle and $q = e^{\varepsilon}$ and $\Psi(z) \in K_{\mathsf{T}}(X)[[z]]$. The theory of quantum difference equations for Nakajima varieties was developed in [51]. In particular, the operators $\mathbf{M}_{\mathcal{L}}(z) \in \mathrm{End}(K_{\mathsf{T}}(X))$ were constructed for an arbitrary line bundle $\mathcal{L}$. These operators are the *q*-deformations of (77), that is, in the cohomological limit they behave as

$$\mathbf{M}_{\mathcal{L}}(z) = 1 + \lambda \star + \cdots,$$

where ... stands for the terms vanishing the cohomological limit and $\lambda = c_1(\mathcal{L})$.

In *K*-theory, the sum over roots in (77) is substituted by a product:

$\mathbf{M}_{\mathcal{L}}(z) = \mathcal{L}_w \mathbf{B}_w(z)$ The singularities of the quantum difference

equations, that is, the singularities

over certain set of affine root hyperplanes of an affine algebra $\mathfrak{g}_X$.

of matrix $\mathbf{M}_{\mathcal{L}}(z)$ are located in the union of singularities of $\mathbf{B}_w(z)$. The wall crossing operators $\mathbf{B}_w(z)$ are constructed in [51, Section 5.3]. In particular, if $z^{\alpha} = 1$ are the singularities of the quantum differential equation in cohomology, then the singularities of (78) can only be located at $z^{\alpha}q^{p}\hbar^{-s} = 1$ for some integral p,s. This, together with Corollary 4 gives

Proposition 8. The singularities of the quantum difference equation associated with the Nakajima variety X are located at

$$z_i z_{i+1} \cdots z_j q^{p} \hbar^{-s} = 1, \qquad 1 \leq i < j \leq n,\ p,s \in \mathbb{Z}$$

9.4.4 *Pole subtraction theorem*

The elliptic stable envelopes describes the monodromy of q-difference equations. More precisely, the q-difference equation (78) has two distinguished fundamental solution matrices, indexed by fixed points X^{T}. The z-solutions Ψ^{z} form a basis of solutions, which are holomorphic in the Kähler parameters in the neighborhood $|z_i| < 1$. Similarly, the a-solutions Ψ^{a} for a basis of solutions holomorphic in $|a| < 1$. By general theory of q-difference equations, every two bases of solutions are related by a transition matrix:

$$\Psi^{a} = W(z)\Psi^{z}, \tag{79}$$

known as the monodromy matrix from the solutions Ψ^{z} to Ψ^{a}. (Let us clarify the meaning of terms in (79): here Ψ^{z} denotes the fundamental solution matrix - the $|X^{\mathsf{T}}| \times |X^{\mathsf{T}}|$ dimensional matrix with columns of Ψ^{z} satisfying the quantum difference equation. The set of $|X^{\mathsf{T}}|$ columns of Ψ^{z} forms a basis in the space of solutions. The elements of this basis are holomorphic in variables z. Similarly, Ψ^{a} is a matrix whose columns form a basis of solutions, which are holomorphic in parameters a. The theorem above says that $W(z) = \{W(z)_{\lambda,\mu}\}_{\lambda,\mu \in X^{\mathsf{T}}}$ coincides with $\Psi^{a}(\Psi^{z})^{-1}$.) The central result of [1] (in the case when X^{T} is finite) is the following:

Theorem 8 ([1, Theorem 5]). Let X be a Nakajima variety and let

$$T_{\lambda,\mu}(z) = \mathrm{Stab}(\lambda)|_{\mu}, \qquad \lambda,\mu \in X^{\mathsf{T}}$$

be the restriction matrix for elliptic stable envelope in the basis of fixed points. Then, the matrix $W(z) = \{W(z)_{\lambda,\mu}\}_{\lambda,\mu\in X^T}$ from (79) in takes the form:

$$W(z)_{\lambda,\mu} = \frac{T_{\lambda,\mu}(z)}{\Theta(T^{1/2}X_{\mu})},$$

where $\Theta(T^{1/2}X_{\mu})$ is given by (11) applied to Laurent polynomial $T^{1/2}X_{\mu}$. In particular, $\Theta(T^{1/2}X_{\mu})$ does not depend on the Kähler parameters.

The singularities of solutions ^{a} and ^{z} are supported on the singularities of the corresponding q-difference equation. It implies that the transition matrix also may have only these singularities (if $W(z)$ is singular at a hyperplane h, which is not a singularity of q-difference equation then, by (79) Ψ^{a} is also singular along h, which is not possible).

In particular, combining the last theorem with Proposition 8, we obtain

Corollary 5. Let $T_{\lambda,\mu}$ be the restriction matrix of the elliptic stable envelope for the Nakajima variety X in the basis of fixed points. Then, the singularities of $T_{\lambda,\mu}$ are supported to the set of hyperplanes:

$$z_i z_{i+1} \cdots z_j q^p h^{-s} = 1, \qquad 1 \leq i < j \leq n,\ p,s \in \mathbb{Z}$$

This implies that the poles of the restriction matrix $T_{\lambda,\mu}$ in the coordinates u_i related to Kähler variables (55) are of the form:

$$\frac{u_i}{u_j} h^{-s} q^p, \qquad i = j,\ p,s \in \mathbb{Z} \tag{80}$$

9.4.5 *Holomorphicity of stable envelope*

Let us return to the Nakajima varieties X and X defined in Sections 3 and 4, respectively. We identify the fixed points as in Section 6.1 and identify the equivariant and Kähler parameters by (55). Let $T_{\lambda,\mu}$

and $T_{\mathbf{p},\mathbf{q}}$ be the restriction matrices of the elliptic stable envelopes for the Nakajima varieties X and X, respectively.

Theorem 9. The functions

$$T_{\mathbf{p},p}\, T_{\lambda,\mu}$$

are holomorphic in parameters u_i.

Proof. By (80), we need to show that the denominators of functions $T_{\mathbf{p},p}T_{\lambda,\mu}$ do not contain factors of the form

$$\theta\left(\frac{u_i}{u_j}\hbar^{s_{ij}}\right), \quad i \neq j.$$

On the other hand, by Proposition 5, the explicit formula for the elliptic stable envelope on X has the form

$$T'_{\lambda,\mu} = \epsilon(\lambda)\Theta(N'^{-}_{\lambda})_{\mu} \cdot \sum_{\sigma\in} \frac{\mathcal{N}^{\sigma}_{\bar\lambda}}{\mathcal{D}^{\sigma}_{\bar\lambda}} \mathcal{R}^{\sigma}(\bar t)\, \mathcal{W}^{\sigma}(\bar t)\Big|_{\bar t = s_{\mu\lambda}},$$

where $\Theta(N'^{-}_{\lambda})_{\mu}, \mathcal{N}^{\sigma}_{\bar\lambda}, \mathcal{D}^{\sigma}_{\bar\lambda}, \mathcal{R}^{\sigma}(\bar t)$ are independent of u_i, and

$$\mathcal{W}_{\sigma}(\bar t) = \prod \frac{\theta\left(\frac{x_{\sigma(r)}}{x_{\sigma(l)}}\hbar\,\frac{u_{c_l}}{u_{c_{l+1}}}\right)}{\theta\left(\frac{u_{c_l}}{u_{c_{l+1}}}\right)} \prod_{e\in} \frac{\theta\left(\frac{x_{\sigma(t(e))}}{x_{\sigma(h(e))}}\phi_h^{\lambda}(e)\,\frac{u_{c_l}}{u_{c_{l+1}}}\right)}{\theta\left(\frac{u_{c_l}}{u_{c_{l+1}}}\right)}.$$

Therefore, we conclude that among (80) only factors with $s_{ij} = 0$ may appear. To show that those are actually not poles, it suffices to prove that

$$\theta\, u \_\_ u_i\, T_{\mathbf{p},p} T_{\lambda,\mu}\ {}_i\ {}_j = 0.$$

$$j \qquad u = u$$

As discussed before, the only possible nontrivial terms of the left side come from trees ¯t which contains some (i,j)-strip B.

If $j \in \mathbf{p}$, one can see that $\bar\lambda \backslash B$ contains a path in ¯t admitting a box with local maximal content, which is not allowed. In other words, contributions from all ¯t are zero in this case.

If $i \in \mathbf{n}\backslash\mathbf{p}$, then the boxes above and to the left of the root of B both lie in $\bar\lambda$, and the involution inv(t¯) is also a tree in $\bar\lambda$. By the cancellation Lemma 7, contribution from t¯ cancels with that from inv(t¯). Sum over all t¯ gives 0.

If $i \in \mathbf{p}$ and $j \in \mathbf{n}\backslash\mathbf{p}$, then $T_{\mathbf{p},p}$ contains a factor $\theta\_\_u^i$, and nontrivial terms come u_j

from trees t¯ that contains at least two (i,j)-strips, for example, B_1, B_2. At least one of them, say B_1, is not contained in the boundary of $\bar\lambda$ and hence the involution of t¯ with respect to B_1 is well defined. Contribution from t¯ then cancels with that from inv(t¯). Therefore, we exclude all possible poles $\theta(u_i/u_j)$, and $T_{\mathbf{p},p} T_{\lambda,\mu}$ is holomorphic in u_i.

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