THE HESSIAN POLYNOMIAL AND THE JACOBIAN IDEAL OF A REDUCED HYPERSURFACE IN $\mathbb{P}^n$

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Abstract. For a reduced hypersurface $V(f) \subseteq \mathbb{P}^n$ of degree d, the Castelnuovo-Mumford regularity of the Milnor algebra $M(f)$ is well understood when $V(f)$ is smooth, as well as when $V(f)$ has isolated singularities. We study the regularity of $M(f)$ when $V(f)$ has a positive dimensional singular locus. In certain situations, we prove that the regularity is bounded by $(d-2)(n+1)$, which is the degree of the Hessian polynomial of f. However, this is not always the case, and we prove that in $\mathbb{P}^n$ the regularity of the Milnor algebra can grow quadratically in d.

1. Introduction

Let $S = \oplus_k S_k = \mathbb{C}[x_0, ..., x_n]$ be the graded polynomial ring, where S_k denotes the vector space of degree k homogeneous polynomials. For a homogeneous polynomial $f \in S_d$, the Jacobian ideal J_f is generated by the partial derivatives of f, and the Milnor algebra $M(f)$ is the graded ring S/J_f.

From a geometric standpoint $M(f)$ is of interest because it encodes the singular subscheme $\Sigma = \Sigma(f)$ of the projective hypersurface $V(f) \subseteq \mathbb{P}^n$. When $V(f)$ is smooth, $M(f)$ is an Artinian complete intersection and plays a central role in the Hodge theory of $V(f)$. A landmark result of Griffiths [17] shows that the Hodge numbers of $V(f)$ can be extracted from $M(f)$, and recent work of Dimca [8] shows that one can obtain related results for an even dimensional nodal hypersurface. The Milnor algebra also has applications in physics, where it is known as the Chiral ring [5], in the study of Bernstein-Sato polynomials (recent work of Walther [29]), in the study of multiplier ideals [13], and in Torelli type theorems.

One case where $M(f)$ has been the object of intense investigation is when $V(f)$ is an arrangement of hypersurfaces. Of course, in this setting $V(J_f)$ is of codimension two. Even the simplest case, where f is a product of distinct linear forms, is highly nontrivial. A landmark result is Terao's theorem [25] relating the Cohen-Macaulay property of J_f (often referred to as *freeness*, because in this case the syzygy module of J_f is free) to the topology of the complement of the arrangement. A second case where much is known about $M(f)$ is when the singularities of $V(f)$ are isolated, see [7].

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Beyond the cases above, in general little is known about algebraic properties of $M(f)$. Our aim in this paper is to investigate one of the fundamental algebro-geometric invariants associated to $M(f)$-the *Castelnuovo-Mumford regularity*. The importance of regularity is that it measures, in a precise sense, the complexity of a finitely generated graded S-module M; it is determined by the "shape" of a minimal free resolution of M. For us, the module in question will be the Milnor algebra $M(f)$:

Definition 1.1. Let

$$0 \to F_m \to \cdots \to F_0 \to M \to 0,$$

be a minimal graded free resolution of the graded S-module M, where

$$F_k = \bigoplus_j S(-a_{k,j}) \text{ for } k = 0, \dots, m.$$

By the Hilbert Syzygy Theorem, $m \leq n+1$ and $m = \operatorname{pd} M$ is the projective dimension of M. The Castelnuovo-Mumford regularity of M is

$$\text{regularity } M = \max_{i,j}\{a_{i,j} - i\}.$$

Two other central algebraic invariants are the Hilbert function and Hilbert polynomial:

Definition 1.2. For a finitely generated graded S-module M, the Hilbert function is $H(M,t) = \dim_{\mathbb{C}} M_t$, for $t \in \mathbb{Z}$. Hilbert proved that for $t \gg 0$, the Hilbert function is given by the Hilbert polynomial $P(M,t) \in \mathbb{Q}[t]$.

$P(M(f),t)$ encodes information about $V(f)$; for example, if V has isolated singularities then $P(M(f),t)$ is a constant which is equal to the sum of the Tjurina numbers at the singular points of $V(f)$. Castelnuovo-Mumford regularity and the Hilbert function are related via the stability threshold, defined as

$$st(f) = \min\{q \ : \ H(M(f),t) = P(M(f),t) \text{ for all } t \geq q\}.$$

By [15, Theorem 4.2], $st(f) \leq \text{regularity } M(f) + m - n$.

When the singular subscheme Σ is composed of finitely many points, there is a sharp upper bound for the stability threshold [4], but there are very few results when Σ is of positive dimension. A main motivation for our work comes from the study of arrangements of hypersurfaces in $\mathbb{P}^n$; clearly when $n \geq 3$ a hypersurface arrangement will have a positive dimensional singular locus.

The square matrix of second order partial derivatives of f is the *Hessian* matrix, and we write hess_f to denote the determinant of the Hessian. The polynomial hess_f has degree $T = (n+1)(d-2)$, and it is easy to show that $\mathrm{hess}_f \in (J_f : \mathfrak{m})$ where $\mathfrak{m}$ denotes the irrelevant ideal $(x_0, \dots, x_n)$. In fact, Spodzieja shows in [24] (see also Vasconcelos [28]) that for any hypersurface $V(f)$, then $\mathrm{hess}_f \in J_f$ iff $V(f)$ is singular. As noted earlier, when $V(f)$ is smooth, $M(f)$ is an Artinian complete intersection, so $M(f)_T$ is a one-dimensional $\mathbb{C}$-vector space generated by the class of hess_f in $M(f)$ and $st(f) = T+1$. The objective of this paper is to investigate the following question:

Question 1.3. Find upper bounds on the regularity $M(f)$, where $V(f)$ is a reduced singular hypersurface in $\mathbb{P}^n$ of degree d.

In §2 and §3, we prove regularity bounds related to T. It is known that the regularity of $M(f)$ is bounded above by T for a general hypersurface, as well as for hypersurfaces that have at most finitely many singular points. This property, which we recall with more detail in §2, suggests investigating hypersurfaces with positive dimensional singular loci.

Theorem: For the following classes of reduced hypersurfaces, regularity $M(f) < T$:

(1) $V(f)$ is a generic hyperplane arrangement in any $\mathbb{P}^n$.
(2) $V(f)$ is a generic (or generic symmetric) determinantal hypersurface in any $\mathbb{P}^n$.
(3) $V(f)$ is a hypersurface of degree $d \geq 3$ in $\mathbb{P}^n$ which is free or nearly free, or a cone over a plane curve.
(4) $V(f)$ is a generic arrangement of surfaces with isolated singularities in $\mathbb{P}^3$.

The proof of all but (4) is covered in §2; in these cases a graded finite free resolution of $M(f)$ is available. The proof of (4) involves a delicate spectral sequence argument, and we tackle it in §3. Finally, in §4 we show that the hope of finding an upper bound on regularity that is linear in d is vain. We prove:

Theorem: There exist reduced, irreducible hypersurfaces of degree d in $\mathbb{P}^n$ for which

$$\text{regularity}\, M(f) \sim \mathcal{O}(d^2).$$

For a family of bigraded degree d surfaces in $\mathbb{P}^3$, we give an explicit first syzygy of degree $\frac{d^2+d-2}{3}$; the result follows by coning over such a surface.

2. Basic definitions and a first analysis of regularity

In this section, we first recall the definitions we shall need. Then, we provide a precise statement for the upper bounds of the stability threshold and regularity of the Milnor algebra that are known for reduced hypersurfaces that are smooth or that have isolated singularities. We will end this section by discussing classes of reduced hypersurfaces where it is relatively easy to show that regularity $M(f) < T$ because a graded finite free resolution of the corresponding Jacobian ideal is known.

2.1. Basic definitions and properties.

The stability threshold and the Castelnuovo-Mumford regularity associated to $M(f)$ were defined in the previous section. In this preliminary section, we come back briefly on these two important invariants with a particular emphasis on their connection to the local cohomology modules $H^i_{\mathfrak{m}}(M(f))$, $i \geq 0$.

Let I_f be the saturation of the Jacobian ideal J_f with respect to the ideal $\mathfrak{m}$ in the graded ring S, i.e. $I_f = (J_f : \mathfrak{m}^\infty)$. The 0^{th} local cohomology module measures how far J_f is from being saturated; we have

$$H^0_{\mathfrak{m}}(M(f)) = I_f/J_f$$

and we denote it by $N(f)$ for simplicity. We notice that when $V(f)$ is smooth we have that $N(f) = M(f)$ so that $N(f)_T \neq 0$ and $N(f)_k = 0$ for all $k > T$.

Coming back to the regularity, one has [15, Theorem 4.3]

$$\text{regularity}\, M(f) \geq \max\{e \mid N(f)_e \neq 0\},$$

which is in fact a consequence of the following characterization of regularity (see for instance [4, Fact 6]) in terms of local cohomology modules:

$$\text{regularity}\, M(f) = \min\{e \mid H^i_{\mathfrak{m}}(M(f))_{>e-i} = 0 \text{ for all non-negative integers } i\}.$$

The Hilbert function $H(M(f)) : \mathbb{N} \to \mathbb{N}$ of the graded S-module $M(f)$ is defined by

$$H(M(f))(k) = \dim M(f)_k,$$

and there is a unique polynomial $P(M(f))(t) \in \mathbb{Q}[t]$, called the Hilbert polynomial of $M(f)$, and an integer $k_0 \in \mathbb{N}$ such that [14]

$$H(M(f))(k) = P(M(f))(k)$$

for all $k \geq k_0$. The stability threshold is defined as

$$st(f) = \min\{q \;:\; H(M(f), t) = P(M(f), t) \text{ for all } t \geq q\}$$

and it is connected to local cohomology modules by means of the Grothendieck-Serre Formula [3, Theorem 4.3.5]: for all $k \in \mathbb{Z}$ we have

$$H(M(f))(k) = P(M(f))(k) + \sum_{i \geq 0} (-1)^i H(H^i_{\mathfrak{m}}(M(f)))(k).$$

Finally, recall that the depth of $M(f)$ is linked to the projective dimension of $M(f)$ by the Auslander-Buchsbaum formula, see [15, Theorem A2.15]:

$$\operatorname{depth} M(f) = \operatorname{depth} S - \operatorname{pd} M(f) = n + 1 - \operatorname{pd} M(f).$$

Moreover, [15, Theorem A2.14] tells us that

$$\operatorname{depth} M(f) = \inf\{k \mid H^k_{\mathfrak{m}}(M(f)) \neq 0\},$$

and that $\operatorname{depth} M(f) = 0$ if and only if $\mathfrak{m}$ is an associated prime of $M(f)$. In particular, $\operatorname{depth} M(f) > 0$ if and only if $N(f) = 0$.

2.2. **Isolated singularities.** The behavior of the stability threshold and regularity for reduced hypersurfaces having isolated singularity have been widely studied and are well understood, in particular because the graded S-module $N(f)$ has a nice duality in this situation, see [10, 23, 26].

Given a graded module M we will denote by $\operatorname{indeg}(M)$ the initial degree of M, that is the infimum of the degrees of its nonzero elements. We have the following result.

Proposition 2.3. *If the hypersurface $V(f)$ has isolated singularities then*

$$st(f) \leq T - \operatorname{indeg}(N(f)) + 1$$

and

$$\text{regularity}\, M(f) \leq T - \min\{d-1, \operatorname{indeg}(N(f))\}$$

unless $I_f = J_f$ is a complete intersection and $\deg(J_f) := P(M(f)) = (d-1)^n$, *in which case* regularity $M(f) \leq T - \min\{d-2, \operatorname{indeg}(N(f))\}$.

In particular, regularity $M(f) < T$ *except if $d = 2$ and J_f is a complete intersection defining a simple point, in which case* regularity $M(f) = T$.

Proof. Under our assumption the ideal J_f is an almost complete intersection of dimension 1 and the claimed inequalities follows by applying the results in [27, Appendix]. □

The above property has naturally consequences for hypersurfaces whose singular locus can be described from hypersurfaces in smaller dimension and having isolated singularities, as the following result for instance.

Corollary 2.4. *For any surface in $\mathbb{P}^3$ of degree $d \geq 3$ which is a cone over a plane curve,* regularity $M(f) < T$.

Proof. If $V(f) \subset \mathbb{P}^3$ is a cone over a reduced plane curve C, we can assume that f depends only on x_1, x_2, x_3 and that $g = 0$ is an equation for C in $\mathbb{P}^2$, with $g = f$. As explained in [12, Section 3.6], it follows that $M(f) = M(g) \otimes_{\mathbb{C}} \mathbb{C}[x_0]$, and hence the minimal resolution of $M(f)$ as a graded module over $S = \mathbb{C}[x_0, x_1, x_2, x_3]$ and of $M(g)$ as a graded module over $R = \mathbb{C}[x_1, x_2, x_3]$ have the same numerical invariants. This fact implies that regularity $M(f) =$ regularity $M(g)$ and the claimed result follows from Proposition 2.3. □

2.5. **Some particular classes of hypersurfaces.** We begin with the classes of free and nearly free hypersurfaces in $\mathbb{P}^n$ $(n \geq 2)$. A hypersurface is free if the syzygy module of J_f is free [12]. The notion of *nearly free* appears for curves and surfaces in [11], [12], and was subsequently generalized by Abe in [1], who also defined plus-one generated arrangements. The natural generalization of this to hypersurfaces is

Definition 2.6. A hypersurface is nearly free if its Milnor algebra $M(f)$ admits a graded free resolution of the form

$$0 \to S(-d_n - d) \to S(-d_n - (d-1)) \oplus (\oplus_{i=1}^n S(-d_i - (d-1))) \to S(-(d-1))^{n+1} \to S \tag{2.1}$$

for some integers $d_1 \leq d_2 \leq \ldots \leq d_n$.

Proposition 2.7. *For any free or nearly free hypersurface in $\mathbb{P}^n$ of degree $d \geq 3$, we have* regularity $M(f) < T$.

Proof. Assume first that $V(f)$ is a free hypersurface in $\mathbb{P}^n$ with exponents $(d_1, \ldots, d_n)$, and $d_1 \leq \cdots \leq d_n$. Since the syzygy module of J_f is free, the Hilbert-Burch Theorem [14] shows that $\sum d_i = d - 1$, and hence $d_n \leq d - 1$. We deduce that

$$\text{regularity } M(f) \leq d - 1 + d_n - 2 \leq 2d - 4 < T$$

where the last inequality holds for all $d \geq 3$.

Now, assume that $V(f)$ is a nearly free hypersurface and that $M(f)$ admits a free resolution of the form (2.1) for some integers $d_1 \leq \ldots \leq d_n$. The Hilbert polynomial of $M(f)$ has degree at most $n-2$ so its computation as the alternate sum of the dimensions of the graded slices of (2.1) in sufficiently high degree yields a condition corresponding to the vanishing of the coefficient of degree $n-1$ of the Hilbert polynomial. A straightforward computation shows that this condition is $\sum d_i = d$ (see also [1, Proposition 4.1]). It follows that $d_n \leq d$ and hence

$$\text{regularity}\, M(f) \leq d + d_n - 3 \leq 2d - 3 < T$$

where the last inequality holds for all $d \geq 3$. □

We next discuss the case when $V(f)$ is a generic hyperplane arrangement $\mathcal{A}$, with $d = |\mathcal{A}| > n \geq 2$.

Proposition 2.8. *Let $\mathcal{A}$ be a generic hyperplane arrangement in the projective space $\mathbb{P}^n$, with $d = |\mathcal{A}| > n \geq 2$. Then*

$$\text{regularity}\, M(f) = 2d - n - 3 < T.$$

Proof. Using the resolution for $M(f)$ when $V(f)$ is a generic hyperplane arrangement, with $d > n \geq 2$ given in [22, Corollary 4.5.4], it follows that depth $M(f) = 0$, a fact stated in [22, Corollary 4.5.5], and regularity $M(f) = 2d - n - 3$, since the differences $a_{i,j} - i$ are 0 for $i = 0$, $d - 2$ for $i = 1$ and $2d - n - 3$ for $i = 2, \ldots, n + 1$. By our assumption, $2d - n - 3 \geq d - 2$. The inequality regularity $M(f) < T$ is equivalent to $(n-1) < (n-1)d$, which clearly holds. See Ziegler [30] and Mustata-Schenck [21] for related results. □

A much studied class of hypersurfaces are determinantal hypersurfaces; see Beauville [2] for results and open problems. We close this section with a result in this direction.

Proposition 2.9. *For an $n \times n$ generic matrix A_n whose (i, j) entry is x_{ij}, or for a generic symmetric matrix B_n whose $(i, j) = (j, i)$ entry is x_{ij}, we have*

$$\text{regularity}\, M(det(A_n)) = n - 1 \textit{ and } \text{regularity}\, M(det(B_n)) = 2n - 4.$$

So generic and generic symmetric determinantal hypersurfaces have regularity $M(f) < T$.

Proof. Let $a = \det(A_n)$ and $b = \det(B_n)$. Then

$$J_a = I_{n-1}(A_n) \text{ and } J_b = I_{n-1}(B_n).$$

The resolution of the ideal of submaximal minors for a generic matrix A_n is determined by Gulliksen-Negård in [18]: the ideal is Gorenstein of codimension four, and has

$$\text{regularity}\, I_{n-1}(A_n) = 2n - 4.$$

In similar fashion, the ideal of submaximal minors for a generic symmetric matrix B_n is determined by Józefiak in [19] (and by Lascoux in characteristic zero in [20]): the ideal is Cohen-Macaulay of codimension three, and has an Eagon-Northcott resolution with

$$\text{regularity}\, I_{n-1}(B_n) = n - 1.$$

Since for J_a we have $T = n^2(n - 2)$, and for J_b we have $T = (n)(n + 1)(n - 2)/2$, in both cases the regularity of the Milnor ring is much smaller than T. □

3. On generic arrangements of surfaces with isolated singularities in $\mathbb{P}^3$

In this section we show that the regularity of the Milnor algebra is bounded above by T for surfaces in $\mathbb{P}^3$ that are obtained as unions of surfaces with isolated singularities and in general position to some extent. These surfaces have a one-dimensional singular locus and they can be seen as a generalization of generic plane arrangements. We set $S = \mathbb{C}[x_0, \dots, x_3]$ and consider $f \in S_d$ and its Jacobian ideal $J_f \subset S$ which is generated in degree $d-1$ by the four partial derivatives $f_0, \dots, f_3$ of f.

Recall that, according to Hilbert-Burch Theorem (see [14, Theorem 20.15]), a perfect graded ideal $I \subset S$ of codimension 2 admits a minimal free resolution of the form

$$0 \to \oplus_{i=1}^{r-1} S(-l_i) \xrightarrow{\Psi} \oplus_{i=1}^{r} S(-e_i) \to I \to 0 \tag{3.1}$$

where Ψ corresponds to an homogeneous matrix. In addition, we have the equality

$$\sum_{i=1}^{r-1} l_i = \sum_{i=1}^{r} e_i =: \sigma$$

and without loss of generality, one can assume that $1 \le e_1 \le e_2 \le \cdots \le e_r$ and $l_1 \le l_2 \le \cdots \le l_{r-1}$. We notice that the minimality assumption implies that $e_1 < l_1$. We begin with the following rather general result.

Theorem 3.1. *Suppose $I = (g_1, \dots, g_r)$ be a perfect ideal in S of codimension 2 with a minimal free resolution of the form (3.1), and $f \in S$ is such that $J_f \subset I$ and that the ideal $(J_f : I)$ defines a 0-dimensional subscheme in $\mathbb{P}^3$ (possibly empty). Then*

$$st(f) \le \max\{4d - 7 - 2e_1, l_1 - 3\}$$

and

$$\text{regularity } M(f) \le \begin{cases} \max\{4d - 8 - 2e_1, l_1 - 2\} \text{ if } e_1 < d-1, \\ \max\{2d - 5, l_1 - 2\} \text{ if } e_1 = d-1. \end{cases}$$

Moreover, in these conditions, $st(f) \le T$, and regularity $M(f) < T$ *providing $l_1 \le T + 1$.*

Remark 3.2. Since $\sum_{i=1}^{r-1} l_i = \sum_{i=1}^{r} e_i$ we have $(r-1)l_1 \le re_r$. Therefore the condition $l_1 \le T+1$ is satisfied if $e_r \le \frac{r-1}{r}(T+1)$.

Proof. By the Grothendieck-Serre Formula we have that for all $k \in \mathbb{Z}$ one has

$$H(M(f))(k) = P(M(f))(k) + \sum_{i \ge 0} (-1)^i H(H^i_{\mathfrak{m}}(M(f)))(k).$$

Moreover, since $\dim(M(f)) \le 2$ we have $H^i_{\mathfrak{m}}(M(f)) = 0$ for all $i > 2$ by [3, Theorem 3.5.7]. Therefore, to prove the claimed result we need to examine the vanishing of the graded components of the local cohomology modules $H^i_{\mathfrak{m}}(M(f))$ for $i = 0, 1, 2$.

The hypotheses of Theorem 3.1 implies that the following graded complex is a minimal free resolution of I:

$$F_\bullet \ : \ F_2 := \oplus_{i=1}^{r-1} S(-l_i) \xrightarrow{\Psi} F_1 := \oplus_{i=1}^{r} S(-e_i) \xrightarrow{(g_1 \ \cdots \ g_r)} S.$$

By examining the two spectral sequences corresponding to the filtrations by rows and columns of the double complex $\mathcal{C}^\bullet_{\mathfrak{m}}(F_\bullet)$, we deduce immediately that $H^0_{\mathfrak{m}}(S/I) = H^1_{\mathfrak{m}}(S/I) = 0$ and the graded isomorphism

$$H^2_{\mathfrak{m}}(S/I) \simeq \ker(H^4_{\mathfrak{m}}(F_2) \to H^4(F_1))$$

where the map $H^4_{\mathfrak{m}}(F_2) \to H^4(F_1)$ is the canonical one induced by the Koszul complex. In particular, we have $H^2_{\mathfrak{m}}(S/I)_k = 0$ for all $k > l_1 - 4$.

Now, consider the canonical exact sequence

$$0 \to I/J_f \to M(f) = S/J_f \to S/I \to 0. \tag{3.2}$$

We know that $H^i_{\mathfrak{m}}(S/I) = 0$ for $i = 0, 1$. We also have $H^i_{\mathfrak{m}}(I/J_f) = 0$ for all $i > 1$ because $\operatorname{ann}_S(I/J_f) = (J_f : I)$ and $(J_f : I)$ is assumed to define a 0-dimensional subscheme in $\mathbb{P}^3$. Therefore, the long exact sequence of local cohomology of (3.2) implies that

$$H^0_{\mathfrak{m}}(M(f)) \simeq H^0_{\mathfrak{m}}(I/J_f), \quad H^1_{\mathfrak{m}}(M(f)) \simeq H^1_{\mathfrak{m}}(I/J_f)$$
$$\text{and } H^2_{\mathfrak{m}}(M(f)) \simeq H^2_{\mathfrak{m}}(S/I).$$

An immediate consequence is that

$$H^2_{\mathfrak{m}}(M(f))_k = 0 \text{ for all } k > l_1 - 4 =: \eta_2. \tag{3.3}$$

To examine $H^i_{\mathfrak{m}}(I/J_f)$, $i = 0, 1$, we proceed as follows.

From the inclusion $J_f \subset I$, one can decompose the f_i's on g and g' to get a $r \times 4$-matrix H such that

$$(f_0\ f_1\ f_2\ f_3) = (g_1\ \cdots\ g_r)\, H = (g_1\ \cdots\ g_r) \begin{pmatrix} h_{0,1} & h_{1,1} & h_{2,1} & h_{3,1} \\ h_{0,2} & h_{1,2} & h_{2,2} & h_{3,2} \\ \vdots & \vdots & \vdots & \vdots \\ h_{0,r} & h_{1,r} & h_{2,r} & h_{3,r} \end{pmatrix}.$$

This latter corresponds to an homogeneous map

$$K_1 = S(-(d-1))^4 \xrightarrow{H} F_1$$

that gives rise to a finite free graded presentation of the quotient I/J_f, namely the graded exact sequence

$$K_1' = F_2 \oplus K_1 \xrightarrow{\varphi} F_1 \xrightarrow{(g_1,\ldots,g_r)} I/J_f \to 0$$

where the map $\varphi : K_1' \to F_1$ is defined by the $r \times (r+3)$ matrix

$$\left(\begin{array}{c|cccc} & h_{0,1} & h_{1,1} & h_{2,1} & h_{3,1} \\ \Psi & \vdots & \vdots & \vdots & \vdots \\ & h_{0,r} & h_{1,r} & h_{2,r} & h_{3,r} \end{array}\right).$$

The Buchsbaum-Rim complex $E_\bullet$ associated to φ, that belongs to the family of generalized Koszul complexes [14, Appendix A.2.6], is the following graded complex

$$E_4 = S_2(F_1^*) \otimes \wedge^{r+3}(K_1')(\sigma) \to E_3 = S_1(F_1^*) \otimes \wedge^{r+2}(K_1')(\sigma) \to$$
$$E_2 = S_0(F_1^*) \otimes \wedge^{r+1}(K_1')(\sigma) \to E_1 = K_1' \xrightarrow{\varphi} E_0 = F_1$$

where F_1^* denotes the dual of F_1 and $\sigma = \sum_{i=1}^r e_i = \sum_{i=1}^{r-1} l_i$. It is a classical property that the homology of $E_\bullet$ is supported on

$$\mathrm{ann}_S(\mathrm{coker}(\varphi)) = \mathrm{ann}_S(I/J_f) = (J_f :_S I)$$

and by the hypotheses of Theorem 3.1 this is a finite subscheme in $\mathbb{P}^3$.

Now, consider the double complex $\mathcal{C}^\bullet_\mathfrak{m}(E_\bullet)$. The spectral sequence corresponding to its filtration by rows converges at the second step and is of the form

$$\begin{array}{ccccc}
H^0_\mathfrak{m}(H_4(E_\bullet)) & H^0_\mathfrak{m}(H_3(E_\bullet)) & H^0_\mathfrak{m}(H_2(E_\bullet)) & H^0_\mathfrak{m}(H_1(E_\bullet)) & H^0_\mathfrak{m}(I/J_f) \\
H^1_\mathfrak{m}(H_4(E_\bullet)) & H^1_\mathfrak{m}(H_3(E_\bullet)) & H^1_\mathfrak{m}(H_2(E_\bullet)) & H^1_\mathfrak{m}(H_1(E_\bullet)) & H^1_\mathfrak{m}(I/J_f) \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

(observe that $H_0(E_\bullet) = I/J_f$). The spectral sequence corresponding to the filtration by columns of $\mathcal{C}^\bullet_\mathfrak{m}(E_\bullet)$ also converges at the second step with a single non-zero row: $H_\bullet(H^4_\mathfrak{m}(E_\bullet))$. Comparing these two spectral sequences, we deduce that

- $H^0_\mathfrak{m}(I/J_f)_k = 0$ for all k such that $H^4_\mathfrak{m}(E_4)_k = 0$ and
- $H^1_\mathfrak{m}(I/J_f)_k = 0$ for all k such that $H^4_\mathfrak{m}(E_3)_k = 0$.

But from the description of $E_\bullet$ we have

$$\begin{aligned} E_4 &= S_2(F_1^*) \otimes \wedge^{r+3}(K_1')(\sigma) \\ &\simeq \oplus_{1 \le i \le j \le r} S(-4(d-1) + e_i + e_j) \end{aligned}$$

from we deduce that $H^4_\mathfrak{m}(E_4)_k = 0$, hence $H^0_\mathfrak{m}(M(f))_k \simeq H^0_\mathfrak{m}(I/J_f)_k = 0$, for all integers

$$k > 4(d-1) - 4 - 2e_1 = 4d - 8 - 2e_1 =: \eta_0, \tag{3.4}$$

We also have

$$\begin{aligned} E_3 &= S_1(F_1^*) \otimes \wedge^{r+2}(K_1')(e + e') \\ &\simeq \oplus_{i=1}^r S(-3(d-1) + e_i)^4 \oplus_{i=1}^r \oplus_{j=1}^{r-1} S(-4(d-1) + l_j + e_i) \end{aligned}$$

from we deduce that $H^4_\mathfrak{m}(E_3)_k = 0$, hence $H^1_\mathfrak{m}(M(f))_k \simeq H^1_\mathfrak{m}(I/J_f)_k = 0$, for all integer

$$k > 3d - 7 - e_1 + (d - 1 - l_1)_+ =: \eta_1, \tag{3.5}$$

where $(d - 1 - l_1)_+ = \max\{0, d - 1 - l_1\}$.

From here, the claimed results follows by comparing the constraints given by (3.4), (3.5) and (3.3). Indeed, we have

$$\eta_0 - \eta_1 = (d - 1 - e_1) - (d - 1 - l_1)_+ = \begin{cases} l_1 - e_1 & \text{if } l_1 \le d - 1, \\ d - 1 - e_1 & \text{if } l_1 \ge d - 1. \end{cases}$$

Since $e_1 \le d - 1$ ($J_f \subset I$), and $l_1 > e_1$ (for otherwise the first column of Ψ would be identically zero), the quantity $\eta_0 - \eta_1$ is always non-negative. More precisely, $\eta_0 > \eta_1$ if $1 \le e_1 < d - 1$ and $\eta_0 = \eta_1 = 2d - 6$ if $e_1 = d - 1$.

Gathering all the previous results, we deduce the following properties. First, $N(f)_k = 0$ for all $k \geq T$ because $\eta_0 < T$, since $e_1 \geq 1$ (I is of codimension 2). Second, we have

$$st(V) \leq \max\{\eta_0, \eta_1, \eta_2\} + 1 = \max\{\eta_0 + 1, \eta_2 + 1\}.$$

Finally, since regularity $M(f) = \max\{\eta_0, \eta_1 + 1, \eta_2 + 2\}$ the claimed inequalities follows from the two cases $e_1 < d-1$, for which $\max\{\eta_0, \eta_1 + 1\} = \eta_0 = T - 2e_1$, and $e_1 = d-1$, for which $\max\{\eta_0, \eta_1 + 1\} = \eta_0 + 1 = 2d - 5$. Therefore regularity$(M(f)) < T$, if $d \geq 2$ and $\eta_2 + 2 < T$, this latter condition being equivalent to $l_1 \leq T + 1$. □

Now we introduce the generic arrangements of surfaces with isolated singularities in $\mathbb{P}^3$ which appear in the title of this section.

Theorem 3.3. *Suppose given a collection of r surfaces $D_i = V(f_i)$, $i = 1, \ldots, r$, in $\mathbb{P}^3$ of positive degree $1 \leq d_1 \leq \ldots \leq d_r$ respectively and consider the surface $V = \cup_{i=1}^r D_i = V(f)$, where $f := \prod_{i=1}^r f_i$, in $\mathbb{P}^3$ of degree $d = \sum_{i=1}^r d_i \geq 2$. Assume that*

- *f is a reduced polynomial,*
- *D_i has only finitely many singular points for all i,*
- *the intersection between any two distinct surfaces D_i and D_j is transverse, except at finitely many points,*
- *the intersections between any three distinct surfaces D_i consists in finitely many points.*

Then

$$st(f) \leq 2d + 2d_r - 7 \tag{3.6}$$

and hence $st(f) \leq T$ providing $d \geq 3$.

Moreover, in the generic setting, more precisely if the surfaces D_i, $i = 1, \ldots, r$, are all smooth surfaces that intersect transversally at all their intersection points (in particular any four distinct surfaces do not intersect), then we have $I_f = (g_1, \ldots, g_r)$, where $g_i := f/f_i$ for all $i = 1, \ldots, r$, and (3.6) is an equality.

Proof. Let I be the ideal of S generated by $g_1, \ldots, g_r$. A straightforward computation shows that $J_f \subset I$. Actually, we have the equality

$$(\partial_0 f \ \partial_1 f \ \partial_2 f \ \partial_3 f) = (g_1 \ g_2 \ \cdots \ g_r) \cdot \begin{pmatrix} \partial_0 f_1 & \partial_1 f_1 & \partial_2 f_1 & \partial_3 f_1 \\ \partial_0 f_2 & \partial_1 f_2 & \partial_2 f_2 & \partial_3 f_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_0 f_r & \partial_1 f_r & \partial_2 f_r & \partial_3 f_r \end{pmatrix}$$

where the matrix on the right is the Jacobian matrix of the f_i's; we denote it H. It defines a graded map

$$R(-d+1)^4 \xrightarrow{H} \oplus_{i=1}^r R(-d + d_i).$$

On the other hand, consider the following matrix:

$$\Psi = \begin{pmatrix} f_1 & 0 & & & \cdots & 0 \\ -f_2 & f_2 & 0 & & \cdots & \\ 0 & -f_3 & f_3 & \ddots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 & \\ \vdots & 0 & \ddots & \ddots & f_{r-2} & 0 \\ & \cdots & & 0 & -f_{r-1} & f_{r-1} \\ 0 & \cdots & & & 0 & -f_r \end{pmatrix}.$$

It is of size $r \times (r-1)$ and its $(r-1)$-minors coincide with the g_i's up to sign. Therefore, assuming that I has codimension 2, i.e. that f is a reduced polynomial, then Hilbert-Burch Theorem implies that I admits the following minimal free resolution

$$0 \to \oplus_{i=1}^{r-1} S(-d) \xrightarrow{\Psi} \oplus_{i=1}^{r} S(-d+d_i) \to S \to S/I \to 0 \tag{3.7}$$

Now, as already used in the proof of Theorem 3.1, the concatenation of the matrices Ψ and H provides a free presentation of I/J_f. Therefore, we deduce that the ideal $I_r(\Psi \oplus H)$ of $r-$minors of this concatenated matrix has the same radical as the ideal $(J_f : I)$ (using a classical property of Fitting ideals [14, Proposition 20.7]). In addition, examining the matrix $\Psi \oplus H$ we notice that

- If $p \in D_i \setminus \cup_{j \neq i} D_j$ and $p \notin \mathrm{Sing}(D_i)$, then $p \notin V(J_f : I)$. This is because the partials $\partial_i(f)$ are obtained as r-minors of $\Psi \oplus H$ (take Ψ and add a column of H).
- If for all points $p \in D_i \cap D_j$, except finitely many, the intersection of D_i and D_j at p is transverse and p is not contained in any other surfaces D_l, then there exists an r-minor of $\Psi \oplus H$ that does not vanish: take Ψ, remove number i and j and replace them by those 2 columns of H corresponding to ∂_k and ∂_l such that the Jacobian minor $\partial_k f_i \partial_l f_j - \partial_k f_j \partial_l f_i$ is nonzero.

Under our assumptions, we deduce that $(J_f : I)$ is supported on finitely many points and hence one can apply Theorem 3.1 with the data: $e_1 = d - d_r$ and $l_1 = d$. The conclusion (3.6) follows because we always have $2d + 2d_r - 7 \geq d - 3$ as $d \geq 2$ and $d_r \geq 1$.

Now, we turn to the proof of the second part of this theorem. Pushing further the above analysis of r-minors of the matrix $\Psi \oplus H$, one can show in the same way that for all point $p \in V$ there exists an r-minor of the matrix $\Psi \oplus H$ that does not vanish at p, under our genericity assumptions. Therefore, the ideal $(J_f : I)$ defines an empty algebraic variety, which means that the ideals J_f and I have the same saturation, so that $J_f^{\mathrm{sat}} = I_f = I^{\mathrm{sat}} = I$. Thus, taking again the proof of Theorem 3.1, this latter property implies that

$$H^0_{\mathfrak{m}}(I/J_f) \simeq \ker\left(H^4_{\mathfrak{m}}(E_4) \to H^4_{\mathfrak{m}}(E_3)\right)$$

because $H^1_{\mathfrak{m}}(H_i(E_\bullet)) = 0$ for all $i \geq 0$. If $\eta_0 > \eta_1$, then $H^4_{\mathfrak{m}}(E_3)_{\eta_0} = 0$ whereas $H^4_{\mathfrak{m}}(E_4)_{\eta_0} \neq 0$ so we deduce that $H^0_{\mathfrak{m}}(I/J_f)_{\eta_0} \neq 0$. As we already observed, the condition $\eta_0 > \eta_1$ holds if and only if $e_1 < d-1$, and since $e_1 = d - d_r$, it holds if and only if $d_r \geq 2$. It remains to consider the case $d_1 = \cdots = d_r = 1$, for which we have $l_1 = \ldots = l_{r-1} = d = r$ and

$\eta_0 = \eta_1 = 2d - 6$. Again, from the proof of Theorem 3.1 we have

$$E_4 \simeq S(-2(d-1))^{\binom{r+1}{2}} \text{ and } E_3 \simeq S(-2(d-1))^4 \oplus S(-2(d-1)+1)^{r(r-1)}.$$

It follows that

$$H^4_{\mathfrak{m}}(E_4)_{\eta_0} \simeq H^4_{\mathfrak{m}}(S)_{-4}^{\binom{r+1}{2}} \simeq \mathbb{C}^{\binom{r+2}{2}}$$

and

$$H^4_{\mathfrak{m}}(E_3)_{\eta_0} \simeq H^4_{\mathfrak{m}}(S)^4_{-4} \oplus H^4_{\mathfrak{m}}(S)^{r(r-1)}_{-3} \simeq \mathbb{C}^4.$$

Therefore, if $\binom{r+1}{2} > 4$, i.e. if $r > 2$, then necessarily $H^0_{\mathfrak{m}}(I/J_f)_{\eta_0} \neq 0$. In the end, the theorem is proved except for the case where f is the product of two linearly independent planes ($d = 2$, $r = 2$, $d_1 = d_2 = 2$). But this case can be treated directly from the definitions: $M(f)$ is isomorphic to a graded polynomial ring in two variables, hence $H(M(f))(k) = \max\{k+1, 0\}$ and $P(M(f))(k) = k+1$ for all $k \in \mathbb{Z}$, so that $st(f) = -1 = 2\cdot 2 + 2\cdot 1 - 7$. $\square$

Example 3.4. In this example we show that the inequality for $st(f)$ in Theorem 3.3 can be either strict, or an equality, even for $r = 2$ surfaces. First we consider the family of surfaces $V_{d+1} = V(f_1 f_2)$, where $f_1 = x_3$ and $f_2 = x_0^d + x_1^d + x_2^d$ with $d \geq 2$. Then $D_1 = V(f_1)$ is a plane, $D_2 = V(f_2)$ is a surface with a singular point at $p = (0:0:0:1)$ with local Tjurina number $\tau(D_2, p) = (d-1)^3$, and the intersection $C = D_1 \cap D_2$ is transverse. Hence Theorem 3.3 applies and gives the inequality

$$st(f) \leq 2(d+1) + 2d - 7 = 4d - 5.$$

Using that the Hilbert polynomial $P(M(f))$ is a linear function of the form $ak + b$ and the values for a, b given in [12, Formula (2.6) and subsection (3.1)], we see that

$$P(M(f))(k) = ak + b = \deg(C)k + \chi(C, \mathcal{O}_C) + \tau_0(V) = dk - \frac{d(d-3)}{2} + (d-1)^3,$$

since in this case $\tau_0(V) = \tau(D_2, p)$. A direct computation of the Hilbert series for V_{d+1} when $2 \leq d \leq 5$ using SINGULAR, shows that in all these cases

$$st(f) = 3d - 5 < 4d - 5.$$

Next we consider the family of surfaces $V'_{2d} = V(f_1 f_2)$, where $f_1 = x_1^d + 2x_2^d + x_3^d$ and $f_2 = x_0^d + x_1^d + x_2^d$ with $d \geq 2$. Then $D_1 = V(f_1)$ is a surface with a singular point at $q = (1:0:0:0)$ with local Tjurina number $\tau(D_1, q) = (d-1)^3$, the surface D_2 is as above, and the intersection $C = D_1 \cap D_2$ is transverse. Hence Theorem 3.3 applies again and gives the inequality

$$st(f) \leq 2(2d) + 2d - 7 = 6d - 7.$$

As above, for the Hilbert polynomial, we get

$$P(M(f))(k) = d^2 k + d^3 - 4d^2 + 6d - 2,$$

since in this case $\tau_0(V) = \tau_q(D_1, q) + \tau_p(D_2, p)$. A direct computation of the Hilbert series for V'_{2d} when $2 \leq d \leq 5$ shows that in all these cases $st(f) = 6d - 7$.

Remark 3.5. By Theorem 3.3 when $r = 2$ then $st(f) = 2d + 2d_2 - 7$, the surfaces D_1 and D_2 are smooth, and the intersection $C = D_1 \cap D_2$ is transverse; see [12, Question 3.2].

4. Regularity of $M(f)$ for hypersurfaces in $\mathbb{P}^n$ singular in codimension one

The main result in this section is in contrast to the results in the previous sections: we prove that a reduced hypersurface $V(f) \subseteq \mathbb{P}^n$ of degree d which has a non-reduced codimension one singular component can have

$$\text{regularity } M(f_d) \sim \mathcal{O}(d^2).$$

We prove the result for $n = 3$; the general case then follows by coning. In fact, Theorem 4.1 below yields a stronger result, giving an explicit minimal first syzygy of high degree. The surfaces which we analyze are bigraded, so have a singular locus containing two disjoint lines. We utilize the extra structure provided by the bigrading in an essential way.

Theorem 4.1. *Let $f_{(k,d-k)} \in \mathbb{C}[x_0, \dots, x_3]$ be a general bihomogeneous polynomial of bidegree $(k, d-k)$ with $k, d-k \geq 1$; note that $f \in (x_0, x_1)^k \cap (x_2, x_3)^{d-k}$. Let $D = V(f_{(k,d-k)})$ be the corresponding surface in $\mathbb{P}^3$, which is of degree d in the $\mathbb{Z}$-grading. Then D is reduced, and if $d-k$ is odd, $M(f_{(k,d-k)})$ has a minimal first syzygy of bidegree*

$$(2k, 0) + \frac{d-k-1}{2}(3k-2, 3).$$

Hence, in the $\mathbb{Z}$-grading

$$\text{regularity } M(f_{(k,d-k)}) \geq \frac{3k+1}{2}d - \frac{3k^2+5}{2}$$

and taking $k = \frac{d-1}{3}$ yields

$$\text{regularity } M(f_{(\frac{d-1}{3}, \frac{2d+1}{3})}) \geq \frac{1}{3}(d^2 + d - 8).$$

Proof. The fact that D is reduced follows from the hypothesis that f is general. Define the (standard) graded polynomial ring $A = \mathbb{C}[x_0, x_1]$, so that the polynomial ring $S = \mathbb{C}[x_0, \dots, x_3] = A[x_2, x_3]$ inherits a bigraded structure. In particular, $f_{(k,d-k)} \in S_{k,d-k}$. Denoting by f_i the partial derivative of $f_{(k,d-k)}$ with respect to x_i, we have $f_0, f_1 \in S_{k-1,d-k}$ and $f_2, f_3 \in S_{k,d-k-1}$. In other words, the Jacobian ideal $J_{f_{(k,d-k)}} = J = (f_0, \dots, f_3)$ has a bigraded presentation

$$S(-k+1, -d+k)^2 \oplus S(-k, -d+k+1)^2 \xrightarrow{\phi} S \to S/J \to 0. \tag{4.1}$$

Since $x_0 f_0 + x_1 f_1 = k f_{(k,d-k)}$ and $x_2 f_2 + x_3 f_3 = (d-k) f_{(k,d-k)}$, the ideal J admits the bi-Euler syzygy $((d-k)x_0, (d-k)x_1, kx_2, kx_3)$. This syzygy is of bidegree $(k, d-k)$ in the first syzygy module Syz of J, that is the kernel of ϕ.

Henceforth, we use *minimal* to mean a syzygy of the smallest possible degree with respect to the variables $\{x_2, x_3\}$, which is not the bi-Euler syzygy. Our goal is to find a minimal syzygy of J; to do this we consider the graded slice of (4.1) in degree $\eta \in \mathbb{N}$ with respect to $\{x_2, x_3\}$:

$$0 \to \mathrm{Syz}_{*,\eta} \to A(-k+1)^{2(\eta-d+k+1)} \oplus A(-k)^{2(\eta-d+k+2)} \xrightarrow{\phi_\eta} A^{\eta+1}.$$

By the Hilbert Syzygy Theorem, $\mathrm{Syz}_{*,\eta}$ is a free A-module. Moreover, taking into account the bi-Euler syzygy we have

$$\mathrm{Syz}_{*,\eta} \simeq M_\eta \oplus A(-k)^{\eta-d+k+1}.$$

In order to have control on the degree of a minimal syzygy, we seek the smallest value of η such that M_η is nonzero. If in addition M_η has rank one, then the minimal syzygy will be determinantal. If the maps ϕ_η have expected ranks for small values of η then will occur when

$$2(\eta-d+k+1)+2(\eta-d+k+2)-(\eta-d+k+1)=\eta+2,$$

or equivalently $2\eta = 3(d-k-1)$. So, if $d-k$ is odd, i.e. if $d=k+2\mu+1$ for some integer $\mu \in \mathbb{N}$, then a minimal syzygy is expected in degree $(e, \eta = 3\mu)$, $e \in \mathbb{N}$. We claim that the maps ϕ_η have expected ranks for all $\eta \leq 3\mu$:

$$M_\eta = 0 \text{ for all } \eta < 3\mu \text{ and } M_{3\mu} \simeq A(-e), \tag{4.2}$$

which implies that this syzygy of degree $(e, 3\mu)$ is a minimal syzygy of J. To prove the claim we focus on the particular structure of the maps ϕ_η.

In standard monomial bases, the matrix of ϕ_η is built from Sylvester blocks:

$$\left(\begin{array}{c|c|c|c} \vdots & \vdots & \vdots & \vdots \\ \mathrm{Sylv}_\eta(f_0) & \mathrm{Sylv}_\eta(f_1) & \mathrm{Sylv}_\eta(f_2) & \mathrm{Sylv}_\eta(f_3) \\ \vdots & \vdots & \vdots & \vdots \end{array}\right). \tag{4.3}$$

The polynomials f_i are seen as homogeneous polynomials in x_2, x_3. The blocks $\mathrm{Sylv}_\eta(f_0)$ and $\mathrm{Sylv}_\eta(f_1)$ are of size $(\eta+1)\times(\eta-2\mu)$ since f_0 and f_1 are of degree $2\mu+1$ in x_2, x_3, and the blocks $\mathrm{Sylv}_\eta(f_2)$ and $\mathrm{Sylv}_\eta(f_3)$ are of size $(\eta+1)\times(\eta-2\mu+1)$ since f_2 and f_3 are of degree 2μ in x_2, x_3. In view of these dimensions, it is clear that $M_\eta = 0$ for all $\eta < 2\mu$. Also, the matrix of $\phi_{2\mu}$ has two columns, namely the coefficients of f_2 and f_3 in degree $\eta = 2\mu$; $\phi_{2\mu}$ is clearly injective and hence $M_{2\mu} = 0$.

For $\eta \geq 2\mu+1$ the bi-Euler syzygy appears and yields $\eta - 2\mu$ independent syzygies in degree η that correspond to the following relations among the columns of the matrix of ϕ_η, for all $i = 0, \ldots, \eta-2\mu-1$:

$$\begin{aligned} 0 &= x_2^{\eta-2\mu-1-i}x_3^i\left((d-k)x_0f_0 + (d-k)x_1f_1 - kx_2f_2 - kx_3f_3\right) \\ &= (d-k)x_0(x_2^{\eta-2\mu-1-i}x_3^if_0) + (d-k)x_1(x_2^{\eta-2\mu-1-i}x_3^if_1) \\ &\quad - k(x_2^{\eta-2\mu-i}x_3^if_2) - k(x_2^{\eta-2\mu-1-i}x_3^{i+1}f_3) \end{aligned}$$

Removing these relations from the matrix of ϕ_η, we obtain a new map whose matrix is

$$\left(\begin{array}{c|c|c|c} \vdots & \vdots & \vdots & \vdots \\ \mathrm{Sylv}_\eta(f_0) & \mathrm{Sylv}_\eta(f_1) & \mathrm{Sylv}_\eta(f_2) & x_2^{\eta-2\mu}f_3 \\ \vdots & \vdots & \vdots & \vdots \end{array}\right). \tag{4.4}$$

It is obtained by removing the last $\eta - 2\mu$ columns to (4.3) (the block on the right is a single column). The matrix of (4.4) is of size $(\eta+1) \times (3\eta - 6\mu + 2)$ and its kernel is M_η. When $\eta < 3\mu$ it has less columns than rows and to prove the claim we must show that those columns are A-independent, or equivalently by McCoy's Lemma, that there exists a nonzero minor of size $3\eta - 6\mu + 2$. To achieve this, we examine in more detail the Sylvester blocks. We set (recall $d - k = 2\mu + 1$)

$$f_{(k,d-k)} = x_2^{2\mu+1} h_0(x_0, x_1) + x_2^{2\mu} x_3 h_1(x_0, x_1) + \ldots + x_3^{2\mu+1} h_{2\mu+1}(x_0, x_1)$$

where $h_i(x_0, x_1)$ are homogeneous polynomials of degree k in A. With this notation, the Sylvester blocks can be written more explicitly; for instance we have

$$\mathrm{Sylv}_\eta(f_0) = \begin{pmatrix} \partial_{x_0} h_0 & 0 & \cdots & 0 \\ \partial_{x_0} h_1 & \partial_{x_0} h_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \partial_{x_0} h_{2\mu+1} \end{pmatrix}$$

which is of size $(\eta + 1) \times (\eta - 2\mu)$ and whose rows are indexed from top to bottom by the monomial basis $\{x_2^\eta, x_2^{\eta-1} x_3, \ldots, x_3^\eta\}$. Thus the top maximal minor of this matrix is a lower triangular matrix of determinant $(\partial_{x_0} h_0)^{\eta-2\mu}$. Similarly, the bottom minor of $\mathrm{Sylv}_\eta(f_1)$ is a lower triangular matrix of determinant $(\partial_{x_1} h_{2\mu+1})^{\eta-2\mu}$. Removing the rows of these two minors from (4.4) we are left with the following submatrix of size

$$(4\mu - \eta + 1) \times (\eta - 2\mu + 2)$$

whose rows are indexed from top to bottom by the monomial basis $\{x_2^{2\mu} x_3^{\eta-2\mu}, \ldots, x_2^{\eta-2\mu} x_3^{2\mu}\}$:

$$\left(\begin{array}{ccc|c} (4\mu-\eta)h_{\eta-2\mu} & \ldots & (2\mu+1)h_0 & (\eta-2\mu+1)h_{\eta-2\mu+1} \\ (4\mu-\eta-1)h_{\eta-2\mu+1} & \ldots & (2\mu)h_1 & (\eta-2\mu)h_{\eta-2\mu+2} \\ \vdots & & \vdots & \ldots \\ h_{2\mu} & \ldots & (\eta-2\mu+1)h_{4\mu-\eta} & (2\mu+1)h_{2\mu+1} \end{array}\right).$$

From here, one can choose the $(\eta - 2\mu + 2)$-minor built with the top minor of maximal size $(\eta - 2\mu + 1)$ in the left block, whose determinant is denoted by Δ, and the bottom minor of size 1 in the right block, whose determinant is obviously equal to $(2\mu + 1)h_{2\mu+1}$. In the end, we have selected a $(3\eta - 6\mu + 2)$-minor in (4.4) whose determinant is equal to

$$(2\mu + 1)h_{2\mu+1} \, (\partial_{x_0} h_0)^{\eta-2\mu} \, (\partial_{x_1} h_{2\mu+1})^{\eta-2\mu} \, \Delta. \tag{4.5}$$

To see that Δ is nonzero, we observe that if

$$h_{\eta-2\mu+1} = \cdots = h_{2\mu} = 0$$

then Δ is the determinant of an upper triangular matrix and its diagonal entries are all equal to $h_{\eta-2\mu}$ up to a nonzero multiplicative constant. Therefore, by our genericity assumption the determinant (4.5) is nonzero.

The case $\eta = 3\mu$ follows in similar fashion. The difference in this case is that the matrix (4.4) has one more column than row; it is of size $(\mu + 1) \times (\mu + 2)$ and we have to show that it is of rank $\mu + 1$. We proceed as previously by selecting the same minors in the blocks $\mathrm{Sylv}_\eta(f_0)$ and $\mathrm{Sylv}_\eta(f_1)$. Then, in the remaining part of the matrix, the minor whose determinant is equal to Δ is of size $\mu+1$ and hence there is no need to select an element in the last column. The determinant Δ is nonzero by the same argument and the claimed property (4.2) is proved.

Finally, it remains to determine the degree e with respect to x_0, x_1 of the minimal syzygy we found, i.e. the integer e that appears in the exact sequence

$$0 \to A(-e) \oplus A(-k)^\mu \to A(-k+1)^{2\mu} \oplus A(-k)^{2\mu+2} \xrightarrow{\phi_{3\mu}} A^{3\mu+1}.$$

As explained above, the contribution of the bi-Euler syzygy in this sequence corresponds to a well identified block matrix. It follows that the minimal syzygy we are interested is a generator of the kernel of the matrix (4.4), which corresponds to the following restriction $\overline{\phi}_{3\mu}$ of the map $\phi_{3\mu}$:

$$0 \to A(-e) \to A(-k+1)^{2\mu} \oplus A(-k)^{\mu+1} \oplus A(-k) \xrightarrow{\overline{\phi}_{3\mu}} A^{3\mu+1}.$$

Therefore, we deduce that

$$A(-e) \simeq \wedge^{3\mu+2}\left(A(-k+1)^{2\mu} \oplus A(-k)^{\mu+1} \oplus A(-k)\right) \simeq A(-(3\mu+2)k+2\mu)$$

and hence that

$$e = (3\mu+2)k - 2\mu.$$

Since $d-k = 2\mu+1$, we deduce that $M(f_{(k,d-k)})$ has a minimal first syzygy of bidegree

$$\begin{array}{rcl} (e, 3\mu) & = & ((3\mu+2)k - 2\mu, 3\mu) \\ & = & (2k, 0) + \mu(3k-2, 3) \\ & = & (2k, 0) + \frac{d-k-1}{2}(3k-2, 3) \end{array}$$

as claimed. $\square$

Remark 4.2. From the proof of this theorem, we see that the minimal syzygy of $M(f_{k,d-k})$ we obtained is completely explicit: it is a determinantal syzygy that is built from the minors of maximal size of a submatrix of (4.4).

Example 4.3. Suppose $d = 19$ and $k = 6$. Then the minimal bigraded first syzygies are of bidegree (with exponent denoting the rank in that bidegree)

$$\begin{array}{ccccc} (6,13), & (8,68), & (8,69), & (9,43)^2, & (9,44)^2, \\ (10,26), & (10,37)^5, & (11,25)^4, & (12,24), & (16,24)^{12}, \\ (17,23)^{10}, & (18,23), & (19,22)^8, & (20,22), & (22,21)^5, \\ (23,21)^2, & (28,20)^5, & (41,19)^2, & (42,19), & (108,18) \end{array}$$

The minimal first syzygy of bidegree $(108, 18)$ has degree 126 in the $\mathbb{Z}$-grading, so

$$\text{regularity } M(f_{6,13}) \text{ is } \geq 124.$$

The betti table (see [15]) for the $\mathbb{Z}$-graded minimal free resolution for $M(f_{6,13})$ is

```
           0 1  2   3  4
    total: 1 4 66 107 44
        0: 1 .  .   .  .
         : . .  .   .  .
       17: . 4  1   .  .
         : . .  .   .  .
       34: . .  6   4  1
         : . .  .   .  .
       38: . . 22  33 10
       39: . .  9  18  9
       40: . .  1   2  1
       41: . .  5  10  5
       42: . .  2   4  2
       43: . .  .   .  .
       44: . .  .   .  .
       45: . .  5   6  .
       46: . .  5  10  5
       47: . .  .   .  .
       48: . .  .   .  .
       49: . .  .   .  1
       50: . .  2   4  2
       51: . .  2   4  2
         : . .  .   .  .
       58: . .  2   4  2
       59: . .  1   2  1
         : . .  .   .  .
       74: . .  1   2  1
       75: . .  1   2  1
         : . .  .   .  .
      124: . .  1   2  1
```

Note that the bound on d is achieved. To save space we have replaced large empty stretches in the table with an unlabelled row beginning with `:`

Closing Remarks and Questions: Theorem 4.1 illustrates that the additional algebraic structure of multigraded hypersurfaces provides leverage to obtain interesting results on the regularity of $M(f)$, and we are working on a sequel to this paper to investigate this question. We thank Carlos D'Andrea for pointing out references [24] and [28] to us, and an anonymous referee for helpful comments. Computations in `Macaulay 2` [16] and `Singular` [6] provided evidence for our work.

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