EQUIVARIANT QUANTUM DIFFERENTIAL EQUATION, STOKES BASES, AND K-THEORY FOR A PROJECTIVE SPACE

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Abstract. We consider the equivariant quantum differential equation for the projective space P^{n-1} and introduce a compatible system of *qKZ* difference equations. We prove an equivariant gamma theorem for P^{n-1}, which describes the asymptotics of the differential equation at its regular singular point in terms of the equivariant characteristic gamma class of the tangent bundle of P^{n-1}. We describe the Stokes bases of the differential equation at its irregular singular point in terms of the exceptional bases of the equivariant K-theory algebra of P^{n-1} and a suitable braid group action on the set of exceptional bases.

Our results are an equivariant version of the well-known results of B.Dubrovin and D.Guzzetti.

Key words: Equivariant quantum differential equation, equivariant K-theory, q-hypergeometric solutions, braid group action

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To Boris Dubrovin with admiration

Contents

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1. Introduction

The quantum differential equation of the complex projective space P^{n-1} is an ordinary differential equation

$$\left(p\frac{d}{dp} - x*_p\right) I(p) = 0 \quad , \tag{1.1}$$

where the unknown function $I(p)$ takes values in the cohomology algebra $H^*(P^{n-1};\mathbb{C})$ and $x*_p : H^*(P^{n-1};\mathbb{C}) \to H^*(P^{n-1};\mathbb{C})$ is the operator of *quantum* multiplication by the first Chern class of the tautological line bundle over P^{n-1}. The differential equation has two singular points: a regular singular point at $p = 0$ and an irregular singular point at $p = \infty$.

The quantum differential equation has the following remarkable structures.

The specially normalized asymptotics of its solutions at $p = 0$ can be described in terms of the characteristic gamma class of the tangent bundle of P^{n-1}. This description of the asymptotics by B.Dubrovin in [D1] was the first example of a gamma theorem, which is proved now in many examples and is known as the gamma conjecture, see [D1, D2, KKP, GGI, GI, GZ, CDG, TV3].

The Stokes matrices of the quantum differential equation at the irregular singular point $p = \infty$ are described in terms of the braid group action on the set of full collections of exceptional objects in the derived category $\mathrm{Der}^b(\mathrm{Coh}(P^{n-1}))$ of coherent sheaves on P^{n-1}. That phenomenon was predicted by B.Dubrovin in [D1] for Fano varieties and was proved for P^{n-1} by D.Guzzetti [Gu]. That braid group action is described with the help of a certain non-symmetric bilinear form on the K-theory algebra $K(P^{n-1},\mathbb{C})$.

In this paper we consider the *equivariant* quantum differential equation of the projective space P^{n-1} and establish similar results.

In the equivariant case the torus $T = (\mathbb{C}^\times)^n$ acts on P^{n-1} and the quantum differential equation takes the form

$$\left(p\frac{d}{dp} - x*_{p,z}\right)I(p,z_1,\dots,z_n) = 0 , \qquad (1.2)$$

where $z = (z_1,\dots,z_n)$ are equivariant parameters. We also introduce a system of the *qKZ* difference equations

$$I(p,z_1,\dots,z_i - 1,\dots,z_n) = -K_i(p,z_1,\dots,z_n)I(p,z_1,\dots,z_n), \qquad i = 1,\dots,n, \qquad (1.3)$$

where $K_i(p,z_1,\dots,z_n)$ are suitable linear operators. The joint system of the equivariant quantum differential equation and *qKZ* difference equations is compatible. The space of solutions of this system is a module over the ring of scalar functions in $z_1,\dots,z_n$, 1-periodic with respect to each of the variables $z_1,\dots,z_n$.

We prove an equivariant gamma theorem, which describes the asymptotics of solutions at $p = 0$ of the equivariant quantum differential equation in terms of the equivariant characteristic gamma-class of the tangent bundle of P^{n-1}, see Theorem 4.3.

We describe the Stokes bases of the equivariant quantum differential equation at $p = \infty$. For that we identify the space of solutions of the joint system of equations (1.2) and (1.3) with the space of the equivariant K-theory algebra $K_T(P^{n-1},\mathbb{C})$. We introduce a sesquilinear form on $K_T(P^{n-1},\mathbb{C})$, exceptional bases of $K_T(P^{n-1},\mathbb{C})$, a braid group action on the exceptional bases, and describe the Stokes bases in terms of that braid group action, see Theorem 7.1.

To prove these results we use integral representations for solutions of the joint system of equations (1.2) and (1.3) obtained in [TV3]. In [TV3] we constructed q-hypergeometric integral representations for solutions of the joint systems of equivariant quantum differential equations and associated *qKZ* difference equations for the cotangent bundle $T^*\mathcal{F}_\lambda$ of a partial flag variety $\mathcal{F}_\lambda$. In a suitable limit those solutions become solutions of the corresponding equations for the partial flag variety $\mathcal{F}_\lambda$. In this paper we use the special case of $\mathcal{F}_\lambda = P^{n-1}$.

The important role in this paper is played by the identification of the space of solutions of the joint system of equations (1.2) and (1.3) with the space of the K-theory algebra $K_T(P^{n-1},\mathbb{C})$. This identification also comes from [TV3]. Earlier examples of such an identification see in [TV1, TV2].

We would like to stress that the equivariant case is simpler than the corresponding nonequivariant case. The equivariant case is more rigid because, in addition to the quantum differential equation, we also have the compatible system of difference equations, and therefore there are less problems with choices of normalizations of solutions.

The paper is organized as follows. In Section 2 we introduce the equivariant cohomology and K-theory algebra of P^{n-1}. In Section 3 we introduce the equivariant quantum differential equation and *qKZ* difference equations. In Section 4 we describe the integral representations for solutions and asymptotics of solutions at $p = 0$. In Section 5 we discuss asymptotics of solutions at $p = \infty$, introduce Stokes bases in the space of solutions. In Section 6 we introduce exceptional bases in the space of solutions and a braid group action on the set of exceptional bases. In Section 7 we describe the Stokes bases of the equivariant quantum differential equation at $p = \infty$. Our proofs in Section 7 are similar to the corresponding proofs in [Gu].

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2. Projective space

2.1. **Equivariant cohomology.** For $n > 2$, let P^{n-1} be the projective space parametrizing one-dimensional subspaces $F \subset \mathbb{C}^n$.

Let $\{u_1,...,u_n\}$ be the standard basis of $\mathbb{C}^n$. For $I \in \{1,...,n\}$, let $pt_I \in P^{n-1}$ be the point corresponding to the coordinate line spanned by u_I. The complex torus T^n acts diagonally on $\mathbb{C}^n$, and hence on P^{n-1}. The points pt_I, $I = 1,...,n$, compose the fixed point set.

We consider the equivariant cohomology algebra $H_{T^n}(P^{n-1};\mathbb{C})$. Denote by x the equivariant Chern root of the tautological line bundle $\mathcal{L}$ over P^{n-1} with fiber F. Denote by $y = (y_1,...,y_{n-1})$

the equivariant Chern roots of the vector bundle over P^{n-1} with fiber $\mathbb{C}^n/F$. Denote by $\boldsymbol{z} = (z_1,\dots,z_n)$ the Chern roots corresponding to the factors of the torus T^n. Then

$$
\begin{aligned}
H_{T^n}(P^{n-1};\mathbb{C}) \;&=\; \mathbb{C}[x,\boldsymbol{z}] \Big/ \Big\langle \prod_{a=1}^{n} (x - z_a) \Big\rangle \\
&= \mathbb{C}[x,\boldsymbol{y},\boldsymbol{z}]^{S_{n-1}} \Big/ \Big\langle (u-x)\prod_{j=1}^{n-1}(u-y_j) - \prod_{a=1}^{n}(u-z_a) \Big\rangle,
\end{aligned}
\tag{2.1}
$$

where $\mathbb{C}[x,\boldsymbol{y},\boldsymbol{z}]^{S_{n-1}}$ is the algebra of polynomials in $x,\boldsymbol{y},\boldsymbol{z}$ symmetric in the variables $y_1,\dots,y_{n-1}$, and the isomorphism of the first quotient to the second one sends an element $f(x,\boldsymbol{z})$ of the first quotient to the element $f(x,\boldsymbol{z})$ of the second.

The cohomology algebra $H_{T^n}(P^{n-1};\mathbb{C})$ is a module over $H^*_{T^n}(pt;\mathbb{C}) = \mathbb{C}[\boldsymbol{z}]$.

2.2. **Symmetric functions.** Consider the algebra $\mathbb{C}[\boldsymbol{Z}^{\pm1}] = \mathbb{C}[Z_1^{\pm1},\dots,Z_n^{\pm1}]$ of Laurent polynomials and its elements

$$
\begin{aligned}
e_k(\boldsymbol{Z}) &= \sum_{1\leqslant i_1<\dots<i_k\leqslant n} Z_{i_1}\cdots Z_{i_k}, & k &= 1,\dots,n, \\
h_k(\boldsymbol{Z}) &= \sum_{\substack{i_1\geqslant 0,\dots,i_n\geqslant 0\\ i_1+\dots+i_n=k}} Z_1^{i_1}\cdots Z_n^{i_n}, & k &\in \mathbb{Z}_{>0}.
\end{aligned}
\tag{2.2}
$$

Put $e_0 = 1$, $h_0 = 1$. Then

$$
\sum_{i=0}^{k} (-1)^i\, h_i(\boldsymbol{Z})\, e_{k-i}(\boldsymbol{Z}) = 0, \qquad k \in \mathbb{Z}_{>0}.
\tag{2.3}
$$

For $f(Z_1,\dots,Z_n) \in \mathbb{C}[\boldsymbol{Z}^{\pm1}]$ denote $f(\boldsymbol{Z}^{-1}) = f(Z_1^{-1},\dots,Z_n^{-1})$.

2.3. **Equivariant K-theory.** Consider the equivariant K-theory algebra $K_{T^n}(P^{n-1};\mathbb{C})$. We have

$$
K_{T^n}(P^{n-1};\mathbb{C}) = \mathbb{C}[X^{\pm1},\boldsymbol{Z}^{\pm1}] \Big/ \Big\langle \prod_{a=1}^{n} (X - Z_a) \Big\rangle.
\tag{2.4}
$$

Here the variable X corresponds to the tautological line bundle $\mathcal{L}$ over P^{n-1}; the variables $Z_1,\dots,Z_n$ are the equivariant parameters corresponding to the factors of T^n; $\mathbb{C}[X^{\pm1},\boldsymbol{Z}^{\pm1}]$ is the algebra of Laurent polynomials in $X, Z_1,\dots,Z_n$.

The algebra $K_{T^n}(P^{n-1};\mathbb{C})$ is a module over $K_{T^n}(pt;\mathbb{C}) = \mathbb{C}[\boldsymbol{Z}^{\pm1}]$.

We have a map

$$
\rho \;:\; K_{T^n}(P^{n-1},\mathbb{C}) \to K_{T^n}(P^{n-1},\mathbb{C}), \qquad f(X,\boldsymbol{Z}) \mapsto f(X^{-1},\boldsymbol{Z}^{-1}),
$$

which sends the class of a vector bundle to the class of the dual vector bundle.

The map to a point $\psi : P^{n-1} \to pt$ gives us the push-forward map $\psi_* : K_{T^n}(P^{n-1},\mathbb{C}) \to \mathbb{C}[Z^{\pm 1}]$ defined by the formula

$$\psi_* f(X, \boldsymbol{Z}) \,=\, \sum_{a=1}^{n} \frac{f(Z_a, \boldsymbol{Z})}{\prod_{j\neq a}(1 - Z_a/Z_j)} \,=\, -\sum_{a=1}^{n} \mathrm{Res}_{X=Z_a} \frac{f(X, \boldsymbol{Z})}{X\prod_{j=1}^{n}(1 - X/Z_a)} \tag{2.5}$$

The push-forward map ψ_* gives us a symmetric bilinear form on $K_{T^n}(P^{n-1},\mathbb{C})$ defined by the formula $(f,g) = \psi_*(fg)$. We are interested in its non-symmetric sesquilinear version, (2.6)

$$\begin{aligned} A(f,g) \,&=\, \psi_*(X^n \rho(f) g)\, \frac{(-1)^{n-1}}{\prod_{j=1}^{n} Z_j} \\ &= \sum_{a=1}^{n} \frac{f(Z_a^{-1}, \boldsymbol{Z}^{-1})\, g(Z_a, \boldsymbol{Z})}{\prod_{j\neq a}(1 - Z_j/Z_a)} \,=\, \sum_{a=1}^{n} \mathrm{Res}_{X=Z_a} \frac{f(X^{-1}, \boldsymbol{Z}^{-1})\, g(X, \boldsymbol{Z})}{X\prod_{a=1}^{n}(1 - Z_a/X)}\,. \end{aligned}$$

Lemma 2.1. *For $i,j \in \mathbb{Z}$, we have $A(X^i,X^j) = h_{j-i}(Z)$ if $i \leq j$, and $A(X^i,X^j) = 0$ if $j < i < j + n$.*

3. Quantum equivariant differential equation and *qKZ* difference equations

3.1. **Quantum multiplication.** In enumerative geometry the multiplication in the equivariant cohomology algebra $H_{T^n}(P^{n-1},\mathbb{C})$ is deformed. The deformed *quantum* multiplication depends on the *quantum* parameter p and equivariant parameters z. The quantum multiplication is determined by the $\mathbb{C}[z]$-linear operator

$$x*_{p,\boldsymbol{z}} : H_{T^n}(P^{n-1},\mathbb{C}) \to H_{T^n}(P^{n-1},\mathbb{C})$$

of multiplication by the generator $x \in H_{T^n}(P^{n-1},\mathbb{C})$. In the basis $\{1,x,...,x^{n-1}\}$, we have

$$x *_{p,\boldsymbol{z}} x^j \,=\, x^{j+1}, \qquad j = 0,\dots,n-2\,,$$

$$x *_{p,\boldsymbol{z}} x^{n-1} \,=\, p + x^n \,=\, p + \sum_{i=1}^{n} (-1)^{i-1} e_i(\boldsymbol{z})\, x^{n-i}\,,$$

where $e_i(z)$ are the elementary symmetric functions in z.[3]

We also use the basis $\{g_1,...,g_n\}$,

$$g_i = \prod_{a=i+1}^{n} (x - z_a), \quad i = 1,\dots,n-1\,, \quad \text{and} \quad g_n = 1.$$

In this basis we have

[3] These formulas were explained to us by A.Givental

$$x *_{p,z} g_i = z_i g_i + g_{i-1}, \; i = 2,\dots,n, \quad x *_{p,z} g_1 = z_1 g_1 + p g_n .$$

3.2. **R-matrices and *qKZ* operators.** For $a,b \in \{1,\dots,n\}$, $a \neq b$, define a $\mathbb{C}[z]$-linear operator

$$R_{ab}(u) : H_{T^n}(\mathbb{P}^{n-1},\mathbb{C}) \to H_{T^n}(\mathbb{P}^{n-1},\mathbb{C}),$$

depending on $u \in \mathbb{C}$ and called the *R-matrix*, by the formula

$$R_{ab}(u) g_i = g_i , \qquad i \neq a,b,$$

$$R_{ab}(u) g_b = g_a , \qquad R_{ab}(u) g_a = g_b + u g_a .$$

These R-matrices satisfy the Yang-Baxter equation

$$R_{ab}(u-v) R_{ac}(u) R_{bc}(v) = R_{bc}(v) R_{ac}(u) R_{ab}(u-v),$$

for all distinct a,b,c, and the inversion relation

$$R_{ab}(u) R_{ba}(-u) = 1.$$

Define the operators $E_1,\dots,E_n$ such that

$$E_i g_j = \delta_{ij} g_i.$$

Define the *qKZ* operators $K_1,\dots,K_n$ by the formula

$$K_i = R_{i,i-1}(z_i - z_{i-1} - 1)\dots R_{i,1}(z_i - z_1 - 1) p^{-E_i} R_{i,n}(z_i - z_n) \dots R_{i,i+1}(z_i - z_{i+1}).$$

3.3. **Isomorphisms $\Theta_{i,z}$.** The basis $\{g_1,\dots,g_n\}$ allows us to define the isomorphisms $\Theta_{i,z}$, $i = 1,\dots,n$, of the vector spaces,

$$\Theta_{i,(z_1,\dots,z_n)} : H_{T^n}(\mathbb{P}^{n-1};\mathbb{C})|_{(z_1,\dots,z_n)} \to H_{T^n}(\mathbb{P}^{n-1};\mathbb{C})|_{(z_1,\dots,z_i-1,\dots,z_n)} \; g_j(x,z_1,\dots,z_n) \mapsto g_j(x,z_1,\dots,z_i - 1,\dots,z_n), \; j = 1,\dots,n.$$

Remark. Let $T^*\mathbb{P}^{n-1}$ be the cotangent bundle of $\mathbb{P}^{n-1}$. The elements $g_1,\dots,g_n \in H_{T^n}(\mathbb{P}^{n-1},\mathbb{C})$ are the limits of the stable envelopes for $T^*\mathbb{P}^{n-1}$, in the limit when the cotangent bundle $T^*\mathbb{P}^{n-1}$ turns into the projective space $\mathbb{P}^{n-1}$. See [RTV] on the stable envelopes for the cotangent bundle $T^*\mathbb{P}^{n-1}$, see [GRTV, Section 7] and [TV3, Section 11.4] on this limit. The cotangent bundle of the projective space is an example of a quiver variety. Stable envelopes were introduced in this generality by Maulik and Okounkov in [MO] together with the systems of equivariant quantum differential equations and compatible difference equations generalizing the *qKZ* equations.

3.4. **Quantum differential equation and *qKZ* difference equations.** The *equivariant quantum differential equation* is the differential equation

$$\left(p\frac{d}{dp} - x*_{p,z}\right) I(p, z_1, \dots, z_n) = 0 \qquad (3.1)$$

The system of the *qKZ difference equations* is the system of difference equations

$$I(p,z_1,\dots,z_i - 1,\dots,z_n) = -[\Theta_{i,z} \circ K_i(p,z_1,\dots,z_n)] I(p,z_1,\dots,z_n), \qquad i = 1,\dots,n. \qquad (3.2)$$

In these equations the unknown function $I(p,z)$ takes values in the cohomology algebra $H_{T_n}(P^{n-1},\mathbb{C})$ extended by functions in p,z.

Theorem 3.1. *The joint system of equations* (3.1) *and* (3.2) *is compatible.*

Proof. The proof is straightforward.

4. Integral representations for solutions

The quantum differential equation (3.1) was solved by A.Givental in [Gi]. In this section we follow [TV3] and describe the integral representations for solutions of the joint system of equations (3.1) and (3.2).

Notice that the space of solutions to the joint system of equations (3.1) and (3.2) is a module over the ring of scalar functions in $z_1,\dots,z_n$, 1-periodic with respect to each of the variables $z_1,\dots,z_n$.

4.1. **Master and weight functions.** Consider the variables t,p, $z = (z_1,\dots,z_n)$. Define the *master function* Φ and $H_{T_n}(P^{n-1};\mathbb{C})$-valued *weight function* W by the formulas:

$$\Phi(t,p,\boldsymbol{z}) = (e^{\pi\sqrt{-1}(2-n)}p)^t \prod_{a=1}^{n} \Gamma(z_a - t), \qquad W(t,\boldsymbol{y}) = \prod_{j=1}^{n-1} (y_j - t), \qquad (4.1)$$

where Γ is the gamma function.

4.2. **Solutions as Jackson integrals.** Consider $\mathbb{C}$ with coordinate p and $\mathbb{C}^n$ with coordinates $z = (z_1,\dots,z_n)$.

Let L^0 be the p-line $\mathbb{C}$ with a cut to fix the argument of p, that is, we delete from $\mathbb{C}$ a ray from 0 to ∞ and fix the argument of p on the complement.

Let L^{00} be the complement in $\mathbb{C}^n$ to the union of the hyperplanes

$$z_a - z_b = m \qquad \text{for all } a,b = 1,\dots,n,\ a \neq b, \text{ and all } m \in \mathbb{Z}. \qquad (4.2)$$

Set $L = L^0 \times L^{00} \subset \mathbb{C} \times \mathbb{C}^n$. For $J = 1,\dots,n$ define

$$\Psi_J(p,y,z) = -\sum \mathrm{Res}_{t=z_J+r}\, \Phi(t,p,z) W(t,y). \qquad (4.3)$$

$r\in\mathbb{Z}_{>0}$

These sums are called the *Jackson integrals*.

Theorem 4.1 ([TV3]). *The functions $\Psi_J(p,y,z)$, $J = 1,...,n$, belong to the extension of $H_{T^n}(P^{n-1},\mathbb{C})$ by holomorphic functions in p,z on the domain $L \subset \mathbb{C}\times\mathbb{C}^n$. Each of the functions is a solution to the joint system of equations* (3.1) *and* (3.2)*. These functions form a basis of solutions.*

Proof. The theorem is proved in [TV3, Section 11.4], see formula (11.18) in there. In particular, the fact that the functions form a basis follows from the determinant formula (11.23).

In fact, in Section 11.4 the solutions to the joint system of the equivariant quantum differential equations and associated *qKZ* equations are described for an arbitrary partial flag variety.

The solutions $\Psi_J(p,y,z)$, $J = 1,...,n$, are called the *q-hypergeometric* solutions.

4.3. **Asymptotics as $p \to 0$ and equivariant gamma theorem.**

Corollary 4.2 ([TV3, Formula (11.19)]). *As $p \to 0$, we have*

$$\Psi_J(p,\boldsymbol{y},\boldsymbol{z}) = (e^{\pi\sqrt{-1}\,(2-n)}p)^{z_J}\prod_{a\neq J}^{n}\Gamma(1+z_a-z_J)\Big(\Delta_J+\sum_{k=1}^{\infty}p^k\Psi_{J,k}(\boldsymbol{y},\boldsymbol{z})\Big), \tag{4.4}$$

where the equivariant class Δ_J restricts to 1 at the fixed point pt_J and restricts to zero at all other fixed points pt_I with I 6= J. The classes $\Psi_{J,k}(y,z)$ are suitable rational functions in z regular on L^{00}.

Recall that $\prod_{j=1}^{n-1}(y_j-x)\in H_{T^n}(P^{n-1},\mathbb{C})$ is the equivariant total Chern class of the tangent bundle of P^{n-1} and $x \in H_{T^n}(P^{n-1},\mathbb{C})$ is the equivariant first Chern class $c_1(\mathcal{L})$ of the tautological line bundle $\mathcal{L}$ over P^{n-1}. The function $\hat\Gamma_{P^{n-1}} = \prod_{a=1}^{n-1}\Gamma(1+y_a-x)$ is called the *equivariant gamma class* of the tangent bundle of P^{n-1}. Corollary 4.2 can be reformulated as the following statement.

Theorem 4.3. *The leading term of the asymptotics as $p \to 0$ of the q-hypergeometric solutions $(\Psi_J(p,\boldsymbol{y},\boldsymbol{z}))_{J=1}^n$ is the product of the equivariant gamma class of the tangent bundle of P^{n-1} and the exponential of the equivariant first Chern class of the tautological line bundle $\mathcal{L}$:*

$$\big(e^{\pi\sqrt{-1}\,(2-n)}p\big)^{c_1(\mathcal{L})}\,\hat\Gamma_{P^{n-1}}. \tag{4.5}$$

This assertion is an equivariant analog of Dubrovin's gamma theorem for P^{n-1}, see [D1, D2] and also [KKP, GGI, GI, GZ, CDG].

4.4. **Solutions as elements of the equivariant K-theory.** Introduce new functions:

$$\acute{T} = e^{2\pi\sqrt{-1}\,t}, \quad \acute{Z}_j = e^{2\pi\sqrt{-1}\,z_j}, \quad j = 1,\dots,n. \tag{4.6}$$

Denote $\acute{Z} = (\acute{Z}_1,...,\acute{Z}_n)$.

Let $Q(X,Z) \in \mathbb{C}[X^{\pm 1}, Z^{\pm 1}]$ be a Laurent polynomial. Define

$$\Psi_Q(p,\boldsymbol{y},\boldsymbol{z}) = \sum_{J=1}^{n} Q(\acute{Z}_J, \acute{\boldsymbol{Z}})\, \Psi_J(p,\boldsymbol{y},\boldsymbol{z}) . \tag{4.7}$$

Clearly, $\Psi_Q(p,y,z)$ is a solution on the domain $L \subset \mathbb{C} \times \mathbb{C}^n$ of the joint system (3.1) and (3.2), as a linear combination of solutions $\Psi_J(p,y,z)$ with coefficients, 1-periodic with respect to $z_1,\dots,z_n$ and independent of p.

It is also clear that if Q lies in the ideal in $\mathbb{C}[X^{\pm 1}, Z^{\pm 1}]$ generated by the polynomial $\prod_{a=1}^{n}(X - Z_a)$, then $\Psi_Q(p,y,z)$ is the zero solution. Hence formula (4.7) defines a map $Q \mapsto \Psi_Q(p,y,z)$ from the equivariant K-theory algebra $K_{T_n}(P^{n-1},\mathbb{C})$ to the space of solutions on the domain L to the joint system (3.1) and (3.2).

4.5. **Solutions Ψ^m.** For $m \in \mathbb{Z}$, denote by $\Psi^m(p,y,z)$ the solution $\Psi_Q(p,y,z)$ corresponding to the Laurent polynomial $Q = X^{m-1}$.

Corollary 4.4. *For any $k \in \mathbb{Z}$, we have*

$$\sum_{i=0}^{n} (-1)^{n-i} e_{n-i}(\acute{\boldsymbol{Z}})\, \Psi^{k+i}(p,\boldsymbol{y},\boldsymbol{z}) = 0 , \tag{4.8}$$

where $e_0(Z'),\dots,e_n(Z')$ are the elementary symmetric functions in Z'.

Theorem 4.5 ([TV3, Theorem 11.3])**.** *For any $k \in \mathbb{Z}$, the solutions $\Psi^{k+m}(p,y,z)$, $m = 0,\dots,n-1$, form a basis of the space of solutions on the domain L of the joint system* (3.1) *and* (3.2)*.*

4.6. **Module S_n.** The space of solutions of the joint system of equations (3.1) and (3.2) is a module over the algebra of functions in $z_1,\dots,z_n$, which are 1-periodic with respect to each variable.

We will consider the space S_n of solutions of the form

$$\sum_{m=1}^{n} Q_m(\acute{\boldsymbol{Z}})\, \Psi^m(p,\boldsymbol{y},\boldsymbol{z}) , \qquad \text{where} \qquad Q_m(Z) \in \mathbb{C}[Z^{\pm 1}]. \tag{4.9}$$

This space is a $\mathbb{C}[Z^{\pm 1}]$-module, in which multiplication by $Q(Z)$ is defined as multiplication by $Q(Z')$. With this choice of the space of solutions, we allow ourselves to multiply solutions $\Psi^m(p,y,z)$ only by 1-periodic functions of the form $Q_m(Z')$, where $Q(Z) \in \mathbb{C}[Z^{\pm 1}]$.

By Corollary 4.4, the module S_n contains all solutions $\Psi^m(p,y,z)$, $m \in \mathbb{Z}$.

Corollary 4.6. *The module S_n contains a basis of solutions of the joint system* (3.1) *and* (3.2)*. Moreover, the map $\theta : K_{T_n}(P^{n-1},\mathbb{C}) \to S_n$, defined by*

$$\theta : X^{m-1} \mapsto \Psi^m(p,y,z), \qquad m \in \mathbb{Z}, \tag{4.10}$$

is an isomorphism of the $\mathbb{C}[Z^{\pm 1}]$-*modules.*

Proof. The corollary follows from Theorem 4.5.

Using the isomorphism θ we define a sesquilinear form A on $\mathcal{S}_n$ as the image of the form A on $K_{T_n}(P^{n-1},\mathbb{C})$.

4.7. **Monodromy of the quantum differential equation.** The equivariant quantum differential equation (3.1) has two singular points. A regular singular point at $p = 0$ and an irregular singular point at $p = \infty$.

Fix (p,z) and increase the argument of p by 2π. The analytic continuation of the solutions along this curve will produce the *monodromy* operator $M(z)$ on the space of solutions.

Theorem 4.7. *For every $m \in \mathbb{Z}$ we have $M(z) : \Psi^m(p,y,z) \mapsto \Psi^{m+1}(p,y,z)$. In particular, for any $k \in \mathbb{Z}$, the matrix of the monodromy operator in the basis $\{\Psi^{k+m}(p,y,z) \,|\, m = 0,\dots,n-1\}$ is the companion matrix of the polynomial $X^n - e_1(Z')X^{n-1} + \dots + (-1)^n e_n(Z')$, that defines the relation in the equivariant K-theory algebra,*

$$\begin{pmatrix} 0 & 0 & \dots & \dots & 0 & (-1)^{n+1}e_n(Z') \\ 1 & 0 & \dots & \dots & 0 & (-1)^n e_{n-1}(Z') \\ 0 & 1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & -e_2(Z') \\ 0 & 0 & \dots & \dots & 1 & e_1(Z') \end{pmatrix}$$

Proof. The shift of the argument of p by 2π leads to multiplication by $e^{2\pi\sqrt{-1}z_J}$ of each term in the sum in (4.3). This means that $M(z) : \Psi_J(p,y,z) \mapsto e^{2\pi\sqrt{-1}z_J}\Psi_J(p,y,z)$, and hence $M(z) : \Psi^m(p,y,z) \mapsto \Psi^{m+1}(p,y,z)$ for any $m \in \mathbb{Z}$. Now the shape of the monodromy matrix in the basis $\{\Psi^{k+m}(p,y,z) \,|\, m = 0,\dots,n-1\}$ follows from relation (2.4) in the K-theory algebra.

4.8. **Solutions as integrals over a parabola.** For $A \in \mathbb{C}$, let $C(A) \subset \mathbb{C}$ be the parabola with the following parametrization:

$$C(A) = \{ (A + s^2 + s\sqrt{-1}) \mid s \in \mathbb{R} \}. \tag{4.11}$$

Given z, take A such that all the points $z_1,\dots,z_n$ lie inside $C(A)$. The integral (4.12) below does not depend on a particular choice of A, so we will denote $C(A)$ by $C(z)$.

Lemma 4.8 ([TV3, Lemma 11.5]). *For any Laurent polynomial $Q(X,Z)$ we have*

$$\Psi^{Q}(p,\boldsymbol{y},\boldsymbol{z}) \,=\, \frac{1}{2\pi\sqrt{-1}}\int_{C(\boldsymbol{z})} Q(\acute{T};\acute{\boldsymbol{Z}})\,\Phi(t,p,\boldsymbol{z})\,W(t,\boldsymbol{y})\,dt, \tag{4.12}$$

where the integral converges for any $(p,z) \in L$.

In particular, we have

$$\Psi^{m}(p,\boldsymbol{y},\boldsymbol{z}) \,=\, \frac{1}{2\pi\sqrt{-1}}\int_{C(\boldsymbol{z})} e^{2\pi\sqrt{-1}\,mt}e^{-\pi\sqrt{-1}\,nt}p^t \prod_{a=1}^{n}\Gamma(z_a-t)\prod_{j=1}^{n-1}(y_j-t)\,dt\,. \tag{4.13}$$

5. Asymptotics as $p \to \infty$

5.1. **Asymptotics of Ψ^m.** We make the change of variables:

$$p=s^n,\qquad s=re^{-2\pi\sqrt{-1}\varphi},\qquad -s=re^{-\pi\sqrt{-1}-2\pi\sqrt{-1}\varphi},\qquad r\geqslant 0\,,\quad \varphi\in\mathbb{R}$$

$$\omega=e^{2\pi\sqrt{-1}/n}\,. \tag{5.1}$$

Denote

Lemma 5.1. *For $m \in \mathbb{Z}$, $\phi \in \mathbb{R}$, and*

$$\frac{m}{n}-1\,<\,\varphi\,<\,\frac{m}{n}\,, \tag{5.2}$$

we have the asymptotic expansion as $r \to \infty$,

$$\Psi^m(s^n,\boldsymbol{y},\boldsymbol{z}) \,=\, \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}\,e^{n\omega^m s}(-\omega^m s)^{\sum_{a=1}^n z_a+(1-n)/2}\prod_{j=1}^{n-1}(y_j-\omega^m s)\,\bigl(1+\mathcal{O}(1/s)\bigr)\,. \tag{5.3}$$

where $\arg(-\omega^m s) = 2\pi m/n - \pi - 2\pi\phi$, *so that* $|\arg(-\omega^m s)| < \pi$ *.*

Proof. The proof of this lemma is a modification of the proof of [Gu, Lemma 5]. Consider the logarithm of the integrand in (4.13),

$$\Upsilon(t,\boldsymbol{y},\boldsymbol{z}) \,=\, \log\Bigl(e^{2\pi\sqrt{-1}\,mt}e^{-\pi\sqrt{-1}\,nt}p^t\prod_{a=1}^{n}\Gamma(z_a-t)\prod_{j=1}^{n-1}(y_j-t)\Bigr),$$

and apply to Υ the Stirling formula

$$\log\,\Gamma(u) \,=\, u\log u\,-\,u\,+\,\frac12\log(2\pi/u)+\mathcal{O}(1/u), \qquad \text{as } u\to\infty,\ |\arg u|<\pi\,.$$

As $t \to \infty$ and $|\arg(-t)| < \pi$, we have

$$\Upsilon = 2\pi\sqrt{-1}\,mt - \pi\sqrt{-1}\,nt + n\left(\log r - 2\pi\sqrt{-1}\,\varphi\right)t + \sum_{a=1}^{n}\Big((z_a - t)\log(z_a - t) - (z_a - t) + \frac{1}{2}\log\Big(\frac{2\pi}{z_a - t}\Big)\Big) + \sum_{j=1}^{n-1}\log(y_j - t) + \mathcal{O}(1/t).$$

The critical point equation of this expression with respect to t yields

(5.4) $$\log(-t) = 2\pi\sqrt{-1}\,\frac{m}{n} - \pi\sqrt{-1} + \log r - 2\pi\sqrt{-1}\,\varphi + \mathcal{O}(1/t).$$

This implies that for

(5.5) $$-\pi < 2\pi\frac{m}{n} - \pi - 2\pi\varphi < \pi\,,$$

the function $\Upsilon(t,y,z)$ has a critical point $t_m \in \mathbb{C}$, with respect to t, such that

(5.6) $$\log(-t_m) = \log(-\omega^m s) + O(1/r),$$

where $\arg(-\omega^m s) = 2\pi m/n - \pi - 2\pi\phi$, so that $|\arg(-\omega^m s)| < \pi$. Inequalities (5.5) give us relations between m and ϕ, which are exactly the inequalities in (5.2). We also have

$$\Upsilon(t_m, \boldsymbol{y}, \boldsymbol{z}) = n\omega^m s + \sum_{a=1}^{n} z_a \log(-\omega^m s) + \frac{n}{2}\log\Big(\frac{2\pi}{-\omega^m s}\Big) + \sum_{j=1}^{n-1}\log(y_j - \omega^m s) + \mathcal{O}(1/r),$$

$$\frac{d^2}{dt^2}\Upsilon(t_m, \boldsymbol{y}, \boldsymbol{z}) = \frac{n}{-\omega^m s} + \mathcal{O}(1/r^2).$$

We apply the steepest descent method to the integral in (4.13) as in [Gu, Appendix 1] and obtain

$$\begin{aligned}\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z}) &= \sqrt{\frac{-\omega^m s}{2\pi n}}\; e^{n\omega^m s}\,(-\omega^m s)^{\sum_{a=1}^n z_a - n/2}\,(2\pi)^{n/2}\prod_{j=1}^{n-1}(y_j - \omega^m s)\,\big(1 + \mathcal{O}(1/r)\big)\\ &= \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}\; e^{n\omega^m s}\,(-\omega^m s)^{\sum_{a=1}^n z_a + (1-n)/2}\prod_{j=1}^{n-1}(y_j - \omega^m s)\,\big(1 + \mathcal{O}(1/s)\big),\end{aligned}$$

which proves the lemma.

5.2. **Admissible ϕ and m.**

Corollary 5.2. *If the argument ϕ of s satisfies the inequalities*

(5.7) $$\frac{k}{n} < \varphi < \frac{k+1}{n}, \qquad \text{for some} \quad k \in \mathbb{Z},$$

then there are exactly n integers satisfying (5.2)*. They are $k+1,\dots,k+n$. Hence each element of the basis $\{\Psi^{k+m}(s^n, \boldsymbol{y}, \boldsymbol{z}) \mid m = 1, \dots, n\}$ of the space of solutions of the joint system of equations* (3.1) *and* (3.2) *has the asymptotic expansion* (5.3)*.*

Corollary 5.3. *If $\phi = k/n$ for some $k \in \mathbb{Z}$, then there are exactly $n-1$ integers m satisfying* (5.2)*.*

They are $m = k+1,\dots,k+n-1$.

We say that $\phi \in \mathbb{R}$ is *resonant*, if $\phi = k/n$ for some $k \in \mathbb{Z}$.

Corollary 5.4. *Given $m \in \mathbb{Z}$, the function $\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z})$ has the asymptotic expansion* (5.3) *if the argument ϕ of s satisfies the inequalities*

$$\frac{m}{n} - 1 < \varphi < \frac{m}{n}, \tag{5.8}$$

cf. (5.2)*. Thus, the function $\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z})$ has the asymptotic expansion* (5.3) *on* $\mathbb{C}$ *with the ray $\phi = m/n$ deleted and the argument of s fixed by* (5.8)*.*

5.3. **Stokes rays.** The *Stokes rays* in $\mathbb{C}$ with coordinate $s = re^{-2\pi\sqrt{-1}\phi}$ are the rays defined by the equations

$$\varphi = \frac{k}{2n}, \qquad k \in \mathbb{Z}. \tag{5.9}$$

The rays with k even (resp., odd) will be called *even* (resp., *odd*).

Consider an interval $k/n < \phi < (k+1)/n$ between consecutive even rays. Then each element of the basis $\{\Psi^{k+m}(s^n, \boldsymbol{y}, \boldsymbol{z}) \mid m = 1, \dots, n\}$ has the asymptotic expansion (5.3) on that interval, see Corollary 5.2.

For given $k/n < \phi < (k+1)/n$ and $r \to \infty$, the absolute value of a basis solution $\Psi^{k+m}(s^n, \boldsymbol{y}, \boldsymbol{z})$ is determined by the real number $\mathrm{Re}(\omega^{k+m}s)$. Namely, if $\mathrm{Re}(\omega^{k+m_1}s) < \mathrm{Re}(\omega^{k+m_2}s)$ for some $1 \leqslant m_1, m_2 \leqslant n$, then

$$|\Psi^{k+m_1}(s^n, \boldsymbol{y}, \boldsymbol{z})| \ll |\Psi^{k+m_2}(s^n, \boldsymbol{y}, \boldsymbol{z})| \qquad \text{as} \quad r \to \infty,$$

see formula (5.3).

The meaning of Stokes rays is explained by the following lemma.

Lemma 5.5. *A number $\phi \in \mathbb{R}$ is of the form $\phi = k/2n$ for some $k \in \mathbb{Z}$, if and only if there are m_1, m_2 such that* $\mathrm{Re}(\omega^{m_1}s) = \mathrm{Re}(\omega^{m_2}s)$ *and $m_1 \not\equiv m_2$* (mod n)*.*

5.4. **Definition of Stokes bases.**

Definition 5.6. *Let $\{I_1(s^n, \boldsymbol{y}, \boldsymbol{z}), \dots, I_n(s^n, \boldsymbol{y}, \boldsymbol{z})\}$ be a basis of solutions of the joint system of equations* (3.1) *and* (3.2)*. Let $a < b$ be real numbers. We say that the basis is a Stokes basis on an interval (a,b) if the basis can be reordered so that for every $m = 1,\dots,n$ and every non-resonant $\phi \in (a,b)$, we have*

$$I_m(s^n, \boldsymbol{y}, \boldsymbol{z}) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}\, e^{n\omega^m s}\, (-\omega^m s)^{\sum_{a=1}^n z_a + (1-n)/2} \prod_{j=1}^{n-1} (y_j - \omega^m s)\, \big(1 + \mathcal{O}(1/s)\big) \tag{5.10}$$

as $s \to \infty$. Here for every m, the argument of $-\omega^m s$ is chosen so that $|\arg(-\omega^m s)| < \pi$ when ϕ tends to b inside (a,b), and the argument of $-\omega^m s$ is continuous when $\phi \in (a,b)$.

For example, for $k \in \mathbb{Z}$, the basis $\{\Psi^{k+m}(s^n, \boldsymbol{y}, \boldsymbol{z}) \mid m = 1, \dots, n\}$ is a Stokes basis on the interval $(k/n, (k+1)/n)$, see Lemma 5.1.

For any ray $\phi = a$, which is not a Stokes ray, we will construct a Stokes basis on the interval $(a - 1/2 - \epsilon, a + \epsilon)$, where is a small positive number. We will formulate the result in terms of a suitable braid group action.

6. Exceptional bases and braid group

6.1. **Braid group action.** Let M_n be a free $\mathbb{C}[Z^{\pm 1}]$-module with basis $\{w_1,...,w_n\}$. Define a sesquilinear form A on M_n by the formulas:

$$A(w_i,w_i) = 1, \qquad A(w_i,w_j) = 0 \text{ for } i > j, \qquad A(w_i,w_j) = h_{j-i}(Z) \text{ for } i < j,$$

$$A(a(Z)x,b(Z)y) = a(Z^{-1})b(Z)A(x,y) \qquad \text{for } a,b \in \mathbb{C}[Z^{\pm 1}],\ x,y \in M_n.$$

Here the elements $h_k(Z) \in \mathbb{C}[Z^{\pm 1}]$ for $k \in \mathbb{Z}_{>0}$ are defined in (2.2). Cf. Section 2.3.

The matrix of A in the basis $\{w_1,...,w_n\}$ will be called *canonical*.

A basis $\{v_1,...,v_n\}$ of M_n will be called *exceptional* if

$$A(v_i,v_i) = 1, \qquad A(v_i,v_j) = 0 \text{ for } i > j.$$

In particular the basis $\{w_1,...,w_n\}$ is exceptional.

Let B_n be the braid group on n strands with standard generators $\tau_1,...,\tau_{n-1}$. The element

(6.1) $\quad C = \tau_1\tau_2 ... \tau_{n-1} \in B_n$ is called the *Coxeter element*.

Lemma 6.1. *The braid group acts on the set of exceptional bases by the formula,*

$$Q = \{v_1,...,v_n\} \mapsto \tau_i Q = \{...,v_{i-1},v_{i+1} - A(v_i,v_{i+1})v_i,v_i,v_{i+2},...\}.$$

Proof. The fact that the basis $\tau_i Q$ is exceptional, if Q is exceptional, and the equality $\tau_i\tau_{i+1}\tau_i Q = \tau_i\tau_{i+1}\tau_i Q$ are checked by direct calculations.

Lemma 6.2. *Let $Q = \{v_1,...,v_n\}$ be an exceptional basis in which the matrix of A is canonical. Then*

$$CQ = \{v_n - e_1(Z)v_{n-1} + ... + (-1)^n e_n(Z)v_1, v_1, v_2, ..., v_{n-1}\}. \tag{6.2}$$

Moreover, if we multiply the first element of the basis CQ by $(-1)^{n+1}e_n(Z^{-1})$, then the basis will remain exceptional and the matrix of A in this new basis

$$\{(-1)^{n+1}e_n(Z^{-1})(v_n - e_1(Z)v_{n-1} + ... + (-1)^n e_n(Z)v_1), v_1, v_2, ..., v_{n-1}\} \tag{6.3}$$

is canonical.

Proof. By induction we observe that $\tau_i\tau_{i+1} ... \tau_{n-1}Q = (v_1,...,v_{i-1},v_n - e_1(Z)v_{n-1} + ... + (-1)^{n-i}e_{n-i}(Z)v_i, v_i, v_{i+1},...,v_n\}$.

Then we calculate the matrix of A relative to the basis C^0Q from the definitions. In these calculations we use relations (2.3).

The map of bases

$\{v_1,\dots,v_n\}$ 7→ $\{(-1)^{n+1}e_n(Z^{-1})(v_n - e_1(Z)v_{n-1} + \dots + (-1)^n e_n(Z)v_1)\ ,v_1,v_2,\dots,v_{n-1}\}$ will be called the *modified Coxeter map* and denoted by C^0.

6.2. **The element** $\gamma_n \in \mathrm{B}_n$**.** Let \` $= n-1$ for $n$ odd and \` $= n-2$ for n even. Thus \` is always even. Set $\gamma_2 = 1$, and for $n > 3$,

$$\beta_k = \tau_k\tau_{k+1}\dots\tau_{n-1}, \qquad \gamma_n = \beta_{`}\beta_{`-2}\dots\beta_2.$$

For example, $\gamma_3 = \tau_2$, $\gamma_4 = \tau_2\tau_3$, $\gamma_5 = (\tau_4)(\tau_2\tau_3\tau_4)$, $\gamma_6 =$

$$(\tau_4\tau_5)(\tau_2\tau_3\tau_4\tau_5).$$

Define

$$\delta_{n,\mathrm{odd}} = \tau_1\tau_3\dots\tau_{n-2}, \qquad \delta_{n,\mathrm{even}} = \tau_2\tau_4\dots\tau_{n-1}, \qquad \text{for } n \text{ odd},$$
$$\delta_{n,\mathrm{odd}} = \tau_1\tau_3\dots\tau_{n-1}, \qquad \delta_{n,\mathrm{even}} = \tau_2\tau_4\dots\tau_{n-2}, \qquad \text{for } n \text{ even}.$$

Lemma 6.3. *We have the following identity in* B_n*:*

$$\delta_{n,\mathrm{even}}\,\delta_{n,\mathrm{odd}}\,\gamma_n = \gamma_n\,C\,. \tag{6.4}$$

Proof. The proof is straightforward.

6.3. **Bases** Q^0 **and** Q^{00}**.** Let $n = 2k+1$. Let $Q = \{v_1,\dots,v_n\}$ be a basis of M_n. For $1\ 6\ l\ 6\ m\ 6\ n$ denote

$$v_m(l) = v_m - e_1(Z)v_{m-1} + \cdots + (-1)^{m-l}e_{m-l}(Z)v_l. \tag{6.5}$$

Introduce a basis Q^0 in which the vectors $v_1,\dots,v_{k+1}$ stay at the positions 1, 3, 5,..., $2k+1$, respectively, and the vectors $v_{2k+1}(2)$, $v_{2k}(3),\dots,v_{k+2}(k+1)$ stay at the positions $2,4,6,\dots,2k$, respectively.

Introduce a basis Q^{00} in which the vectors $v_1,\dots,v_{k+1}$ stay at the positions 2,4,6,..., $2k,2k+1$, respectively, and the vectors $v_{2k+1}(1)$, $v_{2k}(2),\dots,v_{k+2}(k)$ stay at the positions $1,3,5,\dots,2k-1$, respectively.

For example for $n = 5$, we have

$$Q^0 = \{v_1,v_5 - e_1(Z)v_4 + e_2(Z)v_3 - e_3(Z)v_2,v_2,v_4 - e_1(Z)v_3,v_3\}, \tag{6.6}$$

$$Q^{00} = \{v_5 - e_1(Z)v_4 + e_2(Z)v_3 - e_3(Z)v_2 + e_4(Z)v_1,v_1,\ v_4 - e_1(Z)v_3 + e_2(Z)v_2,v_2,v_3\}.$$

Let $n = 2k$. Let $Q = \{v_1,...,v_n\}$ be a basis of M_n. Introduce a basis Q^0, in which the vectors $v_1,...,v_{k+1}$ stay at the positions $1,3,5,...,2k-1$, $2k$, respectively, and the vectors $v_{2k}(2)$, $v_{2k-1}(3),...,v_{k+2}(k)$ stay at the positions $2,4,6,...,2k-2$, respectively.

Introduce a basis Q^{00}, in which the vectors $v_1,...,v_k$ stay at the positions $2,4,6,...,2k$, respectively, and the vectors $v_{2k}(1)$, $v_{2k-1}(2),...,v_{k+1}(k)$ stay at the positions $1,3,5,..., 2k-1$, respectively.

For example for $n = 6$, we have

$$Q^0 = \{v_1, v_6 - e_1(Z)v_5 + e_2(Z)v_4 - e_3(Z)v_3 + e_4(Z)v_2, v_2, v_5 - e_1(Z)v_4 + e_2(Z)v_3, v_3, v_4\}, \tag{6.7}$$

$$Q^{00} = \{v_6 - e_1(Z)v_5 + e_2(Z)v_4 - e_3(Z)v_3 + e_4(Z)v_2 - e_5(Z)v_1,\ v_1, v_5 - e_1(Z)v_4 + e_2(Z)v_3 - e_3(Z)v_2, v_2, v_4 - e_1(Z)v_3, v_3\}.$$

Lemma 6.4. *Let $n > 1$. Let $Q = \{v_1,...,v_n\}$ be a basis of M_n such that the matrix of A relative to Q is canonical. Then*

$$\gamma_n Q = Q^0, \qquad \delta_{n,\mathrm{odd}} Q_0 = Q_{00}.$$

Proof. The proof is straightforward.

6.4. **Modules** M_n**,** $K_{T_n}(P^{n-1}, \mathbb{C})$**, and** $\mathbb{S}_n$**.**

Lemma 6.5. *The map $\iota : M_n \to K_{T^n}(P^{n-1}, \mathbb{C})$, defined by*

$$\iota : w_j \mapsto X^{j-1}, \qquad j = 1,...,n, \tag{6.8}$$

is an isomorphism of $\mathbb{C}[Z^{\pm 1}]$-modules, which identifies the form A on M_n with the form A on $K_{T_n}(P^{n-1}, \mathbb{C})$.

Recall the isomorphism $\theta : K_{T_n}(P^{n-1}, \mathbb{C}) \to \mathbb{S}_n$. The composition isomorphism $\theta \circ \iota : M_n \to \mathbb{S}_n$ is defined by

$$\theta \circ \iota : w_m \mapsto \Psi^m, \qquad m = 1,...,n. \tag{6.9}$$

Using the isomorphism $\theta \circ \iota$ we define exceptional bases of $\mathbb{S}_n$ with the action of the braid group $\mathbb{B}_n$ on them.

6.5. **Exceptional bases of** $\mathbb{S}_n$**.**

Lemma 6.6. *For every $k \in \mathbb{Z}$, the basis $Q_k = \{\Psi^{k+1},...,\Psi^{k+n}\}$ of $\mathbb{S}_n$ is an exceptional basis, in which the matrix of A is canonical. We also have $C^0 Q_k = Q_{k-1}$.*

Proof. The first statement follows from Lemma 2.1. The second statement follows from Lemma 6.2 and formula (4.8).

Using the formulas of Section 6.3 we assign to every basis Q_k two exceptional bases Q'_k and Q''_k. For example for $n = 5$, we define

$$Q'_k = \{\Psi_{k+1}, \Psi_{k+5} - e_1(Z)\Psi_{k+4} + e_2(Z)\Psi_{k+3} - e_3(Z)\Psi_{k+2}, \Psi_{k+2}, \Psi_{k+4} - e_1(Z)\Psi_{k+3}, \Psi_{k+3}\},$$

$$Q''_k = \{\Psi_{k+5} - e_1(Z)\Psi_{k+4} + e_2(Z)\Psi_{k+3} - e_3(Z)\Psi_{k+2} + e_4(Z)\Psi_{k+1}, \Psi_{k+1}, \Psi_{k+4} - e_1(Z)\Psi_{k+3} + e_2(Z)\Psi_{k+2}, \Psi_{k+2}, \Psi_{k+3}\}.$$

cf. (6.6). For any n and k we have

$$\gamma_n Q_k = Q'_k, \qquad \delta_{n,\mathrm{odd}} Q'_k = Q''_k, \tag{6.10}$$

by Lemma 6.4.

Lemma 6.7. *For any n and $k \in \mathbb{Z}$, multiplying the first basis vector of the basis $\delta_{n,\mathrm{even}}Q''_k$ by $(-1)^{n+1}e_n(Z^{-1})$ yields the basis Q'_{k-1}.*

Proof. The lemma follows from Lemmas 6.3 and 6.6.

7. Stokes bases

7.1. Main theorem.

Theorem 7.1. *The basis Q'_k is a Stokes basis on the interval $(a-1/2-\epsilon, a+\epsilon)$ if $a \in ((2k+1)/2n, (k+1)/n)$ and $\epsilon > 0$ is small enough. The basis Q''_k is a Stokes basis on the interval $(a-1/2-\epsilon, a+\epsilon)$ if $a \in (k/n, (2k+1)/2n)$ and $\epsilon > 0$ is small enough.*

The smallness of ϵ means that the intervals $(a, a+\epsilon)$ and $(a-1/2-\epsilon, a-1/2)$ do not contain points of the form $r/2n$ where $r \in \mathbb{Z}$.

Corollary 7.2. *Consider the three consecutive asymptotic bases Q'_k, Q''_k, Q'_{k-1}. Then $Q''_k = \delta_{n,\mathrm{odd}}Q'_k$, and Q'_{k-1} is obtained from the basis $\delta_{n,\mathrm{even}}Q''_k$ by multiplying the first basis vector of $\delta_{n,\mathrm{even}}Q''_k$ by $(-1)^{n+1}e_n(Z^{-1})$.*

It is enough to prove Theorem 7.1 for $k = 0$, since the case of arbitrary k is obtained from the case of $k = 0$ by the change of variables $m \mapsto k+m$ and $\phi \mapsto \phi + k/n$ in the integral (4.13). Theorem 7.1 for $k = 0$ is proved in Section 7.4.

7.2. **Paths and functions.** For integers $l \leqslant m$ we define the *path* $C^m(l)$ on the regular n-gone Δ with vertices $\{\omega^1, \omega^2, \dots, \omega^n\}$ as the path along the boundary of Δ, which starts at the vertex ω^l and goes to the vertex ω^m through the vertices $\omega^{l+1}, \dots, \omega^{m-1}$. The vertices ω^m and ω^l are the *head* and *tail* of the path. The number $m - l$ is the *length* of the path. The path $C^m(l)$ goes around Δ counterclockwise.

All our paths will be of length less than n.

Let $l \leqslant m$ and $m - l < n$. Define the *reflected path* $\bar{C}^m(l)$ to be the path along the boundary of Δ, which goes from the vertex ω^{l-1} to the vertex $\omega^{m-n} = \omega^m$ through the vertices $\omega^{l-2}, \omega^{l-3}, \dots, \omega^{m-n+1}$. The reflected path $\bar{C}^m(l)$ goes around Δ clockwise.

Both $C^m(l)$ and $\bar{C}^m(l)$ have the same heads. The sum of lengths of $C^m(l)$ and $\bar{C}^m(l)$ equals $n - 1$.

Definition 7.3. *Let $l \leqslant m$ and $m - l < n$. Assign to the path $C^m(l)$ the function*

$$\Psi^m(l) = \Psi^m - e_1(Z)\Psi^{m-1} + \dots + (-1)^{m-l}e_{m-l}(Z)\Psi^l, \tag{7.1}$$

and to the reflected path $\bar{C}^m(l)$ the function

$$\bar{\Psi}^m(l) = (-1)^{n-1}e_n(Z)\Psi^{m-n} + (-1)^{n-2}e_{n-1}(Z)\Psi^{m-n+1} + \dots + (-1)^{m-l}e_{m-l+1}(Z)\Psi^{l-1}. \tag{7.2}$$

Notice that the functions $\Psi^m(l)$ and $\bar{\Psi}^m(l)$ are equal by formula (4.8), while the summands in $\Psi^m(l)$ correspond to the vertices of the path $C^m(l)$ and the summands in $\bar{\Psi}^m(l)$ correspond to the vertices of the path $\bar{C}^m(l)$.

Consider the rotated n-gone $e^{-2\pi\sqrt{-1}\phi}\Delta$ and rotated paths $e^{-2\pi\sqrt{-1}\phi}C^m(l)$, $e^{-2\pi\sqrt{-1}\phi}\bar{C}^m(l)$.

We say that the path $e^{-2\pi\sqrt{-1}\phi}C^m(l)$ is *admissible* if the number $\mathrm{Re}(e^{-2\pi\sqrt{-1}\phi}\omega^m)$ is greater than the number $\mathrm{Re}(e^{-2\pi\sqrt{-1}\phi}\omega^k)$ for any other vertex of the path $C^m(l)$, and we say that the path $e^{-2\pi\sqrt{-1}\phi}\bar{C}^m(l)$ is *admissible* if the number $\mathrm{Re}(e^{-2\pi\sqrt{-1}\phi}\omega^m)$ is greater than the number $\mathrm{Re}(e^{-2\pi\sqrt{-1}\phi}\omega^k)$ for any other vertex of the path $\bar{C}^m(l)$.

7.3. **Bases Q'_0, Q''_0.** We have

$$Q'_0 = \{\Psi^1, \Psi^n - e_1(Z)\Psi^{n-1} + \dots + (-1)^{n-2}e_{n-2}(Z)\Psi^2, \Psi^2, \Psi^{n-1} - e_1(Z)\Psi^{n-2} + \dots + (-1)^{n-4}e_{n-4}(Z)\Psi^3, \Psi^3, \dots\}, \tag{7.3}$$

$$Q''_0 = \{\Psi^n - e_1(Z)\Psi^{n-1} + \dots + (-1)^{n-1}e_{n-1}(Z)\Psi^1, \Psi^1, \Psi^{n-1} - e_1(Z)\Psi^{n-2} + \dots + (-1)^{n-2}e_{n-3}(Z)\Psi^2, \Psi^2, \dots\}. \tag{7.4}$$

7.4. **Proof of Theorem 7.1 for** $k = 0$**.** We will prove the theorem for Q'_0. The proof for Q''_0 is completely similar.

We will prove that the basis Q'_0 is a Stokes basis on the interval $(a - 1/2 - \epsilon, a + \epsilon)$, if $a \in (1/2n, 1/n)$, where ϵ is small. The Stokes rays divide the non-resonant points of the interval $(a - 1/2 - \epsilon, a + \epsilon)$ into the subintervals $(1/2n, a + \epsilon), (0, 1/2n), (-1/2n, 0), \dots$. The first and last of these subintervals are shorter than the intervals between the Stokes rays, since they have boundary points $a + \epsilon$, $a - 1/2 - \epsilon$ lying in between Stokes rays. We will prove that Q'_0 is a Stokes basis on each of these subintervals.

We start with the first two subinterval $(1/2n, a + \epsilon)$ and $(0, 1/2n)$. We assume that ϵ is small so that $(1/2n, a + \epsilon) \subset (1/2n, 1/n)$.

The functions $\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z})$, $m = 1, \dots, n$, appearing in (7.3) are all admissible for the interval $(0, 1/n)$ in the sense of Corollary 5.2. For $\phi \in (0, 1/n)$ each of these functions has an asymptotic expansion with the leading term $\exp(nre^{-2\pi\sqrt{-1}\phi}\omega^m)$. The magnitude of a function $\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z})$ is determined by the real part of the number $e^{-2\pi\sqrt{-1}\phi}\omega^m$. Hence to order the magnitudes of the solutions $\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z})$, $m = 1, \dots, n$, for $\varphi \in (0, 1/n)$ we need to consider the rotated n-gone $e^{-2\pi\sqrt{-1}\phi}\Delta$ and order the real parts of its vertices. Using notations of Section 7.2 we write

$$Q'_0 = \{\Psi^1(1), \Psi^n(2), \Psi^2(2), \Psi^{n-1}(3), \Psi^3(3), \dots\}. \tag{7.5}$$

These functions are the functions, which were assigned to the sequence of paths $\{C^1(1), C^n(2), C^2(2), C^{n-1}(3), C^3(3), \dots\}$ in Definition 7.3. Each of these paths is admissible with respect to $e^{-2\pi\sqrt{-1}\phi}\Delta$ for $\phi \in (0, 1/n)$. Hence each linear combination $\Psi^m(l)$ appearing in this sequence has asymptotic expansion with leading term $\exp(nre^{-2\pi\sqrt{-1}\phi}\omega^m)$, coming from the summand Ψ^m of $\Psi^m(l)$, corresponding to the head of the path $C^m(l)$. Therefore the basis Q'_0 is an asymptotic basis on the two subintervals $(1/2n, a + \epsilon)$ and $(0, 1/2n)$.

Consider the next two subintervals $(-1/2n, 0)$ and $(-1/n, -1/2n)$. On the interval $(-1/n, 0)$ the admissible functions are $\Psi^0, \dots, \Psi^{n-1}$. For $\phi \in (-1/n, 0)$ each of these functions has an asymptotic expansion with the leading term $\exp(nre^{-2\pi\sqrt{-1}\phi}\omega^m)$.

In formula (7.3) the function $\Psi^n(2)$ is the only function that uses the non-admissible function Ψ^n. We replace the presentation of $\Psi^n(2)$ in (7.3) by the equal sum

$$\bar\Psi^n(2) = (-1)^{n-1}e_n(Z)\Psi^0 + (-1)^{n-2}e_{n-1}(Z)\Psi^1,$$

which uses only the admissible functions $\Psi^0, \dots, \Psi^{n-1}$. On the interval $(-1/n, 0)$ we have

$$Q'_0 = \{\Psi^1(1), \bar\Psi^n(2), \Psi^2(2), \Psi^{n-1}(3), \Psi^3(3), \dots\}, \tag{7.6}$$

where the dots indicates the same functions as in (7.3). This new presentation of the basis Q_0' corresponds to the sequence of paths $\{C^1(1), \bar C^n(2), C^2(2), C^{n-1}(3), C^3(3), \dots\}$.

Each of these paths is admissible with respect to $e^{-2\pi\sqrt{-1}\phi}\Delta$ for $\phi \in (-1/n, 0)$. Hence each linear combination of the functions $\Psi^0, \dots, \Psi^{n-1}$ appearing as a basis vector in (7.6) has asymptotic expansion with leading term $\exp(nre^{-2\pi\sqrt{-1}\phi}\omega^m)$, coming from the summand Ψ^m corresponding to the head of the corresponding path. Therefore the basis Q_0' is an asymptotic basis on the two subintervals $(-1/2n, 0)$ and $(-1/n, -1/2n)$.

On the next two subintervals $(-3/2n, -1/n)$ and $(-2/n, -3/2n)$ the admissible functions are $\Psi^{-1}, \dots, \Psi^{n-2}$. For $\phi \in (-2/n, -1/n)$ each of these functions has an asymptotic expansion with the leading term $\exp(nre^{-2\pi\sqrt{-1}\phi}\omega^m)$.

In formula (7.6) the function $\Psi^{n-1}(3)$ is the only function that uses the non-admissible Ψ^{n-1}. We replace the presentation of $\Psi^{n-1}(3)$ in (7.3) by the equal sum

$$\bar\Psi^{n-1}(3) = (-1)^{n-1}e_n(Z)\Psi^{-1} + (-1)^{n-2}e_{n-1}(Z)\Psi^0 + (-1)^{n-3}e_{n-2}(Z)\Psi^1 + (-1)^{n-4}e_{n-3}(Z)\Psi^2,$$

which uses only the admissible functions $\Psi^{-1}, \dots, \Psi^{n-2}$. On the interval $(-2/n, -1/n)$ we have

$$Q_0' = \{\Psi^1(1), \bar\Psi^n(2), \Psi^2(2), \bar\Psi^{n-1}(3), \Psi^3(3), \dots\}, \tag{7.7}$$

where the dots indicates the same functions as in (7.3). This new presentation of the basis Q_0' corresponds to the sequence of paths $\{C^1(1), \bar C^n(2), C^2(2), \bar C^{n-1}(3), C^3(3), \dots\}$.

Each of these paths is admissible with respect to $e^{-2\pi\sqrt{-1}\phi}\Delta$ for $\phi \in (-2/n, -1/n)$. Hence each linear combination of the functions $\Psi^{-1}, \dots, \Psi^{n-2}$ appearing as a basis vector in (7.7) has asymptotic expansion with leading term $\exp(nre^{-2\pi\sqrt{-1}\phi}\omega^m)$, coming from the summand Ψ^m corresponding to the head of the corresponding path. Therefore the basis Q_0' is an asymptotic basis on the two subintervals $(-3/2n, -1/n)$ and $(-2/n, -3/2n)$.

Repeating this procedure we prove Theorem 7.1 for Q_0'. See a similar reasoning in [Gu].

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