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Space-curve Cartan matrix and exact differentiability of the curvature and torsion

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ABSTRACT

In formulation of beam-vibration equations, *curvature* of space curves is used to define the strain energy and elastic forces. Only in special cases, the curvature and torsion can be associated with derivatives of angles. Furthermore, curve twist is result of coupled in-plane and out-of-plane bending modes. While this mode coupling can be represented by two rotations, curve curvature or torsion cannot, in general, be associated with single rotation. Curvature and torsion, in their most general forms, are defined using skew-symmetric *Cartan matrix*, which leads to the *Serret-Frenet equations*. This paper uses two different sequences of rotation to discuss *exact differentiability* of curvature and torsion and demonstrate that curve torsion cannot, in general, be defined as derivative of uniquely-defined angle performed about curve tangent vector. *Frenet angles* are used to develop simple and general expressions for elements of curve Cartan matrix. The analysis and results presented show the fundamental difference between *Bishop shear angle*, which is not unique and does not enter into definition of curve geometry; and *Frenet bank angle*, which is unique and enters into definition of curve geometry.

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1. Introduction

Curvature of space curves is widely used in formulation of beam vibration equations and in definition of beam strain energy and elastic forces (Antman 1973; Dym and Shames 1973; Timoshenko, Young, and Weaver 1974; Meirovitch 1986; Goyal, Perkins, and Lee 2008; Goyal and Perkins 2008; O'Reilly 2016; Khalid Jawed, Novelia, and O'Reilly 2018; Shabana 2019). Curve torsion is result of out-of-plane bending that leads to twist that requires two independent variables to completely define curve geometry. In case of a general space curve, curvature and torsion expressions can assume complex forms (Guggenheimer 1963; O'Neill 1966; Goetz 1970; Kreyszig 1991; Piegl and Tiller 1997; Farin 1999; Rogers 2001; Gallier 2011). Only in special cases, the curvature and torsion can be represented as derivatives of angles. For a planar curve, the torsion is zero and the curvature can be defined as derivative of a single angle performed about a fixed axis. For a space curve, on the other hand, curve curvature and torsion (twist) are result of coupled in-plane and out-of-plane bending modes. While this mode coupling can be represented by two rotations, curve torsion cannot, in general, be associated with a single rotation about curve tangent vector, as will be discussed in this paper.

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1.1. Curve Cartan matrix and exact differentiability

The curvature and torsion can be defined in their most general forms using the skew-symmetric *Cartan matrix*, which leads to definition of the *Serret-Frenet equations*. The Cartan matrix associated with a three-dimensional orthogonal matrix $\mathbf{A} = \mathbf{A}(t)$, where t is a parameter, is defined as $\mathbf{A}^T\mathbf{A}_t$, where $\mathbf{A}_t = \partial\mathbf{A}/\partial t$ (Guggenheimer 1963; O'Neill 1966). It is to be noted that "Cartan matrix" is used in the literature to refer to different matrix forms. It is used in this paper to refer to the skew-symmetric matrix $\mathbf{A}^T\mathbf{A}_t$, as previously used in the literature (Bishop 1975). For a general orthogonal matrix with no additional conditions imposed on its columns, except for the six orthonormality conditions, the skew-symmetric Cartan matrix has three independent elements. For a space curve, the orthogonal matrix that defines the orientation of the curve Frenet frame is obtained by imposing the condition that the second column of the matrix is curve normal vector, which is the normalized vector obtained from differentiating the first column, which is unit vector tangent to the curve, with respect to curve arc length. Because of this condition, number of Cartan-matrix independent elements reduces to two elements: curvature and torsion. Nonetheless, because different frames were introduced to overcome problem of defining Frenet frame at curvature-vanishing points, Cartan matrix can have different structure if the tangent or normal vector is not used in curve-framing method. Both curvature and torsion, however, are not in general *exact differentials* and cannot, in general, be defined as derivative of uniquely-defined angle performed about a single axis. Furthermore, a curve can be twisted without performing any rotation about the curve tangent vector. For example, a straight beam can be bent by performing a rotation about the vertical axis to define planar space curve. The curve can then be twisted by performing a second rotation about curve normal vector (lateral axis). That is, twisted-beam configuration can be achieved without any rotation about beam longitudinal axis. This fact, which sheds light on torsion definition used in the literature as the derivative of an angle about longitudinal beam axis, is examined in this study using two different sequences of rotations. Both rotation sequences demonstrate that the curvature and torsion are not exact differentials and none of them can, in general, be integrated to determine an angle.

1.2. Curvature singularity and framing methods

Frenet frame of space curve $\mathbf{r} = \mathbf{r}(t)$, where t is curve parameter, is defined by three orthogonal unit vectors: unit tangent vector $\mathbf{t} = \mathbf{r}_t/|\mathbf{r}_t| = \mathbf{r}_s$, unit normal vector $\mathbf{n} = \mathbf{t}_s/|\mathbf{t}_s| = \mathbf{r}_{ss}/|\mathbf{r}_{ss}|$, and bi-normal vector $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, where $a_\chi = \partial a/\partial\chi$, $\mathbf{t}_s = \partial^2\mathbf{r}/\partial s^2 = \mathbf{r}_{ss}$ is the curvature vector, and s is the curve arc length (Guggenheimer 1963; O'Neill 1966; Goetz 1970; Kreyszig 1991; Piegl and Tiller 1997; Farin 1999; Rogers 2001; Gallier 2011). The curve *curvature* κ is norm of curvature vector defined as $\kappa = |\mathbf{r}_{ss}| = 1/R$, where R is curve *radius of curvature*. For straight sections of space curve, curvature κ is equal to zero, and consequently, normal vector $\mathbf{n}$ and bi-normal vector $\mathbf{b}$ used as axes of Frenet frame cannot be defined using classical differential-geometry procedures.

Because of curvature singularity and need for framing space curves in a wide range of engineering and physics applications, alternate frames were introduced (Yung-Chow Wong 1963; 1972; Bishop 1975; Andrew and Ma 1995; Bahaddin Bukcu and Karacan 2008; 2009). While axes of these frames do not have same geometric interpretation as Frenet-frame axes, the goal was to introduce well-defined frames at curvature-vanishing points. Some of these frames differ from Frenet frame by single rotation about tangent vector, and therefore, there is infinite number of frames that differ from Frenet frame by such a single rotation. Curvature vector $\mathbf{r}_{ss}$, for example, can have a component along horizontally-oriented axis equal to what is referred to in the literature as *horizontal curvature* (Ling and Shabana 2021; Shabana, 2021a, 2021b). Using such a frame or other frames, however, has a clear disadvantage compared to Frenet frame which has normal vector $\mathbf{n}$ as one of its axes. Normal vector defines direction of centrifugal force of a particle or a vehicle tracing space curve. Furthermore, in case of motion-trajectory curves, Frenet-frame *osculating plane* (OP), defined by unit vectors $\mathbf{t}$ and $\mathbf{n}$ tangent and normal to the curve, respectively, represents *motion plane*, which contains absolute velocity and acceleration vectors.

Introducing different methods for framing space curves was motivated by lack of definition of Frenet frame at zero-curvature points when using the classical differential-geometry procedures based on definition of curvature vector $\mathbf{r}_{ss}$. These procedures also fail to define Serret-Frenet equations at the zero-curvature points. At these points, curvature vector is zero and cross product $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ fails to determine bi-normal vector $\mathbf{b}$ (Bishop 1975; Shabana, 2021a).

1.3. Out-of-plane bending and torsion

In differential geometry, *torsion* is used to refer to curve twist (Guggenheimer 1963; O'Neill 1966; Goetz 1970; Kreyszig 1991; Piegl and Tiller 1997; Farin 1999; Rogers 2001; Gallier 2011). This type of torsion is to be distinguished from torsion due to shear. Curve torsion is an out-of-plane bending mode, and consequently, is attributed to second and independent bending mode that does not lead to any shearing between cross sections. In some investigations, frames different from curve Frenet frame were introduced by performing a rotation about curve tangent vector. This rotation does not have an effect on curve geometry. To distinguish such a rotation and make clear that it does not influence curve geometry, the rotation used in other framing methods will be referred to as *shear angle* and not torsion angle. The difference between twist and shear modes is often overlooked in computational mechanics literature as result of improper use of angles to describe deformation modes.

1.4. Frenet angles and existence of Frenet frame

Another approach is to view curvature vector $\mathbf{r}_{ss} = \kappa\mathbf{n}$ as a vector along well-defined unit normal vector $\mathbf{n}$, and avoid using curvature vector $\mathbf{r}_{ss}$ to define normal vector. Magnitude of curvature vector $\mathbf{r}_{ss}$ along $\mathbf{n}$ can assume zero value at zero-curvature points without having effect on definition of the normal vector $\mathbf{n}$. For space curves, normal vector can be defined everywhere using concept of *Frenet angles*, which are set of Euler angles performed according to Euler sequence $Z, -Y, -X$. This sequence of rotations can be used to introduce three Frenet angles: *curvature angle* ψ, *vertical-development-angle* θ, and *bank angle* ϕ (Ling and Shabana 2021; Shabana, 2021a, 2021b). In particular, bank angle ϕ defines *super-elevation* of curve osculating plane. In this study, another rotation sequence is used to confirm conclusions obtained using sequence $Z, -Y, -X$ used in railroad literature (Shabana, 2021b). Normal vector $\mathbf{n}$ obtained using Frenet angles is continuous, does not flip over in neighborhood of zero-curvature points, and coincides with conventional normal vector $\mathbf{n}_c$ over some curve segments and is opposite to $\mathbf{n}_c$ over other curve segments when sense of curvature changes. That is, normal vector determined using Frenet angles is not always directed along curve center of curvature.

1.5. Scope and contributions of this investigation

The curvature and torsion, in their most general forms, are defined, as previously mentioned, using skew-symmetric *Cartan matrix*, which leads to the *Serret-Frenet equations*. Use of different curve framing methods leads to different structure of Cartan matrix which has number of independent elements that depends on number of conditions imposed on the orthogonal matrix used to define its Cartan matrix. This paper discusses this fundamental issue and its relationship to *exact differentiability* of the curvature and torsion. It is demonstrated that the curvature or torsion cannot, in general, be defined as derivative of uniquely-defined angle performed about curve tangent. The fact that the curvature and torsion are not exact differentials is demonstrated using two different rotation sequences; the first of which is widely used in railroad literature, and therefore, it is discussed in more detail. Specific contributions and organization of this paper are summarized as follows:

1. *Cartan matrix structure:* Skew-symmetric Cartan matrix is presented in its most general form in terms of three independent elements to discuss conditions of exact differentiability of these elements. Conditions that can be imposed on columns of orthogonal transformation matrix to reduce number of independent elements in its Cartan matrix are formulated in Section 2. In Section 3, special case of Frenet frame which has Cartan matrix with two independent elements, curvature and torsion, is discussed.
2. *Curve framing methods and curvature-vanishing points:* Conditions used to define curve frames different from Frenet frame to avoid its discontinuity at curvature-vanishing points are discussed in Section 4 to explain theoretical foundation of *Bishop frame* discussed in Section 5. Integrability condition used in developing Bishop frame in view of non-exact differentiability of the torsion and non-uniqueness of this frame is discussed. Particular attention is given to definition of *Bishop-torsion angle* and its uniqueness. Given the fact that Frenet frame exists at curvature-vanishing points, discussion of Section 6 sheds light on whether there is need for introducing other framing methods.
3. *Exact differentiability of the torsion:* A curve can be twisted without performing any rotation about its longitudinal axis. The twist is result of two coupled bending modes. In this study, general torsion expression is used to demonstrate dependence of curve twist on two non-commutative rotations and to demonstrate twist without performing a rotation about longitudinal axis. Special cases of torsion exact differentiability are presented using Frenet angles introduced in Section 7 and used in torsion analysis of Section 8.
4. *Demonstration Examples:* Three examples are used in Section 9 to discuss conditions of exact differentiability of curvature and torsion. The first example is *helix* curve equation that depends on one parameter-dependent angle. This example shows a special case of twisted curve with exact-differential curvature and torsion. The second example demonstrates fundamental difference between Frenet bank and Bishop shear angles. The third example is an example with non-zero torsion.
5. *Rotation Sequences:* Two rotation sequences are used in this paper to confirm that curvature and torsion are not exact differentials and none of them can be, in general, integrated to define a curvature or torsion angle. The first sequence involves three rotations $\psi, \theta,$ and ϕ about axes $Z, -Y,$ and $-X,$ respectively. The second sequence involves three angles $\bar{\psi}, \bar{\phi},$ and $\bar{\theta}$ about axes $Z, X,$ and $Y,$ respectively. Analytical results obtained using these two different rotation sequences are compared and used in Section 10 to confirm that curvature and torsion are not, in general, exact differentials.
6. *Relevance to Mechanics Problems:* Relevance of the analysis presented in this paper to mechanics problems is explained in Section 11.

Summary and conclusions drawn from this study are presented in Section 12. It is to be noted that in each of the two rotation sequences used in this study, three angles are used despite curve geometry can be completely described using only two independent angles that define in-plane and out-of-plane bending modes. The reason for using three angles instead of two angles is to avoid using derivatives in definition of Frenet frame transformation matrix. This is with the understanding that one of the three angles is dependent on the other two angles (Shabana and Ling 2019; Ling and Shabana 2021; Shabana, 2021a; Shabana 2022).

2. Cartan matrices: background

Space-curve skew-symmetric *Cartan matrix* is written in terms of curve curvature κ and curve torsion τ (Guggenheimer 1963; O'Neill 1966). For a three-dimensional orthogonal matrix $\mathbf{A} = \mathbf{A}(t) = \begin{bmatrix} \mathbf{a}_1(t) & \mathbf{a}_2(t) & \mathbf{a}_3(t) \end{bmatrix}$ where t is arbitrary parameter, product $\mathbf{A}^T\mathbf{A}_t$ defines skew-symmetric matrix $\tilde{\bar{\boldsymbol{\omega}}}$ written in terms of three entries ω_{12}, ω_{13}, and ω_{23}, which define three-dimensional vector $\bar{\boldsymbol{\omega}}$. Matrix $\tilde{\bar{\boldsymbol{\omega}}}$ and vector $\bar{\boldsymbol{\omega}}$ are defined, respectively, as

$$\tilde{\bar{\boldsymbol{\omega}}} = \begin{bmatrix} 0 & -\bar{\omega}_3 & \bar{\omega}_2 \\ \bar{\omega}_3 & 0 & -\bar{\omega}_1 \\ -\bar{\omega}_2 & \bar{\omega}_1 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{\omega}} = \begin{bmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \\ \bar{\omega}_3 \end{bmatrix} = \begin{bmatrix} -\bar{\omega}_{23} \\ \bar{\omega}_{13} \\ -\bar{\omega}_{12} \end{bmatrix} = \begin{bmatrix} -\mathbf{a}_2^T\mathbf{a}_{3t} \\ \mathbf{a}_1^T\mathbf{a}_{3t} \\ -\mathbf{a}_1^T\mathbf{a}_{2t} \end{bmatrix} \tag{1}$$

Alternate form of vector $\bar{\boldsymbol{\omega}}$ is $\bar{\boldsymbol{\omega}} = \begin{bmatrix} \mathbf{a}_3^T\mathbf{a}_{2t} & -\mathbf{a}_3^T\mathbf{a}_{1t} & \mathbf{a}_2^T\mathbf{a}_{1t} \end{bmatrix}^T$. Using equation $\mathbf{A}^T\mathbf{A}_t = \tilde{\bar{\boldsymbol{\omega}}}$, one can write derivatives of columns $\mathbf{a}_k$, in terms of $\mathbf{a}_k$, $k = 1, 2, 3$ as

$$\left.\begin{aligned} \mathbf{a}_{1t} &= \bar{\omega}_3\mathbf{a}_2 - \bar{\omega}_2\mathbf{a}_3 = -\left(\mathbf{a}_1^T\mathbf{a}_{2t}\right)\mathbf{a}_2 - \left(\mathbf{a}_1^T\mathbf{a}_{3t}\right)\mathbf{a}_3, \\ \mathbf{a}_{2t} &= \bar{\omega}_1\mathbf{a}_3 - \bar{\omega}_3\mathbf{a}_1 = -\left(\mathbf{a}_2^T\mathbf{a}_{3t}\right)\mathbf{a}_3 + \left(\mathbf{a}_1^T\mathbf{a}_{2t}\right)\mathbf{a}_1, \\ \mathbf{a}_{3t} &= \bar{\omega}_2\mathbf{a}_1 - \bar{\omega}_1\mathbf{a}_2 = \left(\mathbf{a}_1^T\mathbf{a}_{3t}\right)\mathbf{a}_1 + \left(\mathbf{a}_2^T\mathbf{a}_{3t}\right)\mathbf{a}_2 \end{aligned}\right\} \tag{2}$$

In this equation, $\mathbf{a}_{kt} = \partial\mathbf{a}_k/\partial t$, $k = 1, 2, 3$.

Three-dimensional unit vector $\hat{\mathbf{a}} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix}^T$ has at most two independent elements because of constraint $|\hat{\mathbf{a}}| = \sqrt{\hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2} = 1$. A planar unit vector has at most one independent element and can always be written in terms of single angle γ as $\hat{\mathbf{a}} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \end{bmatrix}^T$. For general orthogonal matrix, there are three independent elements in its Cartan matrix defined by the vector $\bar{\boldsymbol{\omega}} = \begin{bmatrix} \bar{\omega}_1 & \bar{\omega}_2 & \bar{\omega}_3 \end{bmatrix}^T = \begin{bmatrix} -\bar{\omega}_{23} & \bar{\omega}_{13} & -\bar{\omega}_{12} \end{bmatrix}^T$.

An orthogonal matrix can always be expressed in terms of orientation parameters that can be angles (Goldstein 1950; Greenwood 1988; Roberson and Schwertassek 1988; Shabana and Ling 2019). In case of general three-dimensional rotations, elements of vector $\bar{\boldsymbol{\omega}}$ are not *exact differentials*, and therefore, cannot be integrated. Equating any of the elements $\bar{\omega}_1$, $\bar{\omega}_2$, and $\bar{\omega}_3$ to derivative of a variable β does not imply that β_t can be integrated. The variable can assume any value, while its derivative is uniquely defined by equation $\beta_t = \bar{\omega}_k$, for a given $k = 1, 2, 3$. This is case of nonholonomic constraint equations encountered in mechanics. Curve curvature and torsion that appear in curve Cartan matrix are, in general, non-integrable, and each of which cannot be used to uniquely define an angle.

3. Frenet-frame constraint and normal vector

In this section, Frenet-frame constraint condition, Cartan matrix general form, definition of normal vector, and sequence of rotations are discussed.

3.1. Frenet-Frame constraint

If last two columns of matrix $\mathbf{A} = \mathbf{A}(t) = \begin{bmatrix} \mathbf{a}_1(t) & \mathbf{a}_2(t) & \mathbf{a}_3(t) \end{bmatrix}$ are determined from first column, skew-symmetric matrix $\tilde{\bar{\boldsymbol{\omega}}}$ has only two independent entries. This can be demonstrated by an example in which $\mathbf{a}_2 = \mathbf{a}_{1t}/|\mathbf{a}_{1t}|$ and $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$. It follows that $\mathbf{a}_{3t} = \mathbf{a}_{1t} \times \mathbf{a}_2 + \mathbf{a}_1 \times \mathbf{a}_{2t} = \mathbf{a}_1 \times \mathbf{a}_{2t}$, and $\mathbf{a}_1^T\mathbf{a}_{3t} = 0$. In this case, vector $\bar{\boldsymbol{\omega}}$ reduces to $\bar{\boldsymbol{\omega}} = -\begin{bmatrix} \mathbf{a}_2 \cdot \mathbf{a}_{3t} & 0 & \mathbf{a}_1 \cdot \mathbf{a}_{2t} \end{bmatrix}^T$, which has only two independent elements.

Second and third columns $\mathbf{n}$ and $\mathbf{b}$ of orthogonal matrix $\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix}$ that defines space-curve Frenet frame are determined from unit tangent vector $\mathbf{t}$ which is the first column, as previously discussed. For this reason, one obtains *Serret-Frenet equations* $\mathbf{t}_s = \kappa\mathbf{n}$, $\mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}$, and

$\mathbf{b}_s = -\tau\mathbf{n}$, which correspond to Cartan matrix with two independent elements $\kappa = \mathbf{a}_1 \cdot \mathbf{a}_{2t}$ and $\tau = \mathbf{a}_2 \cdot \mathbf{a}_{3t}$ that represent, respectively, curvature and torsion and uniquely define curve geometry. Uniqueness of curve geometry for given curvature and torsion is associated with differential-geometry local theory of curves and such a geometry is invariant under rigid-body transformation.

3.2. Cartan-matrix general form

For a more general orthogonal matrix expressed in terms of three independent angles that define orientation of a coordinate system; Cartan matrix, obtained by differentiation with respect to time t as parameter, defines components of angular velocity vector. These three angular-velocity components are independent and can assume any values that depend on motion of the coordinate system. Therefore, in case of Frenet frame, resulting Cartan matrix is such that $\mathbf{t}_s = \kappa\mathbf{n}$ and $\mathbf{b}_s = -\tau\mathbf{n}$ which describe sense of rotations and frame axes that make derivatives of two vectors $\mathbf{t}$ and $\mathbf{b}$ in a direction along third vector $\mathbf{n}$, while derivative of $\mathbf{n}$ is linear combination of the other two vectors $\mathbf{t}$ and $\mathbf{b}$ as $\mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}$. For a curve, another coordinate system can be selected such that derivatives of unit vectors that define coordinate-system axes have different forms and two independent parameters that appear in Cartan matrix associated with this frame have different interpretation depending on type of constraint imposed.

3.3. Continuous normal vector

Unit tangent vector $\mathbf{t} = \mathbf{a}_1$ defines identity $\mathbf{a}_1^T\mathbf{a}_1 = 1$. Differentiation of this equation with respect to arc length parameter s leads to $\mathbf{a}_{1s}^T\mathbf{a}_1 + \mathbf{a}_1^T\mathbf{a}_{1s} = 0$, which shows that both $\mathbf{a}_{1s}$ and $-\mathbf{a}_{1s}$ are orthogonal to $\mathbf{a}_1$. In description of conventional Frenet frame in neighborhood of zero-curvature points, normal vector $\mathbf{n}_c = \mathbf{a}_{1s}$ flips over by 180° leading to discontinuity point at which the frame is not defined. Use of Frenet angles allows alleviating this problem by defining continuous normal vector $\mathbf{n}$ in neighborhood of zero-curvature points. On some curve segments, $\mathbf{n}$ and $\mathbf{n}_c$ are in same direction; and on some other curve segments, $\mathbf{n}$ and $\mathbf{n}_c$ are in opposite directions. That is, normal vector $\mathbf{n}$ is not always directed to curve center of curvature. In both representations, however, curve curvature is defined using equation $\kappa = \sqrt{\mathbf{a}_{1s} \cdot \mathbf{a}_{1s}}$, regardless of whether $\mathbf{a}_{1s}$ or $-\mathbf{a}_{1s}$ is used. It is to be noted that the approach based on Frenet angles is fundamentally different from attempts previously made in the literature to address Frenet-frame discontinuity using Bishop frame which does not have unique definition (Carroll, Kose, and Sterling 2013).

3.4. Different rotation sequences

The condition that $\mathbf{n} = \kappa\mathbf{t}_s$ in addition to the orthonormality of matrix $\mathbf{A}_f$ demonstrates that Frenet transformation matrix can always be written in terms of two independent orientation parameters. In this study, as in previous studies (Shabana and Ling 2019; Ling and Shabana 2021; Shabana 2021a, 2022), three orientation parameters are used in order to avoid use of derivatives in definition of normal vector $\mathbf{n}$ and bi-normal vector $\mathbf{b}$. This is with the understanding that one of the angles used in three-angle rotation sequence is dependent on the other two angles. Two different rotation sequences are used to confirm the conclusions drawn in this study. The first sequence involves three rotations $\psi, \theta,$ and ϕ about axes $Z, -Y,$ and $-X,$ respectively, with angle ϕ considered as dependent angle. In this first sequence, in-plane bending is produced by rotation ψ and out-of-plane bending and twist is produced by rotation θ. None of these two independent rotations is performed about the longitudinal axis tangent to the curve. The second sequence, on the other hand, involves three angles $\bar{\psi}, \bar{\phi},$ and $\bar{\theta}$ about axes $Z, X,$ and $Y,$ respectively, with angle $\bar{\theta}$ considered as dependent angle. In this sequence, curve twist is produced by a rotation about longitudinal axis tangent to the curve. This second sequence, however, demonstrates that the torsion is still not an exact differential, confirming the conclusions drawn in this study.

4. Curve framing

Instead of using Frenet frame which is defined by the matrix $\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix}$ and is associated with singularity at vanishing-curvature points, other frames can be introduced to describe motion along the curve despite these frames do not have the same geometric interpretation as Frenet frame. There are infinite number of choices of such frames that can have the first axis defined by unit tangent vector $\mathbf{t}$, that is, $\mathbf{a}_1 = \mathbf{t}$. Because $\mathbf{a}_1$ is perpendicular to $\mathbf{a}_{1s}$ and is perpendicular to the plane formed by $\mathbf{a}_2$ and $\mathbf{a}_3$, $\mathbf{a}_{1s}$ can always be written as linear combination of $\mathbf{a}_2$ and $\mathbf{a}_3$ which are not derived from $\mathbf{a}_1$. That is, one can always write $\mathbf{t}_s = \mathbf{r}_{ss} = \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3$. Because $\mathbf{a}_2 = -\mathbf{t} \times \mathbf{a}_3$ and $\mathbf{a}_3 = \mathbf{t} \times \mathbf{a}_2$, one has

$$\left.\begin{aligned} \mathbf{a}_{2s} &= -\mathbf{t}_s \times \mathbf{a}_3 - \mathbf{t} \times \mathbf{a}_{3s} = -(\alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3) \times \mathbf{a}_3 - \mathbf{t} \times \mathbf{a}_{3s} \\ &= -\alpha_2\mathbf{a}_2 \times \mathbf{a}_3 - \mathbf{t} \times \mathbf{a}_{3s} = -\alpha_2\mathbf{t} - \mathbf{t} \times \mathbf{a}_{3s} \\ \mathbf{a}_{3s} &= \mathbf{t}_s \times \mathbf{a}_2 + \mathbf{t} \times \mathbf{a}_{2s} = (\alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3) \times \mathbf{a}_2 + \mathbf{t} \times \mathbf{a}_{2s} \\ &= \alpha_3\mathbf{a}_3 \times \mathbf{a}_2 + \mathbf{t} \times \mathbf{a}_{2s} = -\alpha_3\mathbf{t} + \mathbf{t} \times \mathbf{a}_{2s} \end{aligned}\right\} \tag{3}$$

Derivatives $\mathbf{a}_{2s}$ and $\mathbf{a}_{3s}$ do not, in general, lie in curve normal plane, and there are infinite choices of frames in the form $\mathbf{A}_B = \begin{bmatrix} \mathbf{t} & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ that differ from Frenet frame $\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix}$ by a simple rotation θ_B about the local $\bar{\mathbf{t}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ axis. In this case, one has

$$\left.\begin{aligned} \mathbf{a}_2 &= \mathbf{n}\cos\theta_B + \mathbf{b}\sin\theta_B \\ \mathbf{a}_3 &= -\mathbf{n}\sin\theta_B + \mathbf{b}\cos\theta_B \end{aligned}\right\} \tag{4}$$

Because θ_B can assume any value, to define specific frame that shares tangent vector with Frenet frame, a condition must be imposed. Since $\mathbf{a}_{3s}$ is orthogonal to $\mathbf{a}_3$, imposing the condition that $\mathbf{a}_{3s}$ is also perpendicular to $\mathbf{a}_2$ ensures that $\mathbf{a}_{3s}$ is directed along tangent vector $\mathbf{t}$. Since $\mathbf{a}_2^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{2s}$, one concludes that $\mathbf{a}_{2s}$ is also parallel to tangent vector $\mathbf{t}$.

This geometric explanation can be demonstrated mathematically because equation $\mathbf{a}_2^T\mathbf{a}_{3s} = 0$ leads to

$$\bar{\boldsymbol{\omega}} = \begin{bmatrix} -\mathbf{a}_2^T\mathbf{a}_{3s} & \mathbf{t}^T\mathbf{a}_{3s} & -\mathbf{t}^T\mathbf{a}_{2s} \end{bmatrix}^T = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \end{bmatrix}^T \tag{5a}$$

This equation leads to the following counterpart of the Serret-Frenet equations:

$$\mathbf{t}_s = \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3, \quad \mathbf{a}_{2s} = -\alpha_2\mathbf{t}, \quad \mathbf{a}_{3s} = -\alpha_3\mathbf{t} \tag{5b}$$

These equations define *Bishop frame*, which will be discussed further in following sections of this paper (Bishop 1975). It is important, however, to note that different framing methods used for a curve do not enter into definition of an already-defined curve geometry. Therefore, angle θ_B should not be confused with Frenet angles used to define curve geometry.

5. Bishop frame

The frame defined by the three equations, $\mathbf{t}_s = \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3$, $\mathbf{a}_{2s} = -\alpha_2\mathbf{t}$, and $\mathbf{a}_{3s} = -\alpha_3\mathbf{t}$ are the basis for defining Bishop frame. According to these equations, normal vector can be written as $\mathbf{n} = \mathbf{t}_s/\kappa = (\alpha_2/\kappa)\mathbf{a}_2 + (\alpha_3/\kappa)\mathbf{a}_3$, where κ is curve curvature. Normal vector can then be written in terms of angle θ_B as $\mathbf{n} = \mathbf{a}_2\cos\theta_B + \mathbf{a}_3\sin\theta_B$, where $\tan\theta_B = \alpha_3/\alpha_2$ and $\kappa = \sqrt{\alpha_2^2 + \alpha_3^2}$.

Derivative of $\mathbf{n} = \mathbf{a}_2 \cos\theta_B + \mathbf{a}_3 \sin\theta_B$ with respect to arc length s leads to

$$\mathbf{n}_s = -\kappa\mathbf{t} + \theta_{Bs}(-\mathbf{a}_2 \sin\theta_B + \mathbf{a}_3 \cos\theta_B) = -\kappa\mathbf{t} + \theta_{Bs}\mathbf{b} \tag{6}$$

In developing the preceding equation, the fact that unit bi-normal $\mathbf{b} = (-\mathbf{a}_2 \sin\theta_B + \mathbf{a}_3 \cos\theta_B)$ is utilized. It is clear that this expression for $\mathbf{b}$ satisfies $\mathbf{t} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{b} = 0$. Equating the preceding equation with the second Serret-Frenet equation $\mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}$ leads to

$$\theta_{Bs} = \partial\theta_B/\partial s = \tau \tag{7}$$

Bishop determined θ_B using integral $\theta_B = \int \tau ds$. Nonetheless, torsion and curvature are not, in general, exact differentials, and therefore, the condition $\theta_{Bs} = \partial\theta_B/\partial s = \tau$ cannot be used to define unique value for angle θ_B. This is mainly due to the fact that constraints on derivatives do not always define constraints on coordinates. That is, number of independent derivatives can be less than number of independent coordinates. This case of non-integrable constrains, encountered in mechanics, is referred to as *nonholonomic constraints*. Therefore, definition of Bishop frame is not unique, and any angle of rotation about curve unit tangent $\mathbf{t}$ can be used to define a frame different from Frenet frame. Furthermore, a curve can be twisted without performing a rotation about tangent vector. An example is twisted curve obtained by performing two successive rotations; the first is rotation ψ about vertical axis Z to define a curve that lies in the horizontal plane, followed by a rotation θ about vector normal to the planar curve to achieve the twist. The twist, therefore, can be result of two rotations, none of them is performed about curve tangent. The first rotation is about a vertical axis, while the second rotation is about an axis that lies in the horizontal plane. Use of Frenet angles to define curve geometry demonstrates this important fact and sheds light on the approach used to define Bishop frame, which was mainly introduced based on the assumption that Frenet frame is not defined at curvature-vanishing points. This assumption is briefly discussed in the following section since existence of Frenet frame at curvature-vanishing points can be demonstrated (Shabana, 2021a; Shabana, 2022). It is also to be noted that structure of Bishop-frame Cartan matrix is different from that of Frenet-frame Cartan matrix.

6. Curvature-vanishing points

Bishop frame was introduced as alternate frame to circumvent problem of existence of Frenet frame and Serret-Frenet equations at curve points with zero curvature. There are, however, concerns regarding Bishop framing method since angle θ_B cannot be uniquely defined from integral $\theta_B = \int \tau ds$ since the torsion is not, in general, an exact differential and constraints on derivatives, such as $\theta_{Bs} = \partial\theta_B/\partial s = \tau$, do not imply constraints on coordinates, as previously discussed. Furthermore, torsion τ is not defined at curvature vanishing points. For a curve $\mathbf{r} = \mathbf{r}(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T$, torsion τ is defined as

$$\begin{aligned}\tau &= (\mathbf{r}_t \times \mathbf{r}_{tt}) \cdot \mathbf{r}_{ttt}/|\mathbf{r}_t \times \mathbf{r}_{tt}|^2 \\ &= \frac{x_{ttt}(y_t z_{tt} - y_{tt} z_t) + y_{ttt}(z_t x_{tt} - x_t z_{tt}) + z_{ttt}(x_t y_{tt} - y_t x_{tt})}{(y_t z_{tt} - y_{tt} z_t)^2 + (z_t x_{tt} - x_t z_{tt})^2 + (x_t y_{tt} - y_t x_{tt})^2}\end{aligned} \tag{8}$$

At points with zero curvature, one has $x_{tt} = y_{tt} = z_{tt} = 0$, and therefore, torsion τ is not defined at curvature-vanishing points if conventional differential-geometry procedures are used. It is also demonstrated that Frenet frame exists at curvature-vanishing points by writing curvature vector $\mathbf{r}_{ss}$ as linear combination of two axes $\mathbf{n}_1$ and $\mathbf{n}_2$ that define *pre-super-elevated osculating* (PSEO) plane as $\mathbf{r}_{ss} = \alpha_h\mathbf{n}_1 + \alpha_v\mathbf{n}_2$, where α_h and α_v are defined in (Ling and Shabana 2021; Shabana, 2021a). Curve curvature is defined as $\kappa = \sqrt{\alpha_h^2 + \alpha_v^2}$ and normal vector can be written as $\mathbf{n} = (\alpha_h/\kappa)\mathbf{n}_1 + (\alpha_v/\kappa)\mathbf{n}_2$. At curvature-vanishing points, ratio α_v/α_h is defined using L'Hopital rule, and therefore, normal vector and Frenet frame are defined at curvature-vanishing points (Shabana, 2021a; Shabana, 2022).

7. Frenet-angle representation of the torsion

Frenet angles are Euler angles used to define curve geometry (Ling and Shabana 2021a; Shabana, 2021b). Three Frenet angles are *curvature angle* ψ, *vertical-development angle* θ, and *bank angle* ϕ. These angles are defined using sequence of rotations Z, $-Y$, and $-X$, respectively. Using these three angles, Frenet frame can be defined as (Ling and Shabana 2021; Shabana 2021a, Shabana, 2021b)

$$\mathbf{A}_f = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix} \\ = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & -\sin\psi\sin\phi - \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\psi\sin\phi - \sin\psi\sin\theta\cos\phi \\ \sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \tag{9}$$

For this matrix to define Frenet frame, one has the condition $\mathbf{t}_s = \kappa\mathbf{n}$, that is, second column of $\mathbf{A}_f$ is obtained by differentiation of first column with respect to arc length s. This condition is also equivalent to $\mathbf{a}_1^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{1s} = 0$ as previously discussed. Using this Frenet-angle representation, the curvature and torsion can be written, respectively, as (Ling and Shabana 2021)

$$\left.\begin{aligned} \kappa &= \mathbf{a}_1^T\mathbf{a}_{2s} = -\mathbf{a}_2^T\mathbf{a}_{1s} = \psi_s\cos\phi\cos\theta - \theta_s\sin\phi \\ \tau &= \mathbf{a}_2^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{2s} = \psi_s\sin\theta - \phi_s \end{aligned}\right\} \tag{10}$$

The condition $\mathbf{a}_1^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{1s} = 0$ can be expressed in terms of Frenet angles as

$$\mathbf{a}_1^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{1s} = \psi_s\sin\phi\cos\theta + \theta_s\cos\phi = 0 \tag{11}$$

This condition, which is result of defining curvature vector $\mathbf{r}_{ss}$ as derivative of tangent vector with respect to s, reduces number of independent elements of Cartan matrix $\mathbf{A}_f^T\mathbf{A}_{fs}$ from three to two, and these two independent elements are curvature κ and torsion τ. This condition also demonstrates that curve geometry can be completely described using two independent angles only since Frenet bank angle ϕ can be expressed in terms of curvature angle ψ and vertical-development angle θ as $\phi = \tan^{-1}(-\theta_s/\psi_s\cos\theta)$. This equation shows that if Frenet bank angle ϕ is zero, $\theta_s = 0$, and consequently, vertical-development angle θ is constant, a condition satisfied for helix curve, for which tangent and normal vectors can be written, respectively, as

$$\mathbf{t} = \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} \tag{12}$$

In this special case, the curvature is defined in terms of Frenet angles as $\kappa = \psi_s\cos\theta$. Curve binormal vector $\mathbf{b}$ and its derivative are defined in this special case as

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} = \begin{bmatrix} -\cos\psi\sin\theta \\ -\sin\psi\sin\theta \\ \cos\theta \end{bmatrix}, \quad \mathbf{b}_s = \psi_s\sin\theta\begin{bmatrix} \sin\psi \\ -\cos\psi \\ 0 \end{bmatrix} = -(\psi_s\sin\theta)\mathbf{n} \tag{13}$$

This equation defines torsion τ in this special case as $\tau = \psi_s\sin\theta$. If $\psi_s = 0$ or $\theta = 0$, the curve is planar with zero torsion. Therefore, the helix is a twisted curve with constant Frenet vertical-development angle and zero Frenet bank angle.

Curve twist is the result of coupled in-plane and out-of-plane bendings, as previously mentioned. These two bending modes can be represented by two independent rotations. A single rotation leads to planar curve with zero torsion. The twist, on the other hand, requires two rotations to achieve both in-plane and out-of-plane bending. A curve can be twisted without any rotation about tangent vector $\mathbf{t}$. For example, rotation ψ about Z-axis creates a curve that lies in $X-Y$ plane, and such a curve has zero torsion. Unit vector $\mathbf{n}$ normal to the planar curve lies in the horizontal plane. A second rotation θ about vector $\mathbf{n}$ normal to the planar curve produces the twist which depends on the two rotations. In this case, no rotation about tangent vector $\mathbf{t}$ is performed. This fact is clear from definition of curve torsion τ in terms of Frenet angles as $\tau=\psi_s\sin\theta-\phi_s$. If no rotation is performed about tangent vector $\mathbf{t}$, the curve is still twisted and has a torsion defined by the equation $\tau=\psi_s\sin\theta$. The helix example, previously discussed in this section, can be used to provide a demonstration of this special case.

The expression $\kappa=\psi_s\cos\phi\cos\theta-\theta_s\sin\phi$ for curve curvature is not in general an exact differential. Curve curvature can be written in terms of two independent angles ψ and θ as $\kappa=\kappa(\psi,\theta)=\sqrt{(\psi_s\cos\theta)^2+(\theta_s)^2}$. This equation shows that curvature κ cannot in general be written as $\kappa=\kappa_1\psi_s+\kappa_2\theta_s$ with $\partial\kappa_1/\partial\theta=\partial\kappa_2/\partial\psi$, which is exact-differentiability condition. This implies that curvature κ cannot be written directly as derivative of an angle. An exception to this rule is planar curves or curves that have special geometry as helix curve for which Frenet vertical-development angle θ is constant, as will be discussed in a later section of this paper. Curve torsion is result of two bending modes that define also curve curvature. Torsion definition is function of curvature vector and its derivative. Derivative of curvature κ with respect to arc length can be written as $\kappa_s=(1/\kappa)(\psi_{ss}\cos\theta-\psi_s\theta_s\sin\theta+\theta_{ss})$. Derivatives of Frenet vertical-development and curvature angles are given, respectively, as $\theta_s=(\dot{s}z_{tt}-\ddot{s}z_t)/(\dot{s}^3\cos\theta)$, and $\psi_s=(\dot{s}y_{tt}\cos\theta-y_t(\ddot{s}\cos\theta-\dot{s}^2\theta_s\sin\theta))/\ (\dot{s}^3\cos\psi\cos^2\theta)$, where dot in this case refers to differentiation with respect to time in case of motion-trajectory curves. These derivatives can also be used to define curve curvature.

One can also show that, in this case of Frenet angles, projection of vector $\bar{\boldsymbol{\omega}}=\begin{bmatrix}\tau & 0 & \kappa\end{bmatrix}^T$ along tangent vector is $\bar{\boldsymbol{\omega}}\cdot\mathbf{t}=\tau\cos\psi\cos\theta+\kappa\sin\theta$, which demonstrates that projection of the vector that defines elements of skew-symmetric Cartan matrix cannot, in general, be written as derivative of a single angle along curve tangent vector. This is despite Frenet bank angle ϕ is performed about tangent vector $\mathbf{t}$. That is, while ϕ_s enters into torsion definition, it is not in general equal to the torsion, and distinction needs to be made between local (relative) and global geometry definitions.

8. Frame uniqueness

Infinite number of frames can be defined by a simple rotation θ_B about curve tangent vector $\mathbf{t}$. If there are no conditions imposed on definition of these frames, frame Cartan matrix has three independent elements; $\mathbf{a}_1^T\mathbf{a}_{2s}=-\mathbf{a}_2^T\mathbf{a}_{1s}$, $\mathbf{a}_1^T\mathbf{a}_{3s}=-\mathbf{a}_3^T\mathbf{a}_{1s}$, and $\mathbf{a}_2^T\mathbf{a}_{3s}=-\mathbf{a}_3^T\mathbf{a}_{2s}$. The rotations ϕ and θ_B are commutative because they are performed about the same axis defined by tangent vector $\mathbf{t}$. Performing rotation θ_B about the tangent vector $\mathbf{t}$ defines the frame

$$\begin{aligned}\mathbf{A}_B&=\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\\&=\begin{bmatrix}\cos\psi\cos\theta & -\sin\psi\cos\beta+\cos\psi\sin\theta\sin\beta & -\sin\psi\sin\beta-\cos\psi\sin\theta\cos\beta\\ \sin\psi\cos\theta & \cos\psi\cos\beta+\sin\psi\sin\theta\sin\beta & \cos\psi\sin\beta-\sin\psi\sin\theta\cos\beta\\ \sin\theta & -\cos\theta\sin\beta & \cos\theta\cos\beta\end{bmatrix}\end{aligned}\tag{14}$$

In this equation

$$\beta=\phi+\theta_B\tag{15}$$

If no conditions are imposed, then Cartan matrix associated with frame $\mathbf{A}_B$ has three independent elements. Frame $\mathbf{A}_B$ can be defined using the condition

$$\mathbf{a}_2^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{2s} = 0 \tag{16}$$

Differentiating vector $\mathbf{a}_2$ with respect to arc length s, one obtains

$$\begin{aligned}\mathbf{a}_{2s} = \psi_s\begin{bmatrix} -\cos\psi\cos\beta - \sin\psi\sin\theta\sin\beta \\ -\sin\psi\cos\beta + \cos\psi\sin\theta\sin\beta \\ 0\end{bmatrix} + \theta_s\sin\beta\begin{bmatrix}\cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta\end{bmatrix} \\ + \beta_s\begin{bmatrix}\sin\psi\sin\beta + \cos\psi\sin\theta\cos\beta \\ -\cos\psi\sin\beta + \sin\psi\sin\theta\cos\beta \\ -\cos\theta\cos\beta\end{bmatrix}\end{aligned} \tag{17}$$

which can be written as

$$\mathbf{a}_{2s} = \psi_s\begin{bmatrix} -\cos\psi\cos\beta - \sin\psi\sin\theta\sin\beta \\ -\sin\psi\cos\beta + \cos\psi\sin\theta\sin\beta \\ 0\end{bmatrix} + \mathbf{a}_1\theta_s\sin\beta - \mathbf{a}_3\beta_s \tag{18}$$

It follows that

$$\mathbf{a}_3^T\mathbf{a}_{2s} = -\mathbf{a}_2^T\mathbf{a}_{3s} = \psi_s\sin\theta - \beta_s = 0 \tag{19}$$

This equation shows that

$$\theta_{Bs} = \psi_s\sin\theta - \phi_s = \tau \tag{20}$$

For a general space curve, $\tau = \psi_s\sin\theta - \phi_s$ is not exact differential, that is, there is no unique value for angle θ_B, and consequently, there are infinite number of arrangements for the frame defined by the matrix $\mathbf{A}_B$. In this case in which the condition $\mathbf{a}_2^T\mathbf{a}_{3s} = -\mathbf{a}_3^T\mathbf{a}_{2s} = 0$ is imposed, Cartan matrix defined by $\mathbf{A}_B^T\mathbf{A}_{Bs}$ leads to

$$\mathbf{t}_s = \mathbf{a}_{1s} = k_1\mathbf{a}_2 + k_2\mathbf{a}_3, \quad \mathbf{a}_{2s} = -k_1\mathbf{t}, \quad \mathbf{a}_{2s} = -k_2\mathbf{t} \tag{21}$$

where

$$\left.\begin{aligned} k_1 &= -(\psi_s\sin\beta\cos\theta + \theta_s\cos\beta) \\ k_2 &= -(\psi_s\cos\beta\cos\theta - \theta_s\sin\beta)\end{aligned}\right\} \tag{22}$$

A constant vertical-development angle θ, as in case of the helix curve, implies $\phi = 0$. In this special case, one has $\theta_{Bs} = \psi_s\sin\theta = \tau$, which leads to *coordinate-level* (*holonomic*) *equation* $\theta_B = \theta_{Bo} + (\psi - \psi_o)\sin\theta$, where subscript o refers to initial value. Constants k_1 and k_2 reduce, respectively, to

$$\left.\begin{aligned} k_1 &= -\psi_s\sin\beta\cos\theta = -\kappa\sin\theta_B \\ k_2 &= -\psi_s\cos\beta\cos\theta = -\kappa\cos\theta_B\end{aligned}\right\} \tag{23}$$

which shows that $\kappa = \sqrt{k_1^2 + k_2^2}$.

9. Examples and results

Three different examples of curves with different geometries are considered in this section. The first example is helix curve which has zero Frenet bank angle ϕ and constant vertical-development angle θ. The helix has constant curvature and torsion, and its twist is not attributed to rotation about curve tangent vector. The second example is a curve with non-constant curvature and zero torsion. This second example is used to demonstrate the difference between Frenet bank angle which is oscillatory and Bishop shear angle which increases linearly with arc length. The third example is for a curve with non-zero torsion and is used to demonstrate that the torsion is not in general an exact differential.

9.1. Helix curve

Parametric equation of helix curve is

$$\mathbf{r}(s) = \begin{bmatrix} a\cos\alpha & a\sin\alpha & b\alpha \end{bmatrix}^T \tag{24}$$

where $\alpha = s/r$, s is arc-length parameter, a is helix radius, b/a is helix slope, and $r = \sqrt{a^2 + b^2}$. Using the preceding equation, it can be shown that curvature κ and torsion τ are constant and defined, respectively, as $\kappa = |a|/r$ and $\tau = b/r$.

Unit tangent vector is $\mathbf{t} = (1/r)\begin{bmatrix} -a\sin(s/r) & a\cos(s/r) & b \end{bmatrix}^T$. The matrix that defines orientation of helix Frenet frame can be written as product of two matrices as

$$\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} (-a/r)\sin(s/r) & -\cos(s/r) & (b/r)\sin(s/r) \\ (a/r)\cos(s/r) & -\sin(s/r) & -(b/r)\cos(s/r) \\ (b/r) & 0 & a/r \end{bmatrix} = \mathbf{A}_{f1}\mathbf{A}_{f2} \tag{25}$$

where

$$\mathbf{A}_{f1} = \begin{bmatrix} -\sin(s/r) & -\cos(s/r) & 0 \\ \cos(s/r) & -\sin(s/r) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_{f2} = \begin{bmatrix} a/r & 0 & -b/r \\ 0 & 1 & 0 \\ b/r & 0 & a/r \end{bmatrix} \tag{26}$$

Frenet-angle representation of tangent vector $\mathbf{t}$ is

$$\mathbf{t} = (1/r)\begin{bmatrix} -a\sin(s/r) \\ a\cos(s/r) \\ b \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix} \tag{27}$$

Using this equation, one can define Frenet curvature and vertical-development angles ψ and θ, respectively, as

$$\left.\begin{aligned} \cos\psi = -\sin(s/r), \quad \sin\psi = \cos(s/r), \\ \cos\theta = a/r, \quad \sin\theta = b/r, \end{aligned}\right\} \tag{28}$$

These equations lead to

$$\psi_s = 1/r = (r/b)\tau, \quad \tau = \psi_s \sin\theta = b/r^2 = b/(a^2 + b^2) \tag{29}$$

This equation shows that torsion τ depends only on derivative of one angle, curvature angle ψ. This is a case of a single parameter-dependent rotation. In this special case, torsion τ is an exact differential and its integration leads to

$$\int \tau ds = b(s - s_o)/r^2 = (b/r)(\psi - \psi_o) \tag{30}$$

This equation shows that

$$\psi = s/r, \quad \psi_o = s_o/r \tag{31}$$

That is, helix equation can be written in terms of Frenet curvature angle as

$$\mathbf{r}(s) = \begin{bmatrix} a\cos\psi & a\sin\psi & b\psi \end{bmatrix}^T \tag{32}$$

Because the torsion is constant, in this special case of helix curve, derivate of Bishop shear angle θ_{Bs} is constant.

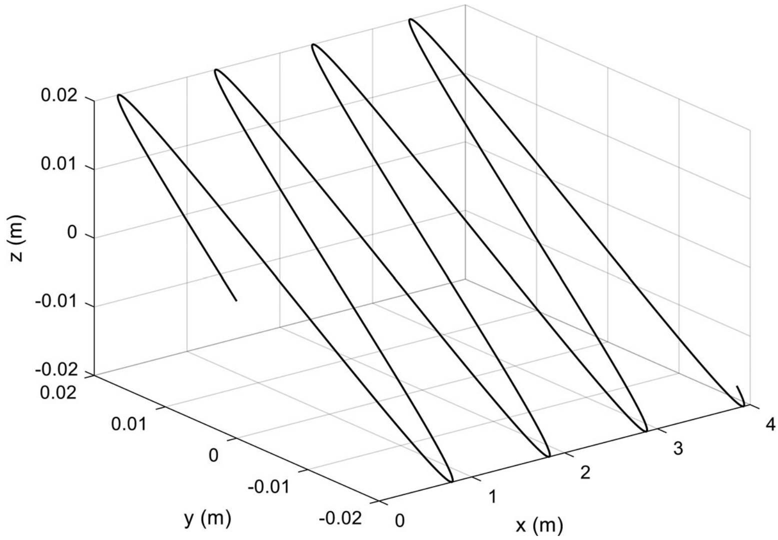


Figure 1. Curve plot.

9.2. Bishop shear and Frenet angles

In case of zero torsion, Bishop angle can assume any constant value, while Frenet bank angle can be oscillatory. The example considered in this section demonstrates difference between Frenet bank angle and Bishop shear angle in case of zero-torsion curve. As discussed in this paper, Frenet bank angle enters into definition of curve geometry and defines direction of curve normal vector. For helix curve, for example, Frenet bank angle is zero, while Bishop shear angle is non-zero and increases as the arc length increases since the helix has constant torsion τ. The curve considered in this section is shown in Figure 1 and is defined as

$$\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} t & Y_o \sin \omega t & Z_o \sin \omega t \end{bmatrix}^T \tag{33}$$

In this equation, curve parameter t is time, $Y_o = Z_o = 0.02$ m, and $\omega = 6$ rad/s. Curve tangent vector is defined as

$$\mathbf{r}_t = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T = \begin{bmatrix} 1 & \omega Y_o \cos \omega t & \omega Z_o \cos \omega t \end{bmatrix}^T \tag{34}$$

Norm of this tangent vector is $|\mathbf{r}_t| = \sqrt{1 + (\omega \cos \omega t)^2 (Y_o^2 + Z_o^2)}$. For this curve, Frenet curvature and vertical-development angles can be defined using the equations:

$$\left.\begin{aligned} \tan \psi &= y_t / x_t = \omega Y_o \cos \omega t \\ \tan \theta &= z_t \Big/ \sqrt{x_t^2 + y_t^2} = \omega Z_o \cos \omega t \Big/ \sqrt{1 + (\omega Y_o \cos \omega t)^2} \end{aligned}\right\} \tag{35}$$

These equations show oscillatory nature of the two Frenet angles ψ and θ. Curve second and third derivatives with respect to parameter t are defined, respectively, as

$$\left.\begin{aligned} \mathbf{r}_{tt} &= \begin{bmatrix} x_{tt} & y_{tt} & z_{tt} \end{bmatrix}^T = \omega^2 \begin{bmatrix} 0 & -Y_o \sin \omega t & -Z_o \sin \omega t \end{bmatrix}^T \\ \mathbf{r}_{ttt} &= \begin{bmatrix} x_{ttt} & y_{ttt} & z_{ttt} \end{bmatrix}^T = \omega^3 \begin{bmatrix} 0 & -Y_o \cos \omega t & -Z_o \cos \omega t \end{bmatrix}^T \end{aligned}\right\} \tag{36}$$

The curvature components α_h and α_v are defined for this curve as (Shabana, 2021a; Shabana 2022)

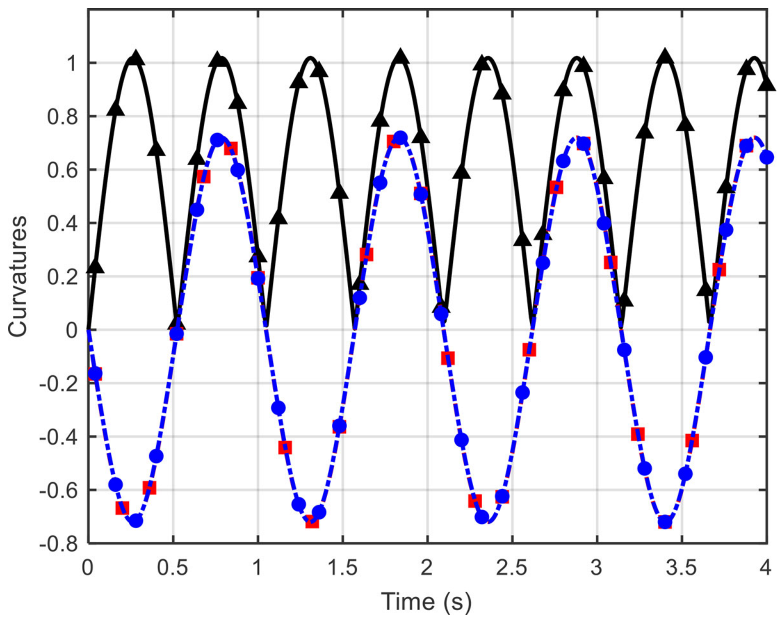


Figure 2. Curve curvatures (Black-triangle: Curvature κ; Red-square: Curvature component α_h; Blue-circle: Curvature component α_v).

$$\left.\begin{aligned}\alpha_h &= (y_{tt}x_t - x_{tt}y_t)\Big/|\mathbf{r}_t|^2\sqrt{x_t^2+y_t^2}\\ &= \left(-\omega^2 Y_o \sin\omega t\right)\Big/\left(1+(\omega\cos\omega t)^2\left(Y_o^2+Z_o^2\right)\right)\sqrt{1+(\omega Y_o\cos\omega t)^2},\\ \alpha_v &= \left(z_{tt}\left(x_t^2+y_t^2\right) - z_t(x_t x_{tt}+y_t y_{tt})\right)\Big/\left(|\mathbf{r}_t|^3\sqrt{x_t^2+y_t^2}\right)\\ &= Y_o Z_o\left(-\sin\omega t\left(1+(\omega Y_o\cos\omega t)^2\right)+Y_o(\sin\omega t\cos\omega t)\right)\Big/|\mathbf{r}_t|^3\sqrt{1+(\omega Y_o\cos\omega t)^2}\end{aligned}\right\} \tag{37}$$

Figure 2 shows the curvature $\kappa = \sqrt{\alpha_h^2+\alpha_v^2}$ and curvature components α_h and α_v. The results of this figure show that all curvature components are oscillatory and the curve has curvature-vanishing points. Figure 3 shows that Frenet bank angle ϕ is also oscillatory around a nominal value which indicates that super-elevation of Frenet osculating plane is not zero. Figure 4 shows that Frenet curvature and vertical-development angles of this curve are equal and both angles are oscillatory. Figure 5 demonstrates that normal vector exists at curvature-vanishing points for this curve with the longitudinal component having smallest absolute value. Curve unit normal and bi-normal vectors are defined, respectively, as

$$\left.\begin{aligned}\mathbf{n} &= \left(1\Big/\left(|\mathbf{r}_t|\sqrt{Y_o^2+Z_o^2}\right)\right)\left[-\omega\left(Y_o^2+Z_o^2\right)\cos\omega t \quad Y_o \quad Z_o\right]^T\\ \mathbf{b} &= \left(1\Big/\sqrt{Y_o^2+Z_o^2}\right)\left[0 \quad -Z_o \quad Y_o\right]^T\end{aligned}\right\} \tag{38}$$

This equation shows that bi-normal vector $\mathbf{b}$ is constant, and therefore, this curve is a planar curve with zero torsion. Bishop shear angle θ_B, therefore, is constant, which when compared with the results presented in Figure 3 demonstrates difference between Bishop shear angle θ_B and Frenet bank angle ϕ.

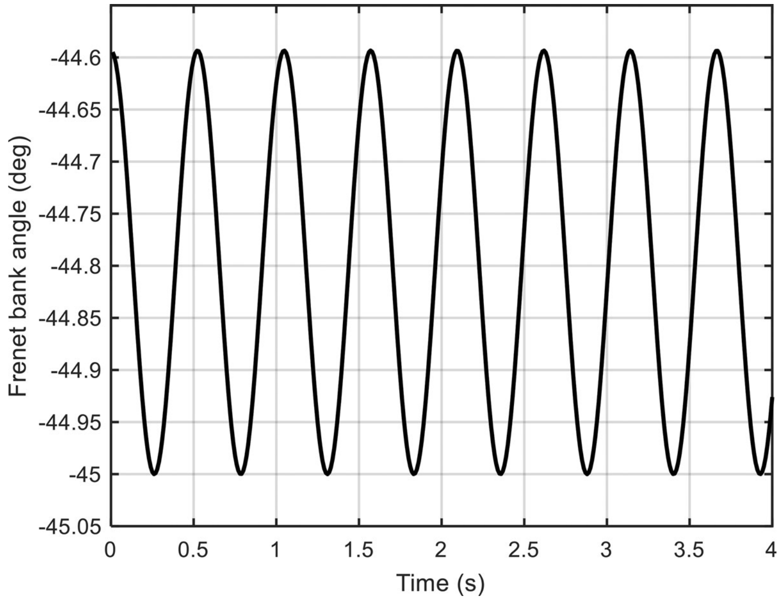


Figure 3. Frenet bank angle ϕ.

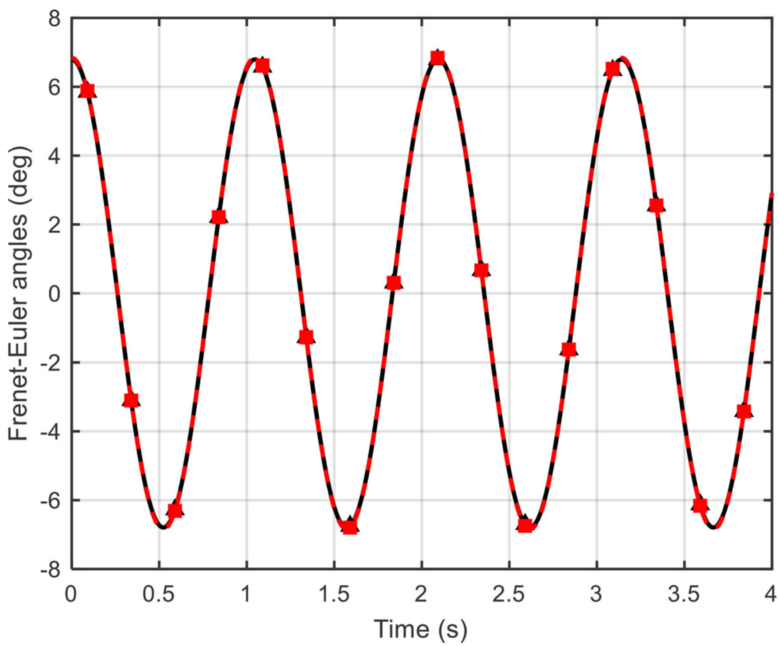
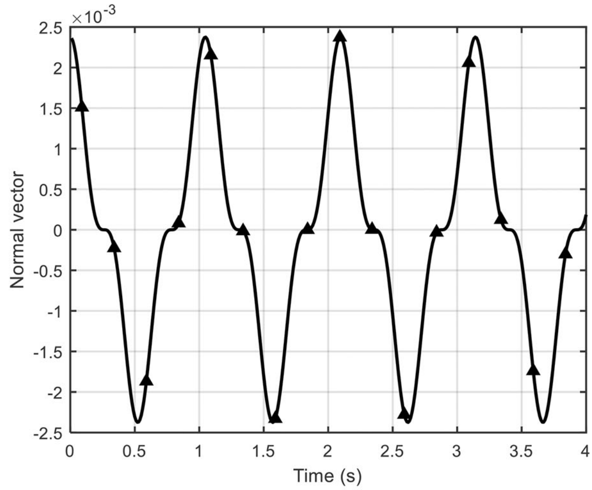


Figure 4. Frenet curvature angle ψ and Frenet vertical-Development angle θ (Black-triangle: ψ; Red-square: θ).

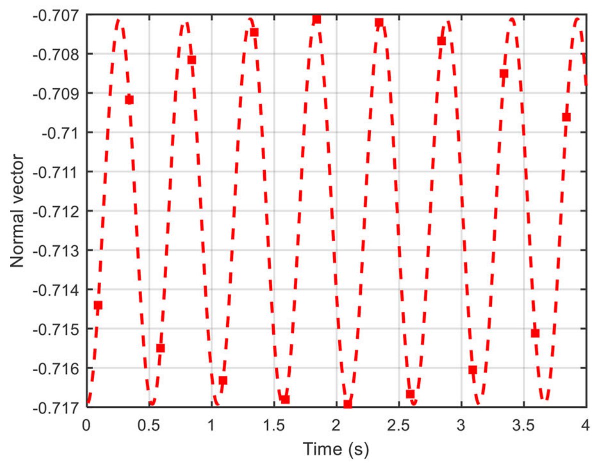
9.3. Non-zero torsion

Curve equations considered in this example is

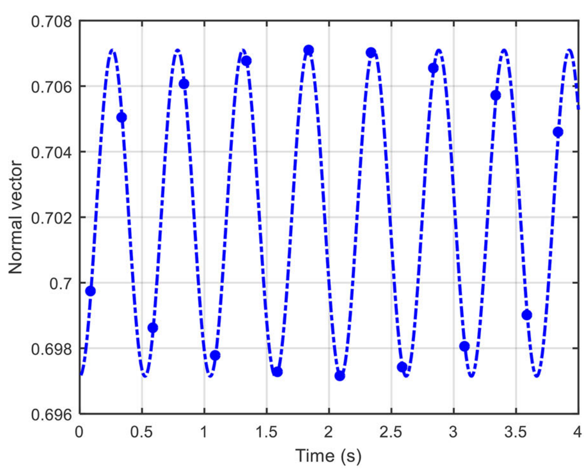
$$\mathbf{r}(t) = \begin{bmatrix} \cos t & \sin t & bt^2/2 \end{bmatrix}^T \tag{39}$$



(a)



(b)



(c)

Figure 5. Normal vector components (Black-triangle: n_1; Red-square: n_2; Blue-circle: n_3).

Tangent vector is

$$\mathbf{t} = (1/|\mathbf{r}_t|)\begin{bmatrix} -\sin(t) & \cos(t) & bt \end{bmatrix}^T \tag{40}$$

where $|\mathbf{r}_t| = \sqrt{1 + (bt)^2}$. Curve Frenet-angle representation is

$$\mathbf{t} = (1/|\mathbf{r}_t|)\begin{bmatrix} -\sin t \\ \cos t \\ bt \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix} \tag{41}$$

It follows that

$$\left.\begin{array}{ll} \cos\psi = -\sin t, & \sin\psi = \cos t, \\ \cos\theta = 1/|\mathbf{r}_t|, & \sin\theta = bt/|\mathbf{r}_t|, \end{array}\right\} \tag{42}$$

The expression for torsion τ is

$$\theta_{Bs} = \tau = \psi_s \sin\theta - \phi_s \tag{43}$$

If τ were an exact differential in this example, then

$$\left.\begin{array}{l} \theta_{Bs} = \tau = (\partial\theta_B/\partial\psi)\psi_s + (\partial\theta_B/\partial\theta)\theta_s \\ \left(\partial^2\theta_B/\partial\psi\,\partial\theta\right) = \left(\partial^2\theta_B/\partial\theta\,\partial\psi\right) \end{array}\right\} \tag{44}$$

where θ_B is Bishop shear angle. The above exact-differentiability condition, however, is not satisfied for this curve since

$$\left.\begin{array}{l} (\partial\theta_B/\partial\psi) = \sin\theta = bt/|\mathbf{r}_t|, \quad (\partial\theta_B/\partial\theta) = -1 \\ \left(\partial^2\theta_B/\partial\psi\partial\theta\right) \neq \left(\partial^2\theta_B/\partial\theta\partial\psi\right) \end{array}\right\} \tag{45}$$

These equations imply that the torsion τ is not an exact differential.

10. Rotation sequences

Two of the three Frenet angles used to describe curve geometry are sufficient since third angle ϕ can be written in terms of the other two angles ψ and θ. This fact is clear from tangent vector definition $\mathbf{t} = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & \sin\theta \end{bmatrix}^T$. Using this equation, curvature vector can be written as

$$\begin{aligned} \mathbf{t}_s &= \psi_s\cos\theta\begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix}^T \\ &= \theta_s\begin{bmatrix} -\cos\psi\sin\theta & -\sin\psi\sin\theta & \cos\theta \end{bmatrix}^T \end{aligned} \tag{46}$$

which defines curve curvature as $\kappa = \sqrt{(\psi_s\cos\theta)^2 + (\theta_s)^2}$ and curve normal vector $\mathbf{n}$ as

$$\begin{aligned} \mathbf{n} &= \left(\psi_s\cos\theta/\kappa\right)\begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix}^T \\ &= (\theta_s/\kappa)\begin{bmatrix} -\cos\psi\sin\theta & -\sin\psi\sin\theta & \cos\theta \end{bmatrix}^T \end{aligned} \tag{47}$$

Bi-normal vector $\mathbf{b}$ can be obtained from cross-product $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Therefore, Frenet frame transformation matrix $\mathbf{A}_f$ can be written in terms of two independent angles only. However, derivatives of these angles with respect to the arc length s appear in this matrix. Definition of normal vector $\mathbf{n}$ in the preceding equation, however, shows that angle $\phi = \tan^{-1}(-\theta_s/\psi_s\cos\theta)$ can be used to eliminate explicit dependence of Frenet-frame transformation matrix on derivatives.

Another set of angles that can be used are three rotations $\bar{\psi}$, $\bar{\phi}$, and $\bar{\theta}$ about local axes $Z, X,$ and Y, respectively. This sequence is selected to examine effect of applying rotation $\bar{\phi}$ about tangent vector before applying third rotation $\bar{\theta}$. Using this new sequence, one has

$$\mathbf{A}_f = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \cos\bar{\psi} & -\sin\bar{\psi}\cos\bar{\phi} & \sin\bar{\psi}\sin\bar{\phi} \\ \sin\bar{\psi} & \cos\bar{\psi}\cos\bar{\phi} & -\cos\bar{\psi}\sin\bar{\phi} \\ 0 & \sin\bar{\phi} & \cos\bar{\phi} \end{bmatrix} \begin{bmatrix} \cos\bar{\theta} & 0 & \sin\bar{\theta} \\ 0 & 1 & 0 \\ -\sin\bar{\theta} & 0 & \cos\bar{\theta} \end{bmatrix} \tag{48}$$

One can show that if first and second columns of this matrix define tangent and normal vector, respectively, one has the condition

$$\bar{\psi}_s \sin\bar{\phi} + \bar{\theta}_s = 0 \tag{49}$$

The curvature and torsion that form skew-symmetric Cartan matrix are given in this case, respectively, as

$$\left.\begin{aligned} \kappa &= \bar{\psi}_s \cos\bar{\theta}\cos\bar{\phi} + \bar{\phi}_s \sin\bar{\theta}, \\ \tau &= -\bar{\psi}_s \sin\bar{\theta}\cos\bar{\phi} + \bar{\phi}_s \cos\bar{\theta} \end{aligned}\right\} \tag{50}$$

These definitions of the curvature and torsion demonstrate again that curve geometric invariants are not exact differentials. That is, exact differentiability of Cartan-matrix elements does not depend on rotation sequence. Comparing these expressions with the rotation sequence previously used indicates that the first rotation sequence leads to simple expressions. Furthermore, this first sequence is more relevant to railroad track construction procedures used in practice.

In order to check the results obtained in this section using an alternate approach, tangent and normal vectors are written, respectively, using the matrix $\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix}$ as

$$\mathbf{t} = \begin{bmatrix} \cos\bar{\psi}\cos\bar{\theta} - \sin\bar{\psi}\sin\bar{\phi}\sin\bar{\theta} \\ \sin\bar{\psi}\cos\bar{\theta} + \cos\bar{\psi}\sin\bar{\phi}\sin\bar{\theta} \\ -\cos\bar{\phi}\sin\bar{\theta} \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} -\sin\bar{\psi}\cos\bar{\phi} \\ \cos\bar{\psi}\cos\bar{\phi} \\ \sin\bar{\phi} \end{bmatrix} \tag{51}$$

This equation shows that curvature vector $\mathbf{t}_s = \mathbf{r}_{ss}$ can be written as

$$\begin{aligned} \mathbf{t}_s = {} & \bar{\psi}_s \begin{bmatrix} -\sin\bar{\psi}\cos\bar{\theta} - \cos\bar{\psi}\sin\bar{\phi}\sin\bar{\theta} \\ \cos\bar{\psi}\cos\bar{\theta} - \sin\bar{\psi}\sin\bar{\phi}\sin\bar{\theta} \\ 0 \end{bmatrix} + \bar{\phi}_s \sin\bar{\theta} \begin{bmatrix} -\sin\bar{\psi}\cos\bar{\phi} \\ \cos\bar{\psi}\cos\bar{\phi} \\ \sin\bar{\phi} \end{bmatrix} \\ & + \bar{\theta}_s \begin{bmatrix} -\cos\bar{\psi}\sin\bar{\theta} - \sin\bar{\psi}\sin\bar{\phi}\cos\bar{\theta} \\ -\sin\bar{\psi}\sin\bar{\theta} + \cos\bar{\psi}\sin\bar{\phi}\cos\bar{\theta} \\ -\cos\bar{\phi}\cos\bar{\theta} \end{bmatrix} \end{aligned} \tag{52}$$

Using condition $\bar{\psi}_s \sin\bar{\phi} + \bar{\theta}_s = 0$ to eliminate θ_s, the preceding equation can be written as

$$\mathbf{t}_s = \left(\bar{\psi}_s \cos\bar{\phi}\cos\bar{\theta} + \bar{\phi}_s \sin\bar{\theta}\right) \begin{bmatrix} -\sin\bar{\psi}\cos\bar{\phi} \\ \cos\bar{\psi}\cos\bar{\phi} \\ \sin\bar{\phi} \end{bmatrix} = \kappa\mathbf{n} \tag{53}$$

This result confirms curvature expression previously obtained. It is to be noted, however, that for this rotation sequence, unlike the first rotation sequence, use of two angles $\bar{\psi}$ and $\bar{\phi}$ without introducing third angle $\bar{\theta}$ is not sufficient to properly define unit tangent vector. This is mainly due to the fact that rotation $\bar{\phi}$ about curve tangent vector in the horizontal plane does not change orientation of this vector and this tangent vector remains function of angle $\bar{\psi}$ only.

11. Relevance to mechanics problems

The analysis presented in this paper is focused on curve geometry, and consequently, it is relevant to both motion trajectory curves and one-dimensional beam theory. No restriction is imposed in this study on space-curve geometry, and therefore, the analysis is applicable to recorded motion trajectories with arbitrary geometry and large deformation of beams. Because curve geometry is defined in terms of two independent Frenet angles; curvature angle ψ and vertical-development angle θ, it is clear, as previously mentioned, that curve twist (torsion) is due to out-of-plane bending mode and should be distinguished from torsion due to shear. It is demonstrated that curve torsion due to out-of-plane bending cannot be defined as derivative of an angle, shedding light on issues that need to be addressed when considering Bishop frame in which definition of angle θ_B is not unique and depends on assumed initial value. Because curve torsion is in general not an exact differential, such a torsion due to bending cannot be integrated to define a frame.

Bishop frame is also to be distinguished from frames introduced in mechanics-literature to account for torsion due to shear. von Dombrowski introduced a frame that has beam-centerline tangent as its first axis (von Dombrowski 2002). This frame also differs from Frenet frame by single rotation about beam-centerline tangent vector. Nonetheless, von Dombrowski properly introduced additional degree of freedom to describe torsion due to shear. This degree of freedom is associated with unique initial conditions required for solving resulting second-order ordinary differential equations of motion. The frame introduced by von Dombrowski, therefore, should not be confused with Bishop frame which does not have unique definition. Nonetheless, if Bishop frame is associated with beam cross section, both θ_{Bs} and $\dot{\theta}_B$ can be integrated since θ_B represents a rotation about single axis and is uniquely defined by initial conditions. In this case, Bishop frame and the frame used in (von Dombrowski 2002) are identical; and $\theta_{Bs} = \partial\theta_B/\partial s$ can be used to define strain energy resulting from torsion due to shear. Bending strain energy that accounts for beam twist can be defined using beam-centerline curvature κ.

Regarding relevance of the analysis presented in this study to beam vibration problems, curve equation used in this study represents a specific configuration in which s is curve arc length. That is, the curve can be viewed as a snapshot of a configuration that can be reference or current configuration regardless of how beam equation is formulated. Using general curve description in terms of Frenet angles which define two bending modes; in-plane and out-of-plane bending; nonlinear beam equations can be systematically formulated to obtain two partial differential equations associated with $\psi = \psi(s)$ and $\theta = \theta(s)$. Kinetic energy can be defined in terms of these two filed variables, and bending strain energy can be defined using general curvature expression. Resulting partial differential equations can be highly nonlinear if small displacement assumptions are not made in formulating beam strain energy and definition of curvature. Nonetheless, using two Frenet angles defines an inextensible beam since longitudinal tangent determined by differentiation with respect to arc length s remains unit vector. To account for beam extension, longitudinal field variable can be introduced. A snapshot of beam configuration defines a space curve of extensible beam with arc length s defined in current configuration. Formulation of beam equation using Frenet angles is beyond the scope of this investigation which is focused on definition of space-curve Cartan matrix and demonstrating that curve curvature and torsion are not in general exact differentials.

12. Conclusions

Space-curve curvature is used in linear and nonlinear formulations of beam vibration equations to formulate strain energy and elastic forces. Curve torsion is result of out-of-plane bending that produces twist to be distinguished from continuum-mechanics shear mode. Nonetheless, the curvature and torsion are not, in general, associated with derivatives of angles because they are elements of Cartan matrix and are not exact differentials. Curve twist, for example, is result of coupled in-plane and out-of-plane bending modes, which can be described mathematically using two rotations. As discussed in this paper, a curve can be twisted without performing a rotation about curve tangent; example of such a curve is the helix curve. Because different frames were introduced to overcome problem of defining Frenet frame at curvature-vanishing points, Cartan matrix can have different structures that depend on the condition used to define the frame. *Frenet angles* are used in this study to develop simple and general expressions for the curvature and torsion. Two different rotation sequences are used to demonstrate that the curvature and torsion are not exact differentials. The analysis and results presented in this investigation demonstrate fundamental difference between Bishop shear angle and Frenet bank angle. Uniqueness of Bishop shear angle and Bishop frame are discussed

Compliance with ethical standards

The author declares that they have no conflict of interest

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Data availability statement

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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