



Cyclotomic expansion of generalized Jones polynomials

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Abstract

In (Compos. Math. 152(7): 1333–1384, 2016), Berest and Samuelson proposed a conjecture that the Kauffman bracket skein module of any knot in S^3 carries a natural action of a rank 1 double-affine Hecke algebra SH_{q,t_1,t_2} depending on 3 parameters q, t_1, t_2 . As a consequence, for a knot K satisfying this conjecture, we defined a three-variable polynomial invariant $J_n^K(q, t_1, t_2)$ generalizing the classical coloured Jones polynomials $J_n^K(q)$. In this paper, we give explicit formulas and provide a quantum group interpretation for the polynomials $J_n^K(q, t_1, t_2)$. Our formulas generalize the so-called cyclotomic expansion of the classical Jones polynomials constructed by Habiro (Invent. Math. 171(1): 1–81, 2008): as in the classical case, they imply the integrality of $J_n^K(q, t_1, t_2)$ and, in fact, make sense for an arbitrary knot K independent of whether or not it satisfies the conjecture of Berest and Samuelson (Compos. Math. 152(7): 1333–1384, 2016). When one of the Hecke deformation parameters is set to be 1, we show that the coefficients of the (generalized) cyclotomic expansion of $J_n^K(q, t_1)$ are expressed in terms of Macdonald orthogonal polynomials.

Keywords Jones polynomial · Kauffman skein bracket module · Double affine Hecke algebra · Macdonald polynomial · Volume conjecture

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1 Introduction and statement of results

One of the most interesting ‘quantum’ invariants of an oriented 3-manifold M studied extensively in recent years is the Kauffman bracket skein module $K_q(M)$. This invariant—introduced by Przytycki [21] and Turaev [26] in the early 1990s—is defined topologically as the quotient vector space spanned by all (framed unoriented) links in M modulo the Kauffman skein relations depending on a parameter q . In [1], the first and third authors conjectured that the skein module $K_q(M_K)$ of the complement $M_K := S^3 \setminus K$ of a knot in S^3 carries a *natural* action of a rank one (spherical) double-affine Hecke algebra $\mathcal{SH}_{q,t_1,t_2}$, which depends—in addition to the ‘quantum’ parameter q —on two new ‘Hecke’ parameters t_1 and t_2 (see Conjecture 2.12). Our conjecture boils down to the assumption that $K_q(M_K)$ possesses a certain symmetry of algebraic nature that allows one to deform the topological action of the skein algebra $K_q(\partial M_K)$ of the boundary 2-torus into the action of $\mathcal{SH}_{q,t_1,t_2}$. We verified our conjecture in a number of nontrivial cases, including torus knots and some (nonalgebraic) 2-bridge knots (see [1,2]). An important consequence of this conjecture is the existence of polynomial knot invariants $J_n^K(q, t_1, t_2) \in \mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$ depending on the three variables q, t_1, t_2 , which specialize (when $t_1 = t_2 = 1$) to the classical ($\mathfrak{sl}_2$, coloured) Jones polynomials $J_n^K(q)$. We call $J_n^K(q, t_1, t_2)$ the *generalized Jones polynomials* of K .

The goal of this paper is to give an explicit formula for the polynomials $J_n^K(q, t_1, t_2)$ generalizing the so-called *cyclotomic expansion* of the coloured Jones polynomials $J_n^K(q)$ discovered by K. Habiro. We recall that Habiro proved in [8] the following remarkable theorem.

Theorem 1.1 ([8]) *For any knot K in S^3 , the n -th coloured Jones polynomial of K can be written in the form*

$$J_n^K(q) = \sum_{i=1}^n c_{n,i-1}(q) H_{i-1}^K(q) \quad (1.1)$$

where $H_{i-1}^K(q) \in \mathbb{Z}[q^{\pm 1}]$ are integral Laurent polynomials depending on the knot K (but not on the ‘colour’ n), and the coefficients $c_{n,i-1}(q)$ are independent of K and given by the elementary formulas

$$c_{n,i-1}(q) := \frac{1}{q^2 - q^{-2}} \prod_{p=n-i+1}^{n+i-1} (q^{2p} - q^{-2p}), \quad 1 \leq i \leq n \quad (1.2)$$

Following [6], we refer to $H_{i-1}^K(q)$, $i \geq 1$, as the *Habiro polynomials* of K . It is not hard to show that the $H_{i-1}^K(q)$ ’s always exist as rational functions in $\mathbb{Q}(q)$; the nontrivial part of Theorem 1.1 is that these rational functions are actually Laurent polynomials in $\mathbb{Z}[q^{\pm 1}]$. The coefficients $c_{n,i-1}(q)$ are called the *cyclotomic coefficients*: besides explicit formulas (1.2) it is often convenient to define them in terms of

generating functions which can be written in the following form:

$$\sum_{n=0}^{\infty} c_{n,i-1}(q) \lambda^n = \frac{\det B_{2i}(q; \lambda)}{\prod_{N=1}^{2i-1} \gamma_N}, \quad i \geq 1, \quad (1.3)$$

where $B_{2i}(q; \lambda)$ is the $(2i \times 2i)$ -matrix

$$\begin{pmatrix} 0 & \alpha_1^{(i)} & 0 & \alpha_2^{(i)} & \cdots & 0 & \alpha_i^{(i)} \\ \beta_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 \\ \beta_2 & 0 & \gamma_2 & 0 & \cdots & 0 & 0 \\ \beta_3 & 0 & 0 & \gamma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{2i-1} & 0 & 0 & 0 & \cdots & 0 & \gamma_{2i-1} \end{pmatrix} \quad (1.4)$$

with entries

$$\alpha_k^{(i)} := (-1)^{i-k} \begin{bmatrix} 2i-1 \\ i-k \end{bmatrix}_{q^2}, \quad \beta_N := [N]_{q^2}, \quad \gamma_N := q^{2N} + q^{-2N} - \lambda - \lambda^{-1}. \quad (1.5)$$

(See Sect. 2 for notation.)

Next, we recall the construction of the generalized Jones polynomials $J_n^K(q, t_1, t_2)$ from [1]. The starting point for this construction is a well-known topological formula (due to Kirby and Melvin [12]) that expresses the classical coloured Jones polynomials $J_n^K(q)$ in terms of the Kauffman bracket skein module $K_q(M_K)$. To give this formula we note that $K_q(M_K)$ carries a natural action $K_q(T^2) \times K_q(M_K) \xrightarrow{\cdot} K_q(M_K)$ of the skein algebra $K_q(T^2)$ of the boundary torus $T^2 = \partial M_K$, and there is a natural $\mathbb{C}[q^{\pm 1}]$ -linear pairing

$$\langle -, - \rangle : K_q(S^1 \times D^2) \otimes_{K_q(T^2)} K_q(M_K) \rightarrow K_q(S^3) \cong \mathbb{C}[q^{\pm 1}] \quad (1.6)$$

induced topologically by gluing the solid torus $S^1 \times D^2$ to the knot complement M_K along the common boundary $T^2 = S^1 \times S^1$ to obtain the S^3 (see Sect. 2.1.2 for more details). The Kirby–Melvin formula reads (cf. Theorem 2.7):

$$J_n^K(q) = (-1)^{n-1} \langle \emptyset, S_{n-1}(L) \cdot \emptyset \rangle, \quad (1.7)$$

where $S_{n-1}(L) \in K_q(T^2)$ is the $(n-1)$ -th Chebyshev polynomial evaluated at the (0-framed) longitude L of T^2 viewed as an operator in $K_q(T^2)$ acting on a distinguished element (the ‘empty link’) $\emptyset$ in $K_q(M_K)$. Now, the main conjecture of [1] (see Conjecture 2.12 in Sect. 2.3) asserts that the action of $K_q(T^2)$ on $K_q(M_K)$ admits a canonical deformation to an action of the double-affine Hecke algebra $\mathcal{SH}_{q,t_1,t_2}$, and pairing (1.6) deforms to a $\mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$ -linear pairing balanced over $\mathcal{SH}_{q,t_1,t_2}$

(cf. Lemma 3.1):

$$\langle -, - \rangle : K_q(S^1 \times D^2) \otimes_{\mathcal{SH}_{q,t_1,t_2}} K_q(M_K) \rightarrow \mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}] \quad (1.8)$$

Moreover, the longitude operator $L \in K_q(T^2)$ appearing in (1.7) has a natural analogue (deformation) in $\mathcal{SH}_{q,t_1,t_2}$ —the so-called *Askey-Wilson operator* L_{t_1,t_2} (see [19])—that specializes to L when $t_1 = t_2 = 1$. Having all ingredients in hand, we can deform Kirby–Melvin formula (1.7) and define polynomials (cf. Definition 2.13)

$$J_n^K(q, t_1, t_2) := (-1)^{n-1} \langle \emptyset, S_{n-1}(L_{t_1,t_2}) \cdot_t \emptyset \rangle, \quad (1.9)$$

where $\cdot_t$ stands for the deformed action of $\mathcal{SH}_{q,t_1,t_2}$ on $K_q(M_K)$. By construction, $J_n^K(q, t_1, t_2)$ specializes to $J_n^K(q)$ when $t_1 = t_2 = 1$; however, formula (1.9) defining $J_n^K(q, t_1, t_2)$ makes sense only if Conjecture 2.12 holds true.

The main result of the present paper can now be stated as follows.

Theorem 1.2 *Assume Conjecture 2.12 holds for a knot $K \subset S^3$. Then generalized Jones polynomials (1.9) can be written in the form*

$$J_n^K(q, t_1, t_2) = \sum_{i=1}^n \tilde{c}_{n,i-1}(q, t_1, t_2) H_{i-1}^K(q) \quad (1.10)$$

where $H_{i-1}^K(q)$ are the Habiro polynomials of K . The coefficients $\tilde{c}_{n,i-1}(q, t_1, t_2)$ are independent of K and determined by the following generating functions:

$$\sum_{n=0}^{\infty} \tilde{c}_{n,i-1}(q, t_1, t_2) \lambda^n = \frac{\det \tilde{B}_{2i}(q, t_1, t_2; \lambda)}{\prod_{N=1}^{2i-1} \tilde{\gamma}_N}, \quad i \geq 1 \quad (1.11)$$

where $\tilde{B}_{2i}(q, t_1, t_2; \lambda)$ is the $(2i \times 2i)$ -matrix

$$\begin{pmatrix} 0 & \alpha_1^{(i)} & 0 & \alpha_2^{(i)} & \cdots & 0 & \alpha_i^{(i)} \\ \beta_1 & \tilde{\gamma}_1 & 0 & 0 & \cdots & 0 & 0 \\ \beta_2 & b_{21} & \tilde{\gamma}_2 & 0 & \cdots & 0 & 0 \\ \beta_3 & b_{31} & b_{32} & \tilde{\gamma}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{2i-1} & b_{2i-1,1} & b_{2i-1,2} & b_{2i-1,3} & \cdots & b_{2i-1,2i-2} & \tilde{\gamma}_{2i-1} \end{pmatrix} \quad (1.12)$$

with entries $\alpha_k^{(i)}$ and β_N the same as in (1.5) and $b_{p,N}$ and $\tilde{\gamma}_N$ given by

$$\begin{aligned} b_{p,N} &:= (-1)^k (\{p+N\}_q - \{p-N\}_q) (t_k - t_k^{-1}), \quad k \equiv p - N + 1 \pmod{2} \\ \tilde{\gamma}_N &:= q^{2N} t_1^{-1} + q^{-2N} t_1 - \lambda - \lambda^{-1}, \quad 1 \leq p, N \leq 2i - 1 \end{aligned} \quad (1.13)$$

(In (1.13) we use the notation $\{n\}_q := q^n - q^{-n}$, cf. Sect. 2.)

Note that, for $t_1 = t_2 = 1$, matrix (1.12) reduces to (1.4), so comparing generating functions (1.3) and (1.10) shows $\tilde{c}_{n,i-1}(q, 1, 1) = c_{n,i-1}(q)$ as required.

One important consequence of formula (1.11) is that the generalized cyclotomic coefficients are integral, i.e. $\tilde{c}_{n,i-1} \in \mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$ (cf. Remark 3.12). Theorem 1.2 thus says that each $J_n^K(q, t_1, t_2)$ is a linear combination of the Habiro polynomials with integral coefficients¹. Together with Habiro's Theorem 1.1, this implies

Corollary 1.3 *The generalized Jones polynomials are integral: for all $n \geq 0$*

$$J_n^K(q, t_1, t_2) \in \mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$$

In the special case when $t_2 = 1$, we can compute the (generalized) cyclotomic coefficients $\tilde{c}_{n,k-1}(q, t_1, t_2)$ in a simple closed form using the classical Macdonald orthogonal polynomials.

Theorem 1.4 *For $(t_1, t_2) = (t, 1)$, the (generalized) cyclotomic coefficients in (1.10) are given by*

$$\tilde{c}_{n,i-1}(q, t) = \frac{p_{n-i}(q^{2i}t^{-1}; q^4|q^4)}{p_{n-i}(q^{2i}; q^4|q^4)} \left(\prod_{k=2}^i \frac{q^{2k-1}t^{-1} - q^{-2k+1}t}{q^{2k-1} - q^{-2k+1}} \right) c_{n,i-1}(q) \quad (1.14)$$

where $p_n(z; \beta|q)$ are the Macdonald symmetric polynomials of type A_1 and $c_{n,i-1}(q)$ are classical cyclotomic coefficients (1.2).

We remark that the Macdonald polynomials $p_n(z; \beta|q)$ can be expanded in terms of q -binomial coefficients, so formulas (1.14) are entirely explicit (see Remark 3.12). The Habiro polynomials are known for certain families of knots (see, e.g. [8] and [16]). In those cases, Theorem 1.4 gives a closed-form expression for generalized Jones polynomials.

Example 1.5 1. For the unknot, $H_0^U = 1$ and $H_n^U = 0$ for $n \geq 1$. In this case,

$$J_n^U(q, t) = \tilde{c}_{n,0}(q, t) = \frac{p_{n-1}(q^2t^{-1}; q^4|q^4)}{p_{n-1}(q^2; q^4|q^4)} \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}} = \frac{(q^2t^{-1})^n - (q^2t^{-1})^{-n}}{q^2t^{-1} - q^{-2}t}$$

where we have used a well-known evaluation formula for Macdonald polynomials $p_{n-1}(z; q^4|q^4) = (z^n - z^{-n})/(q^2 - q^{-2})$ (see [3, p. 202]). This recovers the result of [1, Thm. 6.10].

2. For the figure eight knot, $H_n^K = 1$ for all $n \geq 0$. Hence, by Theorem 1.4,

$$J_n^K(q, t) = \sum_{i=1}^n \frac{p_{n-i}(q^{2i}t^{-1}; q^4|q^4)}{p_{n-i}(q^{2i}; q^4|q^4)} \left(\prod_{k=2}^i \frac{q^{2k-1}t^{-1} - q^{-2k+1}t}{q^{2k-1} - q^{-2k+1}} \right) c_{n,i-1}$$

¹ By contrast, the polynomials $J_n^K(q, t_1, t_2)$ cannot be written, in general, as linear combinations of the classical coloured Jones polynomials $J_n^K(q)$ with coefficients in $\mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$ (cf. Remark 3.4 in Sect. 3.2).

Note that when $t = 1$, this formula specializes to the well-known formula for the Jones polynomials of the figure 8 knot,

$$J_n(q) = \sum_{i=1}^n c_{n,i-1} = \frac{1}{q^2 - q^{-2}} \sum_{i=1}^n \prod_{p=n-i+1}^{n+i-1} (q^{2p} - q^{-2p})$$

The last result that we want to state in Introduction provides an interpretation of our generalized Jones polynomials $J_n^K(q, t_1, t_2)$ in terms of quantum groups: more precisely, we express $J_n^K(q, t_1, t_2)$ via the universal $\mathfrak{sl}_2$ invariant J^K of the knot K introduced by R. Lawrence [14, 15] (see also [7, 8]). Recall that J^K takes values in the centre $\mathcal{Z}(\mathcal{U}_h)$ of the (h -adically) complete quantized enveloping algebra $\mathcal{U}_h := \mathcal{U}_h(\mathfrak{sl}_2)$ defined over the formal power series ring $\mathbb{Q}_h = \mathbb{Q}[[h]]$ (see Sect. 3.4). We set $q = e^{h/4}$ and let $\mathcal{R}_{q,t_1,t_2} := K_0(\text{Rep } \mathcal{U}_h) \otimes_{\mathbb{Z}} \mathbb{Q}(q)[t_1^{\pm 1}, t_2^{\pm 1}]$ denote the representation ring of the category $\text{Rep}(\mathcal{U}_h)$ of finite-dimensional $\mathcal{U}_h$ -modules over the commutative ring $\mathbb{Q}(q)[t_1^{\pm 1}, t_2^{\pm 1}]$. The ring $\mathcal{R}_{q,t_1,t_2}$ is a free module over $\mathbb{Q}(q)[t_1^{\pm 1}, t_2^{\pm 1}]$ generated by the classes $\{[V_n]\}_{n \geq 1}$ of irreducible representations of $\mathcal{U}_h$; it comes together with a natural bilinear map

$$\text{tr}_q(-, -) : \mathcal{U}_h \times \mathcal{R}_{q,t_1,t_2} \rightarrow \mathbb{Q}_h[t_1^{\pm 1}, t_2^{\pm 1}] \quad (1.15)$$

defined by quantum traces of elements of $\mathcal{U}_h$ acting on finite-dimensional modules (see Sect. 3.4). If $z \in \mathcal{Z}(\mathcal{U}_h)$ is a central element of $\mathcal{U}_h$, we write

$$\hat{z} := \text{tr}_q(z, -) : \mathcal{R}_{q,t_1,t_2} \rightarrow \mathbb{Q}_h[t_1^{\pm 1}, t_2^{\pm 1}] \quad (1.16)$$

and note that, by the Schur lemma,

$$\hat{z}([V_n]) = [n]_{q^2} z_n \quad (1.17)$$

where z_n is the scalar in $\mathbb{Q}_h$ by which z acts on the irreducible representation V_n .

Now, to state our theorem we define a sequence of functions $a_{n,p} \in \mathbb{Q}(q)[t_1^{\pm 1}, t_2^{\pm 1}]$ (indexed by the integers $n \geq 1$ and $p \geq 0$) inductively, using the recurrence relation:

$$a_{n+1,p} = A_p a_{n,p-1} + (A_p - A_{p+1}) a_{n,p} + A_{-p} a_{n,p+1} - a_{n-1,p} \quad (1.18)$$

with ‘boundary’ conditions

$$a_{1,1} = 1, \quad a_{n,0} = 0, \quad a_{n,p} = 0 \quad (n \geq p), \quad (1.19)$$

where

$$A_p := \frac{q^{2p-1} t_1^{-1} - q^{1-2p} t_1 + t_2 - t_2^{-1}}{q^{2p-1} - q^{1-2p}}$$

Theorem 1.6 For $n \geq 1$, let $[\tilde{V}_n]$ denote the class in $\mathcal{R}_{q,t_1,t_2}$ given by the formula

$$[\tilde{V}_n] := \sum_{p=1}^n (-1)^{n+p} a_{n,p} [V_p] \quad (1.20)$$

where the coefficients $a_{n,p} = a_{n,p}(q, t_1, t_2)$ are defined by (1.18) and (1.19). Then

$$J_n^K(q, t_1, t_2) = \hat{J}^K[\tilde{V}_n], \quad (1.21)$$

where $\hat{J}^K$ is quantum trace map (1.16) defined by the universal invariant J^K .

Note that when $t_1 = t_2 = 1$, we have $A_p = 1$ for all p , and it follows easily from (1.18) and (1.19) that $a_{n,p}$ is equal to 1 for $p = n$ and is 0 otherwise. Formula (1.21) thus reduces to $J_n^K(q) = \hat{J}^K[V_n]$, which is a well-known formula for the coloured Jones polynomials. For arbitrary $t_1, t_2 \in \mathbb{C}^*$, one can easily compute from (1.18) the first ‘top’ terms of the sequence $\{a_{n,p}\}$:

$$\begin{aligned} a_{n,n} &= A_2 A_3 \cdots A_n, \quad (n \geq 2) \\ a_{n,n-1} &= A_2 A_3 \cdots A_{n-1} (A_1 - A_n) \end{aligned}$$

By (1.20), this gives

$$[\tilde{V}_1] = [V_1], \quad [\tilde{V}_2] = A_2[V_2] + (A_2 - A_1)[V_1] \quad (1.22)$$

In general, for $n \geq 3$, the recursive formulas for $a_{n,p}$ are more complicated: in fact, we could not find a nice closed-form expression for these coefficients (which seems like an interesting problem). The origin of recurrence Eqs. (1.18) and (1.19) and their relation to the double-affine Hecke algebra $\mathcal{H}_{q,t_1,t_2}$ is explained in the proof of Lemma 3.3.

We conclude this Introduction with some questions and motivation for studying our generalized Jones polynomials $J_n^K(q, t_1, t_2)$. The main result of this paper (Theorem 1.2) shows that $J_n^K(q, t_1, t_2)$ can be expressed in terms of Habiro polynomials $H_{i-1}^K(q)$, and hence, strictly speaking, are not new knot invariants. Although this is certainly disappointing, there are still good reasons for studying these polynomials.

First of all, notice that the right-hand side of formula (1.10) of Theorem 1.2 makes sense for an arbitrary knot $K \subset S^3$, even though the polynomials $J_n^K(q, t_1, t_2)$ are defined only under the assumption that K satisfies Conjecture 2.12. Hence, the very existence of a formula like (1.10) for $J_n^K(q, t_1, t_2)$ can be viewed as a further evidence for the main conjecture of [1].

Next, we remark that the relation of $J_n^K(q, t_1, t_2)$ to the classical Jones polynomials $J_n^K(q)$ is somewhat more subtle than that to the Habiro polynomials. While the values of $J_n^K(q)$ for all $q \in \mathbb{C}^*$ can be directly recovered from $J_n^K(q, t_1, t_2)$ (by simply specializing $t_1 = t_2 = 1$), the converse is not true²: for generic t_1 and t_2 , the values of $J_n^K(q, t_1, t_2)$ at roots of unity, $q = e^{\pi i N/2n}$, are *not* determined by the values of $J_n^K(q)$ at these q 's. This is because universal formulas (3.3), expressing $J_n^K(q, t_1, t_2)$ in terms of $J_n^K(q)$, involve rational functions as coefficients which have poles exactly at $q \in \{e^{\pi i N/2n} : N \geq 0\}$ (cf. Remark 3.4). The nontriviality of this relationship between $J_n^K(q, t_1, t_2)$ and $J_n^K(q)$ was already demonstrated in [1], where we used the properties of $J_n^K(q, t_1, t_2)$ to prove some conjectures about the classical Jones polynomials $J_n^K(q)$ (see, e.g. *loc.cit.*, Theorem 2 and Theorem 3).

Finally, we mention that the values of the coloured Jones polynomials $J_n^K(q)$ at roots of unity—called the Kashaev invariants—play an important role in quantum topology. In particular, they appear in the celebrated *Volume Conjecture* that predicts a deep connection between quantum and geometric invariants of knots (see [9], [17] and also [6], [18] for more recent results). In its original form, this conjecture reads:

Conjecture ([9,17]) *For any hyperbolic knot K in S^3 ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{J_n^K(e^{\pi i/2n})}{J_n^U(e^{\pi i/2n})} \right| = \frac{1}{2\pi} \text{Vol}(M_K) \quad (1.23)$$

where $\text{Vol}(M_K)$ is the (hyperbolic) volume of the knot complement $M_K = S^3 \setminus K$.

The existence and the integrality property of the generalized Jones polynomials $J_n^K(q, t_1, t_2)$ established in this paper naturally lead us to the following

Question *Does the Volume Conjecture limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{J_n^K(e^{\pi i/2n}, t_1, t_2)}{J_n^U(e^{\pi i/2n}, t_1, t_2)} \right| \quad (1.24)$$

exist for some $(t_1, t_2) \in (\mathbb{C}^)^2$ other than $(t_1, t_2) = (1, 1)$? If so, what is its geometric meaning?*

The fact that the values of $J_n^K(q, t_1, t_2)$ at roots of unity are not determined by those of $J_n^K(q)$ seems to indicate that the above question is interesting and far from being trivial. The closed formulas for $J_n^K(q, t_1, t_2)$ given in this paper open the way for studying limit (1.24) analytically, at least in case of simple knots when the Habiro polynomials are explicitly known (see, e.g. Example 1.5). We plan to address this question in our future work.

² For this reason and also to avoid confusion with terminology of [1] we will keep referring to $J_n^K(q, t_1, t_2)$ as ‘generalized’ Jones polynomials, even though they are not generalizing the classical Jones polynomials in the same way as other multivariable polynomial knot invariants (arising, for example, from Khovanov homology).

The paper is organized as follows. In Sect. 2, we introduce notation and review basic results of [1], including the main conjecture of [1] (see Sect. 2.3) and the definition of the generalized Jones polynomials $J_n^K(q, t_1, t_2)$ (see Sect. 2.4). Section 3 contains the proofs of the 3 theorems stated in Introduction; it also fills in some details and provides definitions needed for the precise statements of these theorems.

2 Preliminaries

In this section we provide some background material needed for the present paper. This includes basic properties of Kauffman bracket skein modules and double-affine Hecke algebras, as well as a summary of main results of [1]. Throughout we use the following standard notation:

$$\{n\}_q := q^n - q^{-n}, \quad [n]_q := \frac{\{n\}_q}{\{1\}_q}, \quad \left[\begin{matrix} n \\ m \end{matrix} \right]_q := \prod_{k=1}^m \frac{[n-k+1]_q}{[m-k+1]_q}, \quad (n, m \in \mathbb{N})$$

2.1 Kauffman bracket skein modules

A *framed link* in an oriented 3-manifold M is an embedding of a disjoint union of annuli $S^1 \times [0, 1]$ into M , considered up to ambient isotopy. In what follows, the letter q will denote either a nonzero complex number or a formal parameter generating the field

$$\mathbb{C}_q := \mathbb{C}(q)$$

(we will specify which when it matters).

Let $\mathcal{L}(M)$ be the vector space over $\mathbb{C}_q$ spanned by the set of ambient isotopy classes of framed unoriented links in M (including the empty link $\emptyset$). Let $\mathcal{L}'(M)$ denote the smallest subspace of $\mathcal{L}(M)$ containing the skein expressions in Fig. 1

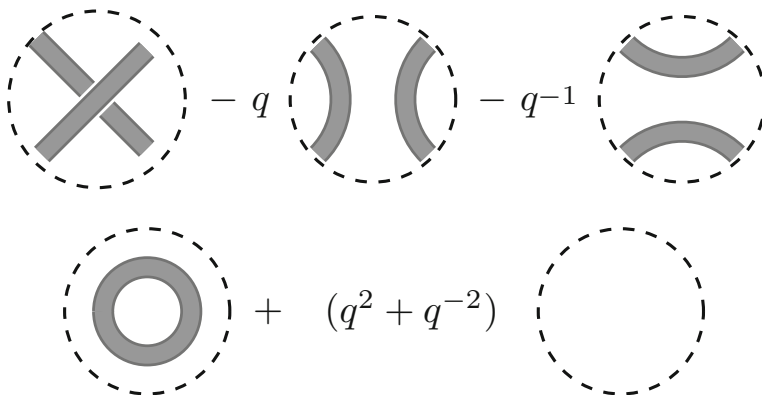


Fig. 1 Framed skein relations

where the diagrams represent embeddings of annuli which are identical outside of the oriented 3-ball represented by the dotted circle.

Definition 2.1 ([21]) The *Kauffman bracket skein module* of an oriented 3-manifold M is the quotient vector space $K_q(M) := \mathcal{L}(M)/\mathcal{L}'(M)$. It contains a canonical element $\emptyset \in K_q(M)$ corresponding to the empty link.

Remark 2.2 If F is a surface, we will often write $K_q(F)$ for the skein module $K_q(F \times [0, 1])$ of the cylinder over F .

In general, $K_q(M)$ carries only a linear structure. However, the assignment $M \rightarrow K_q(M)$ is functorial with respect to oriented embeddings, which implies the following facts:

- (1) If $M = M_1 \sqcup M_2$, then $K_q(M) \cong K_q(M_1) \otimes K_q(M_2)$.
- (2) For any surface F , the embedding $[0, \frac{1}{3}] \sqcup [\frac{2}{3}, 1] \rightarrow [0, 1]$ induces a map

$$\mu : K_q(F) \otimes K_q(F) \rightarrow K_q(F)$$

which make $K_q(F)$ an associative unital algebra (with unit $\emptyset$).

- (3) If $\partial M \cong F$ and if $M = F \times [0, 1] \sqcup N$ represents a decomposition of M into a tubular neighbourhood of the boundary and a retract $N \cong M$, the map

$$m : K_q(F) \otimes K_q(M) \rightarrow K_q(M)$$

gives M the structure of a left module over $K_q(F)$.

Example 2.3 An original motivation for defining $K_q(M)$ was a theorem of Kauffman [10] asserting that the natural map

$$\mathbb{C}_q \xrightarrow{\sim} K_q(S^3), \quad 1 \mapsto \emptyset$$

is an isomorphism of vector spaces, and that the inverse image of a link L in S^3 under this isomorphism is the Jones polynomial of L . Clearly $K_q(S^3)$ is of dimension at most 1 over $\mathbb{C}_q$ thanks to the skein relations; the key point of Kauffman's theorem is that this map is injective.

Example 2.4 Let $M = S^1 \times D^2$ be the solid torus, or complement of the unknot. If x is the nontrivial loop, then the map $\mathbb{C}_q[x] \rightarrow K_q(S^1 \times D^2)$ sending x^n to n parallel copies of x is surjective (because all crossings and trivial loops can be removed using the skein relations). Less obvious is the fact that this map is injective and thus an isomorphism (see, e.g. [24]).

2.1.1 The Kauffman bracket skein module of the torus

Recall that the *quantum Weyl algebra* (or *quantum torus*) is defined by

$$A_q := \frac{\mathbb{C}[q^{\pm 1}]\langle X^{\pm 1}, Y^{\pm 1} \rangle}{(XY - q^2 YX)}$$

Note that this algebra carries a $\mathbb{Z}_2$ action defined by the automorphism $(X, Y) \mapsto (X^{-1}, Y^{-1})$.

We now recall a theorem of Frohman and Gelca [5] that gives a connection between $K_q(T^2)$ and the invariant subalgebra $A_q^{\mathbb{Z}_2}$. Let $T_n \in \mathbb{C}[x]$ be the Chebyshev polynomials defined by

$$T_0 = 2, \quad T_1 = x, \quad T_{n+1} = xT_n - T_{n-1}.$$

If m, l are relatively prime, write (m, l) for the m, l curve on the torus (the simple curve wrapping around the torus l times in the longitudinal direction and m times in the meridian's direction). It is clear that the links $(m, l)^n$ span $K_q(T^2)$, and it follows from [24] that this set is actually a basis. However, a more convenient basis is given by the elements $(m, l)_T := T_d((\frac{m}{d}, \frac{l}{d}))$ (where $d = \gcd(m, l)$). Define $e_{r,s} = q^{-rs} X^r Y^s \in A_q$, which form a linear basis in A_q .

Theorem 2.5 ([5]) *The map $K_q(T^2) \rightarrow A_q^{\mathbb{Z}_2}$ given by $(m, l)_T \mapsto e_{m,l} + e_{-m,-l}$ is an isomorphism of algebras.*

Remark 2.6 If K is an oriented knot, then the meridian/longitude pair (m, l) gives a canonical identification of $S^1 \times S^1$ with the boundary of $S^3 \setminus K$. If the orientation of K is reversed, this identification is twisted by the ‘hyper-elliptic involution’ of $S^1 \times S^1$ (which negates both components). However, this induces the identity isomorphism on $K_q(T^2 \times [0, 1])$, so the $A_q^{\mathbb{Z}_2}$ -module structure on $K_q(S^3 \setminus K)$ is canonical and does not depend on the choice of orientation of K .

2.1.2 Topological pairings and coloured Jones polynomials

Let M be any closed 3-manifold. If (M_1, M_2) represents a Heegaard splitting of M , that is, $M_1, M_2 \subset M$ are oriented submanifolds with boundary satisfying

$$M_1 \cup M_2 = M \quad M_1 \cap M_2 = \partial M_1 = \partial M_2 = F,$$

the inclusion $\iota : M_1 \sqcup M_2 \rightarrow M$ determines by functoriality the map

$$K_q(\iota) : K_q(M_1) \otimes K_q(M_2) \rightarrow K_q(M), \quad [L_1] \otimes [L_2] \mapsto [L_1 \cup L_2] \quad (2.1)$$

where $[L_1]$ and $[L_2]$ are isotopy classes of links in M_1 and M_2 , respectively. Now put an orientation on F as the boundary of M_2 , and let $N_F \subset M$ be a tubular neighbourhood of F with respect to this orientation. Let $\iota_i : N_F \rightarrow M_i, i \in \{1, 2\}$ be the natural inclusions. As usual, ι_2 gives M_2 the structure of a left module over $K_q(N_F)$. However, as the orientation of F is reversed from that of ∂M_1 , the map ι_1 gives $K_q(M_1)$ the structure of a *right* module over $K_q(N_F)$. As a skein in N_F can be pushed into either M_1 or M_2 , this tells us that (2.1) actually factors as a map

$$K_q(\iota) : K_q(M_1) \otimes_{K_q(F)} K_q(M_2) \rightarrow K_q(M).$$

If $M = S^3$, then M_1 is the tubular neighbourhood of a knot K and $M_2 = S^3 \setminus K$, and we refer to this map as the *topological pairing*

$$\langle -, - \rangle : K_q(S^1 \times D^2) \otimes_{K_q(T^2)} K_q(S^3 \setminus K) \rightarrow K_q(S^3) \cong \mathbb{C}_q \quad (2.2)$$

The coloured Jones polynomials $J_n^K(q) \in \mathbb{C}[q^{\pm 1}]$ of a knot $K \subset S^3$ were originally defined by Reshetikhin and Turaev in [22] using the representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$. Here we recall a theorem of Kirby and Melvin that shows how $J_n^K(q)$ can be computed in terms of the topological pairing.

If $D^2 \times S^1$ is a tubular neighbourhood of the knot K , then we identify $K_q(D^2 \times S^1) \cong \mathbb{C}_q[u]$, where $u \in K_q(D^2 \times S^1)$ is the image of the (0-framed) longitude $l \in K_q(\partial(S^3 \setminus K))$. Let $S_n \in \mathbb{C}_q[u]$ be the Chebyshev polynomials of the second kind, which satisfy the initial conditions $S_0 = 1$ and $S_1 = u$, and the recursion relation $S_{n+1} = uS_n - S_{n-1}$.

Theorem 2.7 ([12]) *If $\emptyset \in K_q(S^3 \setminus K)$ is the empty link, we have*

$$J_n^K(q) = (-1)^{n-1} \langle S_{n-1}(u), \emptyset \rangle$$

As the zero-framed longitude l considered as an element in the skein module of the boundary torus $K_q(T^2)$ is identified with $Y + Y^{-1}$ under Theorem 2.5, we have

$$(-1)^{n-1} J_n^K(q) = \langle \emptyset \cdot S_{n-1}(Y + Y^{-1}), \emptyset \rangle = \langle \emptyset, S_{n-1}(Y + Y^{-1}) \cdot \emptyset \rangle \quad (2.3)$$

Remark 2.8 The sign correction is chosen so that for the unknot we have $J_n(q) = [n]_{q^2} = (q^{2n} - q^{-2n})/(q^2 - q^{-2})$. Also, with this normalization, $J_0^K(q) = 0$ and $J_1^K(q) = 1$ for every knot K . This agrees with the convention of labelling irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ by their dimension.

2.2 The double-affine Hecke algebra

In this section we define a 5-parameter family of algebras $\mathcal{H}_{q,\underline{t}}$ —called the double-affine Hecke algebra of type $C^\vee C_1$ —originally introduced in [23] (see also [19] and [1] for our present notation). This family represents the universal deformation of the algebra $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}] \rtimes \mathbb{Z}_2$, the crossed product of the Laurent polynomial ring $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$, with $\mathbb{Z}_2$ acting by the natural involution (see [20]). The algebra $\mathcal{H}_{q,\underline{t}}$ for $q \in \mathbb{C}^*$ and $\underline{t} = (t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ is generated by the elements T_1, T_2, T_3 , and T_4 subject to the five relations

$$\begin{aligned} (T_1 - t_1)(T_1 + t_1^{-1}) &= 0 \\ (T_2 - t_2)(T_2 + t_2^{-1}) &= 0 \\ (T_3 - t_3)(T_3 + t_3^{-1}) &= 0 \\ (T_4 - t_4)(T_4 + t_4^{-1}) &= 0 \\ T_4 T_3 T_1 T_2 &= q \end{aligned} \quad (2.4)$$

Recall that, by definition, the crossed product algebra $A_q \rtimes \mathbb{Z}_2$ is generated by X, Y, s , satisfying

$$sX = X^{-1}s, \quad sY = Y^{-1}s, \quad s^2 = 1, \quad XY = q^2YX.$$

Let $D_q := \mathbb{C}_q(X)[Y^{\pm 1}]/(XY - q^2YX)$ denote the *localized* quantum Weyl algebra obtained from A_q by inverting all (nonzero) polynomials in X . Note that the action of $\mathbb{Z}_2$ extends to D_q so that we can form the crossed product $D_q \rtimes \mathbb{Z}_2$. Now, consider the following elements in $D_q \rtimes \mathbb{Z}_2$:

$$\begin{aligned} \hat{T}_1 &:= t_1sY + \frac{q\bar{t}_1X + \bar{t}_2}{qX - q^{-1}X^{-1}}(1 - sY) \\ \hat{T}_3 &:= t_3s + \frac{\bar{t}_3 + \bar{t}_4X}{1 - X^2}(1 - s) \end{aligned} \quad (2.5)$$

These elements are called the Dunkl-Cherednik and Demazure-Lusztig operators, respectively. The next proposition establishes the relation between the algebras $\mathcal{H}_{q,\underline{t}}$ and $A_q \rtimes \mathbb{Z}_2$.

Proposition 2.9 ([23], see also [19, Thm. 2.22]) *The assignment*

$$T_1 \mapsto \hat{T}_1, \quad T_3 \mapsto \hat{T}_3, \quad T_2 \mapsto q\hat{T}_1^{-1}X, \quad T_4 \mapsto X^{-1}\hat{T}_3^{-1} \quad (2.6)$$

extends to an injective algebra homomorphism $\mathcal{H}_{q,\underline{t}} \hookrightarrow D_q \rtimes \mathbb{Z}_2$.

Note that $A_q \rtimes \mathbb{Z}_2$ embeds in $D_q \rtimes \mathbb{Z}_2$ via the natural localization map. When $\underline{t} = \underline{1}$, the assignment in (2.6) becomes

$$T_1 \mapsto sY, \quad T_3 \mapsto s, \quad T_2 \mapsto qsYX, \quad T_4 \mapsto sX \quad (2.7)$$

and the image of $\mathcal{H}_{q,\underline{1}}$ coincides with the image of $A_q \rtimes \mathbb{Z}_2$. Thus, using (2.7), we can identify $\mathcal{H}_{q,\underline{1}} \cong A_q \rtimes \mathbb{Z}_2$.

Remark 2.10 The algebra $\mathcal{H}_{q,\underline{t}}$ is also generated by the (invertible) elements

$$X := q^{-1}T_1T_2, \quad Y := T_3T_1, \quad T := T_3$$

which satisfy the relations

$$\begin{aligned} XT &= T^{-1}X^{-1} - \bar{t}_4 \\ T^{-1}Y &= Y^{-1}T + \bar{t}_1 \\ T^2 &= 1 + \bar{t}_3T \\ TXY &= q^2T^{-1}YX - q^2\bar{t}_1X - q\bar{t}_2 - \bar{t}_4Y \end{aligned} \quad (2.8)$$

where $\bar{t}_i = t_i - t_i^{-1}$. With this presentation it is immediate that $\mathcal{H}_{q,\underline{1}} \cong A_q \rtimes \mathbb{Z}_2$. Note that while the operator X does not depend on $\underline{t}$, the operator Y does. We will write this

last operator as $Y_{\underline{t}}$ when we want to stress its dependence on $\underline{t}$. Explicitly, we have

$$Y_{\underline{t}} = \hat{T}_3 \hat{T}_1$$

where $\hat{T}_1$ and $\hat{T}_3$ are given by formulas (2.5).

The following simple observation can be regarded as a motivation for the main conjecture of [1]. For $f(X) \in \mathbb{C}(X)$, define the operators

$$Y \cdot f(X) = f(q^{-2}X), \quad X \cdot f(X) = Xf(X), \quad s \cdot f(X) = f(X^{-1})$$

These operators give $\mathbb{C}(X)$ the structure of a left $D_q \rtimes \mathbb{Z}_2$ -module. The subspace $\mathbb{C}[X^{\pm 1}] \subset \mathbb{C}(X)$ is obviously preserved by $A_q \rtimes \mathbb{Z}_2$ and is called the *polynomial representation*. A remarkable fact (which can be checked by direct calculation) is that $\mathbb{C}[X^{\pm 1}]$ is also preserved by $\mathcal{H}_{q,\underline{t}}$ (for all $\underline{t}$) under the action of (2.6). This gives the *polynomial representation* of $\mathcal{H}_{q,\underline{t}}$, which can thus be viewed as a deformation of the polynomial representation of $A_q \rtimes \mathbb{Z}_2$.

The element $\mathbf{e} := (T_3 + t_3^{-1})/(t_3 + t_3^{-1})$ is an idempotent in $\mathcal{H}_{q,\underline{t}}$, and the algebra $S\mathcal{H}_{q,\underline{t}} := \mathbf{e}\mathcal{H}_{q,\underline{t}}\mathbf{e}$ is called the *spherical subalgebra* of $\mathcal{H}_{q,\underline{t}}$. It is easy to check that $\mathbf{e}$ commutes with $X + X^{-1}$ and that the subspace $\mathbf{e} \cdot \mathbb{C}[X^{\pm 1}] \subset \mathbb{C}[X^{\pm 1}]$ is equal to the subspace $\mathbb{C}[X + X^{-1}]$ of symmetric polynomials in $\mathbb{C}[X^{\pm 1}]$. The spherical algebra therefore acts on $\mathbb{C}[X + X^{-1}]$, and this module is called the *symmetric polynomial representation* of $S\mathcal{H}_{q,\underline{t}}$.

2.3 Main conjecture of [1]

We first recall that the algebras $A_q \rtimes \mathbb{Z}_2$ and $A_q^{\mathbb{Z}_2}$ are Morita equivalent. More precisely, if $q^4 - 1$ is invertible, then the functors

$$\begin{aligned} \mathbf{e}A \otimes_A - : \text{Mod}(A) &\rightarrow \text{Mod}(\mathbf{e}A\mathbf{e}) \\ A\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} - : \text{Mod}(\mathbf{e}A\mathbf{e}) &\rightarrow \text{Mod}(A) \end{aligned} \quad (2.9)$$

are mutually inverse equivalences of categories.

We can identify $(A_q \rtimes \mathbb{Z}_2)\mathbf{e} = A_q$ as left $A_q \rtimes \mathbb{Z}_2$ -modules and $\mathbf{e}A\mathbf{e} \cong A_q^{\mathbb{Z}_2}$ as $\mathbb{C}_q$ -algebras. Let K be a knot in S^3 , so that $K_q(S^3 \setminus K)$ has the canonical structure of a left $A_q^{\mathbb{Z}_2}$ -module. Applying the previous proposition, we may form the *nonsymmetric skein module* $\widehat{K}_q(S^3 \setminus K)$

$$\widehat{K}_q(S^3 \setminus K) := A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus K).$$

This is naturally a left $A_q \rtimes \mathbb{Z}_2$ -module, and so we may localize it at all nonzero polynomials in X . Call the resulting $D_q \rtimes \mathbb{Z}_2$ -module $\widehat{K}_q^{loc}(S^3 \setminus K)$, i.e.

$$\widehat{K}_q^{loc}(S^3 \setminus K) := (D_q \rtimes \mathbb{Z}_2) \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{K}_q(S^3 \setminus K)$$

By Proposition 2.6, $\widehat{K}_q^{loc}(S^3 \setminus K)$ is then a $\mathcal{H}_{q,(t_1,t_2,t_3,t_4)}$ -module.

Example 2.11 Let K be the unknot. In this case, $\widehat{K}_q(S^3 \setminus K) \cong \mathbb{C}_q[X^{\pm 1}]$ as a $\mathbb{C}_q[X^{\pm 1}]$ -module. The action of the generators $Y, s \in A_q \rtimes \mathbb{Z}_2$ is given by the formulas

$$Y \cdot f(X) := -f(q^{-2}X), \quad s \cdot f(X) = -f(X^{-1})$$

The localized skein module $\widehat{K}_q^{loc}(S^3 \setminus K)$ is simply $\mathbb{C}_q(X)$. Thus in this case the natural localization map

$$\eta : \widehat{K}_q(S^3 \setminus K) \rightarrow \widehat{K}_q^{loc}(S^3 \setminus K)$$

is injective, and we can identify $\widehat{K}_q(S^3 \setminus K)$ with its image under η . We want to know if the $\mathcal{H}_{q,t}$ action preserves this image as in the case of the polynomial representation.

Recall that by Remark 2.10, the algebra $\mathcal{H}_{q,t}$ is generated by the operators X, T_1, T_3 , which act on polynomials by formulas (2.5):

$$\begin{aligned} T_1 \cdot X^n &= t_1 q^{2n} X^{-n} + q^{-2n} X^{-n} (q^2 \overline{t_1} X^2 + q \overline{t_2} X) \frac{1 - q^{2n} X^{2n}}{1 - q^2 X^2} \\ T_3 \cdot X^n &= -t_3 X^{-n} + (\overline{t_3} + \overline{t_4} X) \frac{X^n + X^{-n}}{1 - X^2} \end{aligned}$$

We see that T_1 always preserves $\widehat{K}_q(S^3 \setminus K) \subset \widehat{K}_q^{loc}(S^3 \setminus K)$, while T_3 preserves this subspace only when $t_3 = t_4 = \pm 1$. Conjecturally, this behaviour generalizes to all knots. To be precise, we have

Conjecture 2.12 ([1]) *For all knots $K \subset S^3$, the following are true:*

- (1) *The localization map $\eta : \widehat{K}_q(S^3 \setminus K) \rightarrow \widehat{K}_q^{loc}(S^3 \setminus K)$ is injective.*
- (2) *The natural action of $\mathcal{H}_{q,(t_1,t_2,1,1)}$ on $\widehat{K}_q^{loc}(S^3 \setminus K)$ preserves the subspace $\widehat{K}_q(S^3 \setminus K)$, the image of the localization map η .*

By symmetrization, the second statement of Conjecture 2.12 implies that the spherical subalgebra $S\mathcal{H}_{q,t_1,t_2,1,1}$ acts on the skein module $K_q(S^3 \setminus K)$ itself. It is shown in [1] and [2] that this holds in many cases: for the unknot, figure eight, and $(2, 2p+1)$ -torus knots for generic q , and for 2-bridge knots, all torus knots, and connect sums of such when $q = -1$.

2.4 The generalized Jones polynomials

An interesting consequence of Conjecture 2.12 is the existence of a multivariable generalization of the (coloured) Jones polynomials $J_n^K(q)$. Recall, by Theorem 2.7, $J_n^K(q)$ can be computed using the natural (topological) pairing of the Kauffman bracket skein modules, by the Kirby–Melvin formula (cf. (2.3)):

$$J_n^K(q) = (-1)^{n-1} \langle \emptyset, S_{n-1}(Y + Y^{-1}) \cdot \emptyset \rangle \quad (2.10)$$

Under Morita equivalence 2.9, the topological pairing $\langle -, - \rangle$ extends uniquely to a bilinear pairing of nonsymmetric skein modules³:

$$\langle -, - \rangle : \hat{K}_q(S^1 \times D^2) \times \hat{K}_q(S^3 \setminus K) \rightarrow \mathbb{C}_q \quad (2.11)$$

and formula (2.10) still holds for this extended pairing (see [1, Cor. 5.3]). We note that by construction, this bilinear pairing is in fact balanced over $A_q \rtimes \mathbb{Z}_2$, i.e. it induces a $\mathbb{C}_q$ -linear map

$$\langle -, - \rangle : \hat{K}_q(S^1 \times D^2) \otimes_{A_q \rtimes \mathbb{Z}_2} \hat{K}_q(S^3 \setminus K) \rightarrow \mathbb{C}_q \quad (2.12)$$

The right action of $A_q \rtimes \mathbb{Z}_2$ on $\hat{K}_q(S^1 \times D^2)$ in (2.12) is described explicitly in [1, Lemma 5.5]. Specifically, $\hat{K}_q(S^1 \times D^2)$ can be identified with the space of Laurent polynomials $\mathbb{C}_q[U^{\pm 1}]$ with $A_q \rtimes \mathbb{Z}_2$ acting by

$$\begin{aligned} f(U) \cdot Y &:= f(U) \cdot U^{-1} \\ f(U) \cdot X &:= -f(q^2 U) \\ f(U) \cdot s &:= -f(U^{-1}) \end{aligned} \quad (2.13)$$

The distinguished element ('empty link') $\emptyset$ in $\hat{K}_q(S^1 \times D^2)$ corresponds under this identification to the element $U - U^{-1} \in \mathbb{C}_q[U^{\pm 1}]$, which we still denote by $\emptyset$.

When a knot K satisfies Conjecture 2.12, the nonsymmetric skein module $\hat{K}_q(S^3 \setminus K)$ carries a natural action of the DAHA $\mathcal{H}_{q,t_1,t_2}$ and the 'longitude' operator Y admits a natural deformation to the DAHA operator $Y_{t_1,t_2} := T_3 T_1$ (see Remark 2.10). This motivates the following.

Definition 2.13 ([1]) Assume that $K \subset S^3$ satisfies Conjecture 2.12. Then we define the *generalized Jones polynomial* of K by

$$J_n^K(q, t_1, t_2) := (-1)^{n-1} \langle \emptyset, S_{n-1}(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}) \cdot \emptyset \rangle \quad (2.14)$$

where $\langle -, - \rangle$ is extended topological pairing (2.11).

Note that formula (2.14) makes sense precisely because, by Conjecture 2.12, the skein module $\hat{K}_q(S^3 \setminus K)$ is a module over $\mathcal{H}_{q,t_1,t_2}$. When $t_1 = t_2 = 1$, it reduces to Kirby–Melvin formula (2.10), and we have $J_n^K(q, 1, 1) = J_n^K(q)$. The generalized Jones polynomial $J_n^K(q, t_1, t_2)$ can be thus viewed as a two-parameter ('Hecke') deformation of $J_n^K(q)$.

3 Proofs

In this section, we prove our three main theorems stated in Introduction.

³ Abusing notation, we denote the extended pairing of nonsymmetric skein modules in the same way as the 'symmetric' (topological) one.

3.1 The deformed pairing

To compute generalized Jones polynomials (2.14), we need a ‘deformed’ version of formula (2.3), which leads us to the natural question: Is topological pairing (2.11) balanced over $\mathcal{H}_{q,t_1,t_2}$ for $t_1, t_2 \neq 1$? The (affirmative) answer to this question is the starting point for our calculations:

Lemma 3.1 *Assume that a knot $K \subset S^3$ satisfies Conjecture 2.12. Then, for any $t_1, t_2 \in \mathbb{C}^*$, pairing (2.11) induces a linear map*

$$\langle -, - \rangle : \hat{K}_q(S^1 \times D^2) \otimes_{\mathcal{H}_{q,t_1,t_2}} \hat{K}_q(S^3 \setminus K) \rightarrow \mathbb{C}_q(t_1, t_2) \quad (3.1)$$

where the (right) $\mathcal{H}_{q,t_1,t_2}$ -module structure on $\hat{K}_q(S^1 \times D^2)$ is defined by (2.13) via Demazure-Lusztig and Dunkl-Cherednik operators (2.5).

Proof Recall that pairing (2.11) is balanced over $A_q \rtimes \mathbb{Z}_2$ (see (2.12)). To prove the lemma, it is sufficient to show that it is balanced over an invertible generating set of $\mathcal{H}_{q,(t_1,t_2)}$, which we take to be X, s , and the operator

$$T_1 = t_1 s Y - \frac{q^2 \bar{t}_1 X^2 + q \bar{t}_2 X}{1 - q^2 X^2} (1 - s Y)$$

Since the pairing is already balanced over s and X and Y , it will suffice to show it is balanced with respect to $\frac{1}{1 - q^2 X^2} (1 - s Y)$. Since the image of $\hat{K}_q(S^3 \setminus K)$ is preserved in its localization, if $m \in \hat{K}_q(S^3 \setminus K)$ then there exists a unique $m' \in \hat{K}_q(S^3 \setminus K)$ such that

$$(1 - s Y) \cdot m = (1 - q^2 X^2) \cdot m'$$

Thus we can compute

$$\langle U^k, \frac{1}{1 - q^2 X^2} (1 - s Y) \cdot m \rangle = \langle U^k, m' \rangle.$$

On the other hand, acting on the right by the same operator gives

$$\begin{aligned} \langle U^k \cdot \frac{1}{1 - q^2 X^2} (1 - s Y), m \rangle &= \langle U^k \cdot \frac{1}{1 - q^2 X^2}, (1 - s Y) \cdot m \rangle \\ &= \langle U^k \cdot \frac{1}{1 - q^2 X^2}, (1 - q^2 X^2) m' \rangle \\ &= \langle U^k, m' \rangle \end{aligned}$$

This completes the proof of Lemma 3.1. □

Corollary 3.2 *If K satisfies Conjecture 2.12, then*

$$J_n^K(q, t_1, t_2) = (-1)^{n-1} \langle \emptyset \cdot S_{n-1}(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}), \emptyset \rangle \quad (3.2)$$

Proof Formula (3.2) is immediate from Definition 2.13 and Lemma 3.1. $\square$

3.2 Proof of Theorem 1.2

From now, we fix a knot $K \subset S^3$ and (unless otherwise stated) assume that it satisfies the conditions of Conjecture 2.12.

Lemma 3.3 For all $n \geq 0$,

$$J_n^K(q, t_1, t_2) = \sum_{p=1}^n (-1)^{n+p} a_{n,p}(q, t_1, t_2) J_p^K(q), \quad (3.3)$$

where the coefficients $a_{n,p} = a_{n,p}(q, t_1, t_2)$ are defined in Introduction (see (1.18) and (1.19)).

Remark 3.4 We note that $a_{n,k}(q, t_1, t_2)$ in (3.3) are rational functions of q , and it is by no means obvious that the right-hand side of formula (3.3) is polynomial in q . We will show later—invoking the Habiro Theorem—that this is indeed the case for any knot K , whether or not it satisfies Conjecture 2.12.

Proof of Lemma 3.3 Recall that under the identification $\hat{K}_q(S^1 \times D^2) \cong \mathbb{C}_q[U^{\pm 1}]$ (see (2.14)), the empty link $\emptyset$ in $\hat{K}_q(S^2 \times D^2)$ corresponds to the element $U - U^{-1} \in \mathbb{C}_q[U^{\pm 1}]$. The operators $S_{n-1}(Y_{t_1, t_2} + Y_{t_1, t_2}^{-1})$ are invariant under (i.e. commute with) the action of $\mathbb{Z}_2$ on $\mathbb{C}_q[U^{\pm 1}]$. Hence, for all $n \geq 1$, we can expand $(U - U^{-1}) \cdot S_{n-1}(Y_{t_1, t_2} + Y_{t_1, t_2}^{-1})$ in $\mathbb{C}_q[U^{\pm 1}]$ as

$$(U - U^{-1}) \cdot S_{n-1}(Y_{t_1, t_2} + Y_{t_1, t_2}^{-1}) = \sum_{p=1}^n a_{n,p}(U^p - U^{-p}) \quad (3.4)$$

for some (uniquely determined) coefficients $\tilde{a}_{n,p} \in \mathbb{C}_q(t_1, t_2)$. By Corollary 3.2, this gives

$$\begin{aligned} J_n(q, t_1, t_2) &= (-1)^{n-1} \langle \emptyset \cdot S_{n-1}(Y_{t_1, t_2} + Y_{t_1, t_2}^{-1}), \emptyset \rangle \\ &= (-1)^{n-1} \langle (U - U^{-1}) \cdot S_{n-1}(Y_{t_1, t_2} + Y_{t_1, t_2}^{-1}), \emptyset \rangle \\ &= (-1)^{n-1} \sum_{p=1}^n a_{n,p} \langle U^p - U^{-p}, \emptyset \rangle \\ &= \sum_{p=1}^n (-1)^{n+p} a_{n,p} J_p(q) \end{aligned}$$

where the last equality is the consequence of Kirby–Melvin formula (2.3) (cf. [1, Lemma 5.6]). Thus, to complete the proof of the lemma it suffices to show that the coefficients $a_{n,p}$ in (3.4) are determined precisely by relations (1.18) and (1.19). This

can be done by a lengthy but straightforward induction (in n) using the defining relations $S_n = uS_{n-1} - S_{n-2}$ for the Chebyshev polynomials. We leave this calculation as an exercise for the reader. $\square$

Combining formula (3.3) of Lemma 3.3 with Habiro's expansion of the classical Jones polynomials (see Theorem 1.1), we get

$$\begin{aligned} J_n(q, t_1, t_2) &= \sum_{p=1}^n (-1)^{n+p} a_{n,p} J_p(q) = \sum_{p=1}^n (-1)^{n+p} a_{n,p} \left(\sum_{i=1}^p c_{p,i-1} H_{i-1} \right) \\ &= \sum_{p=1}^n \sum_{i=1}^p (-1)^{n+p} a_{n,p} c_{p,i-1} H_{i-1} = \sum_{i=1}^n \left(\sum_{p=i}^n (-1)^{n+p} a_{n,p} c_{p,i-1} \right) H_{i-1} \end{aligned}$$

where $H_{i-1} = H_{i-1}(q)$ are the Habiro polynomials of the knot K and $c_{p,i-1}$ are the classical cyclotomic coefficients defined by formula (1.2). Since $c_{p,i-1} \equiv 0$ for $p < i$, we can rewrite the last formula in the form

$$J_n(q, t_1, t_2) = \sum_{i=1}^n \tilde{c}_{n,i-1} H_{i-1} \quad (3.5)$$

where

$$\tilde{c}_{n,i-1} := \sum_{p=1}^n (-1)^{n+p} a_{n,p} c_{p,i-1} \quad (3.6)$$

Now, to prove Theorem 1.2 we need to compute the generating functions $G_i(\lambda) := \sum_{n=0}^{\infty} \tilde{c}_{n,i-1} \lambda^n$. Using (3.6) we can write these functions in the form

$$G_i(\lambda) = \sum_{n=0}^{\infty} \sum_{p=1}^n (-1)^p a_{n,p} c_{p,i-1} (-\lambda)^n, \quad i \geq 1 \quad (3.7)$$

Formula (3.6) suggests that $G_i(\lambda)$ may be expressed in a simple way in terms of the generating series of the double sequence $\{a_{n,p}\}$:

$$F(U, \lambda) := \sum_{n=0}^{\infty} \sum_{p=-\infty}^{\infty} a_{n,p} U^p \lambda^n \quad (3.8)$$

which we define by formally extending the functions $p \mapsto a_{n,p}$ to all integers $p \in \mathbb{Z}$ using recurrence relation (1.18) for $p < 0$. Note that, by symmetry of (1.18), we actually have

$$a_{n,-p} = -a_{n,p}, \quad \forall p \in \mathbb{Z} \quad (3.9)$$

Together with ‘boundary’ conditions (1.19) this implies

$$\sum_{p=-\infty}^{\infty} a_{n,p} U^p = \sum_{p=-n}^n a_{n,p} U^p = \sum_{p=1}^n a_{n,p} (U^p - U^{-p})$$

Hence (3.8) can be rewritten in the form

$$F(U, \lambda) = \sum_{n=0}^{\infty} \left(\sum_{p=1}^n a_{n,p} (U^p - U^{-p}) \right) \lambda^n \quad (3.10)$$

Remark 3.5 To reduce notation, for the rest of this section we write

$$\{n\} := \{n\}_q$$

Comparing (3.10) with formula (3.7) for $i = 1$, we see at once that

$$G_1(\lambda) = \frac{1}{\{2\}} F(-q^2, -\lambda)$$

The next lemma extends this observation to all $G_i(\lambda)$ ’s.

Lemma 3.6 For all $i \geq 1$,

$$G_i(\lambda) = \frac{1}{\{2\}} \sum_{k=1}^i \alpha_k^{(i)} F(-q^{2(2k-1)}, -\lambda) \quad (3.11)$$

where

$$\alpha_k^{(i)} = (-1)^{i-k} \left[\begin{matrix} 2i-1 \\ i-k \end{matrix} \right]_{q^2} \quad (3.12)$$

Proof Using the explicit formulas for the cyclotomic coefficients $c_{p,i-1}$ (see (1.2)) and the (skew) symmetry of the $a_{n,p}$ ’s (see (3.10)), we write

$$\begin{aligned} & \sum_{p=1}^n (-1)^p a_{n,p} c_{p,i-1} \\ &= \frac{1}{\{2\}} \sum_{p=1}^n (-1)^p a_{n,p} \{2(p-(i-1))\} \cdots \{2(p-1)\} \{2p\} \{2(p+1)\} \cdots \{2(p+(i-1))\} \\ &= \frac{1}{\{2\}} \sum_{p=-n}^n a_{n,p} \{2(p-(i-1))\} \cdots \{2(p-1)\} (-q^2)^p \{2(p+1)\} \cdots \{2(p+(i-1))\} \\ &= \frac{1}{\{2\}} \left[\sum_{p=-\infty}^{\infty} a_{n,p} U^p \{2(p-(i-1))\} \cdots \{2(p-1)\} \{2(p+1)\} \cdots \{2(p+(i-1))\} \right]_{U=-q^2} \end{aligned}$$

Since $U^p \cdot f(X^{\pm 1}) = U^p f(-q^{-2p})$ for any $f(X) \in \mathbb{C}[X^{\pm 1}]$, we can rewrite the last sum in the form

$$\sum_{p=1}^n (-1)^p a_{n,p} c_{p,i-1} = \frac{1}{\{2\}} \left[\sum_{p=-\infty}^{\infty} a_{n,p} U^p \cdot P^{(i)}(X) \right]_{U=-q^2}$$

where $P^{(i)}(X) \in \mathbb{C}[X^{\pm 1}]$ are the Laurent polynomials defined by

$$P_i(X) := \prod_{k=1}^{i-1} (q^{-2k} X - q^{2k} X^{-1})(q^{2k} X - q^{-2k} X^{-2k}), \quad i \geq 1$$

By formula (3.7), we get

$$G_i(\lambda) = \frac{1}{\{2\}} \left[F(U, -\lambda) \cdot P^{(i)}(X) \right]_{U=-q^2}$$

Writing the polynomials $P^{(i)}(X)$ in the form

$$P^{(i)}(X) = b_0^{(i)} + \sum_{k=1}^{i-1} b_k^{(i)} (X^{2k} + X^{-2k})$$

we compute

$$F(U, -\lambda) \cdot P^{(i)}(X) = b_0^{(i)} F(U, -\lambda) + \sum_{k=1}^{i-1} b_k^{(i)} \left(F(q^{4k} U, -\lambda) + F(q^{-4k} U, -\lambda) \right)$$

Now, substituting $U = -q^2$ and using the skew-symmetry $F(U^{-1}) = -F(U)$ of the generating series, we find

$$\left[F(U, -\lambda) \cdot P^{(i)}(X) \right]_{U=-q^2} = \sum_{k=1}^i (b_{k-1}^{(i)} - b_k^{(i)}) F(-q^{2(2k-1)}, -\lambda)$$

Whence

$$G_i(\lambda) = \frac{1}{\{2\}} \sum_{k=1}^i (b_{k-1}^{(i)} - b_k^{(i)}) F(-q^{2(2k-1)}, -\lambda)$$

To complete the proof of the lemma, it suffices to notice that

$$b_{k-1}^{(i)} - b_k^{(i)} = \alpha_k^{(i)} \quad \forall i \geq 1, \quad 1 \leq k \leq i-1$$

which can be seen easily from the formula

$$(X - X^{-1})P^{(i)}(X) = \prod_{k=-(i-1)}^{i-1} (q^{2k}X - q^{-2k}X^{-1}) = \sum_{k=1}^i \alpha_k^{(i)} (X^{2k-1} - X^{-2k+1}).$$

This finishes the proof of Lemma 3.6. $\square$

Thus, by Lemma 3.6, the generating functions $G_i(\lambda)$ are determined by the values of $F(U, \lambda)$ at $U = -q^{2(2k-1)}$ for $k \geq 1$. To compute these values we will use the functional equation

$$F(U, \lambda) \cdot (Y_{t_1, t_2} + Y_{t_1, t_2}^{-1} - \lambda - \lambda^{-1}) = U^{-1} - U \quad (3.13)$$

which is equivalent to recurrence relations (1.18) defining the coefficients $a_{n,p}$. The equivalence of (3.13) and (1.18) follows easily from formulas (3.4) and (3.10) and the standard generating series of Chebyshev polynomials:

$$\sum_{n=0}^{\infty} S_{n-1}(z + z^{-1})\lambda^n = -(z + z^{-1} - \lambda - \lambda^{-1})^{-1}$$

We need one more technical lemma.

Lemma 3.7 *For any $N \in \mathbb{Z}$ and any $f(U) \in \mathbb{C}_q[U^{\pm 1}]$,*

$$\begin{aligned} & \left[f(U) \cdot (Y_{t_1, t_2} + Y_{t_1, t_2}^{-1}) \right]_{U=-q^{2N}} \\ &= -(t_1 q^{-2N} + t_1^{-1} q^{2N}) f(-q^{2N}) \\ & \quad + \bar{t}_1 \sum_{p=0}^{N-1} \{2p\} f(-q^{2(2p-N)}) \\ & \quad - \bar{t}_2 \sum_{p=0}^{N-1} \{2p+1\} f(-q^{2(2p-N+1)}) \end{aligned} \quad (3.14)$$

Proof Recall that the Dunkl-Cherednik operator $Y_{t_1, t_2} := Y_{t_1, t_2, 1, 1}$ is given explicitly by the formula (cf. (2.5) and Remark 2.10):

$$Y_{t_1, t_2} = t_1 Y - a(X)(Y - s)$$

where

$$a(X) := \frac{q\bar{t}_1 X^{-1} + \bar{t}_2}{qX^{-1} - q^{-1}X} = \frac{\bar{t}_1 + \bar{t}_2(q^{-1}X)}{1 - (q^{-1}X)^2}$$

For any $k \in \mathbb{Z}$, using (2.13) we compute

$$U^k \cdot Y_{t_1, t_2} = t_1 U^{k-1} - a(-q^{2k})(1 + U^{2k-1})U^{-k}$$

Then, evaluating at $U = -q^{2N}$ yields

$$\begin{aligned} [U^k \cdot Y_{t_1, t_2}]_{U=-q^{2N}} &= -t_1 q^{-2N} (-q^{2N})^k - \bar{t}_1 \sum_{p=0}^{N-1} q^{-2p} (-q^{2p-2N})^k \\ &\quad + \bar{t}_2 \sum_{p=0}^{N-1} q^{-2p-1} (-q^{4p-2N+2})^k \end{aligned}$$

Hence, for any $f(U) \in \mathbb{C}[U^{\pm 1}]$ we have

$$\begin{aligned} [f(U) \cdot Y_{t_1, t_2}]_{U=-q^{2N}} &= -t_1 q^{-2N} f(-q^{2N}) \\ &\quad - \bar{t}_1 \sum_{p=0}^{N-1} q^{-qp} f(-q^{2(2p-N)}) \\ &\quad + \bar{t}_2 \sum_{p=0}^{N-1} q^{-2p-1} f(-q^{2(2p-N+1)}) \end{aligned} \quad (3.15)$$

A similar calculation with the inverse operator

$$Y_{t_1, t_2}^{-1} = t_1 Y^{-1} - a(X^{-1})Y^{-1} - s) - \bar{t}_1 s$$

yields

$$\begin{aligned} [f(U) \cdot Y_{t_1, t_2}^{-1}]_{U=-q^{2N}} &= -t_1^{-1} q^{2N} f(-q^{2N}) \\ &\quad + \bar{t}_1 \sum_{p=0}^{N-1} q^{2p} f(-q^{2(2p-N)}) \\ &\quad - \bar{t}_2 \sum_{p=0}^{N-1} q^{2p+1} f(-q^{2(2p-N+1)}) \end{aligned} \quad (3.16)$$

Adding up (3.15) and (3.16) we get formula (3.14). $\square$

Now we are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 Using Lemma 3.7, from functional Eq. (3.13) we get the system of linear equations for the values $F(-q^{2N}, -\lambda)$:

$$\gamma_N F(-q^{2N}) - \bar{t}_1 \sum_{p=1}^{N-1} \{2p\} F(-q^{2(2p-N)}) + \bar{t}_2 \sum_{p=0}^{N-1} \{2p+1\} F(-q^{2(2p-N+1)}) = -\{2N\}$$

where $\gamma_N = -\lambda - \lambda^{-1} + q^{2N}t_1^{-1} + q^{-2N}t_1$. This system can be written in the matrix form

$$\begin{pmatrix} \gamma_1 & 0 & 0 & \cdots \\ b_{2,1} & \gamma_2 & 0 & \cdots \\ b_{3,1} & b_{3,2} & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} F(-q^2) \\ F(-q^4) \\ F(-q^6) \\ \vdots \end{pmatrix} = - \begin{pmatrix} \{2\} \\ \{4\} \\ \{6\} \\ \vdots \end{pmatrix} \quad (3.17)$$

where

$$b_{p,N} := (-1)^i (\{p+N\} - \{p-N\}) \bar{t}_i, \quad \text{with } i \equiv p - N + 1 \pmod{2}$$

By Lemma 3.6, the generating functions $G_i(\lambda)$ are given by linear combinations of solutions of this system, $F(-q^{2N}, -\lambda)$, with $N = 2k - 1$ for $k \geq 1$. Solving (3.17) by Cramer's rule, we can formally express these linear combinations in terms of the matrix $\tilde{B}_{2i}(q, t_1, t_2; \lambda)$ described in Introduction (see (1.12)). This yields required formulas (1.11) for $G_i(\lambda)$, finishing the proof of Theorem 1.2. $\square$

3.3 Proof of Theorem 1.4

In this subsection we specialize $t_2 = 1$ and give an explicit formula for the coefficients $\tilde{c}_{n,i}$ in terms of classical Macdonald polynomials of type A_1 . We begin by recalling the definition.

Definition 3.8 The *Macdonald polynomials* $p_n(x; \beta|q)$, $n \geq 0$ are the symmetric orthogonal polynomials in $\mathbb{C}[q^{\pm 1}, \beta^{\pm 1}][x + x^{-1}]$ satisfying the 3-term recurrence relation

$$p_{n+1} = (x + x^{-1})p_n - \frac{q^{n/2} - q^{-n/2}}{\beta^{1/2}q^{n-2} - \beta^{-1/2}q^{-n/2}} \frac{\beta q^{(n-1)/2} - \beta^{-1}q^{(1-n)/2}}{\beta^{1/2}q^{(n-1)/2} - \beta^{-1/2}q^{(1-n)/2}} p_{n-1}$$

with $p_0 = 1$ and $p_1 = x + x^{-1}$.

After the following renormalization⁴

$$C_n(x; \beta|q) := \frac{(\beta; q)_n}{(q; q)_n} p_n(x; \beta|q)$$

the Macdonald polynomials assemble into the generating series (see, e.g. [11]):

$$\sum_{n=0}^{\infty} C_n(x; \beta|q) z^n = \frac{(z\beta x; q)_{\infty} (z\beta x^{-1}; q)_{\infty}}{(zx; q)_{\infty} (zx^{-1}; q)_{\infty}}, \quad (3.18)$$

⁴ The polynomials $C_n(x; \beta|q)$ are sometimes called the q -ultraspherical (or Rogers) polynomials (cf. [11, Sect. 14.10.1]).

where

$$(a; q)_n := \begin{cases} 1 & n = 0 \\ \prod_{k=0}^{n-1} (1 - aq^k) & 1 \leq n \leq \infty \end{cases}$$

(For $n = \infty$ one assumes that $|q| < 1$.) In fact, these polynomials can be given by

$$C_n(x; \beta|q) = \sum_{k=0}^n \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} x^{n-2k} \quad (3.19)$$

If we specialize $q \mapsto q^4$ and $\beta \mapsto q^{4i}$, then formulas (3.18) and (3.19) become

$$\sum_{n=0}^{\infty} C_n(x; q^{4i}|q^4) z^n = \frac{1}{\prod_{k=0}^{i-1} (1 - q^{4k}zx)(1 - q^{4k}zx^{-1})} \quad (3.20)$$

and

$$C_n(x; q^{4i}|q^4) = \sum_{k=0}^n \begin{bmatrix} k+i-1 \\ i-1 \end{bmatrix}_{q^4} \begin{bmatrix} n-k+i-1 \\ i-1 \end{bmatrix}_{q^4} x^{n-2k} \quad (3.21)$$

Note that the last formula shows that $C_n(x; q^{4i}|q^4) \in \mathbb{Z}[q^{\pm 4}][x + x^{-1}]$ for all $n \geq 0$. To prove Theorem 1.4 we compare (3.20) to the generating function $G_i(\lambda)$. First, we simplify formula (1.11) for $G_i(\lambda)$ given in Theorem 1.2 by explicitly computing the determinant of B_{2i} in the case $t_2 = 1$. The result is given by the following

Proposition 3.9 *For $t_1 = t$ and $t_2 = 1$, we have*

$$G_i(\lambda) = \frac{c_{i,i-1} \prod_{k=2}^i A_k(t)}{\prod_{k=1}^i (\lambda + \lambda^{-1} - q^{2(2k-1)}t^{-1} - q^{-2(2k-1)}t)} \quad (3.22)$$

where

$$A_k(t) = \frac{q^{2k-1}t^{-1} - q^{1-2k}t}{q^{2k-1} - q^{1-2k}}$$

Proof We break up the proof into two steps stated as Lemmas 3.10 and 3.11. First, Lemma 3.10 shows that

$$\tilde{c}_{n,i-1} = \frac{\det[\bar{B}_i]}{\prod_{k=1}^i (\lambda + \lambda^{-1} - q^{2(2k-1)}t^{-1} - q^{-2(2k-1)}t)}$$

where $\bar{B}_i$ is a certain submatrix of $\tilde{B}_{2i}$. Then Lemma 3.11 computes the determinant of $\bar{B}_i$ by induction, showing that

$$\det[\bar{B}_{i+1}] = -\{2(2i)\}\{2(2i+1)\}A_{i+1} \det[\bar{B}_i] \quad (3.23)$$

Together with (1.11), this gives formula (3.22). $\square$

Lemma 3.10 For all $i \geq 1$,

$$G_i(\lambda) = \frac{\det[\bar{B}_i]}{\prod_{k=1}^i (\lambda + \lambda^{-1} - q^{2(2k-1)}t^{-1} - q^{-2(2k-1)}t)}$$

where $\bar{B}_i$ is the matrix

$$\bar{B}_i := \begin{pmatrix} 0 & \alpha_1^{(i)} & \alpha_2^{(i)} & \cdots & \alpha_{i-1}^{(i)} & 1 \\ \beta_1 & \gamma_1 & 0 & \cdots & & \\ \beta_3 & b_{3,1} & \gamma_3 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & 0 & \cdots \\ \beta_{2j-1} & b_{2j-1,1} & b_{2j-1,2} & \cdots & \gamma_{2j-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \\ \beta_{2i-1} & b_{2i-1,1} & b_{2i-1,2} & \cdots & \cdots & b_{2i-1,2i-2} & \gamma_{2i-1} \end{pmatrix} \quad (3.24)$$

Proof Note that if $t_2 = 1$ and $i - j$ is even, then $b_{i,j} = 0$. This means that the second-to-last column in $\bar{B}_{2i}$ has exactly one nonzero entry, which is γ_{2i-2} , located on the diagonal. Expanding the determinant along this column, we see that the same is true with the resulting $(2i - 1) \times (2i - 1)$ matrix. Then induction shows

$$\det(\bar{B}_{2i}) = \det(\bar{B}_i) \prod_{j=2}^i \gamma_{2j-2}$$

The result then follows from this identity combined with Theorem 1.2. $\square$

Lemma 3.11 $\det[\bar{B}_i] = (-1)^i \left(\prod_{N=1}^{2i-1} \{2N\} \right) \left(\prod_{k=2}^i A_k(t) \right).$

Proof The proof consists of a sequence of row and column operations to show that $\det[\bar{B}_{i+1}] = -\{2(2i)\}\{2(2i+1)\}A_{i+1}\det[\bar{B}_i]$. First, we kill all entries in the first row of $\bar{B}_{i+1}$ except for the last using column operations to obtain the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ \beta_1 & \gamma_1 & 0 & \cdots & \\ \vdots & \vdots & \ddots & \cdots & \\ \beta_{2i-1} & b_{2i-1,1} & b_{2i-1,2} & \cdots & \gamma_{2j-1} & 0 \\ \beta_{2i+1} & b_{2i+1,1} - \alpha_1^{(i+1)}\gamma_{2i+1} & b_{2i+1,2} - \alpha_2^{(i+1)}\gamma_{2i+1} & \cdots & \cdots & \gamma_{2i+1} \end{pmatrix}$$

Then we reduce the size of this matrix by one, expanding the determinant along its first row. Next, we add $\alpha_k^{(i+1)}$ multiples of the first i rows to the last row to obtain the matrix

$$(-1)^{i+3} \gamma_{2i+1} \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \beta_{2i-1} & b_{2i-1,1} & b_{2i-1,2} & \cdots & \gamma_{2j-1} \\ \tilde{\beta}_{2i+1} & \tilde{b}_{2i+1,1} & \tilde{b}_{2i+1,2} & \cdots & \tilde{b}_{2i+1,i} \end{pmatrix}$$

where

$$\begin{aligned} \tilde{\beta}_{2i+1} &= \alpha_1^{(i+1)} \beta_1 + \cdots + \alpha_i^{(i+1)} \beta_{2i-1} + \beta_{2i+1} \\ \tilde{b}_{2i+1,k} &= b_{2i+1,k} + \alpha_k^{(i+1)} (\gamma_k - \gamma_{2i+1}) + \sum_{j=k+1}^i \alpha_j^{(i+1)} b_{2j-1,k} \end{aligned}$$

Now, observe that by (3.17) we have $\tilde{\beta}_{2i+1} = 0$, so we move the last row to the top and divide it by its last entry, which is $\tilde{b}_{2i+1,i}$. Finally, by a straightforward computation, we check that the resulting matrix is *exactly* $\tilde{B}_i$. $\square$

Proof of Theorem 1.4 It follows from Lemma 3.9 that

$$\tilde{c}_{n,i-1} = \left(\frac{c_{i,i-1} \prod_{k=2}^i A_k}{\prod_{k=0}^{i-1} (\lambda + \lambda^{-1} - q^{2(2k+1)} t^{-1} - q^{-2(2k+1)} t)} \right) [\lambda^n] \quad (3.25)$$

where the notation $[\lambda^n]$ means the coefficient of λ^n in the preceding expression. If we change variables $z = q^{-2(i-1)} \lambda$ and $x = q^{2i} t^{-1}$ in (3.20) and compare the result with (3.25) we obtain

$$\tilde{c}_{n,i-1} = c_{i,i-1} q^{-2(n-i)(i-1)} C_{n-i}(q^{2i} t^{-1}; q^{4i} | q^4) \prod_{k=2}^i A_k \quad (3.26)$$

By specializing $t = 1$ in (3.26), we see that

$$C_{n-i}(q^{2i}; q^{4i} | q^4) = q^{2(n-i)(i-1)} \frac{c_{n,i-1}}{c_{i,i-1}}$$

Hence, it follows from (3.26) that

$$\frac{\tilde{c}_{n,i-1}}{c_{n,i-1}} = \frac{C_{n-i}(q^{2i} t^{-1}; q^{4i} | q^4)}{C_{n-i}(q^{2i}; q^{4i} | q^4)} \left(\prod_{k=2}^i A_k \right) = \frac{p_{n-i}(q^{2i} t^{-1}; q^{4i} | q^4)}{p_{n-i}(q^{2i}; q^{4i} | q^4)} \left(\prod_{k=2}^i A_k \right) \quad (3.27)$$

This completes the proof of Theorem 1.4. $\square$

Remark 3.12 Using (3.21), we can rewrite formula (3.26) in the following explicit form:

$$\begin{aligned} \tilde{c}_{n,i-1} = & (q^2 t^{-1})^n \prod_{k=1}^{i-1} (q^{2k+1} + q^{-2k-1}) (q^{-2k-1} t - q^{2k+1} t^{-1}) \\ & \times (q^8; q^8)_{i-1} \sum_{k=0}^{n-i} \begin{bmatrix} k+i-1 \\ i-1 \end{bmatrix}_{q^4} \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}_{q^4} (q^{-2i} t)^{i+2k} \end{aligned}$$

which makes the integrality of $\tilde{c}_{n,i-1}$ (Corollary 1.3) obvious.

3.4 Proof of Theorem 1.6

Theorem 1.6 follows easily by comparing our results (specifically Lemma 3.3) with Habiro's results proved in [8]. For the reader's convenience (and to avoid confusion with notation), we will state Habiro's main theorem on universal $\mathfrak{sl}_2$ -invariants below. First, we recall from Introduction that $\mathcal{U}_h = \mathcal{U}_h(\mathfrak{sl}_2)$ stands for the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$: this is an h -adically complete $\mathbb{Q}[[h]]$ -algebra (topologically) generated by elements E, F, H satisfying the relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{K - K^{-1}}{v - v^{-1}} \quad (3.28)$$

where $v := e^{h/2}$ and $K := v^H = e^{hH/2}$. This algebra carries a natural (complete) ribbon Hopf algebra structure with universal R -matrix given by

$$R = v^{H \otimes H/2} \sum_{n \geq 0} v^{n(n-1)/2} \frac{(v - v^{-1})^n}{[n]_v!} E^n \otimes F^n \quad (3.29)$$

(where we have used the notation $[n]_q! := \prod_{k=1}^n [k]_q$). Using R -matrix (3.29), for any (ordered, oriented, framed) link L in S^3 , R. Lawrence [14, 15] constructed a link invariant J^L , called the *universal $\mathfrak{sl}_2$ -invariant*⁵ of L . If L has ℓ components, the Lawrence invariant J^L takes its values in $\mathcal{U}_h^{\hat{\otimes} \ell}$, the h -adically completed tensor product of ℓ copies of $\mathcal{U}_h$. In the case of knots (i.e. a link K with a single component), the Lawrence invariant J^K is contained in the centre $\mathcal{Z}(\mathcal{U}_h)$, which is a complete commutative subalgebra of $\mathcal{U}_h$ (topologically) freely generated by the Casimir element $C = (v - v^{-1})^2 FE + (vK + v^{-1}K^{-1} - v - v^{-1})$.

Habiro found a general formula for J^K expressing it in terms of polynomials $H_k^K(v) \in \mathbb{Z}[v^2, v^{-2}]$:

⁵ Lawrence's universal invariants can be defined for more general Lie algebras than $\mathfrak{sl}_2$ and for more general link-type diagrams (bottom tangles), see [7].

Theorem 3.13 ([8, Theorem 4.5]) *For any (string, 0-framed) knot K , the Lawrence universal $\mathfrak{sl}_2$ -invariant is given by*

$$J^K = \sum_{k=0}^{\infty} H_k^K(v) \sigma_k \quad (3.30)$$

where

$$\sigma_k = \prod_{i=1}^k (C^2 - (v^i + v^{-i})^2) \in \mathcal{Z}(\mathcal{U}_h), \quad k \geq 0$$

Now, Habiro's Theorem 1.1 stated in Introduction follows from Theorem 3.13 by evaluating the elements σ_k on finite-dimensional irreducible representations of $\mathcal{U}_h$ using quantum traces. It is well known that such representations V_n are classified by the nonnegative integers—the dimension (i.e. the rank of V_n as a free module over $\mathbb{Q}[[h]]$). Recall, for a finite-dimensional representation $\rho_V : \mathcal{U}_h \rightarrow \text{End}_{\mathbb{Q}[[h]]}(V)$ and an element $u \in \mathcal{U}_h$, the quantum trace $\text{tr}_q^V(u)$ is defined by

$$\text{tr}_q(u, v) := \text{Tr}_V[\rho_V(Ku)] \quad (3.31)$$

where Tr_V is the usual (matrix) trace on V . For central elements $z \in \mathcal{Z}(\mathcal{U}_h)$ one can compute (3.31) using the Harish-Chandra homomorphism

$$\mathcal{Z}(\mathcal{U}_h) \hookrightarrow \mathcal{U}_h \rightarrow \mathbb{Q}[[h]][H]$$

defined (on the PBW basis of $\mathcal{U}_h$) by

$$\varphi(F^i H^j E^k) = \delta_{i,0} \delta_{k,0} H^j$$

Specifically, for any $n \geq 1$, we have

$$\text{tr}_q(z, V_n) = \dim_v(V_n) \text{ev}_n(\varphi(z)) = [n]_v \text{ev}_n(\varphi(z)) \quad (3.32)$$

where $\text{ev}_n : \mathbb{Q}[[h]][H] \rightarrow \mathbb{Q}[[h]]$ is the evaluation map $f(H) \mapsto f(n)$.

Using formula (3.32), it is straightforward to show that

$$\text{tr}_q(\sigma_k; V_n) = \frac{1}{v - v^{-1}} \prod_{p=n-k}^{n+k} (v^p - v^{-p}), \quad \forall k \geq 1$$

Thus, setting $v = q^2$, we obtain⁶

$$\text{tr}_q(\sigma_k, V_n) = c_{n,k}(q) \quad (3.33)$$

⁶ We warn the reader that our q differs from the q in [8]: in fact, the q in [8] equals v^2 , which is our q^4 .

where $c_{n,k}(q)$ are precisely cyclotomic coefficients (1.2). It follows from Theorem 3.13 and formula (3.33) that

$$\hat{J}^K(V_n) = \mathrm{tr}_q(J^K, V_n) = \sum_{k \geq 0} c_{n,k}(q) H_k^K(q) = J_n^K(q) \quad (3.34)$$

Now, the proof of Theorem 1.6 reduces to the one-line calculation

$$\begin{aligned} \hat{J}^K(\tilde{V}_n) &= \sum_{p=1}^n (-1)^{n+p} a_{n,p} \hat{J}^K(V_n) \\ &\stackrel{(3.34)}{=} \sum_{p=1}^n (-1)^{n+p} a_{n,p} J_p^K(q) \\ &= J_n^K(q, t_1, t_2) \end{aligned}$$

where the last equality is formula (3.3) of Lemma 3.3.

Remark 3.14 In connection with Theorem 1.6, one might wonder why an invariant defined by a DAHA action on a skein module could be expressed in terms of the representation ring of $\mathcal{U}_q(\mathfrak{sl}_2)$. A brief explanation for this is as follows: consider the *Temperley-Lieb category*, which is a monoidal category whose objects are the natural numbers and whose morphisms from m to n are the (m, n) -tangles⁷ in $[0, 1] \times [0, 1]$ regarded modulo the Kauffman bracket skein relations. The monoidal structure comes from addition on objects, and on morphisms is defined using juxtaposition of disks. It is a classical fact (see [4, 13, 25] and references therein) that (the Karoubi envelope of) the Temperley-Lieb category is equivalent to the category $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$ of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. This implies that there is a natural map $H H_0(\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))) \rightarrow K_q(S^1 \times D^2)$ from the Hochschild homology of $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$ to the skein module of (closed) loops in the annulus, which is actually an isomorphism. On the other hand, for any semisimple category C , there is a canonical (Chern character) map $\mathrm{ch} : K_0(C) \rightarrow H H_0(C)$ which becomes an isomorphism upon linearization of $K_0(C)$. This means in our case that we can naturally identify the representation ring $R_q := K_0(\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2)))$ with the skein algebra $K_q(S^1 \times D^2)$. As a result, for a knot K we get a commutative diagram

$$\begin{array}{ccc} R_q & \xrightarrow{\mathrm{ch}} & K_q(S^1 \times D^2) \\ & \searrow \hat{J}^K & \downarrow \langle -, \varnothing \rangle \\ & & \mathbb{Q}[[h]] \end{array}$$

which leads to formula (3.34).

⁷ An (m, n) -tangle is a properly embedded 1-manifold in $[0, 1] \times [0, 1]$ with m endpoints on $\{0\} \times [0, 1]$ and n endpoints on $\{1\} \times [0, 1]$.

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