DYNAMICAL sl_2 BETHE ALGEBRA AND FUNCTIONS ON PAIRS OF QUASI-POLYNOMIALS

A.SLINKIN[1], D.THOMPSON[2], A.VARCHENKO[3]

◇,∗,⋆

National Research University Higher School of Economics

20 Myasnitskaya Street, 101000 Moscow, Russia

[?]Faculty of Mathematics and Mechanics, Lomonosov Moscow State University Leninskiye Gory 1, 119991 Moscow GSP-1, Russia

Moscow Center of Fundamental and Applied Mathematics Leninskiye Gory 1, 119991 Moscow GSP-1, Russia

On the Occasion of the 70th Birthday of Igor Krichiver

Abstract. We consider the space $\mathrm{Fun}_{\mathrm{sl}_2} V[0]$ of functions on the Cartan subalgebra of sl_2 with values in the zero weight subspace $V[0]$ of a tensor product of irreducible finitedimensional sl_2-modules. We consider the algebra B of commuting differential operators on $\mathrm{Fun}_{\mathrm{sl}_2} V[0]$, constructed by V.Rubtsov, A.Silantyev, D.Talalaev in 2009. We describe the relations between the action of B on $\mathrm{Fun}_{\mathrm{sl}_2} V[0]$ and spaces of pairs of quasi-polynomials.

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[1] *E-mail*: slinalex@live.unc.edu

[2] *E-mail*: dthomp@email.unc.edu

[3] *E-mail*: anv@email.unc.edu, supported in part by NSF grants DMS-1665239, DMS-1954266

A.SLINKIN, D.THOMPSON, A.VARCHENKO

1. Introduction

A quantum integrable model is a vector space V and an algebra B of commuting linear operators on V, called the Bethe algebra of Hamiltonians. The problem is to find eigenvectors and eigenvalues. If the vector space is a space of functions, then the Hamiltonians are differential or difference or integral operators.[1]

We say that a quantum integrable model can be geometrized, if there is a topological space (a scheme) X with an algebra O_X of functions on X, an isomorphism of vector spaces $\psi : O_X \to V$, an isomorphism of algebras $\tau : O_X \to B$ such that

$$\psi(fg) = \tau(f)\psi(g), \qquad \forall f,g \in O_X.$$

These objects O_X, ψ, τ identify the B-module V with the regular representation of the algebra O_X of functions.

If a quantum integrable model (V,B) is geometrized, then the eigenvectors of B in V are identified with delta-functions of points of X and the eigenvalues of an eigenvector in V correspond to evaluations of functions on X at the corresponding point of X.

Our motivation to geometrize the Bethe algebras came from the examples considered in [MTV3, MTV5], where the algebra of Hamiltonians acting on a subspace of a tensor product of gl_N-modules was identified with the algebra of functions on the intersection of suitable Schubert cycles in a Grassmannian. That identification gave an unexpected relation between the representation theory and Schubert calculus.

The examples in [MTV3, MTV5] are related to models with a finite-dimensional vector space V. How to proceed in the infinite-dimensional case of commuting differential operators is not clear yet. In this paper we discuss an example. In our infinite-dimensional space V we distinguish a family of finite-dimensional subspaces $E[\mu]$, $\mu \in C$, each of which is invariant with respect to the algebra B of commuting differential operators. We geometrize each of the pairs ($E[\mu], \mathcal{B}|_{E[\mu]}$), thus constructing a family of topological spaces $X[\mu]$, $\mu \in C$. We observe that natural interrelations between the subspaces $E[\mu]$ correspond to natural interrelations between the topological spaces $X[\mu]$. For example, the Weyl involution $V \to V$, available in our case, identifies $E[\mu]$ and $E[-\mu]$. We show that this identification corresponds to a natural isomorphism $X[\mu] \to X[-\mu]$.

Representation theory provides a source of commuting differential or difference operators. In this paper we discuss the construction due to V.Rubtsov, A.Silantyev, D.Talalaev, [RST]. That quantum integrable model is called the *quantum dynamical Gaudin*

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[1] See examples of commuting integral operators in [Ko, EFKa]

model. We study the sl$_2$ trigonometric version of the quantum dynamical Gaudin model, while in [RST] the gl$_N$ elliptic version was considered.

Consider the Lie algebra sl$_2$ and its Cartan subalgebra h $\subset$ sl$_2$, dimh = 1. For s = 1,...,n, let V_{m_s} be the irreducible sl$_2$-module of dimension m_s + 1. Let $V = \otimes_{s=1}^{n} V_{m_s}$,

$$V[0] = \{v \in V \mid hv = 0, \forall h \in \mathsf{h}\},$$

the zero weight subspace. The space $V[0]$ is nontrivial if $M = \sum_{s=1}^{n} m_s$ is even. Let Fun$_{\mathsf{sl}_2}V[0]$ be the space of $V[0]$-valued functions on h. Fix a subset $z = \{z_1,...,z_n\} \subset \mathsf{C}^\times$. Having these data, Rubtsov, Silantyev, and Talalaev construct a family of commuting differential operators acting on Fun$_{\mathsf{sl}_2}V[0]$ as follows.

First, one constructs a 2 × 2-matrix $\begin{bmatrix} x\partial_x & 0 \\ 0 & x\partial_x \end{bmatrix} + L(x) = \begin{bmatrix} \delta_{ij} x\partial_x + L_{ij}(x) \end{bmatrix}$, where x is a parameter, $\partial_x = \frac{\partial}{\partial x}$, and $L_{ij}(x)$ are some special differential operators on Fun$_{\mathsf{sl}_2}V[0]$ depending on x. Let $C = \mathrm{cdet}\begin{bmatrix} \delta_{ij} x\partial_x + L_{ij}(x) \end{bmatrix}$, where *cdet* is the column determinant of the matrix with non-commuting entries, cdet $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb.$ Using the new variable u with $x = e^{-2\pi\sqrt{-1}u}$, the operator C can be written in the form

$$\partial_u^2 + C_1(x)\partial_u + C_2(x),$$

where $C_1(x), C_2(x)$ are differential operators on Fun$_{\mathsf{sl}_2}V[0]$, whose coefficients are rational functions of x. It turns out that for any $a,b \in \mathsf{C} - \{z_1,...,z_n\}$ and i,j = 1,2 the operators $C_i(a)$, $C_j(b)$ commute. The space Fun$_{\mathsf{sl}_2}V[0]$ with the algebra B generated by these commuting differential operators is called the *quantum dynamical Gaudin model.*

We show that the algebra B is generated by the trigonometric differential KZB operators H_0, $H_1(z)$, ..., $H_n(z)$, see them in formula (2.13) and [FW, JV]. The KZB operator H_0 is also known as the trigonometric Hamiltonian operator of the quantum two-particle CalogeroMoser system with spin space V. The operator H_0 is a second order differential operator independent of z.

For any $\mu \notin \mathsf{Z}$ we define in Section 3.2 the subspace $E[\mu] \subset$ Fun$_{\mathsf{sl}_2}V[0]$ as the space of meromorphic eigenfunctions of H_0 with eigenvalue $\pi\sqrt{-1}\frac{\mu^2}{2}$ and prescribed poles. The subspaces $E[\mu]$ were introduced in [FV2] and studied in [JV]. We have dim$E[\mu]$ = dim$V[0]$. The Bethe algebra B preserves each of $E[\mu]$.

The sl$_2$ Weyl involution acts on Fun$_{\mathsf{sl}_2}V[0]$ and induces an isomorphism $E[\mu] \to E[-\mu]$, which is called in [FV2] the scattering matrix. The Bethe algebra B is Weyl group invariant. The scattering matrix $E[\mu] \to E[-\mu]$ is an isomorphism of B-modules.

The basis of the geometrization procedure lies in the following observation. Let $\psi \in E[\mu]$ be an eigenvector of B,

$$C_i(x)\psi = E_i(x,\psi)\psi, \qquad i = 1,2,$$

where $E_i(x,\psi)$ are scalar eigenvalue functions of the eigenvector ψ. We assign to ψ the scalar differential operator

$$\mathcal{E}_\psi = \partial_u^2 + E_1(x,\psi)\partial_u + E_2(x,\psi).$$

We show that the kernel of E_ψ is spanned by two quasi-polynomials $x^{-\mu/2}f(x), x^{\mu/2}g(x)$, where $f(x),g(x)$ are monic polynomials of degree $M/2$, with the property that the Wronskian of the two quasi-polynomials is

(1.1) $$\mathrm{Wr}(x^{-\mu/2}f(x), x^{\mu/2}g(x)) = \frac{\mu}{x}\prod_{s=1}^{n}(x-z_s)^{m_s}.$$

This fact suggests that the space $X[\mu]$ geometrizing $(E[\mu], \mathcal{B}|_{E[\mu]})$ is the space of pairs $(x^{-\mu/2}f(x), x^{\mu/2}g(x))$ of quasi-polynomials with Wronskian given by (1.1).

In this paper we show that this is indeed so. We show that the space of functions $E[\mu]$, with the commutative algebra $\mathcal{B}|_{E[\mu]}$ of differential operators acting on $E[\mu]$, is isomorphic to the algebra of functions on the space $X[\mu]$ of pairs of quasi-polynomials $(x^{-\mu/2}f(x), x^{\mu/2}g(x))$ with Wronskian given by (1.1). In particular, it means that the eigenfunctions of the algebra $\mathcal{B}|_{E[\mu]}$ are in bijection with the points of $X[\mu]$, see Corollary 11.6.

We also show that the scattering matrix isomorphism $E[\mu] \to E[-\mu]$ corresponds to the natural isomorphism $X[\mu] \to X[-\mu]$ defined by the transposition of the quasi-polynomials, $(x^{-\mu/2}f(x),x^{\mu/2}g(x)) \mapsto (x^{\mu/2}g(x),x^{-\mu/2}f(x))$.

The results of this paper indicate the deep relation between the quantum dynamical Gaudin model $(B,\mathrm{Fun}_{sl_2}V[0])$ and the spaces of pairs of quasi-polynomials.

It would be interesting to develop the elliptic version of this correspondence. In the elliptic version the pairs of quasi-polynomials are replaced with pairs of theta-polynomials, see [ThV], but the elliptic KZB operator H_0 does depend on z and does not have apparent analogs of the family of subspaces $E[\mu]$.

The paper is organized as follows. In Section 2 we define the sl_2 quantum dynamical Gaudin model. In Section 3 we discuss properties of the spaces $E[\mu]$. In Section 4 we introduce the quantum trigonometric Gaudin model $(V[v],B(z,\mu,V[v])$ on a weight subspace $V[v] \subset V$ and show that the quantum dynamical Gaudin model $(E[\mu], \mathcal{B}|_{E[v]})$ is isomorphic to the quantum trigonometric Gaudin model $(V[0],B(z,\mu,V[0])$ on the zero weight subspace. In Section 5 we describe the Bethe ansatz for the quantum trigonometric Gaudin model. In Sections 6 and 7 we describe the kernel of the operator E_ψ. In Sections 8 - 11 we develop the geometrization procedure. The constructions of Sections 9 - 11 are parallel to the geometrization constructions in [MTV3, MTV2].

2. Quantum dynamical Gaudin model

2.1. gl_2 *RST*-**operator.** Consider the complex Lie algebra gl_2 with standard basis e_{11}, e_{12}, e_{21}, e_{22}. Denote by h the Cartan subalgebra of gl_2 with basis e_{11}, e_{22} and elements $\lambda_1 e_{11} + \lambda_2 e_{22}$. Denote

$$\lambda := \lambda_1 - \lambda_2.$$

Let $z = \{z_1,...,z_n\} \subset C^\times$ be a set of nonzero pairwise distinct numbers.

Let $V^1,...,V^n$ be gl_2-modules and $V = \otimes^n_{k=1} V^k$. Let $V = \bigoplus_{v \in h^*} V[v]$ be the weight decomposition, where $V[v] = \{v \in V \mid e_{jj}v = v(e_{jj})v \text{ for } j = 1,2\}$. In particular,

$$V[0] = \{v \in V \mid e_{11}v = e_{22}v = 0\}.$$

For $g \in gl_2$, denote $g^{(s)} = 1 \otimes \cdots \otimes g \otimes \cdots \otimes 1 \in \text{End}(V)$, with g in the s-th factor. An element e_{jk} acts on V by $e^{(1)}_{jk} + \cdots + e^{(n)}_{jk}$.

Let u be a variable. Denote

$$= e^{-2\pi \sqrt{-1}u \cdot x}$$

Let $\partial_u = \frac{\partial}{\partial u}, \quad \partial_x = \frac{\partial}{\partial x}, \quad \partial_{\lambda_j} = \frac{\partial}{\partial \lambda_j}$ and so on.

Introduce a 2×2-matrix L,

(2.1)

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} \pi\sqrt{-1} \sum_{s=1}^{n} \frac{z_s+x}{z_s-x} e^{(s)}_{11} + \pi\cot(\pi\lambda)e_{22} & \pi\sqrt{-1}\sum_{s=1}^{n}\frac{z_s+x}{z_s-x} e^{(s)}_{21} - \pi\cot(\pi\lambda)e_{21} \\ \pi\sqrt{-1}\sum_{s=1}^{n}\frac{z_s+x}{z_s-x} e^{(s)}_{12} + \pi\cot(\pi\lambda)e_{12} & \pi\sqrt{-1}\sum_{s=1}^{n}\frac{z_s+x}{z_s-x} e^{(s)}_{22} - \pi\cot(\pi\lambda)e_{11} \end{bmatrix}.$$

The entries of L are $\text{End}(V)$-valued trigonometric functions of u and λ.

The *universal dynamical differential operator* (or the *RST-operator*) is defined by the formula

(2.2)
$$C = \text{cdet}(\delta_{jk}\partial_u - \delta_{jk}\partial_{\lambda_j} + L_{jk}),$$

where for a 2×2-matrix $A = (a_{jk})$ with noncommuting entries the column determinant is defined by the formula

$$\text{cdet}A = a_{11}a_{22} - a_{21}a_{12}.$$

Write the *RST*-operator in the form

(2.3)
$$\mathcal{C} = \partial_u^2 + C_1(x)\,\partial_u + C_2(x),$$

where $C_1(x)$ and $C_2(x)$ are functions in x with values in the space of linear differential operators in variables λ_1, λ_2 with coefficients in $\text{End}(V)$.

Theorem 2.1 ([RST])**.** *Fix $z = \{z_1,...,z_n\} \subset C^\times$. Then for any $a \in C − \{z_1,...,z_n\}$ the operators $C_1(a),C_2(a)$, restricted to $V[0]$-valued functions of λ_1,λ_2, define linear differential operators in λ_1,λ_2 with coefficients in* End$(V[0])$*. Moreover, for any $a,b \in C−\{z_1,...,z_n\}$, the differential operators $C_j(a)$, $C_k(b)$, $j,k = 1,2$, acting on the space of $V[0]$-valued functions of λ_1,λ_2 commute:*

$$(2.4) \qquad\qquad [C_j(a),C_k(b)] = 0, \qquad j,k = 1,2.$$

The elliptic *RST*-operator for gl$_N$ was introduced by V.Rubtsov, A.Silantyev, D.Talalaev in [RST]. The elliptic gl$_2$ *RST*-operator was discussed in [ThV]. The *RST*-operator, defined in (2.3), is the trigonometric degeneration of the elliptic gl$_2$ *RST*-operator.

2.2. **Dynamical Bethe algebra of** Fun$_{sl_2}V[0]$**.** In this paper, we are interested in the sl$_2$ version of the *RST*-operator.

The Lie algebra sl$_2$ is a Lie subalgebra of gl$_2$. We have gl$_2$ = sl$_2 \bigoplus C(e_{11} + e_{22})$, where $e_{11} + e_{22}$ is a central element. Let $V^1,...,V^n$ be sl$_2$-modules, thought of as gl$_2$-modules, where the central element $e_{11} + e_{22}$ acts by zero. Let $V = \otimes_{k=1}^n V^k$ be the tensor product of the sl$_2$-modules.

In this paper *we consider only such tensor products.*

We consider the Cartan subalgebra of sl$_2$ consisting of elements $\lambda_1 e_{11}+\lambda_2 e_{22}$ with $\lambda_1+\lambda_2 = 0$. We identify the algebra of functions on the Cartan subalgebra of sl$_2$ with the algebra of functions in the variable

$$\lambda = \lambda_1 − \lambda_2,$$

since the elements $\lambda_1 e_{11} + \lambda_2 e_{22}$ with $\lambda_1 + \lambda_2 = 0$ are uniquely determined by the difference of coordinates.

Denote by Fun$_{sl_2}V[0]$ the space of $V[0]$-valued meromorphic functions on the Cartan subalgebra of sl$_2$. In other words, Fun$_{sl_2}V[0]$ is the space of $V[0]$-valued meromorphic functions in one variable λ.

Each coefficient $C_1(x),C_2(x)$ of the *RST*-operator, defines a differential operator acting on Fun$_{sl_2}V[0]$. From now on *we consider the coefficients $C_1(x)$, $C_2(x)$ as a family of commuting differential operators on* Fun$_{sl_2}V[0]$, depending on the parameter x.

The commutative algebra of differential operators on Fun$_{sl_2}V[0]$ generated by the identity operator and the operators $\{C_j(a) \mid j = 1,2, a \in C − \{z_1,...,z_n\}\}$ is called the *dynamical Bethe algebra* of Fun$_{sl_2}V[0]$. The dynamical Bethe algebra depends on the choice of the numbers $\{z_1,...,z_n\}$.

2.3. **Tensor product of** sl$_2$-**modules.** Given $m \in Z_{>0}$, denote by V_m the irreducible sl$_2$module with highest weight m. It has a basis $v_0^m, \cdots, v_m^m$ such that

$$(2.5) \quad (e_{11} − e_{22})v_k^m = (m − 2k)v_k^m, \quad e_{21}v_k^m = (k + 1)v_{k+1}^m, \quad e_{12}v_k^m = (m − k + 1)v_{k-1}^m.$$

From now on our tensor product V is of the form

$$(2.6) \qquad V = \otimes_{s=1}^{n} V_{m_s}, \qquad m_s \in \mathbb{Z}_{>0}.$$

We have the weight decomposition $V = \oplus_{v \in \mathbb{Z}} V[v]$ consisting of weight subspaces

$$(2.7) \qquad V[v] = \{v \in V \mid (e_{11} - e_{22})v = vv\}.$$

If $V[v]$ is nonzero, then

$$(2.8) \qquad v = \overset{n}{\underset{s=1}{X}} m_s - 2k,$$

for some nonnegative integer k. The dimension of $V[0]$ is positive if the sum$\sum_{s=1}^{n} m_s$ is even.

2.4. **Operator** $A(\mu)$. The $\mathfrak{sl}_2$ Weyl group W consists of two elements: identity and involution σ. The projective action of W on V_m is given by the formula

$$\sigma : v_k^m \mapsto (-1)^k v_{m-k}^m$$

for any k. We have $\sigma^2 = (-1)^m$. The Weyl group W acts on the tensor product V diagonally.

Following [TV], introduce

$$p(\mu) = \sum_{k=0}^{\infty} e_{21}^k e_{12}^k \frac{1}{k!} \prod_{j=0}^{k-1} \frac{1}{\mu + e_{22} - e_{11} - j}, \qquad \mu \in \mathbb{C}$$

.

The series $p(\mu)$ acts on V_m, since only a finite number of terms acts nontrivially. The formula for the action of $p(\mu)$ on a basis vector v_k^m becomes more symmetric if μ is replaced by $\mu + \frac{v}{2} - 1$, where $v = m - 2k$ is the weight of v_k^m,

$$p\left(\mu + \frac{v}{2} - 1\right) v_k^m = \prod_{j=0}^{k-1} \frac{\mu + \frac{m}{2} - j}{\mu - \frac{m}{2} + j} v_k^m,$$

$$(2.9)$$

see [TV, Section 2.5].

The series $p(\mu)$ acts on the tensor product V in the standard way. Introduce the operator

$$(2.10) \qquad A(\mu) : V \to V, \qquad v \mapsto \sigma p(\mu)v.$$

The operator $A(\mu)$ is a meromorphic function of μ. For any v, we have $A(\mu)V[v] \subset V[-v]$, and $\lim_{\mu \to \infty} A(\mu) = \sigma$. The operator $A(\mu)$ may be considered as a deformation of the Weyl group operator σ.

$$V = \otimes_{s=1}^{n} V_{m_s} \ as \ in \ (2.6) \qquad M = \sum_{s=1}^{n} m_s \qquad \mu \notin \frac{M}{2} + \mathbb{Z}$$

Then for any v the operator

$$\mathcal{A}\left(\mu + \frac{v}{2} - 1\right)\Big|_{V[v]} : V[v] \to V[-v]$$

$$(2.11)$$

is an isomorphism of vector spaces. The composition of the operator $\mathcal{A}\left(\mu + \frac{v}{2} - 1\right)\Big|_{V[v]}$ *and the operator*

$$\mathcal{A}\left(-\mu - \frac{v}{2} + 1\right)\Big|_{V[-v]} : V[-v] \to V[v]$$

$$(2.12)$$

is the scalar operator on $V[\nu]$ of multiplication by $(-1)^{M}\frac{\mu-\nu/2}{\mu+\nu/2}$.

Proof. The $\mathfrak{sl}_2$ irreducible decomposition $V = \oplus_m V_m$ of the tensor product V has the highest weights m of the form $m = M - 2k$ for $k \in \mathbb{Z}_{>0}$, only. Now (2.11) is an isomorphism by formula (2.9). The statement on the composition is [TV, Theorem 10].

Remark. The operator $A(\mu)$ is the (only) nontrivial element of the $\mathfrak{sl}_2$ dynamical Weyl group of V, see definitions in [EV].

2.5. **KZB operators.** Introduce the following elements of $\mathfrak{gl}_2 \otimes \mathfrak{gl}_2$,

$$\Omega_{12} = e_{12} \otimes e_{21}, \qquad\qquad \Omega_{21} = e_{21} \otimes e_{12},$$
$$\Omega_0 = e_{11} \otimes e_{11} + e_{22} \otimes e_{22}, \qquad \Omega = \Omega_0 + \Omega_{12} + \Omega_{21}.$$

The *KZB operators* $H_0, H_1(z),...,H_n(z)$ are the following differential operators in variables λ_1, λ_2 acting on the space $\mathrm{Fun}_{\mathfrak{sl}_2} V[0]$,

(2.13)

$$H_0 = \frac{1}{4\pi\sqrt{-1}}(\partial_{\lambda_1}^2 + \partial_{\lambda_2}^2) + \frac{\pi\sqrt{-1}}{4}\sum_{s,t=1}^{n}\left[\frac{1}{2}\Omega_0^{(s,t)} + \frac{1}{\sin^2(\pi\lambda)}\left(\Omega_{12}^{(s,t)} + \Omega_{21}^{(s,t)}\right)\right],$$

$$H_s(z) = -(e_{11}^{(s)}\partial_{\lambda_1} + e_{22}^{(s)}\partial_{\lambda_2}) + \sum_{t:t\neq s}\left[\pi\sqrt{-1}\frac{z_t + z_s}{z_t - z_s}\Omega^{(s,t)} - \pi\cot(\pi\lambda)\left(\Omega_{12}^{(s,t)} - \Omega_{21}^{(s,t)}\right)\right],$$

cf. formulas in Section 3.4 of [JV]. The elliptic KZB operators were introduced in [FW]. In (2.13) we consider the trigonometric degeneration of the elliptic KZB operators.

By [FW] the operators $H_0, H_1(z),...,H_n(z)$ commute and $\sum_{s=1}^{n}H_s(z) = 0$.

Remark. The differential operator H_0 is the *Hamiltonian operator of the trigonometric quantum two-particle system with spin space V*.

2.6. **Coefficients $C_1(x)$, $C_2(x)$.**

Lemma 2.3. *We have*
$$C_1(x) = \mathcal{L}_{11}^0(x) + \mathcal{L}_{22}^0(x) - \partial_{\lambda_1} - \partial_{\lambda_2}.$$

Hence the coefficient $C_1(x)$ acts by zero on $\mathrm{Fun}_{\mathfrak{sl}_2} V[0]$. **Corollary 2.4.** *The RST-operator* (2.3) *has the form*
(2.14) $\mathcal{C} = \partial_u^2 + C_2(x)$ *as an operator on* $\mathrm{Fun}_{\mathfrak{sl}_2} V[0]$.

Theorem 2.5 ([ThV]). *We have*

(2.15)

$$C_2(x) = -2\pi\sqrt{-1}H_0 - \sum_{s=1}^{n}\left[2\pi\sqrt{-1}\frac{H_s(z)}{1-x/z_s} + 4\pi^2\left(-\frac{c_2^{(s)}}{1-x/z_s} + \frac{c_2^{(s)}}{(1-x/z_s)^2}\right)\right],$$

where $c_2 = e_{11}e_{22} - e_{12}e_{21} + e_{11}$ is a central element of $\mathfrak{gl}_2$.

Proof. This is the trigonometric degeneration of the elliptic version of this theorem, see [ThV, Theorem 4.9].

Corollary 2.6. *The dynamical Bethe algebra of* $\mathrm{Fun}_{\mathfrak{sl}_2}V[0]$ *is generated by the identity operator and the KZB operators* $H_0, H_1(z),...,H_n(z)$.

The commutativity of the KZB operators and formulas (2.14), (2.15) imply the commutativity $[C_2(a), C_2(b)] = 0$ independently of Theorem 2.1.

2.7. $\mathfrak{sl}_2$ **Weyl group invariance.** The Weyl group acts on $V[0]$ as explained in Section 2.4. Hence the Weyl group acts on $\mathrm{Fun}_{\mathfrak{sl}_2}V[0]$ by the formula

(2.16) $$\sigma : \psi(\lambda) \mapsto \sigma(\psi(-\lambda)), \qquad \psi \in \mathrm{Fun}_{\mathfrak{sl}_2}V[0].$$

This action extends to a Weyl group action on $\mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_2}V[0])$, where for $T \in \mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_2}V[0])$ the operator $\sigma(T)$ is defined as the product $\sigma T \sigma^{-1}$ of the three elements of $\mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_2}V[0])$.

Lemma 2.7 ([ThV]). *For any* $a \in \mathbb{C}-\{z_1,...,z_n\}$ *the operator* $C_2(a) \in \mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_2}V[0])$ *is Weyl group invariant.*

Proof. By [FW] the KZB operators $H_0, H_1(z),...,H_n(z)$ are Weyl group invariant. The lemma follows from formula (2.15).

3. Eigenfunctions of H_0

3.1. **Trigonometric Gaudin operators.** The trigonometric r-matrix is defined by

(3.1) $$r(x) = \frac{\Omega_+ x + \Omega_-}{x-1},$$

where $\Omega_+ = \frac{1}{2}\Omega_0 + \Omega_{12}$, $\Omega_- = \frac{1}{2}\Omega_0 + \Omega_{21}$.

For $\mu \in \mathbb{C}$ the trigonometric Gaudin operators acting on V are defined as

(3.2) $$\mathcal{K}_s(z,\mu) = \frac{\mu}{2}(e_{11} - e_{22})^{(s)} + \sum_{t:t\neq s}r^{(s,t)}(z_s/z_t), \qquad s = 1,\ldots,n.$$

Each operator $K_s(z,\mu)$ preserves each of the weight subspaces $V[\nu]$ and

$$[K_s(z,\mu), K_t(z,\mu)] = 0$$

for all s,t, see [Ch, EFK].

3.2. **Dynamical Bethe algebra of** $E(\mu)$**.** Let $\sqrt{}$

Let A be the algebra of functions in λ, which can be represented as meromorphic functions of Λ with poles only at the set $\{\Lambda = 1\}$.

For $\mu \in \mathbb{C}$ introduce the A-module $A[\mu]$ of functions of the form $e^{\pi\sqrt{-1}\mu\lambda}f$, where $f \in A$. This module is preserved by derivatives with respect to λ_1, λ_2. Therefore the KZB operator H_0 preserves the space $A[\mu] \otimes V[0]$. Any $\psi \in A[\mu] \otimes V[0]$ has the form

$$\psi(\lambda) = e^{\pi\sqrt{-1}\mu\lambda}\sum_{k=0}^{\infty}\Lambda^k\psi^k, \qquad \psi^k \in V[0].$$

Theorem 3.1 ([FV2]). *Let $\mu \notin \mathbb{Z}_{>0}$. Then for any nonzero $v \in V[0]$, there exists a unique $\psi \in A[\mu] \otimes V[0]$ such that*

$$H_0\psi = \epsilon\psi,$$

for some $\epsilon \in \mathbb{C}$ and $\psi^0 = v$. Moreover, $\epsilon = \pi\sqrt{-1}\frac{\mu^2}{2}$.

Cf. [JV]. This function ψ is denoted by ψ_v.

We denote by $E[\mu]$ the vector space of functions $\psi \in A[\mu] \otimes V[0]$ such that $H_0\psi = \pi\sqrt{-1}\frac{\mu^2}{2}\psi$. For more information on this space see [JV, Section 9].

For the zero vector 0 in $V[0]$ we define ψ_0 to be the zero function of $A[\mu] \otimes V[0]$.

Corollary 3.2. *For $\mu \notin \mathbb{Z}_{>0}$, the map*

(3.3) $V[0] \to E[\mu]$, $v \mapsto \psi_v$, *is an isomorphism of complex vector spaces.*

Theorem 3.3 ([JV]). *Let $\mu \notin \mathbb{Z}_{>0}$. Then for $s = 1,...,n$, the KZB operators $H_s(z)$ preserve the space $E[\mu]$. Moreover, for any nonzero $v \in V[0]$ we have*

$$H_s(z)\psi_v = \psi_w,$$

where $w = -2\pi\sqrt{-1}K_s(z,\mu)v$.

Theorem 3.4. *Let $\mu \notin \mathbb{Z}_{>0}$, $V = \otimes_{s=1}^{n}V_{m_s}$, and $v \in V[0]$. Then*

$$C_2(x)\psi_v = \psi_w,$$

where

(3.4) $$w = (2\pi\sqrt{-1})^2\left[-\frac{\mu^2}{4} + \sum_{s=1}^{n}\left[\frac{m_s(m_s+2)/4 + \mathcal{K}_s(z,\mu)}{1 - x/z_s} - \frac{m_s(m_s+2)/4}{(1 - x/z_s)^2}\right]\right]v.$$

Proof. One computes the action of $C_2(X)$ on ψ_v using Theorem 2.5. The computation is based on Theorem 3.3 and the fact that c_2 acts on V_{m_s} as multiplication by $-\frac{m_s(m_s+2)}{4}$.

By Theorem 3.3 the subspace $E[\mu] \subset \mathrm{Fun}_{\mathfrak{sl}_2}V\,[0]$ is invariant with respect to the action of the dynamical Bethe algebra. The restriction of the dynamical Bethe algebra to $E[\mu]$ is called the *dynamical Bethe algebra* of $E[\mu]$ and denoted by $\mathrm{B}(z;E[\mu])$.

Notice that $E[\mu]$ is a finite-dimensional vector space of dimension $\dim V\,[0]$. The space $E[\mu]$ does not depend on z, since the KZB operator H_0 does not depend on z. The algebra $\mathrm{B}(z;E[\mu])$ is generated by the identity operator and the KZB operators $H_1(z),...,H_s(z)$ and does depend on z.

3.3. Two-particle scattering matrix.

Theorem 3.5 ([FV2, Lemma 6.2]). *For $\mu \notin \mathbb{Z}$, the action* (2.16) *of the Weyl group involution σ on* $\mathrm{Fun}_{\mathfrak{sl}_2}V\,[0]$ *identifies the spaces $E[\mu]$ and $E[-\mu]$. More precisely, for any $v \in V\,[0]$ we have*

(3.5)
$$\sigma(\psi_v^\mu(-\lambda)) = \psi_{\mathcal{A}(\mu-1)v}^{-\mu}(\lambda)$$

where $\psi_v^\mu(\lambda)$ is the element of $E[\mu]$ with initial term v and $\psi_{\mathcal{A}(\mu-1)v}^{-\mu}(\lambda)$ is the element of $E[-\mu]$ with initial term $\mathrm{A}(\mu-1)v$. Here $\mathrm{A}(\mu-1) : V\,[0] \to V\,[0]$ is the vector isomorphism, defined in (2.10).

Proof. Formula (3.5) is proved in the example next to Lemma 6.2 in [FV2].

4. Quantum trigonometric Gaudin model

4.1. Universal differential operator. Let $V = \otimes_{s=1}^n V_{m_s}$. Introduce a 2 × 2-matrix

(4.1)
$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix} = -2\pi\sqrt{-1}\sum_{s=1}^n r^{(0,s)}(x/z_s)$$

where $r(x)$ is the trigonometric r-matrix defined in (3.1). More explicitly,

$$\mathcal{M} = \begin{bmatrix} 2\pi\sqrt{-1}\sum_{s=1}^n \frac{1}{1-x/z_s}e_{11}^{(s)} - \pi\sqrt{-1}e_{11} & 2\pi\sqrt{-1}\sum_{s=1}^n \frac{1}{1-x/z_s}e_{21}^{(s)} - 2\pi\sqrt{-1}e_{21} \\ 2\pi\sqrt{-1}\sum_{s=1}^n \frac{1}{1-x/z_s}e_{12}^{(s)} & 2\pi\sqrt{-1}\sum_{s=1}^n \frac{1}{1-x/z_s}e_{22}^{(s)} - \pi\sqrt{-1}e_{22} \end{bmatrix}.$$

The *universal (trigonometric) differential operator* for V with parameter $\mu \in \mathbb{C}$ is defined by the formula

(4.2)
$$\mathcal{D} = \mathrm{cdet}\begin{bmatrix} \partial_u - \pi\sqrt{-1}\mu + \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \partial_u + \pi\sqrt{-1}\mu + \mathcal{M}_{22} \end{bmatrix}.$$

Write the operator D in the form

$$\mathcal{D} = \partial_u^2 + D_1(x)\partial_u + D_2(x),$$

where $D_1(x)$, $D_2(x)$ are $\mathrm{End}(V)$-valued functions of x. It is clear that D commutes with the action on V of the Cartan subalgebra of $\mathfrak{sl}_2$. In particular, it means that $D_1(x)$, $D_2(x)$ preserve the weight decomposition of V.

4.2. Coefficients $D_1(x)$, $D_2(x)$ and Gaudin operators.

Theorem 4.1. *We have $D_1(x) = 0$ and $(2\pi\sqrt{-1})^{-2}D_2(x)$ equals*

$$(4.3) \quad -\frac{\mu^2 + \mu(e_{11} - e_{22}) - e_{11}e_{22}}{4} + \sum_{s=1}^{n} \left[\frac{m_s(m_s+2)/4 + \mathcal{K}_s(Z, \mu)}{1 - x/z_s} - \frac{m_s(m_s+2)/4}{(1 - x/z_s)^2} \right].$$

Proof. The proof is by straightforward calculation. We have

$$\mathcal{D} = \left(\partial_u - \pi\sqrt{-1}\mu + 2\pi\sqrt{-1}\sum_{s=1}^{n} \frac{1}{1 - x/z_s} e_{11}^{(s)} - \pi\sqrt{-1}e_{11} \right)$$

$$\times \left(\partial_u + \pi\sqrt{-1}\mu + 2\pi\sqrt{-1}\sum_{s=1}^{n} \frac{1}{1 - x/z_s} e_{22}^{(s)} - \pi\sqrt{-1}e_{22} \right)$$

$$- \left(2\pi\sqrt{-1}\sum_{s=1}^{n} \frac{1}{1 - x/z_s} e_{12}^{(s)} \right) \left(2\pi\sqrt{-1}\sum_{s=1}^{n} \frac{1}{1 - x/z_s} e_{21}^{(s)} - 2\pi\sqrt{-1}e_{21} \right).$$

Then

$$D_1(x) = 2\pi\sqrt{-1}\sum_{s=1}^{n} \frac{e_{11}^{(s)} + e_{22}^{(s)}}{1 - x/z_s} - \pi\sqrt{-1}(e_{11} + e_{22}) = 0.$$

Since $x = e^{-2\pi\sqrt{-1}u}$ and $\partial_u = -2\pi\sqrt{-1}x\partial_x$, the coefficient of $(2\pi\sqrt{-1})^{-2}D_2(x)$ equals

$$(4.4) \quad -\frac{\mu^2}{4} - \sum_{s=1}^{n} \frac{e_{22}^{(s)}}{(1 - x/z_s)^2} + \sum_{s=1}^{n} \frac{e_{22}^{(s)}}{1 - x/z_s} + \frac{\mu}{2}\sum_{s=1}^{n} \frac{e_{11}^{(s)} - e_{22}^{(s)}}{1 - x/z_s} - \frac{\mu}{4}(e_{11} - e_{22})$$

$$+ \sum_{s=1}^{n} \frac{e_{11}^{(s)}e_{22}^{(s)}}{(1 - x/z_s)^2} + \sum_{s=1}^{n} \left(\sum_{t: t\neq s} \frac{e_{11}^{(s)}e_{22}^{(t)} + e_{11}^{(t)}e_{22}^{(s)}}{1 - z_s/z_t} \right) \frac{1}{1 - x/z_s} - \sum_{s=1}^{n} \frac{e_{11}^{(s)}e_{22}^{(s)}}{1 - x/z_s}$$

$$- \sum_{s=1}^{n} \left(\sum_{t: t\neq s} e_{22}^{(t)} \right) \frac{e_{11}^{(s)}}{1 - x/z_s} + \frac{1}{4}e_{11}e_{22} - \sum_{s=1}^{n} \frac{e_{12}^{(s)}e_{21}^{(s)}}{(1 - x/z_s)^2}$$

$$- \sum_{s=1}^{n} \left(\sum_{t: t\neq s} \frac{e_{12}^{(s)}e_{21}^{(t)} + e_{12}^{(t)}e_{21}^{(s)}}{1 - z_s/z_t} \right) \frac{1}{1 - x/z_s} + \sum_{s=1}^{n} \frac{e_{12}^{(s)}e_{21}^{(s)}}{1 - x/z_s} + \sum_{s=1}^{n} \left(\sum_{t: t\neq s} e_{21}^{(t)} \right) \frac{e_{12}^{(s)}}{1 - x/z_s}.$$

The constant term in (4.4) equals $-\frac{\mu^2 + \mu(e_{11} - e_{22}) - e_{11}e_{22}}{4}$. For $s = 1,\dots,n$, the coefficient of $\frac{1}{1 - x/z_s}$ in (4.4) equals

$$-c_2^{(s)} + \frac{\mu}{2}(e_{11} - e_{22})^{(s)} + \sum_{t: t\neq s} \frac{e_{12}^{(s)}e_{21}^{(t)}z_s + e_{12}^{(t)}e_{21}^{(s)}z_t}{z_s - z_t} + e_{22}^{(s)}\left(e_{22} - e_{22}^{(s)}\right)$$

$$= -c_2^{(s)} + \mathcal{K}_s(z, \mu) = m_s(m_s+2)/4 + K_s(z, \mu).$$

The coefficient of $\frac{1}{(1 - x/z_s)^2}$ in (4.4) equals

$$-e_{22}^{(s)} + e_{11}^{(s)}e_{22}^{(s)} - e_{12}^{(s)}e_{21}^{(s)} = (e_{11}e_{22} - e_{12}e_{21} + e_{11})^{(s)} = c_2^{(s)}.$$

The theorem is proved.

Lemma 4.2. *For any $a,b \in$ C–$\{z_1,...,z_n\}$ the operators $D_2(a),D_2(b) \in$ End(V) commute. They also commute with the* sl$_2$ *Cartan subalgebra.*

Proof. It is clear that the trigonometric Gaudin operators $K_s(z,\mu)$ commute with the sl$_2$ Cartan subalgebra. Now the lemma follows from the commutativity of trigonometric Gaudin operators. $\sqrt{\quad}$

Corollary 4.3. *Choose a weight subspace $V[\nu]$ of V. Then $(2\pi-1)^{-2}D_2(x)$ restricted to $V[\nu]$ equals*

$$-\frac{(\mu+\nu/2)^2}{4} + \sum_{s=1}^n \left[\frac{m_s(m_s+2)/4 + \mathcal{K}_s(z,\mu)}{1-x/z_s} - \frac{m_s(m_s+2)/4}{(1-x/z_s)^2} \right].$$

(4.5)

The commutative algebra of operators on $V[\nu]$ generated by the identity operator and the operators $\{D_2(a) \mid a \in$ C $- \{z_1,...,z_n\}\}$ is called the *Bethe algebra* of $V[\nu]$ with parameter μ and denoted by B$(z;\mu;V[\nu])$. The Bethe algebra B$(z;\mu;V[\nu])$ is generated by the identity operator and the trigonometric Gaudin operators $K_1(z,\mu),...,K_n(z,\mu)$.

The pair $(V[\nu],$B$(z;\mu;V[\nu]))$ is called the *trigonometric Gaudin model on $V[\nu]$*.

Corollary 4.4. *If $\mu \not\in$ Z$_{>0}$, the isomorphism $V[0] \to E[\mu]$ in (3.3) induces an isomorphism* B$(z;\mu;V[0]) \to$ B$(z;E[\mu])$ *between the Bethe algebra of $V[0]$ with parameter μ and the dynamical Bethe algebra of the space $E[\mu]$.*

Proof. The corollary is proved by comparing formulas (3.4) and (4.5).

4.3. **Gaudin operators and Weyl group.**

Lemma 4.5 ([TV, Lemma 18], cf. [MV2, Lemma 5.5]). *For any weight subspace $V[\nu]$, any $v \in V[\nu]$, $s = 1,...,n$, we have*

$$\mathcal{A}\left(\mu+\frac{\nu}{2}-1\right)\mathcal{K}_s(z,\mu)v = \mathcal{K}_s(z,-\mu)\mathcal{A}\left(\mu+\frac{\nu}{2}-1\right)v.$$

(4.6)

Theorem 4.6. *For $V = \otimes_{s=1}^n V_{m_s}$ as in (2.6), denote $M = \sum_{s=1}^n m_s$. Assume that $\mu \notin \frac{M}{2}+$Z. Then for any ν the isomorphism of vector spaces*

$$\mathcal{A}\left(\mu+\frac{\nu}{2}-1\right)\Big|_{V[\nu]} : V[\nu] \to V[-\nu]$$

(4.7)

induces an isomorphism of Bethe algebras

(4.8) $$\mathcal{B}(z;\mu;V[\nu]) \to \mathcal{B}(z;-\mu;V[-\nu]), \quad T \mapsto \mathcal{A}\left(\mu+\frac{\nu}{2}-1\right)T\mathcal{A}\left(\mu+\frac{\nu}{2}-1\right)^{-1}.$$

Proof. The theorem is a corollary of Lemmas 2.2 and 4.5.

4.4. **Commutative diagram.** Assume that $\mu \notin \mathsf{Z}$ and M is even. Then $V[0]$ is a nonzero weight subspace.

Consider the $B(z;\mu;V[0])$-module $V[0]$ and $B(z;-\mu;V[0])$-module $V[0]$. Consider the $B(z;E[\mu])$-module $E[\mu]$ and $B(z;E[-\mu])$-module $E[-\mu]$. Consider the diagram relating these modules

$$
\begin{array}{ccc}
(B(z;\mu;V[0]),\, V[0]) & \longrightarrow & (B(z;-\mu;V[0]),\, V[0]) \\
\downarrow & & \downarrow \\
(B(z;E[\mu]),\, E[\mu]) & \longrightarrow & (B(z;E[-\mu]),\, E[-\mu])
\end{array}
$$

(4.9)

Here the map $(B(z;\mu;V[0]),\, V[0]) \to (B(z;-\mu;V[0]),\, V[0])$ is the module isomorphism of Theorem 4.6. The map $(B(z;E[\mu]),\, E[\mu]) \to (B(z;E[-\mu]),\, E[-\mu])$ is the module isomorphism induced by the action of the Weyl involution σ, see Lemma 2.7. The maps $(B(z;\mu;V[0]),\, V[0]) \to (B(z;E[\mu]),\, E[\mu])$ and $(B(z;-\mu;V[0]),\, V[0]) \to (B(z;E[-\mu]),\, E[-\mu])$ are the module isomorphisms of Corollary 4.4.

Theorem 4.7. *Diagram* (4.9) *is commutative.*

Proof. The theorem follows from Theorems 3.3, 3.5, 4.6.

5. Bethe ansatz

5.1. **Bethe ansatz equations for triple** $(z;\mu;V[v])$. Let $V = \otimes_{s=1}^{n} V_{m_s}$, as in (2.6), and $M = \sum_{s=1}^{n} m_s$. Let $V[v]$ be a nonzero weight subspace of V. Then $v = M - 2m$ for some nonnegative integer m.

Let $z = \{z_1,...,z_n\} \subset \mathsf{C}^\times$ be a set of nonzero pairwise distinct numbers, as in Section 2.1. Let $\mu \in \mathsf{C}$.

Introduce the *master function* of the variables $t = (t_1,...,t_m),\mu,z$,

$$
\begin{aligned}
\Phi(t,z,\mu) \;=\; & \left(1 - \mu + \frac{\nu}{2}\right) \sum_{i=1}^{m} \ln t_i + \sum_{s=1}^{n} \frac{m_s}{4}(2\mu + m_s - \nu)\ln z_s \\
& + \; 2 \sum_{1 \leqslant i < j \leqslant m} \ln(t_i - t_j) - \sum_{i=1}^{m}\sum_{s=1}^{n} m_s \ln(t_i - z_s) + \sum_{1 \leqslant s < r \leqslant n} \frac{m_s m_r}{2} \ln(z_s - z_r)
\end{aligned}
$$

The *Bethe ansatz equations* are the critical point equations for the master function $\Phi(t,z,\mu)$ with respect to the variables $t_1,...,t_m$,

$$
\frac{1 - \mu + \nu/2}{t_i} + \sum_{j \neq i} \frac{2}{t_i - t_j} - \sum_{s=1}^{n} \frac{m_s}{t_i - z_s} = 0, \qquad i = 1,\dots,m.
$$

(5.1)

The master function $\Phi(t,z,\mu)$ is the trigonometric degeneration of the elliptic master function considered in Section 5 of [ThV], see also [FV1, MaV].

The symmetric group S_m acts on the critical set. If $(t_1^0, \cdots, t_m^0; z; \mu)$ is a solution of the Bethe ansatz equations, then for any $\rho \in S_m$, the point $(t_{\rho(1)}^0, \cdots, t_{\rho(m)}^0; z; \mu)$ is also a solution.

5.2. **Bethe vectors.** Define

$$\mathcal{C} \;=\; \{\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}_{\geqslant 0} \mid \ell_s \leqslant m_s, \; \ell_1 + \cdots + \ell_n = m\}$$

$$\omega_\ell(t,z) \;=\; \mathrm{Sym} \left[\prod_{s=1}^{n} \prod_{i=\ell_1+\cdots+\ell_{s-1}+1}^{\ell_1+\cdots+\ell_s} \frac{1}{t_i - z_s} \right],$$

where $\mathrm{Sym} f(t_1,\ldots,t_m) = \sum_{\rho \in S_m} f(t_{\rho(1)},\ldots,t_{\rho(m)})$. Introduce the *weight function*

$$(5.2) \qquad\qquad \omega(t,z) = \sum_{\ell \in \mathcal{C}} \omega_\ell(t,z) v_{m_1-\ell_1} \otimes \cdots \otimes v_{m_n-\ell_n},$$

see Section 2.3. This weight function see in [MV2], also in [JV, MaV, SV].

Notice that the weight function is a symmetric function of the variables $t_1,\ldots,t_m$.

If $(t^0; z; \mu)$ is a solution of the Bethe ansatz equations (5.1), then the vector $\omega(t^0, z)$ is called the *Bethe vector*.

Theorem 5.1 ([MTV6, V]). *Let $(t^0; z; \mu)$ be a solution of the Bethe ansatz equations (5.1). Then the Bethe vector $\omega(t^0, z)$ is nonzero.*

Theorem 5.2 ([FV1, JV], cf. [RV]). *Let $(t^0; z; \mu)$ be a solution of the Bethe ansatz equations (5.1). Then the Bethe vector $\omega(t^0, z)$ is an eigenvector of the trigonometric Gaudin operators,*

$$\mathcal{K}_s(z, \mu) \, \omega(t^0, z) \;=\; z_s \frac{\partial \Phi}{\partial z_s}(t^0, z, \mu) \, \omega(t^0, z), \qquad s = 1, \ldots, n.$$

Denote

$$(5.3) \qquad k_s(t^0, z, \mu) \;=\; z_s \frac{\partial \Phi}{\partial z_s}(t^0, z, \mu)$$

$$=\; \frac{m_s}{2} \left[(\mu - \nu/2 + m_s/2) + \sum_{p:\, p \neq s} m_p \frac{z_s}{z_s - z_p} + 2 \sum_{i=1}^{m} \frac{z_s}{t_i^0 - z_s} \right].$$

5.3. **Bethe vectors and coefficient $D_2(x)$.**

Lemma 5.3. *If $(t^0; z; \mu)$ is a solution of the Bethe ansatz equations (5.1), then the Bethe vector $\omega(t^0, z)$ is an eigenvector of all operators of the Bethe algebra $\mathrm{B}(z; \mu; V[\nu])$. In particular, the operator $D_2(x)$ acts on $\omega(t^0, z)$ by multiplication by the scalar*

$$(5.4) \quad (2\pi\sqrt{-1})^2\left[-\frac{(\mu+\nu/2)^2}{4}+\sum_{s=1}^{n}\left[\frac{m_s(m_s+2)/4+k_s(t^0,z,\mu)}{1-x/z_s}-\frac{m_s(m_s+2)/4}{(1-x/z_s)^2}\right]\right].$$

Proof. The lemma follows from Theorem 5.2 and Corollary 4.3.

For a solution $(t^0;z;\mu)$ of the Bethe ansatz equations (5.1), we introduce the *fundamental differential operator*

$$(5.5) \qquad\qquad \mathrm{E}_{t0,z,\mu} = \partial_u{}^2 + E_2(x,t^0,z,\mu),$$

where the function $E_2(x,t^0,z,\mu)$ is given by formula (5.4).

    5.4. **Basis of Bethe vectors.** The Bethe ansatz method is the method to construct eigenvectors of commuting operators, see Lemma 5.3 as an example. The standard problem is to determine if the Bethe ansatz method gives a basis of eigenvectors of the vector space, on which the commuting operators act. In the case of Lemma 5.3 the answer is positive.

Lemma 5.4. *Let* $\mu \notin \frac{\nu}{2} + \mathbb{Z}_{>0}$*. Then for generic* $z = \{z_1,...,z_n\} \subset \mathbb{C}^\times$*, the set of solutions* $(t^0;z;\mu)$ *of system* (5.1) *of the Bethe ansatz equations is such that the corresponding Bethe vectors* $\omega(t^0,z,\mu)$ *form a basis of the space* $V[\nu]$*.*

Proof. Here the word generic means that the subset of all acceptable sets $\{z_1,...,z_n\}$ forms a Zariski open subset in the space of all sets $\{z_1,...,z_n\}$. The proof of the lemma is standard. It is a modification of [ScV, Theorem 8], cf. [MV1, Section 4.4], [MV2, Section 5.4], [MTV1, Section 10.6].

6. Function $w(x)$ in the kernel of $\mathrm{E}_{t0;z;\mu}$

Let $(t^0;z;\mu)$ be a solution of system (5.1) of Bethe ansatz equations, where $t^0 = (t_1^0, \ldots, t_m^0)$. Define

$$(6.1) \qquad y(x) = \prod_{i=1}^{m}(x - t_i^0), \qquad w(x) = y(x)\, x^{\frac{\nu/2-\mu}{2}} \prod_{s=1}^{n}(x - z_s)^{-m_s/2}.$$

Theorem 6.1. *We have*

$$(6.2) \qquad \mathcal{E}_{t^0;z;\mu} = \left(\partial_u + (\ln w)'\right)\left(\partial_u - (\ln w)'\right),$$

where $' = \partial/\partial u$*. In other words,*

$$(6.3) \qquad E_2(x,t^0,z,\mu) = -(\ln w)'' - ((\ln w)')^2.$$

Remark. For $\nu = 0$ this statement is the trigonometric degeneration of its elliptic version [ThV, Theorem 5.3].

Proof. Recall that $\partial_u = -2\pi\sqrt{-1}x\partial_x$. We have

$$
(\ln w)' = -2\pi\sqrt{-1}\left[-\frac{\nu/2+\mu}{2} + \sum_{i=1}^{m}\frac{t_i^0}{x-t_i^0} - \frac{1}{2}\sum_{s=1}^{n}\frac{z_s m_s}{x-z_s}\right],
$$

$$
(\ln w)'' = (2\pi\sqrt{-1})^2\left[-\sum_{i=1}^{m}\frac{t_i^0}{x-t_i^0} - \sum_{i=1}^{m}\frac{(t_i^0)^2}{(x-t_i^0)^2} + \frac{1}{2}\sum_{s=1}^{n}\frac{z_s m_s}{x-z_s} + \frac{1}{2}\sum_{s=1}^{n}\frac{z_s^2 m_s}{(x-z_s)^2}\right].
$$

Hence, $(2\pi\sqrt{-1})^{-2}(-(\ln w)'' - ((\ln w)')^2)$ equals

$$
\sum_{i=1}^{m}\frac{t_i^0}{x-t_i^0} + \sum_{i=1}^{m}\frac{(t_i^0)^2}{(x-t_i^0)^2} - \frac{1}{2}\sum_{s=1}^{n}\frac{z_s m_s}{x-z_s} - \frac{1}{2}\sum_{s=1}^{n}\frac{z_s^2 m_s}{(x-z_s)^2}
$$

$$
-\frac{1}{4}(\mu+\nu/2)^2 - \sum_{i=1}^{m}\frac{(t_i^0)^2}{(x-t_i^0)^2} - 2\sum_{i=1}^{m}\sum_{j:j\neq i}\frac{t_i^0 t_j^0}{t_i^0-t_j^0}\frac{1}{x-t_i^0}
$$

$$
-\frac{1}{4}\sum_{s=1}^{n}\frac{z_s^2 m_s^2}{(x-z_s)^2} - \frac{1}{2}\sum_{s=1}^{n}\sum_{p:p\neq s}\frac{z_s z_p m_s m_p}{z_s-z_p}\frac{1}{x-z_s} + (\mu+\nu/2)\sum_{i=1}^{m}\frac{t_i^0}{x-t_i^0}
$$

$$
-\frac{1}{2}(\mu+\nu/2)\sum_{s=1}^{n}\frac{z_s m_s}{x-z_s} + \sum_{i=1}^{m}\sum_{s=1}^{n}\frac{t_i^0 z_s m_s}{t_i^0-z_s}\frac{1}{x-t_i^0} - \sum_{i=1}^{m}\sum_{s=1}^{n}\frac{t_i^0 z_s m_s}{t_i^0-z_s}\frac{1}{x-z_s}.
$$

In the expression above for each $i = 1,...,m$ the coefficient of $\frac{1}{(x-t_i^0)^2}$ equals zero. The coefficient of $\frac{1}{x-t_i^0}$ equals

$$
(\mu+\nu/2+1)t_i^0 - \sum_{j:j\neq i}\frac{2t_i^0 t_j^0}{t_i^0-t_j^0} + \sum_{s=1}^{n}\frac{t_i^0 z_s m_s}{t_i^0-z_s}
$$

$$
= t_i^0\left[\mu+\nu/2+1 + 2\sum_{j:j\neq i}\frac{t_j^0-t_i^0+t_i^0}{t_j^0-t_i^0} - \sum_{s=1}^{n}\frac{(z_s-t_i^0+t_i^0)m_s}{z_s-t_i^0}\right]
$$

$$
= t_i^0\left[\mu+\nu/2+1 + 2(m-1) + \sum_{j:j\neq i}\frac{2t_i^0}{t_j^0-t_i^0} - \sum_{s=1}^{n}m_s - \sum_{s=1}^{n}\frac{t_i^0 m_s}{z_s-t_i^0}\right]
$$

$$
= -(t_i^0)^2\left[\frac{1-\mu+\nu/2}{t_i^0} + \sum_{j:j\neq i}\frac{2}{t_i^0-t_j^0} - \sum_{s=1}^{n}\frac{m_s}{t_i^0-z_s}\right] = 0,
$$

where the last equality follows from the Bethe ansatz equations (5.1). For each $s = 1,...,n$ the coefficient of $\frac{1}{(1-x/z_s)^2}$ equals $-m_s(m_s+2)/4$. The coefficient of $\frac{1}{1-x/z_s}$ equals

$$\frac{1}{2}m_s + \frac{1}{2}m_s \sum_{p:\,p\neq s} \frac{z_p m_p}{z_s - z_p} + \frac{1}{2}(\mu + \nu/2)m_s + \sum_{i=1}^{m} \frac{t_i^0 m_s}{t_i^0 - z_s}$$

$$= \frac{m_s}{2}\left[1 + \mu + \nu/2 - \sum_{p:\,p\neq s} \frac{(z_p - z_s + z_s)m_p}{z_p - z_s} + 2\sum_{i=1}^{m} \frac{t_i^0 - z_s + z_s}{t_i^0 - z_s}\right]$$

$$= \frac{m_s}{2}\left[1 + \mu + \nu/2 - \sum_{p\neq s} m_p - \sum_{p:\,p\neq s} \frac{z_s m_p}{z_p - z_s} + 2\sum_{i=1}^{m} 1 + 2\sum_{i=1}^{m} \frac{z_s}{t_i^0 - z_s}\right]$$

$$= \frac{m_s}{2}\left[(1 + \mu - \nu/2 + m_s) + \sum_{p:\,p\neq s} \frac{z_s m_p}{z_s - z_p} + 2\sum_{i=1}^{m} \frac{z_s}{t_i^0 - z_s}\right]$$

$$= m_s(m_s + 2)/4 + \frac{m_s}{2}\left[(\mu - \nu/2 + m_s/2) + \sum_{p:\,p\neq s} m_p \frac{z_s}{z_s - z_p} + 2\sum_{i=1}^{m} \frac{z_s}{t_i^0 - z_s}\right]$$

$$= m_s(m_s + 2)/4 + k_s(t^0, z, \mu),$$

where $k_s(t^0,z,\mu)$ are defined in (5.3). Hence, $E_2 = -(\ln w)^{00} - ((\ln w)^0)^2$.

Corollary 6.2. *The function $w(x)$ lies in the kernel of* $E_{t^0;z;\mu}$.

7. Function $\tilde{w}(x)$ in the kernel of $E_{t^0;z;\mu}$

7.1. **Wronskian.** The *Wronskian* of two functions $f(a)$ and $g(a)$ is

(7.1)
$$\mathrm{Wr}^a(f,g) = f\frac{dg}{da} - \frac{df}{da}g.$$

We have

(7.2) $\mathrm{Wr}_a(hf,hg) = h^2\,\mathrm{Wr}_a(f,g)$ for any function $h(a)$.

7.2. **Wronskian and Bethe ansatz equations.**

Lemma 7.1. *The following two statements hold:*

(i) *Let $\mu \notin \frac{\nu}{2} + \mathbb{Z}_{\geqslant 0}$. Let $(t^0;z;\mu)$ be a solution of the Bethe ansatz equations (5.1) and $y(x) = \prod_{i=1}^{m}(x - t_i^0)$. Then there exists a unique monic polynomial $\tilde{y}(x)$ of degree M – m, such that*

(7.3)
$$\mathrm{Wr}_x(y(x),x^{\mu-\nu/2}\tilde{y}(x)) = \mathrm{const}\; x^{\mu-\nu/2-1}\prod_{s=1}^{n}(x - z_s)^{m_s},$$

where const *is a nonzero constant.*

(ii) *Let $\mu \neq \frac{\nu}{2}$. Assume that $y(x) = \prod_{i=1}^{m}(x - t_i^0)$ is a polynomial with distinct roots such that $y(z_s) \neq 0$, s = 1,...,n. Assume that there exists a polynomial $\tilde{y}(x)$ such that equation (7.3) holds. Then $(t_1^0, \ldots, t_m^0; z; \mu)$ is a solution of the Bethe ansatz equations (5.1).*

Proof. This lemma is a reformulation of Theorem 3.2 and Corollary 3.3 in [MV2].

7.3. **Function** $\tilde{w}(x)$. Recall that we have a solution $(t^0;z;\mu)$ of the Bethe ansatz equations, the differential operator $E_{t^0,z,\mu}$ and the function $w(x) = y(x)\,x^{\frac{\nu/2-\mu}{2}}\prod_{s=1}^{n}(x-z_s)^{-m_s/2}$, where $y(x) = \prod_{i=1}^{m}(x - t_i^0)$.

Theorem 7.2. *Let* $\mu \notin \frac{\nu}{2}+\mathbb{Z}_{\geqslant 0}$. *Then there exists a unique monic polynomial* $\tilde{y}(x)$ *of degree* $M - m$, *such that the function*

$$\tilde{w}(x) = \tilde{y}(x)\,x^{\frac{\mu-\nu/2}{2}}\prod_{s=1}^{n}(x - z_s)^{-m_s/2}$$

(7.4)

lies in the kernel of $E_{t^0;z;\mu}$. *The functions* $w(x),\tilde{w}(x)$ *span the kernel of* $E_{t^0;z;\mu}$.

Proof. The differential operator $E_{t^0;z;\mu}$ introduced in (5.5) has no first order term. Hence the kernel of $E_{t^0;z;\mu}$ consists of the functions $\tilde{w}(x)$ satisfying the equation

(7.5) $Wr_u(w(x),\tilde{w}(x)) = \text{const}$.

By Lemma 7.1, there exists a unique monic polynomial $\tilde{y}(x)$ of degree $M - m$, such that equation (7.3) holds. Dividing both sides of (7.3) by $x^{\mu-\nu/2}\,Q_{s=1}^{n}(x - z_s)^{m_s}$ we obtain

(7.6) $Wr = \text{const}\,x^{-1}$.

theorem is proved.

7.4. **Bethe ansatz equations for triples** $(z;\mu;V[\nu])$ **and** $(z;-\mu;V[-\nu])$.
Recall $x\ y(x)x^{\frac{\nu/2-\mu}{2}}Y(x - z_s)^{-m_s/2}$, $\tilde{y}(x)x^{\frac{\mu-\nu/2}{2}}Y(x - z_s)^{-m_s/2}$ that $x = e^{-2\pi^{-1}u}$,

$$\partial_u = -2\pi\sqrt{-1}\,x\partial_x$$

hence   . This implies equation (7.5). The

Lemma 7.3. *Let* $\mu \notin \frac{\nu}{2}+\mathbb{Z}_{\geqslant 0}$. *Let* $(t^0;z;\mu)$ *be a solution of the Bethe ansatz equations* (5.1) *assigned to the triple* $(z;\mu;V[\nu])$ *in Section* 5.1. *Let*

(7.7) $\tilde{y}(x) = \overset{M-m}{\underset{i=1}{Y}}(x - \tilde{t}_i^0)$

be the polynomial assigned to $(t^0;z;\mu)$ *in Theorem* 7.2. *If* $\tilde{y}(x)$ *has distinct roots and* $\tilde{y}(z_s) \neq 0$ *for* $s = 1,...,n$, *then* $(\tilde{t}_1^0,\ldots,\tilde{t}_{M-m}^0;z;-\mu)$ *is a solution of the Bethe ansatz equations* (5.1) *assigned to the triple* $(z;-\mu;V[-\nu])$. *Proof.* Equation (7.3) can be rewritten as

(7.8) $Wr_x(x^{-\mu+\nu/2}y(x),\tilde{y}(x)) = \text{const}\,x^{-\mu+\nu/2-1}\overset{n}{\underset{s=1}{Y}}(x - z_s)^{m_s}$.

Now the lemma follows from the equalities $-\nu = -M + 2m = M - 2(M - m)$ and Lemma

7.1.

Theorem 7.4 ([MV2, Theorem 5.7]). *Under assumptions of Lemma 7.3 consider the Bethe vectors $\omega(t^0,z,\mu) \in V[\nu]$ and $\omega(\tilde{t}^0,z,-\mu) \in V[-\nu]$. Then*

(7.9)
$$\mathcal{A}\Big(\mu + \frac{\nu}{2} - 1\Big)\omega(t^0, z, \mu) = \text{const}\,\omega(\tilde{t}^0,z,-\mu),$$

where const *is a nonzero constant.*

Corollary 7.5. *Under assumptions of Lemma 7.3, for $s = 1,...,n$, the eigenvalue of $K_s(z,\mu)$ on $\omega(t^0,z,\mu)$ equals the eigenvalue of $K_s(z,-\mu)$ on $\omega(\tilde{t}^0,z,\mu)$.*

Proof. The corollary follows from Lemma 4.5 and Theorem 4.6.

8. Conjugates of D and $E_{t^0;z;\mu}$

8.1. Conjugate of D. Recall the universal differential operator $D = \partial_u^2 + D_2(x)$ introduced in (4.2), where the coefficient $D_2(x)$ is determined by formula (4.3). We introduce the operator

(8.1)
$$\mathcal{D}^c = \frac{1}{(2\pi\sqrt{-1}\,x)^2}\,\prod_{s=1}^{n}(x - z_s)^{m_s/2} \cdot \mathcal{D} \cdot \prod_{s=1}^{n}(x - z_s)^{-m_s/2},$$

where the superscript ^{c} stays for the word "conjugated".

Theorem 8.1. *We have*

(8.2)
$$\begin{aligned}
\mathcal{D}^c &= \partial_x^2 + \Big[\frac{1}{x} - \sum_{s=1}^{n}\frac{m_s}{x - z_s}\Big]\partial_x - \frac{1}{x}\sum_{s=1}^{n}\frac{m_s/2}{x - z_s} + \sum_{s=1}^{n}\frac{m_s(m_s + 2)/4}{(x - z_s)^2} \\
&\quad + \sum_{s\neq p}\frac{m_s m_p/4}{(x - z_s)(x - z_p)} - \frac{\mu^2 + \mu(e_{11} - e_{22}) - e_{11}e_{22}}{4x^2} \\
&\quad - \frac{1}{x^2}\sum_{s=1}^{n}\Big[z_s\frac{m_s(m_s + 2)/4 + \mathcal{K}_s(z,\mu)}{x - z_s} + z_s^2\frac{m_s(m_s + 2)/4}{(x - z_s)^2}\Big].
\end{aligned}$$

Proof. Recall that $x = e^{-2\pi\sqrt{-1}u}$, $\partial_u = -2\pi\sqrt{-1}x\partial_x$, $\partial_u^2 = (2\pi\sqrt{-1})^2(x\partial_x + x^2\partial_x^2)$. Denote $f = \prod_{s=1}^{n}(x - z_s)^{-m_s/2}$. We have $f' = -\sum_{s=1}^{n}\frac{m_s/2}{x-z_s}\,f$,

$$f'' = \Big(\sum_{s=1}^{n}\frac{m_s^2/4}{(x - z_s)^2} + \sum_{s\neq p}\frac{m_s m_p/4}{(x - z_s)(x - z_p)} + \sum_{s=1}^{n}\frac{m_s/2}{(x - z_s)^2}\Big)f,$$

where $' = \partial/\partial x$. Therefore,

$$\begin{aligned}
\mathcal{D}^c &= \frac{1}{x^2}f^{-1}\Big[x^2\partial_x^2 + x\partial_x + \frac{1}{(2\pi\sqrt{-1})^2}D_2(x)\Big]f \\
&= \frac{1}{x^2}f^{-1}\Big[x^2\big(f\partial_x^2 + 2f'\partial_x + f''\big) + x\big(f\partial_x + f'\big) + \frac{1}{(2\pi\sqrt{-1})^2}D_2(x)f\Big] \\
&= \partial_x^2 + \Big[2f^{-1}f' + \frac{1}{x}\Big]\partial_x + \Big[f^{-1}f'' + \frac{1}{x}f^{-1}f' + \frac{1}{x^2}\frac{1}{(2\pi\sqrt{-1})^2}D_2(x)\Big],
\end{aligned}$$

which gives the right-hand side of formula (8.2).

8.2. **Conjugate of** $E_{t0;z;\mu}$.

Similarly to the conjugation of D we conjugate $E_{t0,z,\mu}$ and consider the differential operator

(8.3)
$$\mathcal{E}^c_{t^0;z;\mu} = \frac{1}{(2\pi\sqrt{-1}\,x)^2} \prod_{s=1}^{n} (x - z_s)^{m_s/2} \cdot \mathcal{E}_{t^0;z;\mu} \cdot \prod_{s=1}^{n} (x - z_s)^{-m_s/2}.$$

Lemma 8.2. *The kernel of* $\mathcal{E}^c_{t^0;z;\mu}$ *is spanned by quasi-polynomials*

(8.4)
$$x^{\frac{\nu/2-\mu}{2}} y(x), \qquad x^{\frac{\mu-\nu/2}{2}} \tilde{y}(x),$$

where $y(x)$ is the monic polynomial of degree m, defined in (6.1), and $\tilde{y}(x)$ is the monic polynomial of degree M − m, defined in Theorem 7.2.

Lemma 8.3. *Under assumptions of Lemma 7.3, let $(t^0;z;\mu)$ be a solution of the Bethe ansatz equations (5.1) assigned to the triple $(z;\mu;V\,[\nu])$. Assume that the numbers $\tilde{t}^0_1, \dots, \tilde{t}^0_{M-m}$ defined in Lemma 7.3 are such that $(\tilde{t}^0_1, \dots, \tilde{t}^0_{M-m}; z; -\mu)$ is a solution of the Bethe ansatz equations (5.1) assigned to the triple $(z;-\mu;V\,[-\nu])$. Then*

(8.5)
$$\mathcal{E}^c_{\tilde{t}^0;z;-\mu} = \mathcal{E}^c_{t^0;z;\mu}.$$

9. Space of V-valued functions of $z_1,...,z_n$

9.1. **Space** $V_1^{\otimes n}[\nu]$.

Recall the two-dimensional irreducible $\mathfrak{sl}_2$-module V_1 with basis v^1_0, v^1_1, see (2.5). In the rest of the paper we assume that V is the tensor power of V_1,

(9.1)
$$V = V_1^{\otimes n}, \qquad \text{where} \quad n > 1.$$

The space V has a basis of vectors

$$v_I = v^1_{i_1} \otimes \cdots \otimes v^1_{i_n},$$

labeled by partitions $I = (I_1,I_2)$ of $\{1,...,n\}$, where $i_j = 0$ if $j \in I_1$, and $i_j = 1$ if $j \in I_2$.

We have the weight decomposition $V = \oplus_{m=0}^{n} V[n - 2m]$, where $V\,[n - 2m]$ is of dimension $\binom{n}{m}$ and has the basis $\{v_I \mid I = (I_1,I_2), |I_1| = m, |I_2| = n - m\}$.

We use notations $\nu = n - 2m$, $` = n - m$, and hence $m + ` = n$.

9.2. **Space** V^S.

Let $z = (z_1,...,z_n)$ be variables. The symmetric group S_n acts on the algebra $\mathbb{C}[z_1,...,z_n]$ by permuting the variables. Let $\sigma_s(z)$, $s = 1,...,n$, be the s-th elementary symmetric polynomial in $z_1,...,z_n$. The algebra of symmetric polynomials $\mathbb{C}[z_1,...,z_n]^S$ is a free polynomial algebra with generators $\sigma_1(z),...,\sigma_n(z)$.

Let V be the space of polynomials in $z_1,...,z_n$ with coefficients in $V_1^{\otimes n}$,

$$\mathcal{V} = V_1^{\otimes n} \otimes \mathbb{C}[z_1, \dots, z_n].$$

The symmetric group S_n acts on V by permuting the factors of $V_1^{\otimes n}$ and the variables $z_1,...,z_n$ simultaneously, $\rho(v_1 \otimes \cdots \otimes v_n \otimes p(z_1,...,z_n)) = v_{(\rho^{-1})(1)} \otimes \cdots \otimes v_{(\rho^{-1})(n)} \otimes p(z_{\rho(1)},...,z_{\rho(n)})$, $\rho \in S_n$.

We denote by V^S the subspace of S_n-invariants in V.

Lemma 9.1 ([MTV3]). *The space V^S is a free $C[z_1,...,z_n]^S$-module of rank 2^n.*

Consider the grading on $C[z_1,...,z_n]$ such that $\deg z_s = 1$ for all $s = 1,...,n$. We define a grading on V by setting $\deg(v \otimes p) = \deg p$ for any $v \in V_1^{\otimes n}$ and $p \in C[z_1,...,z_n]$. The grading on V induces a grading on End(V).

The Lie algebras $sl_2 \subset gl_2$ naturally act on V^S. We have the weight decomposition
$$\mathcal{V}^S = \oplus_{m=0}^{n} \mathcal{V}^S[n-2m], \quad V^S[n-2m] = (V[n-2m] \otimes C[z_1,...,z_n])^S.$$

Let M be a $Z_{>0}$-graded space with finite-dimensional homogeneous components. Let $M_j \subset M$ be the homogeneous component of degree j. The formal power series in a variable α, $\mathrm{ch}_M(\alpha) = \sum_{j=0}^{\infty} (\dim M_j)\alpha^j$, is called the graded character of M.

Lemma 9.2 ([MTV2]). *The space $V^S[n-2m]$ is a free $C[z_1,...,z_n]^S$-module of rank $\binom{n}{m}$ and*

$$(9.2) \qquad \mathrm{ch.} \quad \mathcal{V}^S[n-2m](\alpha) = \prod_{i=1}^{m} \frac{1}{1-\alpha^i} \cdot \prod_{i=1}^{n-m} \frac{1}{1-\alpha^i}$$

9.3. Bethe algebra of $V^S[v]$. Recall the differential operator D^c introduced in (8.2) for $V = \otimes_{s=1}^{n} V_{m_s}$ and depending on parameter $\mu \in C$. For $V = V_1^{\otimes n}$ the operator D^c takes the form

$$(9.3) \qquad \mathcal{F} = \partial_x^2 + F_1(x)\partial_x + F_2(x),$$

where

$$(9.4) \quad F_1(x) = \frac{1}{x} - \sum_{s=1}^{n} \frac{1}{x-z_s},$$

$$F_2(x) = -\frac{1}{x} \sum_{s=1}^{n} \frac{1/2}{x-z_s} + \sum_{s=1}^{n} \frac{3/4}{(x-z_s)^2} + \sum_{s \neq p} \frac{1/4}{(x-z_s)(x-z_p)}$$
$$- \frac{\mu^2 + \mu(e_{11}-e_{22}) - e_{11}e_{22}}{4x^2} - \frac{1}{x^2} \sum_{s=1}^{n} \left[z_s \frac{3/4 + \mathcal{K}_s(z,\mu)}{x-z_s} + z_s^2 \frac{3/4}{(x-z_s)^2} \right].$$

In formula (8.2) we had $\{z_1,...,z_n\}$ being a subset of $C^\times$. From now on we consider $z_1,...,z_n$ as independent variables.

The operator F in formulas (9.3), (9.4), in which $z_1,...,z_n$ are variables, is called the *universal differential operator* for V^S with parameter $\mu \in C$.

Lemma 9.3 (cf. [MTV3, Section 2.7]). *The Laurent expansions of $F_1(x)$ and $F_2(x)$ at infinity have the form*

$$F_1(x) = \sum_{j=1}^{\infty} F_{1j} x^{-j}, \qquad F_2(x) = \sum_{j=2}^{\infty} F_{2j} x^{-j},$$

(9.5)

where

$$F_{11} = 1 - n, \qquad F_{1j} = -\sum_{s=1}^{n} z_s^{j-1} \qquad \text{for } j > 2.$$

For any $j > 2$, the element F_{2j} is a homogeneous polynomial in $z_1,...,z_n$ of degree $j-2$ with coefficients in End(V). *The element F_{2j} preserves the weight decomposition of* V. *Each of the elements $F_{1j}, j > 1$, $F_{2j}, j > 2$, defines an endomorphism of the* C$[z_1,...,z_n]^S$-module V^S.

Proof. The proof follows from straightforward calculations.

Lemma 9.4. *The elements $F_{1j}, j > 1$, $F_{2j}, j > 2$, considered as endomorphisms of the* C$[z_1,...,z_n]^S$-module V^S, commute.

Proof. The commutativity follows from the commutativity of the trigonometric Gaudin operators in formula (8.2).

For a weight subspace V$^S[v]$, $v = n-2m$, ` $= n-m$, consider the commutative subalgebra B$(\mu;m;$`$)$ of the algebra of endomorphisms of the C$[z_1,...,z_n]^S$-module V$^S[v]$, generated by the elements $F_{1j}, j > 1$, $F_{2j}, j > 2$. The subalgebra B$(\mu;m;$`$)$ is called the *Bethe algebra* of V$^S[v]$ with parameter $\mu \in$ C.

Lemma 9.5. *The Bethe algebra* B$(\mu;m;$`$)$ *contains the subalgebra of operators of multiplication by elements of* C$[z_1,...,z_n]^S$.

Proof. The subalgebra of operators of multiplication by elements of C$[z_1,...,z_n]^S$ is generated by the elements $F_{1j}, j > 1$, see Lemma 9.3.

Lemma 9.5 makes the Bethe algebra B$(\mu;m;$`$)$ a C$[z_1,...,z_n]^S$-module.

9.4. **Weyl group invariance.** For a weight subspace $V[\nu] = V_1^{\otimes n}[\nu]$ recall the linear map $\mathcal{A}(\mu + \nu/2 - 1) : V[\nu] \to V[-\nu]$, defined in (2.10). It is an isomorphism of vector spaces, if $\mu \notin \frac{n}{2} + \mathbb{Z}$. That operator induces an isomorphism of C$[z_1,...,z_n]^S$-modules,

(9.6) $$\mathcal{A}(\mu + \nu/2 - 1) : \mathcal{V}^S[\nu] \to \mathcal{V}^S[-\nu].$$

Lemma 9.6. *Let* $\mu \notin \frac{n}{2} + \mathbb{Z}$. *Let* $F_{ij}(\mu,m,$`$)$ *be the generators of* B$(\mu;m;$`$)$, *defined in* (9.5), *and* $F_{ij}(-\mu,$`$,m)$ *the generators of* B$(-\mu;$`$;m)$. *Then*

(9.7) $$F_{ij}(-\mu,\ell,m) = \mathcal{A}\left(\mu + \frac{\nu}{2} - 1\right) F_{ij}(\mu,m,\ell) \mathcal{A}\left(\mu + \frac{\nu}{2} - 1\right)^{-1}$$

for all i,j. The map

(9.8) $B(\mu;m;\grave{})\to B(\mu;m;\grave{})$, $F_{ij}(\mu;m;\grave{})\mapsto F_{ij}(-\mu,\grave{},m)$,

is an isomorphism of algebras and of $C[z_1,...,z_n]^S$-modules. The maps in (9.6) and (9.8) define an
isomorphism between the $B(\mu;m;\grave{})$-module $V^S[v]$ *and the* $B(-\mu;\grave{};m)$-module $V^S[-v]$.

Proof. The lemma follows from Lemma 4.5.

9.5. **Generic fibers of** $V^S[v]$. Given $a = (a_1,...,a_n) \in C^n$, denote by $I_a \subset C[z_1,...,z_n]$ the ideal
generated by the polynomials $\sigma_s(z) - a_s$, $s = 1,...,n$. Define

(9.9) $I_a V^S[v] := V^S \cap (V[v] \otimes I_a)$.

Assume that a is such that the polynomial $x^n + \sum_{s=1}^{n}(-1)^s a_s x^{n-s}$ has distinct nonzero roots
$b_1,...,b_n$.

Lemma 9.7 ([MTV3, Lemma 2.13]). *The quotient* $V^S[v]/I_a V^S[v]$ *is a finite-dimensional complex
vector space canonically isomorphic to* $V[v]$. *Under this isomorphism the Bethe algebra*
$B(\mu;m;\grave{})$ *induces a commutative algebra of operators on* $V[v]$. *That commutative algebra of
operators is canonically isomorphic to the Bethe algebra* $B(b_1,...,b_n;\mu;V[v])$ *introduced in
Section* 4.2.

10. Functions on pairs of quasi-polynomials

10.1. **Space of pairs of quasi-polynomials.** Let $m,\grave{},n$ be positive integers, $m + \grave{} = n$. Denote
$v = n - 2m$, cf. Section 9.1. Let

$$\zeta \in C - \frac{1}{2}\mathbb{Z}.$$

Let $\Omega(\zeta,m,\grave{})$ be the affine n-dimensional space with coordinates p_i, $i = 1,...,m$, q_j, $j = 1,...,\grave{}$.
Introduce the generating functions

(10.1) $p(x)$ $=$ $x^{-\zeta}(x^m + p_1 x^{m-1} + \cdots + p_m)$,

                              $q(x)$ $=$ $x^{\zeta}(x^{\grave{}} + q_1 x^{\grave{}-1} + \cdots + q_{\grave{}})$.

We identify points U of $\Omega(\zeta,m,\grave{})$ with two-dimensional complex vector spaces generated by
quasi-polynomials

(10.2) $p(x,U)$ $=$ $x^{-\zeta}(x^m + p_1(U)x^{m-1} + \cdots + p_m(U))$,

                              $q(x,U)$ $=$ $x^{\zeta}(x^{\grave{}} + q_1(U)x^{\grave{}-1} + \cdots + q_{\grave{}}(U))$.

Denote by $O(\zeta,m,\grave{})$ the algebra of regular functions on $\Omega(\zeta,m,\grave{})$, $O(\zeta,m,\grave{}) =$

                    $C[p_1,...,p_m,q_1,...,q_{\grave{}}]$.

Define the grading on $O(\zeta,m,\grave{})$ by $\deg p_i = \deg q_i = i$ for all i. **Lemma 10.1.**

The graded character of the algebra $O(\zeta,m,\grave{})$ *equals*

$$(10.3) \qquad \text{ch.} \quad \mathcal{O}_{(\zeta,m,\ell)}(\alpha) = \prod_i \frac{1}{1-\alpha^i} \cdot \prod_{j=1}^{m} \frac{1}{1-\alpha^j}$$

10.2. Wronski map.

Let $p(x), q(x)$ be the generating functions in (10.1). We have

$$(10.4) \qquad \text{Wr}^x(p,q) = \frac{2\zeta + \ell - m}{x} \left(x^n + \sum_{s=1}^{n} (-1)^s \Sigma_s\, x^{n-s} \right),$$

where $\Sigma_1,...,\Sigma_n$ are elements of $O(\zeta, m, \grave{\,})$. Notice that $2\zeta + \grave{\,} - m = 2\zeta + v\, / \in \mathbb{Z}$ according to our assumptions. The elements $\Sigma_1,...,\Sigma_n$ are homogeneous with $\deg \Sigma_s = s$. Define the *Wronski map*

$$\text{Wr} : \Omega(\zeta, m, \grave{\,}) \to \mathbb{C}^n, \qquad U \mapsto (\Sigma_1(U),...,\Sigma_n(U)).$$

Lemma 10.2. *For* $\zeta \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$, *the Wronski map is a map of positive degree.*

Proof. The proof is a slight modification of the proof of [MTV4, Proposition 3.1].

Let $O^S \subset O(\zeta, m, \grave{\,})$ be the subalgebra generated by $\Sigma_1,...,\Sigma_n$. Let $\sigma_1,...,\sigma_n$ be coordinates on $\mathbb{C}^n$, which is the image of the Wronski map. Introduce the grading on $\mathbb{C}[\sigma_1,...,\sigma_n]$ by $\deg \sigma_s = s$ for all s. The Wronski map induces the isomorphism $\mathbb{C}[\sigma_1,...,\sigma_n] \to O^S$, $\sigma_s \mapsto \Sigma_s$, of graded algebras, see Lemma 10.2. This isomorphism makes $O(\zeta, m, \grave{\,})$ a $\mathbb{C}[\sigma_1,...,\sigma_n]$-module.

10.3. Another realization of $O(\zeta, m, \grave{\,})$.

Define the differential operator G by

$$(10.5) \qquad \mathcal{G} = \frac{1}{\text{Wr}_x(p,q)} \text{rdet} \begin{bmatrix} p & p' & p'' \\ q & q & q \\ 1 & \partial_x & \partial_x^2 \end{bmatrix},$$

where rdet is the row determinant. We have

$$(10.6) \qquad G = \partial_x^2 + G_1(x)\partial_x + G_2(x),$$

where $G_1(x)$, $G_2(x)$ are rational functions in x with coefficients in $O(\zeta, m, \grave{\,})$, cf. [MTV3]. Notice that

$$(10.7) \qquad G_1 = - \frac{(\text{Wr}(p,q))^0_x}{\text{Wr}_x(p,q)}.$$

Lemma 10.3 (cf. [MTV3, Section 2.7]). *The Laurent expansions of $G_1(x)$ and $G_2(x)$ at infinity have the form*

$$G_i(x) = \sum_{j=i}^{\infty} G_{ij}x^{-j}, \qquad i = 1, 2$$

(10.8)

,

where for any i,j, the element G_{ij} is a homogeneous element of $O(\zeta, m, \ell)$ of degree j − i.

Proof. The proof is by straightforward calculation.

Lemma 10.4 ([MTV3, Lemma 3.4], [MTV2, Lemma 4.3]). *Let* $\zeta \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$. *Then the elements* G_{ij}, *i = 1,2, j > i, generate the algebra* $O(\zeta, m, \ell)$.

10.4. **Fibers of Wronski map.** Given $a = (a_1,...,a_n) \in \mathbb{C}^n$, denote by $J_a \subset O(\zeta, m, \ell)$ the ideal generated by the elements $\Sigma_s - a_s$, s = 1,...,n. Define

(10.9) $$\mathcal{O}_a(\zeta, m, \ell) := \mathcal{O}(\zeta, m, \ell)/J_a.$$

The algebra $O_a(\zeta, m, \ell)$ is the algebra of functions on the fiber $Wr^{-1}(a)$ of the Wronski map. Let

(10.10) $$x_n + \sum_{s=1}^{n}(-1)_{n-s}\, a_s\, x_{n-s} = \sum_{s=1}^{n}(x - b_s)$$

for some $b_s \in \mathbb{C}$. Let U be a point of $\Omega(\zeta, m, \ell)$ and

$$p(x,U) = x^{-\zeta}\prod_{i=1}^{m}(x - t^0{}_i), \qquad q(x,U) = x^{\zeta}\prod_{i=1}^{\ell}(x - \tilde{t}_i{}^0),$$

for some $t^0{}_i, \tilde{t}_i{}^0 \in \mathbb{C}$.

Lemma 10.5. *Let* $\zeta \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$. *Then there exists a Zariski open subset $X \subset \mathbb{C}^n$ such that for any $a \in X$ all the numbers $b_1,...,b_n$ are nonzero and distinct. Moreover, for any point $U \in Wr^{-1}(a)$ all the numbers* $b_1, \ldots, b_n,\, t_1^0, \ldots, t_m^0,\, \tilde{t}_1^0, \ldots, \tilde{t}_\ell^0$ *are distinct.*

Lemma 10.6. *If $a \in X$ and $U \in Wr^{-1}(a)$, then* $(t_1^0, \ldots, t_m^0; b_1, \ldots, b_n; 2\zeta + \nu/2)$ *is a solution of the Bethe ansatz equations* (5.1) *assigned to the triple $(b_1,...,b_n;2\zeta + \nu/2;V[\nu])$, and* $(\tilde{t}_1^0, \ldots, \tilde{t}_\ell^0; b_1, \ldots, b_n; -2\zeta - \nu/2)$ *is a solution of the Bethe ansatz equations* (5.1) *assigned to the triple $(b_1,...,b_n;-2\zeta - \nu/2;V[-\nu])$.*

Proof. We have

$$\mathrm{Wr}^x(p(x,U), q(x,U)) = \frac{2\zeta + \ell - m}{x}\left(x^n + \sum_{s=1}^{n}(-1)^s\, a_s\, x^{n-s}\right).$$

Now the lemma follows from Lemmas 10.5, 7.1.

For $U \in \Omega(\zeta, m, \grave{})$ denote by $\mathcal{G}_U$ the monic differential operator with kernel $\langle p(x,U), q(x,U) \rangle$,

(10.11) $\mathcal{G}_U = \partial_x^2 + G_{1;U}(x)\partial_x + G_{2,U}(x)$.

The operator $\mathcal{G}_U$ is obtained from the operator $\mathcal{G}$ by evaluating the generating functions p, q at the point U.

Lemma 10.7. *Let $a \in X$ and $U \in \mathrm{Wr}^{-1}(a)$. Let $(t^0; b; 2\zeta + \nu/2)$ be the solution of the Bethe ansatz equations described in Lemma 10.6. Let $\mathcal{E}^c_{t^0; b; 2\zeta+\nu/2}$ be the differential operator defined in (8.3). Then*

$$\mathcal{E}^c_{t^0; b; 2\zeta+\nu/2} = \mathcal{G}_U.$$

Proof. The lemma follows from Lemma 8.2.

11. Isomorphisms

In Section 9 we introduced the $B(\mu, m, \grave{})$-module $V^S[\nu]$, where $\mu \in \mathbb{C}$, $\nu = n - 2m$, $m + \grave{} = n$. In Section 10 we discussed the properties of the algebra $O(\zeta, m, \grave{})$ under the assumption that $\zeta \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$.

We consider $O(\zeta, m, \grave{})$ as the $O(\zeta, m, \grave{})$-module with action defined by multiplication.

In this section we construct an isomorphism between the $B(\mu, m, \grave{})$-module $V^S[\nu]$ and the $O(\zeta, m, \grave{})$-module $O(\zeta, m, \grave{})$ under the assumption that

(11.1) $\zeta = \frac{\mu}{2} - \frac{\nu}{4}$ and $\zeta \in \mathbb{C} - \frac{1}{2}\mathbb{Z}$,

where the last inclusion can be reformulated as

(11.2) $\mu \notin \frac{n}{2} + \mathbb{Z}$,

cf. the assumptions on μ and ζ in Theorems 4.6, 7.2, Lemmas 7.1, 7.3 and Section 10.

The construction of the isomorphism is similar to the constructions in [MTV3, MTV2].

11.1. **Isomorphism of algebras.** Consider the map

$$\tau : O(\zeta, m, \grave{}) \to B(\mu, m, \grave{}), \qquad G_{ij} \mapsto F_{ij}.$$

Theorem 11.1 (cf. [MTV3, Theorem 5.3], [MTV2, Theorem 6.3]). *Under the assumptions* (11.1) *the map τ is a well-defined isomorphism of graded algebras.*

Proof. Let a polynomial $R(G_{ij})$ in generators G_{ij} be equal to zero in $O(\zeta, m, \grave{})$. Let us prove that the corresponding polynomial $R(F_{ij})$ is equal to zero in $B(\mu, m, \grave{})$. Indeed, $R(F_{ij})$ is a polynomial in $z_1, ..., z_n$ with values in $\mathrm{End}(V[\nu])$. By Lemmas 10.5 - 10.7, 5.4, for generic $b_1, ..., b_n$ the value

of the polynomial $R(F_{ij})$ at $z_1 = b_1,...,z_n = b_n$ equals zero. Hence, the polynomial $R(F_{ij})$ equals zero identically and the map τ is a well-defined defined homomorphism of algebras.

The elements G_{ij}, F_{ij} are of the same degree. Hence τ is a graded homomorphism.

Let a polynomial $R(G_{ij})$ in generators G_{ij} be a nonzero element of $O(\zeta,m,\grave{})$. Then the value of $R(G_{ij})$ at a generic point $U \in \Omega(\zeta,m,\grave{})$ is not equal to zero by Lemma 10.7. Then the polynomial $R(F_{ij})$ is not identically equal to zero. Therefore, the map τ is injective.

Since the elements F_{ij} generate the algebra $B(\mu,m,\grave{})$, the map τ is surjective.

The algebra $C[z_1,...,z_n]^S$ is embedded into the algebra $B(\mu,m,\grave{})$ as the subalgebra of operators of multiplication by symmetric polynomials. The algebra $C[z_1,...,z_n]^S$ is embedded into the algebra $O(\zeta,m,\grave{})$, the elementary symmetric polynomials $\sigma_1(z),...,\sigma_n(z)$ being mapped to the elements $\Sigma_1,...,\Sigma_n$. These embeddings give the algebras $B(\mu,m,\grave{})$ and $O(\zeta,m,\grave{})$ the structure of $C[z_1,...,z_n]^S$-modules.

Lemma 11.2 ([MTV3, Lemma 6.4]). *Under assumptions* (11.1) *the map τ is an isomorphism of* $C[z_1,...,z_n]^S$-*modules.*

Proof. The lemma follows from formulas (7.6), (10.7).

11.2. **Isomorphism of modules.** The subspace of $V^S[v]$ of all elements of degree 0 is of dimension one and is generated by the vector

$$v_+ = \sum_{I=(I_1,I_2),|I_1|=m,|I_2|=\`} v_I.$$

The subspace of $O(\zeta,m,\`)$ of all elements of degree 0 is of dimension one and is generated by the element 1. Define the $C[z_1,...,z_n]^S$-linear map

(11.3) $\phi : O(\zeta,m,\`) \to V^S[v], \qquad G \mapsto \tau(G)v_+.$

Theorem 11.3 ([MTV3, Theorem 6.7])**.** *Under assumptions* (11.1)*, the map ϕ is a graded isomorphism of graded $C[z_1,...,z_n]^S$-modules. The maps τ and ϕ intertwine the action of multiplication operators on $O(\zeta,m,\`)$ and the action of operators of the Bethe algebra $B(\mu,m,\`)$ on $V^S[v]$, that is, for any $f,g \in O(\zeta,m,\`)$, we have*

(11.4) $\phi(fg) = \tau(f)\phi(g).$

In other words, the maps τ and ϕ define an isomorphism between the $O(\zeta,m,\`)$-module $O(\zeta,m,\`)$ and the $B(\mu,m,\`)$-module $V^S[v]$.

Proof. First we show that the map ϕ is injective. Indeed, the algebra $O(\zeta,m,\`)$ is a free polynomial algebra containing the subalgebra $C[z_1,...,z_n]^S$. The quotient algebra $O(\zeta,m,\`)/C[z_1,...,z_n]^S$ is finite-dimensional by Lemma 10.2. The kernel of ϕ is a proper ideal I in $O(\zeta,m,\`)$. Then $\tau(I)$ is an ideal in $B(\mu,m,\`)$. Any proper ideal in $B(\mu,m,\`)$ has zero intersection with $C[z_1,...,z_n]^S$. Hence I has zero intersection with $C[z_1,...,z_n]^S$ and therefore is the zero ideal. The injectivity is proved.

   The map ϕ is graded. The graded characters of $V^S[v]$ and $O(\zeta,m,\`)$ are equal by Lemmas 9.2 and 10.1. Hence ϕ is an isomorphism.

Corollary 11.4. *Assume that $a = (a_1,...,a_n) \in C^n$ is such that the polynomial $x^n + \sum_{s=1}^{n}(-1)^s a_s x^{n-s}$ has distinct nonzero roots $b_1,...,b_n$. Then under assumptions* (11.1)*, the isomorphisms τ, ϕ induce the isomorphism of the $B(b_1,...,b_n;\mu;V[v])$-module $V[v]$ and the $O_a(\zeta;m,\`)$-module $O_a(\zeta;m,\`)$, where $O_a(\zeta;m,\`)$ is the algebra of functions on the fiber $Wr^{-1}(a)$ of the Wronski map, see* (10.9)*.*

Proof. The corollary follows from Lemma 9.7 and Theorems 11.1, 11.3. **Corollary 11.5.** *The degree of the Wronski map* Wr *equals* $\dim V[\nu] = \binom{n}{m}$.

11.3. **Dynamical Bethe algebra and quasi-polynomials.** The space $V = V_1^{\otimes n}$ has a nontrivial zero weight subspace if n is even. Let $n = 2m$. For the zero weight subspace $V[0]$, we have $\nu = 0$, $m = \grave{}$, and assumptions (11.1) take the form

$$\text{(11.5)} \qquad\qquad \zeta = \frac{\mu}{2} \quad \text{and} \quad \mu \, / \in \mathbb{Z}.$$

Let $a = (a_1,...,a_n) \in \mathbb{C}^n$ be such that the polynomial $x^n + \sum_{s=1}^{n}(-1)^s a_s x^{n-s}$ has distinct nonzero roots $b_1,...,b_n$. Consider the functional space $E[\mu]$ as the module over the dynamical Bethe algebra $B(b_1,...,b_n;E[\mu])$, see Section 3.2. Consider the $O_a(\zeta;m,m)$-module $O_a(\zeta;m,m)$, where $O_a(\zeta;m,m)$ is the algebra of functions on the fiber $\text{Wr}^{-1}(a)$ of the Wronski map.

Corollary 11.6. *Under assumptions* (11.5)*, the isomorphisms τ, ϕ and the isomorphism $V[0] \to E[\mu]$ in Corollary 4.4 induce the isomorphism of the $B(b_1,...,b_n;E[\mu])$-module $E[\mu]$ and the $O_a(\zeta;m,m)$-module $O_a(\zeta;m,m)$.*

11.4. **Weyl involution and transposition of quasi-polynomials.** Consider the $B(\mu,m,\grave{})$-module $V^S[\nu]$ and $B(-\mu,\grave{},m)$-module $V^S[-\nu]$. Consider the $O(\zeta,m,\grave{})$-module $O(\zeta,m,\grave{})$ and $O(-\zeta,\grave{},m)$-module $O(-\zeta,\grave{},m)$.

Under assumptions (11.1), consider the diagram,

$$\text{(11.6)} \qquad \begin{array}{ccc} (B(\mu,m,\grave{}), V^S[\nu]) & \longrightarrow & (B(-\mu,\grave{},m), V^S[-\nu]) \\ \downarrow & & \downarrow \\ (O(\zeta,m,\grave{}), O(\zeta,m,\grave{})) & \longrightarrow & (O(-\zeta,\grave{},m), O(-\zeta,\grave{},m)) \end{array} \quad .$$

Here $V^S[\nu] \to O(\zeta,m,\grave{})$ and $V^S[-\nu] \to O(-\zeta,\grave{},m)$ are the module isomorphisms of Theorem 11.3. The map $V^S[\nu] \to V^S[-\nu]$ is the module isomorphism of Lemma 9.6. The map $O(\zeta,m,\grave{}) \to O(-\zeta,\grave{},m)$ is the module isomorphism defined by the transposition of the quasi-polynomials p,q.

Theorem 11.7. *The diagram* (11.6) *is commutative.*

Proof. The theorem follows from Lemma 8.3.

The commutativity of diagram (11.6) implies the commutativity of the diagram of fibers over a generic point $a \in \mathbb{C}^n$,

$$\text{(11.7)} \qquad \begin{array}{ccc} (B(b_1,...,b_n;\mu,V[\nu]), V[\nu]) & \longrightarrow & (B(b_1,...,b_n;-\mu,V[-\nu]), V[-\nu]) \\ \downarrow & & \downarrow \end{array} \quad ,$$

$$(O_a(\zeta,m,`),\ O_a(\zeta,m,`)) \longrightarrow (O_a(-\zeta,`,m),\ O_a(-\zeta,`,m))$$

see notations in Section 11.2.

Combining commutative diagrams (11.7) and (4.9) we obtain the commutative diagram

(11.8)
$$\begin{array}{ccc} (B(z;E[\mu]),\ E[\mu]) & \longrightarrow & (B(z;E[-\mu]),\ E[-\mu]) \\ \downarrow & & \downarrow \\ (O_a(\zeta,m,m),\ O_a(\zeta,m,m)) & \longrightarrow & (O_a(-\zeta,m,m),\ O_a(\zeta,m,m)) \end{array} ,$$

which holds if $n = 2m$ is even and $\mu \notin \mathbb{Z}$. The diagram identifies the Weyl involution $E[\mu] \to E[-\mu]$ in the functional spaces of eigenfunctions of the KZB operator H_0 with the isomorphism $O_a(\zeta,m,m) \to O_a(-\zeta,m,m)$ induced by the transposition of quasi-polynomials.

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