

COMMUTATORS OF CAUCHY–SZEGŐ TYPE INTEGRALS FOR DOMAINS IN $\mathbb{C}^n$ WITH MINIMAL SMOOTHNESS

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ABSTRACT. In this paper we study the commutator of Cauchy type integrals $\mathcal{C}$ on a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$ with boundary bD satisfying the minimum regularity condition C^2 as in the recent result of Lanzani–Stein. We point out that in this setting the Cauchy type integrals $\mathcal{C}$ is the sum of the essential part $\mathcal{C}^{\sharp}$ which is a Calderón–Zygmund operator and a remainder $\mathcal{R}$ which is no longer a Calderón–Zygmund operator. We show that the commutator $[b, \mathcal{C}]$ is bounded on $L^p(bD)$ ($1 < p < \infty$) if and only if b is in the BMO space on bD . Moreover, the commutator $[b, \mathcal{C}]$ is compact on $L^p(bD)$ ($1 < p < \infty$) if and only if b is in the VMO space on bD . Our method can also be applied to the commutator of Cauchy–Leray integral in a bounded, strongly $\mathbb{C}$ -linearly convex domain D in $\mathbb{C}^n$ with the boundary bD satisfying the minimum regularity $C^{1,1}$. Such a Cauchy–Leray integral is a Calderón–Zygmund operator as proved in the recent result of Lanzani–Stein. We also point out that our method provides another proof of the boundedness and compactness of commutator of Cauchy–Szegő operator on a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$ with smooth boundary (first established by Krantz–Li).

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The theory of Hardy spaces originated from the study of functions on the complex plane. Denote the open unit disc in the complex plane by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We recall that the classical Hardy space H^p , $0 < p < \infty$, on $\mathbb{D}$ is defined as the space of holomorphic functions f that satisfy $\|f\|_{H^p(\mathbb{D})} < \infty$, where

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}.$$

It is easy to check that the pointwise product of two $H^2(\mathbb{D})$ functions is a function in the Hardy space $H^1(\mathbb{D})$. The converse is not obvious but actually is true and we have the important *Riesz factorization theorem*: “A function f is in $H^1(\mathbb{D})$ if and only if there exist $g, h \in H^2(\mathbb{D})$ with $f = g \cdot h$ and $\|f\|_{H^1(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})} \|h\|_{H^2(\mathbb{D})}$.”

A similar result holds for the Hardy space H^1 on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This factorisation has a key role in proving the equivalence of norms

$$\|[b, H]\|_{L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})} \approx \|b\|_{\text{BMO}(\mathbb{T})},$$

where $[b, H](f) = bH(f) - H(bf)$ is the commutator of a BMO function b and the Hilbert transform H on the unit circle. We note that this result can be interpreted through Hankel operators, and one then recovers a famous result of Nehari [27]. See [19] for the history and literature of the Nehari theorem. See also [5] for the norm equivalence $\|[b, R_j]\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \approx \|b\|_{\text{BMO}(\mathbb{R}^n)}$, where R_j ($j = 1, \dots, n$) is the j th Riesz transform on the Euclidean space $\mathbb{R}^n$, and [4] for the application of commutator to certain version of div-curl lemmas.

Related estimates on commutators have been studied extensively in different settings, see for example [2, 7, 8, 9, 11, 12, 13, 14, 23, 24, 25, 26, 29, 32, 33, 6] and the references therein.

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We now recall the extension of this fundamental commutator result to the setting of several complex variables. Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary bD , $d\sigma$ the Lebesgue surface measure on bD and $L^p(bD)$ the usual Lebesgue space on bD with respect to the measure $d\sigma$. Let $H^p(D)$ be the holomorphic Hardy spaces defined in [16, 30]. Fatou's theorem [15] shows that, for any $0 < p \leq \infty$, a holomorphic function $f \in H^p(D)$ has a radial limit at almost all points on bD . It then follows from the maximum principle that one can identify $H^p(D)$ as a closed subspace of $L^p(bD)$. Let $S : L^2(bD) \rightarrow H^2(D)$ be the orthogonal projection via the reproducing kernel $S(z, w)$ which is known as the Cauchy–Szegő kernel. For a number of special cases and classes of domains D , we may identify the operator S as a singular integral operator on bD ; in fact, in many instances the Cauchy–Szegő kernel $S(z, w)$ is C^∞ on $bD \times bD \setminus \{z = w\}$.

Recall that when D is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, let T_S be the singular integral associated with the Cauchy–Szegő kernel, in fact one has $S(f)(z) = \frac{1}{2}f(z) + cT_S(f)$ for almost every $z \in bD$. Krantz and Li [18] first proved the following result regarding the boundedness of the commutator of T_S with respect to the BMO space on the boundary bD , as well as the compactness of commutator with respect to the VMO space. Here the BMO and VMO spaces were studied in [17]. To be more precise,

Theorem A ([18]). *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and let $b \in L^1(bD)$. Then for $1 < p < \infty$,*

- (i) *$b \in \text{BMO}(bD)$ if and only if $[b, T_S]$ is bounded on $L^p(bD)$.*
- (ii) *$b \in \text{VMO}(bD)$ if and only if $[b, T_S]$ is compact on $L^p(bD)$.*

We note that in the study of boundedness of the commutator $[b, T_S]$, the regularity of the kernel $S(z, w)$ plays a key role. In the proof of Theorem A, it follows from the results in [1, 10, 28] that the Cauchy–Szegő kernel $S(z, w) \in C^\infty(bD \times bD \setminus \{(z, z) : z \in bD\})$ and is a standard Calderón–Zygmund kernel, hence the boundedness and compactness of the commutator follows from the standard approach as in [5] with suitable modifications. Conversely, when $[b, T_S]$ is bounded on $L^p(bD)$, to show $b \in \text{BMO}(bD)$, Krantz–Li used the fact that the reciprocal of the Cauchy–Szegő kernel, $\frac{1}{S(z, w)}$ has a decomposition as a finite sum of holomorphic functions. Hence, by writing $1 = S(z, w) \times \frac{1}{S(z, w)}$ and by the decomposition of $\frac{1}{S(z, w)}$, they could link the BMO norm to the commutator with some additional algebra. By using this technique they also showed that when $[b, T_S]$ is compact, $b \in \text{VMO}(bD)$.

Recently, Lanzani and Stein [21] studied the Cauchy–Szegő projection operator in a bounded domain D in $\mathbb{C}^n$ which is strongly pseudoconvex and its boundary bD satisfies the minimum regularity condition of class C^2 . The measure that they used on the boundary bD is the Leray–Levi measure $d\lambda$ (for the details we refer to Section 2 below). They obtained the $L^p(bD)$ boundedness ($1 < p < \infty$) of a family of Cauchy integrals $\{\mathcal{C}_\epsilon\}_\epsilon$, and since the role of the parameter ϵ is of no consequence here, when denoting a member in this family we will simply write $\mathcal{C}$. Here the space $L^p(bD)$ is with respect to $d\lambda$. We point out that the kernel of these Cauchy integral operators do not satisfy the standard size or smoothness conditions for Calderón–Zygmund operators. To obtain the $L^p(bD)$ boundedness, they decomposed the Cauchy transform $\mathcal{C}$ which is the restriction of such a Cauchy integral on bD into the essential part $\mathcal{C}^\sharp$ and the remainder $\mathcal{R}$, i.e.,

$$\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R},$$

where the kernel of $\mathcal{C}^\sharp$, denoted by $C^\sharp(w, z)$, satisfies the standard size and smoothness conditions for Calderón–Zygmund operators, i.e. there exists a positive constant A_1 such

that for every $w, z \in bD$ with $w \neq z$,

$$(1.1) \quad \begin{cases} a) & |C^\sharp(w, z)| \leq A_1 \frac{1}{d(w, z)^{2n}}; \\ b) & |C^\sharp(w, z) - C^\sharp(w', z)| \leq A_1 \frac{d(w, w')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq cd(w, w'); \\ c) & |C^\sharp(w, z) - C^\sharp(w, z')| \leq A_1 \frac{d(z, z')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq cd(z, z') \end{cases}$$

for an appropriate constant $c > 0$ and where $d(z, w)$ is a quasi-distance suitably adapted to D . And hence, the $L^p(bD)$ boundedness ($1 < p < \infty$) of $C^\sharp$ follows from a version of $T(1)$ theorem. However, the kernel $R(w, z)$ of $\mathcal{R}$ satisfies a size condition and a smoothness condition for only one of the variables as follows: there exists a positive constant C_R such that for every $w, z \in bD$ with $w \neq z$,

$$(1.2) \quad \begin{cases} d) & |R(w, z)| \leq C_R \frac{1}{d(w, z)^{2n-1}}; \\ e) & |R(w, z) - R(w, z')| \leq C_R \frac{d(z, z')}{d(w, z)^{2n}}, \quad \text{if } d(w, z) \geq c_R d(z, z') \end{cases}$$

for an appropriate large constant c_R . It is worth pointing out that in the size condition and smoothness condition above, the dimensions are strictly smaller than the homogeneous dimension $2n$ of the boundary bD . The $L^p(bD)$ boundedness ($1 < p < \infty$) of $\mathcal{R}$ follows from Schur's lemma. It is also worth to point out that the hypothesis of minimal smoothness is sharp, see more explanations and counterexamples in [22] when the boundary bD does not satisfy the C^2 smoothness.

Thus, along the literature of Nehari, Coifman–Rochberg–Weiss, Krantz–Li, it is natural to study the behavior of the commutator of Cauchy type integrals as studied by Lanzani–Stein ([21]), which is not a standard Calderón–Zygmund operator, with a BMO function for a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with minimal smoothness.

The main result of our paper is on the commutator of Cauchy transform $\mathcal{C}$ (as in [21]).

Theorem 1.1. *Suppose $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class C^2 and is strongly pseudoconvex. Suppose $b \in L^1(bD, d\lambda)$. Then for $1 < p < \infty$,*

- (1) $b \in \text{BMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$.
- (2) $b \in \text{VMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$.

We first point out that the method in [18] does not work in this setting, since in general there is no information about the reciprocal of the kernel of Cauchy transform $\mathcal{C}$.

To obtain the necessary condition in (1) above, we use the decomposition $\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}$ from [21]. Then when b is in $\text{BMO}(bD, d\lambda)$, we study the boundedness of commutators $[b, \mathcal{C}^\sharp]$ and $[b, \mathcal{R}]$, respectively. For $[b, \mathcal{C}^\sharp]$, the upper bound follows directly from the known result in [17, Theorem 3.1] since $\mathcal{C}^\sharp$ is a Calderón–Zygmund operator. For $[b, \mathcal{R}]$, although the kernel of $\mathcal{R}$ does not satisfies the smoothness condition for the first variable and the dimension of the size condition does not match the homogeneous dimension, we can still obtain the upper bound by using the condition that D is a bounded domain and using the sharp maximal function and the John–Nirenberg inequality together with a suitable decomposition of the underlying space bD . Thus, combining the above two results, we obtain the upper bound of the commutator of $[b, \mathcal{C}]$.

To prove the sufficient condition in (1) above, we first point out that comparing to the previous results such as in [33, 24, 8, 32], the kernel of the operator here does not satisfy the conditions such as dilation invariance or sign invariance in a collection of chosen balls. Hence we make good use of the explicit kernel condition of the essential part $\mathcal{C}^\sharp$ and the upper bound of the kernel of the remainder $\mathcal{R}$ as in d) above, and then combine an idea from [33] (see also [24]) of using the median value for the definition of BMO space instead

of average, and exploiting a suitable decomposition of the underlying domain to match the kernel condition.

To obtain the necessary condition in (2) above, again we point out that since $\mathcal{C}^\sharp$ is a Calderón–Zygmund operator, the proof follows from [18]. It suffices to prove that $[b, \mathcal{R}]$ is compact when $b \in \text{VMO}(bD, d\lambda)$. This follows from a standard approach via Ascoli–Arzela theorem, together with the specific conditions on the kernel of $\mathcal{R}$.

To prove the sufficient condition in (2) above, we note that the classical approach of [33] does not apply since $\mathcal{C}$ is no longer a Calderón–Zygmund operator. To verify that $b \in \text{VMO}(bD, d\lambda)$ when $[b, \mathcal{R}]$ is compact, the key steps are the following: i) our approach in the proof of the sufficient condition in (1) by using a suitable modification of decompositions; ii) a fundamental fact that there is no bounded operator $T : \ell^p(\mathbb{N}) \rightarrow \ell^p(\mathbb{N})$ with $Te_j = Te_k \neq 0$ for all $j, k \in \mathbb{N}$. Here, e_j is the standard basis for $\ell^p(\mathbb{N})$. It is worth to point out that this proof here is new in the literature of compactness of commutators.

Remark 1.2. In Theorem [1.1] we used the Leray–Levi measure $d\lambda$ for the L^p , BMO and VMO spaces. However, we point out that the result also holds for the L^p , BMO and VMO spaces with respect to any measure $d\omega$ on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD (as stated in [21, Section 1.2]). We provide the explanation after the proof of Theorem [1.1].

Remark 1.3. Recall that the Cauchy integrals studied in [21] are a family of operators, and we only consider one member from this family. However, in [21] they also pointed out that, if we work with a family of Cauchy integrals $\{\mathcal{C}_\epsilon\}_\epsilon$ parameterized by $0 < \epsilon < \epsilon_0$, then by splitting $\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^\sharp + \mathcal{R}_\epsilon$, the operator norm $\|\mathcal{R}_\epsilon\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}$ can go to ∞ as $\epsilon \rightarrow 0$. Here we also point out that, this fact will affect the $L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)$ norm of the commutator $[b, \mathcal{C}_\epsilon]$ as well. We provide the explanation in details at the end of Subsection 2.3.

We also consider the Cauchy–Leray integral in the setting of Lanzani–Stein [20], where they studied the Cauchy–Leray integral in a bounded domain D in $\mathbb{C}^n$ which is strongly $\mathbb{C}$ -linearly convex and the boundary bD satisfies the minimum regularity $C^{1,1}$ (for the details we refer to Section 3 below). They obtained the $L^p(bD)$ boundedness ($1 < p < \infty$) of the Cauchy–Leray transform $\mathcal{C}$ by showing that the kernel $K(w, z)$ of $\mathcal{C}$ satisfies the standard size and smoothness conditions of Calderón–Zygmund operators as in a), b) and c) above (for details of these definitions and notation, we refer the readers to Section 3), and that $\mathcal{C}$ satisfies a suitable version of $T(1)$ theorem. Following a similar approach as in the proof for Theorem [1.1], we arrive at the second main result of this paper on the commutator of the Cauchy–Leray transform (as in [20]).

Theorem 1.4. Let D be a bounded domain in $\mathbb{C}^n$ of class $C^{1,1}$ that is strongly $\mathbb{C}$ -linearly convex and let $b \in L^1(bD, d\lambda)$. Let $\mathcal{C}$ be the Cauchy–Leray transform (as in [20]). Then for $1 < p < \infty$,

- (1) $b \in \text{BMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$.
- (2) $b \in \text{VMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$.

Similar to the remark for Theorem [1.1], we have the following.

Remark 1.5. In Theorem [1.4] we used the Leray–Levi measure $d\lambda$ for the L^p , BMO and VMO spaces. However, we point out that the result also holds for the L^p , BMO and VMO spaces with respect to any measure $d\omega$ on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD . We provide the explanation in details after the proof of Theorem [1.4].

This paper is organised as follows. In Section 2 we recall the notation and definitions related to a family of Cauchy integrals for bounded strongly pseudoconvex domains in $\mathbb{C}^n$

with minimal smoothness, then we prove Theorem 1.1. In Section 3 we recall the notation and definitions related to the Cauchy–Leray integral for bounded $\mathbb{C}$ -linearly convex domains in $\mathbb{C}^n$ with minimal smoothness and we prove Theorem 1.4. In the last section we point out that our method here can provide a different proof of Theorem A for the Cauchy–Szegő operator on a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, which was first obtained in [18].

Throughout this paper, c and $\tilde{c}$ will denote positive constants which are independent of the main parameters, but they may vary from line to line. For every $p \in (1, \infty)$, p' means the conjugate of p , i.e., $1/p + 1/p' = 1$. By $f \lesssim g$, we shall mean $f \leq cg$ for some positive constant c . If $f \lesssim g$ and $g \lesssim f$, we then write $f \approx g$.

2. CAUCHY TYPE INTEGRAL FOR BOUNDED STRONGLY PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$ WITH MINIMAL SMOOTHNESS

2.1. Preliminaries. The submanifolds we shall be interested in are the boundaries of appropriate domains $D \subset \mathbb{C}^n$. More precisely, we consider a bounded domain D with defining function ρ , which means that $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ with $\rho : \mathbb{C}^n \rightarrow \mathbb{R}$ and $bD = \{w \in \mathbb{C}^n : \rho(w) = 0\}$ with $\nabla \rho(w) \neq 0$ for all $w \in bD$.

In this section, we always assume that D is a bounded strongly pseudoconvex domain whose boundary is of class C^2 , that is, $\rho \in C^2(\mathbb{C}^n, \mathbb{R})$.

We now recall the notation from [21]. Since our domain is strongly pseudoconvex, we may assume without loss of generality that its defining function ρ is strictly plurisubharmonic (see [34]). The assumptions regarding the domain D and ρ will be in force throughout in this section and so will not be restated below.

Let $\mathcal{L}_0(w, z)$ be the negative of the Levi polynomial at $w \in bD$, given by

$$\mathcal{L}_0(w, z) = \langle \partial \rho(w), w - z \rangle - \frac{1}{2} \sum_{j,k} \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k} (w_j - z_j)(w_k - z_k),$$

where $\partial \rho(w) = (\frac{\partial \rho}{\partial w_1}(w), \dots, \frac{\partial \rho}{\partial w_n}(w))$ and we have used the notation $\langle \eta, \zeta \rangle = \sum_{j=1}^n \eta_j \zeta_j$ for $\eta = (\eta_1, \dots, \eta_n), \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$. The strict plurisubharmonicity of ρ implies that

$$2 \operatorname{Re} \mathcal{L}_0(w, z) \geq -\rho(z) + c|w - z|^2,$$

for some $c > 0$, whenever $w \in bD$ and $z \in \bar{D}$ is sufficiently close to w . Then a modification of $\mathcal{L}_0$ is as follows

$$(2.1) \quad g_0(w, z) = \chi \mathcal{L}_0 + (1 - \chi)|w - z|^2.$$

Here $\chi = \chi(w, z)$ is a C^∞ -cutoff function with $\chi = 1$ when $|w - z| \leq \mu/2$ and $\chi = 0$ if $|w - z| \geq \mu$. Then for μ chosen sufficiently small (and then kept fixed throughout), we have that

$$\operatorname{Re} g_0(w, z) \geq c(-\rho(z) + |w - z|^2)$$

for z in $\bar{D}$ and w in bD , with c a positive constant.

Note that the modified Levi polynomial g_0 has no smoothness beyond continuity in the variable w . So in [21], for each $\epsilon > 0$ they considered a variant g_ϵ defined as follows: let $\{\tau_{jk}^\epsilon(w)\}$ be an $n \times n$ -matrix of C^1 functions such that

$$\sup_{w \in bD} \left| \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k} - \tau_{jk}^\epsilon(w) \right| \leq \epsilon, \quad 1 \leq j, k \leq n.$$

Set

$$\mathcal{L}_\epsilon(w, z) = \langle \partial \rho(w), w - z \rangle - \frac{1}{2} \sum_{j,k} \tau_{jk}^\epsilon(w) (w_j - z_j)(w_k - z_k),$$

and define

$$g_\epsilon(w, z) = \chi \mathcal{L}_\epsilon + (1 - \chi)|w - z|^2, \quad z, w \in \mathbb{C}^n.$$

Now g_ϵ is C^1 in w and C^∞ in z . We note that

$$|g_0(w, z) - g_\epsilon(w, z)| \lesssim \epsilon |w - z|^2.$$

We shall always assume that ϵ is sufficiently small, we then have

$$|g_\epsilon(w, z)| \approx |g_0(w, z)|,$$

where the equivalence $\approx$ is independent of ϵ .

Now on the boundary bD , define the function $\mathbf{d}(w, z) = |g_0(w, z)|^{\frac{1}{2}}$. According to [21, Proposition 3], $\mathbf{d}$ satisfies the following conditions: there exist constants $A_2 > 0$ and $C_d > 1$ such that for all $w, z, z' \in bD$,

$$(2.2) \quad \begin{cases} 1) & \mathbf{d}(w, z) = 0 \quad \text{iff} \quad w = z; \\ 2) & A_2^{-1} \mathbf{d}(z, w) \leq \mathbf{d}(w, z) \leq A_2 \mathbf{d}(z, w); \\ 3) & \mathbf{d}(w, z) \leq C_d (\mathbf{d}(w, z') + \mathbf{d}(z', z)). \end{cases}$$

Next we recall the Leray–Levi measure $d\lambda$ on bD defined via the $(2n - 1)$ -form

$$\frac{1}{(2\pi i)^n} \partial \rho \wedge (\bar{\partial} \partial \rho)^{n-1}.$$

To be more precise, we have the linear functional

$$(2.3) \quad f \mapsto \frac{1}{(2\pi i)^n} \int_{bD} f(w) j^* (\partial \rho \wedge (\bar{\partial} \partial \rho)^{n-1})(w) =: \int_{bD} f(w) d\lambda(w)$$

defined for $f \in C(bD)$, and this defines the measure $d\lambda$. Then one also has

$$d\lambda(w) = \frac{1}{(2\pi i)^n} j^* (\partial \rho \wedge (\bar{\partial} \partial \rho)^{n-1})(w) = \Lambda(w) d\sigma(w),$$

where j^* denotes the pullback under the inclusion $j : bD \hookrightarrow \mathbb{C}^n$, $d\sigma$ is the induced Lebesgue measure on bD and $\Lambda(w)$ is a continuous function such that $c \leq \Lambda(w) \leq \tilde{c}$, $w \in bD$, with c and $\tilde{c}$ two positive constants.

We also recall the boundary balls $B_r(w)$ determined via the quasidistance $\mathbf{d}$ and their measures, i.e.,

$$(2.4) \quad B_r(w) := \{z \in bD : \mathbf{d}(z, w) < r\}, \quad \text{where } w \in bD.$$

According to [21, p. 139],

$$(2.5) \quad c_\lambda^{-1} r^{2n} \leq \lambda(B_r(w)) \leq c_\lambda r^{2n}, \quad 0 < r \leq 1,$$

for some $c_\lambda > 1$.

In [21] they defined the Cauchy integrals determined by the denominators g_ϵ and studied their properties when ϵ is kept fixed. For convenience of notation we will henceforth drop explicit reference to ϵ . Thus, we will write g for g_ϵ in the following. To study the Cauchy transform $\mathcal{C}$, which is the restriction of such a Cauchy integral on bD , one of the key steps in [21] is that they provided a constructive decomposition of $\mathcal{C}$ as follows:

$$\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R},$$

where the essential part

$$(2.6) \quad \mathcal{C}^\sharp(f)(z) := \int_{w \in bD} C^\sharp(w, z) f(w) d\lambda(w), \quad z \in bD$$

with the kernel

$$C^\sharp(w, z) := \frac{1}{g(w, z)^n},$$

and the reminder

$$\mathcal{R}(f)(z) := \int_{w \in bD} R(w, z) f(w) dw.$$

Thus, if we write

$$\mathcal{C}(f)(z) := \int_{w \in bD} C(w, z) f(w) dw.$$

Then

$$C(w, z) = C^\sharp(w, z) + R(w, z).$$

Moreover, Lanzani–Stein pointed out that the kernel $C^\sharp(w, z)$ satisfies the standard size and smoothness conditions as in (1.1). Furthermore, concerning the function $g(w, z)$ in the size estimates, according to [21, p. 139], there exist $C_g, \tilde{C}_g > 0$ such that

$$(2.7) \quad \tilde{C}_g \mathbf{d}(w, z)^2 \leq |g(w, z)| \leq C_g \mathbf{d}(w, z)^2.$$

The kernel $R(w, z)$ of $\mathcal{R}$ satisfies the size and smoothness conditions as in (1.2).

We now recall the BMO space on bD . Consider $(bD, \mathbf{d}, d\lambda)$ as a space of homogeneous type with bD compact. Then $\text{BMO}(bD, d\lambda)$ is defined as the set of all $b \in L^1(bD, d\lambda)$ such that

$$\|b\|_* := \sup_{z \in bD, r > 0, B_r(z) \subset bD} \frac{1}{\lambda(B_r(z))} \int_{B_r(z)} |b(w) - b_B| d\lambda(w) < \infty,$$

with the balls $B_r(z)$ as in (2.4) and where

$$(2.8) \quad b_B = \frac{1}{\lambda(B)} \int_B b(z) d\lambda(z).$$

And the norm is defined as

$$\|b\|_{\text{BMO}(bD, d\lambda)} := \|b\|_* + \|b\|_{L^1(bD, d\lambda)}.$$

We now recall the VMO space on bD (see for example [17]).

Definition 2.1. Let $f \in \text{BMO}(bD, d\lambda)$. Then $f \in \text{VMO}(bD, d\lambda)$ if and only if f satisfies

$$(2.9) \quad \lim_{a \rightarrow 0} \sup_{B \subset bD: r_B = a} \frac{1}{\lambda(B)} \int_B |f(z) - f_B| d\lambda(z) = 0,$$

where r_B is the radius of B .

We also let $BUC(bD)$ be the space of all bounded uniformly continuous functions on bD . Before we continue, we first point out that the Leray–Levi measure $d\lambda$ on bD is a doubling measure, and satisfies the condition (1.1) in [18]. That is,

Lemma 2.2. The Leray–Levi measure $d\lambda$ on bD is doubling, i.e., there is a positive constant C such that for all $x \in bD$ and $0 < r \leq 1$,

$$0 < \lambda(B_{2r}(x)) \leq C \lambda(B_r(x)) < \infty.$$

Moreover, λ satisfies the condition: there exist a constant $\epsilon_0 \in (0, 1)$ and a positive constant C such that

$$\lambda(B_r(x) \setminus B_r(y)) + \lambda(B_r(y) \setminus B_r(x)) \leq C \left(\frac{\mathbf{d}(x, y)}{r} \right)^{\epsilon_0}$$

for all $x, y \in bD$ and $\mathbf{d}(x, y) \leq r \leq 1$.

Proof. These facts can be verified directly from (2.5). □

Based on Lemma 2.2, we see that the following fundamental lemma from [18, Lemma 1.2] can be applied to our setting $(bD, d\lambda)$, which we restate as follows.

Lemma 2.3. *Let $b \in \text{VMO}(bD, d\lambda)$. Then for any $\xi > 0$, there is a function $b_\xi \in \text{BUC}(bD)$ such that*

$$(2.10) \quad \|b_\xi - b\|_* < \xi.$$

Moreover, b_ξ satisfies the following conditions: there is an $\epsilon \in (0, 1)$ such that

$$(2.11) \quad |b_\xi(w) - b_\xi(z)| < C_\xi \mathbf{d}(w, z)^\epsilon, \quad \forall w, z \in bD.$$

Moreover, based on Lemma 2.2 we also have another fundamental lemma, whose proof follows from Lemma 1.3 in [18]. For each $0 < \eta < 1$, we let $R^\eta(w, z)$ be a continuous extension of the kernel $R(w, z)$ of $\mathcal{R}$ from $bD \times bD \setminus \{(w, z) : \mathbf{d}(w, z) < \eta\}$ to $bD \times bD$ such that

$$\begin{aligned} R^\eta(w, z) &= R(w, z), \quad \text{if } \mathbf{d}(w, z) \geq \eta; \\ |R^\eta(w, z)| &\lesssim \frac{1}{\mathbf{d}(w, z)^{2n-1}}, \quad \text{if } \mathbf{d}(w, z) < \eta; \\ R^\eta(w, z) &= 0, \quad \text{if } \mathbf{d}(w, z) < \eta/c \text{ for some } c > 1. \end{aligned}$$

Now we let $\mathcal{R}^\eta$ be the integral operator associate to the kernel $R^\eta(w, z)$. Then we have the following.

Lemma 2.4. *Let $b \in \text{BUC}(bD)$ satisfy*

$$(2.12) \quad |b(w) - b(z)| < C_\eta \mathbf{d}(w, z)^\epsilon, \quad \text{for some } C_\eta \geq 1, \epsilon \in (0, 1), \forall w, z \in bD.$$

Then

$$\|[b, \mathcal{R}] - [b, \mathcal{R}^\eta]\|_{L^2(bD, d\lambda) \rightarrow L^2(bD, d\lambda)} \rightarrow 0$$

as $\eta \rightarrow 0$.

We should mention that the kernel of $\mathcal{R}$ is not a standard kernel.

2.2. Characterisation of $\text{BMO}(bD, d\lambda)$ via the Commutator $[b, \mathcal{C}]$. The maximal function Mf is defined as

$$Mf(z) = \sup_{z \in B \subset bD} \frac{1}{\lambda(B)} \int_B |f(w)| d\lambda(w).$$

The sharp function $f^\#$ is defined as

$$f^\#(z) = \sup_{z \in B \subset bD} \frac{1}{\lambda(B)} \int_B |f(w) - f_B| d\lambda(w),$$

where f_B is defined in (2.8).

Note that from Lemma 2.2, $d\lambda$ is a doubling measure, which implies that $(bD, d\lambda)$ satisfies [3, Theorems 1.4–1.6]. Hence, we can apply [3, Theorems 1.4–1.6] to our setting as follows.

Lemma 2.5 (Maximal Inequality). *For every $1 < p \leq \infty$ there exists a constant $c(bD, p)$ such that for every $f \in L^p(bD, d\lambda)$,*

$$\|Mf\|_{L^p(bD, d\lambda)} \leq c \|f\|_{L^p(bD, d\lambda)}.$$

Lemma 2.6 (John-Nirenberg Inequality). *For every $1 \leq p < \infty$ there exists a constant $c(bD, p)$ such that for every $f \in \text{BMO}(bD, d\lambda)$, every ball B ,*

$$\left(\frac{1}{\lambda(B)} \int_B |f(z) - f_B|^p d\lambda(z) \right)^{\frac{1}{p}} \leq c \|f\|_{\text{BMO}(bD, d\lambda)}.$$

Lemma 2.7 (Sharp Inequality). *For every $1 \leq p < \infty$ there exists a constant $c(bD, p)$ such that for every $f \in L^p(bD, d\lambda)$,*

$$\left\| f - \frac{1}{\lambda(bD)} \int_{bD} f(w) d\lambda(w) \right\|_{L^p(bD, d\lambda)} \leq c \|f^\#\|_{L^p(bD, d\lambda)}.$$

We note that in the unbounded domain we have $\|f\|_{L^p} \lesssim \|f^\#\|_{L^p}$, however, in the bounded domain, we will need to subtraction of the average of f over the whole domain.

We also need the L^p boundedness of $\mathcal{C}^\sharp$ and $\mathcal{R}$ which can be found in the proof of [21, Theorem 7].

Lemma 2.8. *Suppose $1 < p < \infty$ and $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class C^2 and is strongly pseudoconvex. Then $\mathcal{C}^\sharp$ and $\mathcal{R}$ are bounded operators on $L^p(bD, d\lambda)$.*

We now prove the argument (1) in Theorem 1.1

Proof of (1) in Theorem 1.1. Necessity:

We first prove necessity, namely that $b \in \text{BMO}(bD, d\lambda)$ implies the boundedness of $[b, \mathcal{C}]$. Since $b \in \text{BMO}(bD, d\lambda)$, without loss of generality we assume that

$$\int_{bD} b(z) d\lambda(z) = 0.$$

Otherwise we will just use $b(z) - \frac{1}{\lambda(bD)} \int_{bD} b(w) d\lambda(w)$ to replace b .

We can write

$$[b, \mathcal{C}] = [b, \mathcal{C}^\sharp] + [b, \mathcal{R}].$$

From Lemma 2.3 we can see that $\mathcal{C}^\sharp$ is bounded on $L^p(bD, d\lambda)$. Since the kernel of $\mathcal{C}^\sharp$ is a standard kernel on $bD \times bD$, according to [17, Theorem 3.1], we can obtain that $[b, \mathcal{C}^\sharp]$ is bounded on $L^p(bD, d\lambda)$ and

$$\|[b, \mathcal{C}^\sharp]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \lesssim \|b\|_{\text{BMO}(bD, d\lambda)}.$$

Thus, it suffices to show that

$$(2.13) \quad \|[b, \mathcal{R}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \lesssim \|b\|_{\text{BMO}(bD, d\lambda)}.$$

To see this, we first prove that for every $f \in L^p(bD, d\lambda)$,

$$(2.14) \quad \left\| ([b, \mathcal{R}](f))^\# \right\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \lesssim \|b\|_{\text{BMO}(bD, d\lambda)} \|f\|_{L^p(bD, d\lambda)}.$$

Since bD is bounded, there exists $\overline{C} > 0$ such that for any $B_r(z) \subset bD$ we have $r < \overline{C}$. For any $\tilde{z} \in bD$, let us fix a ball $B_r = B_r(z_0) \subset bD$ containing $\tilde{z}$, and let z be any point of B_r . Now take $j_0 = \lfloor \log_2 \frac{\overline{C}}{r} \rfloor + 1$. Since $\mathbf{d}$ is a quasi-distance, there exists $i_0 \in \mathbb{Z}^+$, independent of z, r , such that $\mathbf{d}(w, z) > c_R r$ whenever $w \in bD \setminus B_{2^{i_0}r}$, where c_R is in (1.2). We then write

$$\begin{aligned} [b, \mathcal{R}](f)(z) &= b(z) \mathcal{R}(f)(z) - \mathcal{R}(bf)(z) \\ &= (b(z) - b_{B_r}) \mathcal{R}(f)(z) - \mathcal{R}((b - b_{B_r})f \chi_{bD \cap B_{2^{i_0}r}})(z) - \mathcal{R}((b - b_{B_r})f \chi_{bD \setminus B_{2^{i_0}r}})(z) \\ &=: I(z) + II(z) + III(z). \end{aligned}$$

For I , by Hölder's inequality and the John-Nirenberg inequality (Lemma 2.6), we have

$$\begin{aligned} &\frac{1}{\lambda(B_r)} \int_{B_r} |I(z) - I_{B_r}| d\lambda(z) \\ &\leq \frac{2}{\lambda(B_r)} \int_{B_r} |I(z)| d\lambda(z) \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\lambda(B_r)} \int_{B_r} |b(z) - b_{B_r}| |\mathcal{R}(f)(z)| d\lambda(z) \\
&\lesssim \left(\frac{1}{\lambda(B_r)} \int_{B_r} |b(z) - b_{B_r}|^{s'} d\lambda(z) \right)^{\frac{1}{s'}} \left(\frac{1}{\lambda(B_r)} \int_{B_r} |\mathcal{R}(f)(z)|^s d\lambda(z) \right)^{\frac{1}{s}} \\
&\lesssim \|b\|_{\text{BMO}(bD, d\lambda)} (M(|\mathcal{R}f|^s)(\tilde{z}))^{\frac{1}{s}},
\end{aligned}$$

where $1 < s < p < \infty$.

For II , since $\mathcal{R}$ is bounded on $L^q(bD, d\lambda)$, $1 < q < \infty$, we have

$$\begin{aligned}
&\frac{1}{\lambda(B_r)} \int_{B_r} |II(z) - II_{B_r}| d\lambda(z) \\
&\leq \frac{2}{\lambda(B_r)} \int_{B_r} |II(z)| d\lambda(z) \\
&= \frac{2}{\lambda(B_r)} \int_{B_r} |\mathcal{R}((b - b_{B_r})f\chi_{bD \cap B_{2^{i_0}r}})(z)| d\lambda(z) \\
&\lesssim \left(\frac{1}{\lambda(B_r)} \int_{B_r} |\mathcal{R}((b - b_{B_r})f\chi_{bD \cap B_{2^{i_0}r}})(z)|^q d\lambda(z) \right)^{\frac{1}{q}} \\
&\lesssim \left(\frac{1}{\lambda(B_r)} \int_{bD \cap B_{2^{i_0}r}} |b(z) - b_{B_r}|^q |f(z)|^q d\lambda(z) \right)^{\frac{1}{q}} \\
&\lesssim \left(\frac{1}{\lambda(B_r)} \int_{bD \cap B_{2^{i_0}r}} |b(z) - b_{B_r}|^{qv'} d\lambda(z) \right)^{\frac{1}{qv'}} \left(\frac{1}{\lambda(B_r)} \int_{bD \cap B_{2^{i_0}r}} |f(z)|^{qv} d\lambda(z) \right)^{\frac{1}{qv}} \\
&\lesssim \|b\|_{\text{BMO}(bD, d\lambda)} (M(|f|^\beta)(\tilde{z}))^{\frac{1}{\beta}},
\end{aligned}$$

where we have chosen $q, v \in (1, \infty)$ such that $1 < qv < p < \infty$ and have set $\beta := qv$.

To estimate III , observe that if $i_0 \geq j_0$, then $bD \setminus B_{2^{i_0}r} = \emptyset$ and $|III(z) - III(z_0)| = 0$. If $i_0 < j_0$, then we have

$$\begin{aligned}
&|III(z) - III(z_0)| \\
&= |\mathcal{R}((b - b_{B_r})f\chi_{bD \setminus B_{2^{i_0}r}})(z) - \mathcal{R}((b - b_{B_r})f\chi_{bD \setminus B_{2^{i_0}r}})(z_0)| \\
&\leq \int_{bD \setminus B_{2^{i_0}r}} |R(w, z) - R(w, z_0)| |b(w) - b_{B_r}| |f(w)| d\lambda(w) \\
&\leq \mathbf{d}(z, z_0) \int_{bD \setminus B_{2^{i_0}r}} \frac{1}{\mathbf{d}(w, z_0)^{2n}} |b(w) - b_{B_r}| |f(w)| d\lambda(w) \\
&\leq r \left(\int_{bD \setminus B_{2^{i_0}r}} \frac{1}{\mathbf{d}(w, z_0)^{2n}} |b(w) - b_{B_r}|^{s'} d\lambda(w) \right)^{\frac{1}{s'}} \left(\int_{bD \setminus B_{2^{i_0}r}} \frac{1}{\mathbf{d}(w, z_0)^{2n}} |f(w)|^s d\lambda(w) \right)^{\frac{1}{s}},
\end{aligned}$$

where $1 < s < p < \infty$. Since bD is bounded, we can obtain

$$\begin{aligned}
\int_{bD \setminus B_{2^{i_0}r}} \frac{1}{\mathbf{d}(w, z_0)^{2n}} |f(w)|^s d\lambda(w) &\leq \sum_{j=i_0}^{j_0} \int_{2^j r \leq d(w, z_0) \leq 2^{j+1}r} \frac{1}{\mathbf{d}(w, z_0)^{2n}} |f(w)|^s d\lambda(w) \\
&\leq \sum_{j=i_0}^{j_0} \frac{1}{(2^j r)^{2n}} \int_{d(w, z_0) \leq 2^{j+1}r} |f(w)|^s d\lambda(w)
\end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{j=i_0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} |f(w)|^s d\lambda(w) \\ &\lesssim j_0 M(|f|^s)(\tilde{z}). \end{aligned}$$

Similarly, by the John–Nirenberg inequality, we have

$$\begin{aligned} \int_{bD \setminus B_{2^{i_0}r}} \frac{1}{d(w, z_0)^{2n}} |b(w) - b_{B_r}|^{s'} d\lambda(w) &\lesssim \sum_{j=i_0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} |b(w) - b_{B_r}|^{s'} d\lambda(w) \\ &\lesssim j_0 \|b\|_{\text{BMO}(bD, d\lambda)}^{s'}. \end{aligned}$$

Thus,

$$|III(z) - III(z_0)| \lesssim r j_0 \|b\|_{\text{BMO}(bD, d\lambda)} (M(|f|^s)(\tilde{z}))^{\frac{1}{s}}.$$

Therefore,

$$\begin{aligned} \frac{1}{\lambda(B_r)} \int_{B_r} |III(z) - III_{B_r}| d\lambda(z) &\leq \frac{2}{\lambda(B_r)} \int_{B_r} |III(z) - III(z_0)| d\lambda(z) \\ &\lesssim r j_0 \|b\|_{\text{BMO}(bD, d\lambda)} (M(|f|^s)(\tilde{z}))^{\frac{1}{s}} \\ &\lesssim r \left(\log_2 \left(\frac{\overline{C}}{r} \right) + 1 \right) \|b\|_{\text{BMO}(bD, d\lambda)} (M(|f|^s)(\tilde{z}))^{\frac{1}{s}} \\ &\lesssim \|b\|_{\text{BMO}(bD, d\lambda)} (M(|f|^s)(\tilde{z}))^{\frac{1}{s}}, \end{aligned}$$

where the last inequality comes from the fact that $r \log_2 \left(\frac{\overline{C}}{r} \right)$ is uniformly bounded.

By the above estimates we obtain that

$$|([b, \mathcal{R}]f)^\#(\tilde{z})| \lesssim \|b\|_{\text{BMO}(bD, d\lambda)} \left((M(|\mathcal{R}f|^s)(\tilde{z}))^{\frac{1}{s}} + (M(|f|^\beta)(\tilde{z}))^{\frac{1}{\beta}} + (M(|f|^s)(\tilde{z}))^{\frac{1}{s}} \right),$$

which, together with Lemma 2.5 and the fact that $\mathcal{R}$ is bounded on $L^p(bD, d\lambda)$, implies that (2.14) holds.

Based on (2.14), we now prove (2.13). We note that in the unbounded domain we have that $\|g\|_{L^p} \lesssim \|g^\#\|_{L^p}$, however, in this bounded domain, we will need a subtraction of the average of g over the whole domain on the left-hand side of this inequality (see Lemma 2.7).

Thus, we have that

$$\begin{aligned} (2.15) \quad \| [b, \mathcal{R}]f \|_{L^p(bD, d\lambda)} &\leq \left\| [b, \mathcal{R}]f - \frac{1}{\lambda(bD)} \int_{bD} [b, \mathcal{R}]f(z) d\lambda(z) \right\|_{L^p(bD, d\lambda)} \\ &\quad + \left\| \frac{1}{\lambda(bD)} \int_{bD} [b, \mathcal{R}]f(z) d\lambda(z) \right\|_{L^p(bD, d\lambda)} \\ &\leq \|([b, \mathcal{R}]f)^\#\|_{L^p(bD, d\lambda)} + \frac{1}{\lambda(bD)} \int_{bD} |[b, \mathcal{R}]f(z)| d\lambda(z) \lambda(bD)^{\frac{1}{p}} \\ &\lesssim \|b\|_{\text{BMO}(bD, d\lambda)} \|f\|_{L^p(bD, d\lambda)} + \frac{1}{\lambda(bD)^{\frac{1}{p'}}} \int_{bD} |[b, \mathcal{R}]f(z)| d\lambda(z), \end{aligned}$$

where the second inequality follows from Lemma 2.7 and the last inequality follows from (2.14). Now it suffice to show that

$$(2.16) \quad \int_{bD} |[b, \mathcal{R}]f(z)| d\lambda(z) \lesssim \lambda(bD)^{\frac{1}{p'}} \|f\|_{L^p(bD, d\lambda)} \|b\|_{\text{BMO}(bD, d\lambda)}.$$

Note that

$$\int_{bD} |[b, \mathcal{R}]f(z)| d\lambda(z) \leq \int_{bD} |\mathcal{R}(f)(z) b(z)| d\lambda(z) + \int_{bD} |\mathcal{R}(bf)(z)| d\lambda(z)$$

$$=: \mathfrak{A}_1 + \mathfrak{A}_2.$$

For the term $\mathfrak{A}_1$, by Hölder's inequality, the John–Nirenberg inequality and recalling that $b_{bD} = 0$, we have that

$$\begin{aligned} \mathfrak{A}_1 &\leq \|\mathcal{R}(f)\|_{L^p(bD, d\lambda)} \|b\|_{L^{p'}(bD, d\lambda)} = \|\mathcal{R}(f)\|_{L^p(bD, d\lambda)} \|b - b_{bD}\|_{L^{p'}(bD, d\lambda)} \\ &\lesssim \|f\|_{L^p(bD, d\lambda)} \cdot \lambda(bD)^{\frac{1}{p'}} \cdot \left(\frac{1}{\lambda(bD)} \int_{bD} |b(z) - b_{bD}|^{p'} d\lambda(z) \right)^{\frac{1}{p'}} \\ &\lesssim \lambda(bD)^{\frac{1}{p'}} \|f\|_{L^p(bD, d\lambda)} \|b\|_{\text{BMO}(bD, d\lambda)}. \end{aligned}$$

For the term $\mathfrak{A}_2$, by Hölder's inequality, the John–Nirenberg inequality and recalling that $b_{bD} = 0$, we have that

$$\begin{aligned} \mathfrak{A}_2 &\leq \lambda(bD)^{\frac{1}{\gamma}} \|\mathcal{R}(bf)\|_{L^\gamma(bD, d\lambda)} \\ &\lesssim \lambda(bD)^{\frac{1}{\gamma}} \|bf\|_{L^\gamma(bD, d\lambda)} \\ &\lesssim \lambda(bD)^{\frac{1}{\gamma}} \|f\|_{L^p(bD, d\lambda)} \cdot \|b\|_{L^\mu(bD, d\lambda)} \\ &= c\lambda(bD)^{\frac{1}{\gamma}} \|f\|_{L^p(bD, d\lambda)} \cdot \|b - b_{bD}\|_{L^\mu(bD, d\lambda)} \\ &\lesssim \lambda(bD)^{\frac{1}{\gamma}} \|f\|_{L^p(bD, d\lambda)} \lambda(bD)^{\frac{1}{\mu}} \|b\|_{\text{BMO}(bD, d\lambda)} \\ &\lesssim \lambda(bD)^{\frac{1}{p'}} \|f\|_{L^p(bD, d\lambda)} \|b\|_{\text{BMO}(bD, d\lambda)}, \end{aligned}$$

where we have chosen $\gamma, \mu > 1$ satisfying $\frac{1}{\gamma} = \frac{1}{p} + \frac{1}{\mu}$. Therefore, (2.16) holds, which, together with (2.15), implies that (2.13) holds. Hence, the proof of the necessity part is complete.

Sufficiency:

We next turn to proving the sufficient condition, namely that if $[b, \mathcal{C}]$ is bounded, then $b \in \text{BMO}(bD, d\lambda)$. Suppose $1 < p < \infty$. Assume that b is in $L^1(bD, d\lambda)$ and that $\|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} < \infty$.

We now denote by $C_1^\sharp(w, z)$ and $C_2^\sharp(w, z)$ the real and imaginary parts of $C^\sharp(w, z)$, respectively. And denote by $R_1(w, z)$ and $R_2(w, z)$ the real and imaginary parts of $R(w, z)$, respectively. Since $C^\sharp(w, z)$ is a standard Calderón–Zygmund kernel (we refer to (1.1)), it is direct to verify the following argument: let

$$(2.17) \quad \gamma_0 := \frac{3}{4^{n+2} 10 C_d^{4n+2} C_g C_R \max\{5A_1 A_2 (2C_d)^{2n+1} C_g, \sqrt{2} C_d A_2 + \sqrt{2} C_d^2\}},$$

then for every ball $B = B_r(z_0) \subset bD$ with $r < \gamma_0$, there exists another ball $\tilde{B} = B_r(w_0) \subset bD$ with $A_3 r \leq d(w_0, z_0) \leq (A_3 + 1)r$, where $A_3 = \max\{5\sqrt{2} A_1 A_2 (2C_d)^{2n+1} C_g, 2C_d A_2 + 2C_d^2\}$, such that at least one of the following properties holds:

a) for every $z \in B$ and $w \in \tilde{B}$, $C_1^\sharp(w, z)$ does not change sign and

$$|C_1^\sharp(w, z)| \geq \frac{3\sqrt{2}}{10(2C_d^2)^{2n} C_g d(w, z)^{2n}};$$

b) for every $z \in B$ and $w \in \tilde{B}$, $C_2^\sharp(w, z)$ does not change sign and

$$|C_2^\sharp(w, z)| \geq \frac{3\sqrt{2}}{10(2C_d^2)^{2n} C_g d(w, z)^{2n}}.$$

Here the constants A_1, C_g, C_R are defined in (1.1), (2.7), (1.2), respectively, and the constants A_2 and C_d are from (2.2).

Without loss of generality, we assume that the property **a)** holds. Then combining with the size estimate of $R(w, z)$ in (1.2), we obtain that for every $z \in B$ and $w \in \tilde{B}$, $C_1^\sharp(w, z) + R_1(w, z)$ does not change sign and that

$$(2.18) \quad |C_1^\sharp(w, z) + R_1(w, z)| \geq \frac{3\sqrt{2}}{20(2C_d^2)^{2n}C_g d(w, z)^{2n}}.$$

We test the $\text{BMO}(bD, d\lambda)$ condition on the case of balls with big radius and small radius.

Case 1: In this case we work with balls with a large radius, $r \geq \gamma_0$.

By (2.5) and by the fact that $\lambda(B) \geq \lambda(B_{\gamma_0}(z_0)) \approx \gamma_0^{2n}$, we obtain that

$$\frac{1}{\lambda(B)} \int_B |b(w) - b_B| d\lambda(w) \lesssim \gamma_0^{-2n} \|b\|_{L^1(bD, d\lambda)}.$$

Case 2: In this case we work with balls with a small radius, $r < \gamma_0$.

We aim to prove that for every fixed ball $B = B_r(z_0) \subset bD$ with radius $r < \gamma_0$,

$$\frac{1}{\lambda(B)} \int_B |b(w) - b_B| d\lambda(w) \lesssim \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}.$$

Now let $\tilde{B} = B_r(w_0)$ be the ball chosen as above, and let $m_b(\tilde{B})$ be the median value of b on the ball $\tilde{B}$ with respect to the measure $d\lambda$ defined as follows: $m_b(\tilde{B})$ is a real number that satisfies simultaneously

$$\lambda(\{z \in \tilde{B} : b(z) > m_b(\tilde{B})\}) \leq \frac{1}{2}\lambda(\tilde{B}) \quad \text{and} \quad \lambda(\{z \in \tilde{B} : b(z) < m_b(\tilde{B})\}) \leq \frac{1}{2}\lambda(\tilde{B}).$$

Then, following the idea in [24, Proposition 3.1] (see also [32]), by the definition of median value, we define $F_1 := \{w \in \tilde{B} : b(w) \leq m_b(\tilde{B})\}$ and $F_2 := \{w \in \tilde{B} : b(w) \geq m_b(\tilde{B})\}$. Then it is direct that $\tilde{B} = F_1 \cup F_2$, and moreover, from the definition of $m_b(\tilde{B})$, we see that

$$(2.19) \quad \lambda(F_i) \geq \frac{1}{2}\lambda(\tilde{B}), \quad i = 1, 2.$$

Next we define

$$E_1 = \{z \in B : b(z) \geq m_b(\tilde{B})\} \quad \text{and} \quad E_2 = \{z \in B : b(z) < m_b(\tilde{B})\},$$

then $B = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Then it is clear that

$$(2.20) \quad \begin{aligned} b(z) - b(w) &\geq 0, \quad (z, w) \in E_1 \times F_1 \\ b(z) - b(w) &< 0, \quad (z, w) \in E_2 \times F_2. \end{aligned}$$

And for (z, w) in $(E_1 \times F_1) \cup (E_2 \times F_2)$, we have

$$(2.21) \quad \begin{aligned} |b(z) - b(w)| &= |b(z) - m_b(\tilde{B}) + m_b(\tilde{B}) - b(w)| \\ &= |b(z) - m_b(\tilde{B})| + |m_b(\tilde{B}) - b(w)| \\ &\geq |b(z) - m_b(\tilde{B})|. \end{aligned}$$

Therefore, from (2.19), (2.18) and (2.21) we obtain that

$$\begin{aligned} &\frac{1}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| d\lambda(z) \\ &\lesssim \frac{1}{\lambda(B)} \frac{\lambda(F_1)}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| d\lambda(z) \\ &\lesssim \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} \frac{1}{d(w, z)^{2n}} |b(z) - b(w)| d\lambda(w) d\lambda(z) \\ &\lesssim \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} |C_1^\sharp(w, z) + R_1(w, z)| (b(z) - b(w)) d\lambda(w) d\lambda(z) \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{\lambda(B)} \int_{E_1} \left| \int_{F_1} C(w, z) (b(z) - b(w)) d\lambda(w) \right| d\lambda(z) \\
&= \frac{1}{\lambda(B)} \int_{E_1} |[b, \mathcal{C}](\chi_{F_1})(z)| d\lambda(z),
\end{aligned}$$

where the last but second inequality follows from the fact that $C_1^\sharp(w, z) + R_1(w, z)$ is the real part of $C(w, z)$.

Then, by using Hölder's inequality and the fact that $\|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} < \infty$, we further obtain

$$\begin{aligned}
&\frac{1}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| d\lambda(z) \\
&\lesssim \frac{1}{\lambda(B)} (\lambda(E_1))^{\frac{1}{p'}} \left(\int_{E_1} |[b, \mathcal{C}](\chi_{F_1})(z)|^p d\lambda(z) \right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{\lambda(B)} (\lambda(E_1))^{\frac{1}{p'}} \|[b, \mathcal{C}]\chi_{F_1}\|_{L^p(bD, d\lambda)} \\
&\lesssim \frac{1}{\lambda(B)} (\lambda(E_1))^{\frac{1}{p'}} (\lambda(F_1))^{\frac{1}{p}} \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \\
&\lesssim \frac{1}{\lambda(B)} (\lambda(E_1) + \lambda(F_1)) \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)},
\end{aligned}$$

where the last inequality follows from a direct calculation: if $\lambda(E_1) \geq \lambda(F_1)$, then

$$(\lambda(E_1))^{\frac{1}{p'}} (\lambda(F_1))^{\frac{1}{p}} \leq (\lambda(E_1))^{\frac{1}{p'}} (\lambda(E_1))^{\frac{1}{p}} \leq \lambda(E_1);$$

if $\lambda(E_1) \leq \lambda(F_1)$, then

$$(\lambda(E_1))^{\frac{1}{p'}} (\lambda(F_1))^{\frac{1}{p}} \leq (\lambda(F_1))^{\frac{1}{p'}} (\lambda(F_1))^{\frac{1}{p}} \leq \lambda(F_1).$$

Similarly, we can obtain that

$$\frac{1}{\lambda(B)} \int_{E_2} |b(z) - m_b(\tilde{B})| d\lambda(z) \lesssim \frac{1}{\lambda(B)} (\lambda(E_2) + \lambda(F_2)) \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\lambda(B)} \int_B |b(z) - m_b(\tilde{B})| d\lambda(z) \\
&= \frac{1}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})| d\lambda(z) + \frac{1}{\lambda(B)} \int_{E_2} |b(z) - m_b(\tilde{B})| d\lambda(z) \\
&\lesssim \frac{1}{\lambda(B)} (\lambda(E_1) + \lambda(F_1) + \lambda(E_2) + \lambda(F_2)) \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \\
&\lesssim \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}.
\end{aligned}$$

Therefore,

$$\frac{1}{\lambda(B)} \int_B |b(z) - b_B| d\lambda(z) \leq \frac{2}{\lambda(B)} \int_B |b(z) - m_b(\tilde{B})| d\lambda(z) \lesssim \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}.$$

This finishes the proof of (1) in Theorem [1.1](#). □

2.3. Characterisation of $\text{VMO}(bD, d\lambda)$ via the Commutator $[b, \mathcal{C}]$. We now prove the argument (2) in Theorem [1.1](#).

Proof of (2) in Theorem [1.1](#) **Sufficient condition:** Assume that $1 < p < \infty$ and that $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$, then $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$. By the argument (1) in Theorem [1.1](#), we have $b \in \text{BMO}(bD, d\lambda)$. Without loss of generality, we may assume that $\|b\|_{\text{BMO}(bD, d\lambda)} = 1$.

To show $b \in \text{VMO}(bD, d\lambda)$, we seek a contradiction. In its simplest form, the contradiction is that there is no bounded operator $T : \ell^p(\mathbb{N}) \rightarrow \ell^p(\mathbb{N})$ with $Te_j = Te_k \neq 0$ for all $j, k \in \mathbb{N}$. Here, e_j is the standard basis for $\ell^p(\mathbb{N})$.

The main step is to construct the approximates to a standard basis in ℓ^p , namely a sequence of functions $\{g_j\}$ such that $\|g_j\|_{L^p(bD, d\lambda)} \simeq 1$, and for a nonzero ϕ , we have $\|\phi - [b, \mathcal{C}]g_j\|_{L^p(bD, d\lambda)} < 2^{-j}$.

Suppose that $b \notin \text{VMO}(bD, d\lambda)$, then there exist $\delta_0 > 0$ and a sequence $\{B_j\}_{j=1}^\infty := \{B_{r_j}(z_j)\}_{j=1}^\infty$ of balls such that

$$(2.22) \quad \frac{1}{\lambda(B_j)} \int_{B_j} |b(z) - b_{B_j}| d\lambda(z) \geq \delta_0.$$

Without loss of generality, we assume that for all j , $r_j < \gamma_0$, where γ_0 is as in [\(2.17\)](#).

Now choose a subsequence $\{B_{j_i}\}$ of $\{B_j\}$ such that

$$(2.23) \quad r_{j_{i+1}} \leq \frac{1}{4c_\lambda} r_{j_i},$$

where c_λ is the constant as in [\(2.5\)](#).

For the sake of simplicity we drop the subscript i , i.e., we still denote $\{B_{j_i}\}$ by $\{B_j\}$.

Following the proof of (1) of Theorem [1.1](#), for each such B_j , we can choose a corresponding $\tilde{B}_j$. Now let $m_b(\tilde{B}_j)$ be the median value of b on the ball $\tilde{B}_j$ with respect to the measure $d\lambda$. Then, by the definition of median value, we can find disjoint subsets $F_{j,1}, F_{j,2} \subset \tilde{B}_j$ such that

$$F_{j,1} \subset \{w \in \tilde{B}_j : b(w) \leq m_b(\tilde{B}_j)\}, \quad F_{j,2} \subset \{w \in \tilde{B}_j : b(w) \geq m_b(\tilde{B}_j)\},$$

and

$$(2.24) \quad \lambda(F_{j,1}) = \lambda(F_{j,2}) = \frac{\lambda(\tilde{B}_j)}{2}.$$

Next we define

$$E_{j,1} = \{z \in B : b(z) \geq m_b(\tilde{B}_j)\}, \quad E_{j,2} = \{z \in B : b(z) < m_b(\tilde{B}_j)\},$$

then $B_j = E_{j,1} \cup E_{j,2}$ and $E_{j,1} \cap E_{j,2} = \emptyset$. Then it is clear that

$$(2.25) \quad \begin{aligned} b(z) - b(w) &\geq 0, & (z, w) &\in E_{j,1} \times F_{j,1}, \\ b(z) - b(w) &< 0, & (z, w) &\in E_{j,2} \times F_{j,2}. \end{aligned}$$

And for (z, w) in $(E_{j,1} \times F_{j,1}) \cup (E_{j,2} \times F_{j,2})$, we have

$$(2.26) \quad \begin{aligned} |b(z) - b(w)| &= |b(z) - m_b(\tilde{B}_j) + m_b(\tilde{B}_j) - b(w)| \\ &= |b(z) - m_b(\tilde{B}_j)| + |m_b(\tilde{B}_j) - b(w)| \\ &\geq |b(z) - m_b(\tilde{B}_j)|. \end{aligned}$$

We now consider

$$\tilde{F}_{j,1} := F_{j,1} \setminus \bigcup_{\ell=j+1}^{\infty} \tilde{B}_\ell \quad \text{and} \quad \tilde{F}_{j,2} := F_{j,2} \setminus \bigcup_{\ell=j+1}^{\infty} \tilde{B}_\ell, \quad \text{for } j = 1, 2, \dots$$

Then, based on the decay condition of the radius $\{r_j\}$, we obtain that for each j ,

$$\begin{aligned}
 \lambda(\tilde{F}_{j,1}) &\geq \lambda(F_{j,1}) - \lambda\left(\bigcup_{\ell=j+1}^{\infty} \tilde{B}_\ell\right) \geq \frac{1}{2}\lambda(\tilde{B}_j) - \sum_{\ell=j+1}^{\infty} \lambda(\tilde{B}_\ell) \\
 (2.27) \quad &\geq \frac{1}{2}\lambda(\tilde{B}_j) - \frac{c_\lambda^2}{(4c_\lambda)^{2n}-1}\lambda(\tilde{B}_j) \geq \frac{1}{4}\lambda(\tilde{B}_j).
 \end{aligned}$$

Now for each j , we have that

$$\begin{aligned}
 &\frac{1}{\lambda(B_j)} \int_{B_j} |b(z) - b_{B_j}| d\lambda(z) \\
 &\leq \frac{2}{\lambda(B_j)} \int_{B_j} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \\
 &= \frac{2}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) + \frac{2}{\lambda(B_j)} \int_{E_{j,2}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z).
 \end{aligned}$$

Thus, combining with (2.22) and the above inequalities, we obtain that at least one of the following inequalities holds:

$$\frac{2}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \geq \frac{\delta_0}{2}, \quad \frac{2}{\lambda(B_j)} \int_{E_{j,2}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \geq \frac{\delta_0}{2}.$$

We may assume that the first one holds, i.e.,

$$\frac{2}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \geq \frac{\delta_0}{2}.$$

Therefore, for each j , from (2.24) and (2.26) we obtain that

$$\begin{aligned}
 \frac{\delta_0}{4} &\leq \frac{1}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \\
 &\lesssim \frac{1}{\lambda(B_j)} \frac{\lambda(\tilde{F}_{j,1})}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \\
 &\lesssim \frac{1}{\lambda(B_j)} \int_{E_{j,1}} \int_{\tilde{F}_{j,1}} \frac{1}{d(w,z)^{2n}} |b(z) - b(w)| d\lambda(w) d\lambda(z).
 \end{aligned}$$

Following the same estimate as in the proof of (1) of Theorem 1.1, we obtain that

$$\begin{aligned}
 \frac{1}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) &\lesssim \frac{1}{\lambda(B_j)} \int_{E_{j,1}} \left| [b, \mathcal{C}](\chi_{\tilde{F}_{j,1}})(z) \right| d\lambda(z) \\
 &= \frac{1}{\lambda(B_j)^{\frac{1}{p'}}} \int_{E_{j,1}} \left| [b, \mathcal{C}]\left(\frac{\chi_{\tilde{F}_{j,1}}}{\lambda(B_j)^{\frac{1}{p}}}\right)(z) \right| d\lambda(z),
 \end{aligned}$$

where in the last equality, we use p' to denote the conjugate index of p .

Next, by using Hölder's inequality we further have

$$\begin{aligned}
 \frac{1}{\lambda(B_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) &\lesssim \frac{1}{\lambda(B_j)^{\frac{1}{p'}}} \lambda(E_{j,1})^{\frac{1}{p'}} \left(\int_{bD} |[b, \mathcal{C}](f_j)(z)|^p d\lambda(z) \right)^{\frac{1}{p}} \\
 &\lesssim \left(\int_{bD} |[b, \mathcal{C}](f_j)(z)|^p d\lambda(z) \right)^{\frac{1}{p}},
 \end{aligned}$$

where in the above inequalities we denote

$$f_j := \frac{\chi_{\tilde{F}_{j,1}}}{\lambda(B_j)^{\frac{1}{p}}}.$$

Thus, combining the above estimates we have that

$$0 < \delta_0 \lesssim \left(\int_{bD} |[b, \mathcal{C}](f_j)(z)|^p d\lambda(z) \right)^{\frac{1}{p}}.$$

Then, from (2.27), we obtain that

$$\frac{1}{4^{\frac{1}{p}}} \lesssim \|f_j\|_{L^p(bD, d\lambda)} \lesssim 1.$$

Thus, it is direct to see that $\{f_j\}_j$ is a bounded sequence in $L^p(bD, d\lambda)$ with a uniform $L^p(bD, d\lambda)$ -lower bound away from zero.

Since $[b, \mathcal{C}]$ is compact, we obtain that the sequence $\{[b, \mathcal{C}](f_j)\}_j$ has a convergent subsequence, denoted by

$$\{[b, \mathcal{C}](f_{j_i})\}_{j_i}.$$

We denote the limit function by g_0 , i.e.,

$$[b, \mathcal{C}](f_{j_i}) \rightarrow g_0 \quad \text{in } L^p(bD, d\lambda), \quad \text{as } i \rightarrow \infty.$$

Moreover, $g_0 \neq 0$.

After taking a further subsequence, labeled g_j , we have

- $\|g_j\|_{L^p(bD, d\lambda)} \simeq 1$,
- g_j are disjointly supported,
- and $\|g_0 - [b, \mathcal{C}]g_j\|_{L^p(bD, d\lambda)} < 2^{-j}$.

Take $a_j = j^{\frac{p}{p+1}}$, so that $\{a_j\} \in \ell^p \setminus \ell^1$. It is immediate that $\gamma = \sum_j a_j g_j \in L^p(bD, d\lambda)$, hence $[b, \mathcal{C}]\gamma \in L^p(bD, d\lambda)$. But, $g_0 \sum_j a_j \equiv \infty$, and yet

$$\left\| g_0 \sum_j a_j \right\|_{L^p(bD, d\lambda)} \leq \|[b, \mathcal{C}]\gamma\|_{L^p(bD, d\lambda)} + \sum_j a_j \|g_0 - [b, \mathcal{C}]g_j\|_{L^p(bD, d\lambda)} < \infty.$$

This contradiction shows that $b \in \text{VMO}(bD, d\lambda)$.

Necessary condition: Recall that $\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}$. Since the kernel $C^\sharp(\cdot, \cdot)$ of $\mathcal{C}^\sharp$ is a standard kernel, by [18, Theorem 1.1], $[b, \mathcal{C}^\sharp]$ is compact on $L^p(bD, d\lambda)$. Therefore, we only need to show that $[b, \mathcal{R}]$ is also compact on $L^p(bD, d\lambda)$.

From Lemma 2.4 for any $\xi > 0$, there exists $b_\xi \in BUC(bD)$ such that $\|b - b_\xi\|_* < \xi$. Then by Theorem 3.1 in [17], we have

$$\|[b, \mathcal{R}]f - [b_\xi, \mathcal{R}]f\|_{L^p(bD, d\lambda)} \leq C_p \|f\|_{L^p(bD, d\lambda)} \|b - b_\xi\|_* < \xi C_p \|f\|_{L^p(bD, d\lambda)}$$

Thus, to prove that $[b, \mathcal{R}]$ is compact on $L^p(bD, d\lambda)$, it suffices to prove that $[b_\xi, \mathcal{R}]$ is compact on $L^p(bD, d\lambda)$. By Lemma 2.3 and (2.11), without loss of generality, we may assume that $b \in BUC(bD)$ and (2.12) holds. By Lemma 2.4, it suffices to prove that for any fixed η satisfying $0 < \eta \ll 1$, $[b, \mathcal{R}^\eta]$ is compact on $L^p(bD, d\lambda)$.

Since $R(w, z)$ is continuous on $bD \times bD \setminus \{(z, z) : z \in bD\}$, for any $f \in L^p(bD, d\lambda)$, we see that $[b, \mathcal{R}^\eta]f$ is continuous on bD . To conclude the proof, we now argue that the image of the unit ball of $L^p(bD, d\lambda)$ under the commutator $[b, \mathcal{R}^\eta]$ is an equicontinuous family. Compactness follows from the Ascoli–Arzela theorem.

It remains to prove equicontinuity. For any $z, w \in bD$ with $d(w, z) < 1$, we have

$$\begin{aligned} & [b, \mathcal{R}^\eta]f(z) - [b, \mathcal{R}^\eta]f(w) \\ &= b(z) \int_{bD} R^\eta(u, z) f(u) d\lambda(u) - \int_{bD} R^\eta(u, z) b(u) f(u) d\lambda(u) \\ & \quad - b(w) \int_{bD} R^\eta(u, w) f(u) d\lambda(u) + \int_{bD} R^\eta(u, w) b(u) f(u) d\lambda(u) \end{aligned}$$

$$\begin{aligned}
&= (b(z) - b(w)) \int_{bD} R^\eta(u, z) f(u) d\lambda(u) + b(w) \int_{bD} (R^\eta(u, z) - R^\eta(u, w)) f(u) d\lambda(u) \\
&\quad + \int_{bD} (R^\eta(u, w) - R^\eta(u, z)) b(u) f(u) d\lambda(u) \\
&= (b(z) - b(w)) \int_{bD} R^\eta(u, z) f(u) d\lambda(u) + \int_{bD} (R^\eta(u, w) - R^\eta(u, z)) (b(u) - b(w)) f(u) d\lambda(u) \\
&=: I(z, w) + II(z, w).
\end{aligned}$$

For $I(z, w)$, by Hölder's inequality, we have

$$\begin{aligned}
|I(z, w)| &= |(b(z) - b(w))| \left| \int_{bD} R^\eta(u, z) f(u) d\lambda(u) \right| \\
&\leq c |b(z) - b(w)| \left(\int_{bD} |R^\eta(u, z)|^{p'} d\lambda(u) \right)^{\frac{1}{p'}} \|f\|_{L^p(bD, d\lambda)} \\
&\leq c \|f\|_{L^p(bD, d\lambda)} \mathbf{d}(w, z)^\epsilon,
\end{aligned}$$

where the last inequality is due to the fact that $R^\eta(u, z) \in C(bD \times bD)$ and bD is bounded.

Since b is bounded, if we let $\mathbf{d}(w, z) < \frac{\eta}{c \cdot c_R}$, by a discussion similar to [18, p. 645], we can obtain that

$$\begin{aligned}
|II(z, w)| &= \left| \int_{bD} (R^\eta(u, w) - R^\eta(u, z)) (b(u) - b(w)) f(u) d\lambda(u) \right| \\
&\leq c \|b\|_{L^\infty(bD, d\lambda)} \int_{bD \setminus B_{\frac{\eta}{c}}(z)} \frac{\mathbf{d}(w, z)}{\mathbf{d}(u, z)^{2n}} |f(u)| d\lambda(u) \\
&\leq c \|b\|_{L^\infty(bD, d\lambda)} \|f\|_{L^p(bD, d\lambda)} \mathbf{d}(w, z) \left(\int_{bD \setminus B_{\frac{\eta}{c}}(z)} \frac{1}{\mathbf{d}(u, z)^{2np'}} d\lambda(u) \right)^{\frac{1}{p'}} \\
&\leq c \|b\|_{L^\infty(bD, d\lambda)} \|f\|_{L^p(bD, d\lambda)} \mathbf{d}(w, z) \left(\frac{\eta}{c} \right)^{-2n} \lambda(bD)^{\frac{1}{p'}} \\
&\leq c_{\eta, p} \|b\|_{L^\infty(bD, d\lambda)} \|f\|_{L^p(bD, d\lambda)} \mathbf{d}(w, z).
\end{aligned}$$

Therefore, $\{[b, \mathcal{R}^\eta](\mathcal{U})\}$ is an equicontinuous family, where $\mathcal{U}$ is the unit ball in $L^p(bD, d\lambda)$. This finishes the proof of (2) in Theorem 1.1. $\square$

We now provide an explanation of Remark 1.2.

Proposition 2.9. *Theorem 1.1 holds for the L^p , BMO and VMO spaces with respect to any measure $d\omega$ on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD .*

Proof. We recall that the Leray–Levi measure $d\lambda$ played an important role in [21] since they needed to handle the adjoint of the Cauchy–Leray transform, which corresponds to $d\lambda$ but not the induced Lebesgue measure $d\sigma$ on bD . However, this distinction will not matter in the final statements of the $L^p(bD)$ -boundedness of $\mathcal{C}$, since $d\lambda$ and $d\sigma$ are equivalent in the following sense (stated in Section 2.1 but we recall again here):

$$d\lambda(w) = \Lambda(w) d\sigma(w),$$

where $\Lambda(w)$ is a continuous function on bD such that $c \leq \Lambda(w) \leq \tilde{c}$ for all $w \in bD$ with $c, \tilde{c}$ two positive absolute constants.

Now suppose $d\omega$ is a measure on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD . Since bD is bounded, we see that $c \leq \omega(w) \leq \tilde{c}$ for all $w \in bD$ with $c, \tilde{c}$ two positive absolute constants. Hence, $d\omega$ and $d\lambda$ are equivalent.

As a consequence, we see that the spaces $L^p(bD, d\omega)$, $\text{BMO}(bD, d\omega)$ and $\text{VMO}(bD, d\omega)$ are equivalent to $L^p(bD, d\lambda)$, $\text{BMO}(bD, d\lambda)$ and $\text{VMO}(bD, d\lambda)$.

Next, we recall that the proof of Theorem 1.1 depends only on the $L^p(bD)$ boundedness of the essential part $\mathcal{C}^\sharp$ and the remainder $\mathcal{R}$, as well as the pointwise estimate of the kernels of $\mathcal{C}^\sharp$ and $\mathcal{R}$. Hence, by repeating the proof of Theorem 1.1, we see that Theorem 1.1 holds for the $L^p(bD, d\omega)$, $\text{BMO}(bD, d\omega)$ and $\text{VMO}(bD, d\omega)$ spaces. $\square$

We now provide an explanation to Remark 1.3.

Proposition 2.10. *Suppose $b \in \text{BMO}(bD, d\lambda)$ and let $\{\mathcal{C}_\epsilon\}_\epsilon$ be a family of Cauchy integrals parameterized by $0 < \epsilon < \epsilon_0$. Then*

$$(2.28) \quad \|[b, \mathcal{C}_\epsilon]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \leq C_\epsilon \|b\|_{\text{BMO}(bD, d\lambda)},$$

where C_ϵ can go to ∞ as $\epsilon \rightarrow 0$.

Proof. Recall that in [21] it was pointed out that, for a family of Cauchy integrals $\{\mathcal{C}_\epsilon\}_\epsilon$ parameterized by $0 < \epsilon < \epsilon_0$, then by splitting $\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^\sharp + \mathcal{R}_\epsilon$, the operator norm $\|\mathcal{R}_\epsilon\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}$ can go to ∞ as $\epsilon \rightarrow 0$.

To be more precise, we first recall that as pointed out in [21],

$$c_\epsilon := \sup_{\substack{w \in bD \\ 1 \leq j, k \leq n}} |\nabla \tau_{j,k}^\epsilon(w)|$$

will in general tend to ∞ as $\epsilon \rightarrow 0$, where $\tau_{j,k}^\epsilon(w)$ is the $n \times n$ -matrix of C^1 function as stated in Subsection 2.1. Next, we recall the Cauchy–Fantappi  integral. Suppose $0 < \epsilon < \epsilon_0$. Consider

$$G(w, z) := \chi(w, z) \left[\partial \rho(w) - \frac{1}{2} \sum_{j,k} \tau_{j,k}^\epsilon(w) (w_j - z_j) dw_k \right] + (1 - \chi) \sum_k (\bar{w}_k - \bar{z}_k) dw_k,$$

where $\chi(w, z)$ is a C^∞ -cutoff function

$$\chi(w, z) = \begin{cases} 1, & \text{when } |w - z| \leq \frac{\mu}{2}; \\ 0, & \text{when } |w - z| \geq \mu \end{cases}$$

(note that this 1-form G was introduced in (3.1) in [21]). Let $g(w, z) = \langle G(w, z), w - z \rangle$. Then the Cauchy–Fantappi  integral ((3.2) in [21]) is defined as

$$\mathbf{C}^1(f)(z) := \int_{w \in bD} C^{(1)}(w, z) f(w), \quad z \in D,$$

where

$$C^{(1)}(w, z) := \frac{1}{(2\pi i)^n} j^* \left(\frac{G \wedge (\bar{\partial} G)^{n-1}(w, z)}{g(w, z)^n} \right).$$

Another correction operator $\mathbf{C}^2$ is needed since $\mathbf{C}^1(f)$ is not holomorphic for general f (for details we refer to [21, Subsection 3.2]). The kernel $C^{(2)}(w, z)$ of $\mathbf{C}^2$ is bounded on $bD \times \bar{D}$ with

$$\sup_{(w,z) \in bD \times \bar{D}} |C^{(2)}(w, z)| \lesssim 1.$$

Then the Cauchy integral $\mathbf{C} = \mathbf{C}^1 + \mathbf{C}^2$.

Then the “essential” part is defined $\mathbf{C}^\sharp$ just by replacing $\frac{1}{(2\pi i)^n} j^* (G \wedge (\bar{\partial} G)^{n-1}(w, z))$ by the Leray–Levi measure $d\lambda = \frac{1}{(2\pi i)^n} j^* (\partial \rho \wedge (\bar{\partial} \partial \rho)^{n-1}(w, z))$, i.e.,

$$\mathbf{C}^\sharp(f)(z) := \int_{w \in bD} \frac{f(w)}{g(w, z)^n} f(w) d\lambda(w), \quad z \in D.$$

Hence, $\mathbf{C}$ can be written as

$$\mathbf{C} = \mathbf{C}^\sharp + \mathbf{R},$$

where the remainder $\mathbf{R}$ is defined by $\mathbf{R} = \mathbf{C}^1 - \mathbf{C}^\sharp + \mathbf{C}^2$ with the explicit expression as

$$\mathbf{R}(f)(z) = \int_{bD} \frac{j^*(\alpha_0(w, z) + \sum_j \alpha_j(w, z)(w_j - z_j))}{g(w, z)^n} f(w) + \mathbf{C}^2(f)(z),$$

where α_0, α_j are $(2n-1)$ -form with coefficient on $w \in bD$ and smooth on $z \in \bar{D}$. Then for the fixed ϵ at the beginning, the remainder $\mathcal{R}_\epsilon$ is defined as

$$\mathcal{R}_\epsilon(f)(z) = \mathbf{R}(f)(z) \Big|_{z \in bD}.$$

From the definition of $\mathbf{R}$ we see that restriction of $\mathbf{C}^2$ on bD is uniformly bounded on $L^p(bD, d\lambda)$. However, for the first term, the upper bound of the coefficients α_0 and α_j involves $|\nabla \tau_{j,k}^\epsilon(w)|$. Hence, the operator norm $\|\mathcal{R}_\epsilon\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}$ is related to c_ϵ .

We now point out that, this fact will affect the $L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)$ norm of the commutator $[b, \mathcal{C}_\epsilon]$ as well. Recalling the proof of the necessity of (1) in Theorem 1.1, suppose $b \in \text{BMO}(bD, d\lambda)$, then by writing $[b, \mathcal{C}_\epsilon] = [b, \mathcal{C}_\epsilon^\sharp] + [b, \mathcal{R}_\epsilon]$, it suffices to verify the $L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)$ norms of $[b, \mathcal{C}_\epsilon^\sharp]$ and $[b, \mathcal{R}_\epsilon]$. Note that the first one is bounded by a constant multiplying $\|b\|_{\text{BMO}(bD, d\lambda)}$ where the constant is independent of ϵ . For the second one, by repeating the proof of 2.13 we see that

$$\|[b, \mathcal{R}_\epsilon]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} \leq C_\epsilon \|b\|_{\text{BMO}(bD, d\lambda)},$$

where C_ϵ is with respect to c_ϵ , and hence it can go to ∞ as $\epsilon \rightarrow 0$. Hence, (2.28) holds. $\square$

2.4. A remark on the Commutator $[b, \mathcal{C}^\sharp]$. We also point out that from the proof of the main result above, we can also deduce that the commutator of the essential part $\mathcal{C}^\sharp$ of $\mathcal{C}$ can also characterise the BMO space on the boundary bD .

To be more precise, we have the following.

Theorem 2.11. *Suppose $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class C^2 and is strongly pseudoconvex, $b \in L^1(bD, d\lambda)$. Then for $1 < p < \infty$,*

- (1) *the function $b \in \text{BMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}^\sharp]$ is bounded on $L^p(bD, d\lambda)$,*
- (2) *the function $b \in \text{VMO}(bD, d\lambda)$ if and only if the commutator $[b, \mathcal{C}^\sharp]$ is compact on $L^p(bD, d\lambda)$.*

Proof. We point out that the proof of Theorem 2.11 follows from the proof of Theorem 1.1, and in fact, it is simpler, since the operator $\mathcal{C}^\sharp$ is a Calderón–Zygmund operator. We only sketch the proof here.

Proof of (1):

It is clear that the necessary condition follows from the necessary part in the proof of Theorem 1.1 above.

The sufficient condition follows from the proof of the sufficiency part of Theorem 1.1 by mainly using the fact that at least one of the arguments **a**) and **b**) holds (where **a**) and **b**) are stated in the proof of the sufficiency part of Theorem 1.1 above)

Proof of (2): the proof follows from the proof of (2) of Theorem 1.1.

This finishes the proof of Theorem 2.11. $\square$

In the end, we point out that similar to Remark 1.2, we see that Theorem 2.11 holds for the L^p , BMO and VMO spaces with respect to any measure $d\omega$ on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD .

3. THE CAUCHY–LERAY INTEGRAL FOR DOMAINS IN $\mathbb{C}^n$ WITH MINIMAL SMOOTHNESS

In this section, we focus on the bounded domain $D \subset \mathbb{C}^n$ which is strongly $\mathbb{C}$ -linearly convex and whose boundary satisfies the minimal regularity condition of class $C^{1,1}$.

3.1. Preliminaries. We now recall the notation from [20]. Suppose D is a bounded domain in $\mathbb{C}^n$ with defining function ρ satisfying

1) D is of class $C^{1,1}$, i.e., the first derivatives of its defining function ρ are Lipschitz, and $|\nabla\rho(w)| > 0$ whenever $w \in \{w : \rho(w) = 0\} = bD$;

2) D is strongly $\mathbb{C}$ -linearly convex, i.e., D is a bounded domain of C^1 , and at any boundary point it satisfies either of the following two equivalent conditions

$$\begin{aligned} |\Delta(w, z)| &\geq c|w - z|^2, \\ d_E(z, w + T_w^{\mathbb{C}}) &\geq \tilde{c}|w - z|^2, \end{aligned}$$

for some $c, \tilde{c} > 0$, where

$$(3.1) \quad \Delta(w, z) = \langle \partial\rho(w), w - z \rangle,$$

and $d_E(z, w + T_w^{\mathbb{C}})$ denotes the Euclidean distance from z to the affine subspace $w + T_w^{\mathbb{C}}$. Note that $T_w^{\mathbb{C}} := \{v : \langle \partial\rho(w), v \rangle = 0\}$ is the complex tangent space referred to the origin, $w + T_w^{\mathbb{C}}$ is its geometric realization as an affine space tangent to bD at w .

On bD there is a quasi-distance $\mathbf{d}$, which is defined as

$$\mathbf{d}(w, z) = |\Delta(w, z)|^{\frac{1}{2}} = |\langle \partial\rho, w - z \rangle|^{\frac{1}{2}}, \quad w, z \in bD.$$

The Leray–Levi measure $d\lambda$ on bD introduced in [20] is defined as that in Section 2. According to [20, Proposition 3.4], $d\lambda$ is also equivalent to the induced Lebesgue measure $d\sigma$ on bD in the following sense:

$$d\lambda(w) = \tilde{\Lambda}(w)d\sigma(w) \quad \text{for } \sigma \text{ a.e. } w \in bD,$$

and there are two strictly positive constants c_1 and c_2 so that

$$c_1 \leq \tilde{\Lambda}(w) \leq c_2 \quad \text{for } \sigma \text{ a.e. } w \in bD.$$

We also denote by $B_r(w) = \{z \in bD : \mathbf{d}(w, z) < r\}$ the boundary balls determined via the quasidistance $\mathbf{d}$. By [20, Proposition 3.5], we also have

$$(3.2) \quad \lambda(B_r(w)) \approx r^{2n}, \quad 0 < r \leq 1.$$

The Cauchy–Leray integral of a suitable function f on bD , denoted $\mathbf{C}(f)$, is formally defined by

$$\mathbf{C}(f)(z) = \int_{bD} \frac{f(w)}{\Delta(w, z)^n} d\lambda(w), \quad z \in D.$$

When restricting z to the boundary bD , we have the Cauchy–Leray transform $f \mapsto \mathcal{C}(f)$, defined as

$$\mathcal{C}(f)(z) = \int_{bD} \frac{f(w)}{\Delta(w, z)^n} d\lambda(w), \quad z \in bD,$$

where the function f satisfies the Hölder-like condition

$$|f(w_1) - f(w_2)| \lesssim \mathbf{d}(w_1, w_2)^\alpha, \quad w_1, w_2 \in bD,$$

for some $0 < \alpha \leq 1$.

The main result in Lanzani–Stein [20] is as follows:

Theorem 3.1 (Lanzani–Stein [20], Theorem 5.1). *The Cauchy–Leray transform $f \mapsto \mathcal{C}(f)$, initially defined for functions f that satisfy the Hölder-like condition for some α , extends to a bounded linear operator on $L^p(bD)$ for $1 < p < \infty$.*

To obtain the $L^p(bD)$ boundedness, the main approach that Lanzani–Stein [20] used is first to obtain the kernel estimate for the Cauchy–Leray transform $\mathcal{C}$ and then to use the $T(1)$ theorem. To be more specific, let us take $K(w, z)$ to be the function defined for $w, z \in bD$, with $w \neq z$, by

$$K(w, z) = \frac{1}{\Delta(w, z)^n}.$$

This function is the “kernel” of the operator $\mathcal{C}$, in the sense that

$$\mathcal{C}(f)(z) = \int_{bD} K(w, z) f(w) d\lambda(w),$$

whenever z lies outside of the support of f and f satisfies the Hölder-like condition for some α . The size and regularity estimates that are relevant for us are:

$$(3.3) \quad \begin{aligned} |K(w, z)| &\lesssim \frac{1}{d(w, z)^{2n}}; \\ |K(w, z) - K(w', z)| &\lesssim \frac{d(w, w')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq c_K d(w, w'); \\ |K(w, z) - K(w, z')| &\lesssim \frac{d(z, z')}{d(w, z)^{2n+1}}, \quad \text{if } d(w, z) \geq c_K d(z, z'), \end{aligned}$$

for an appropriate constant $c_K > 0$. Moreover, for the size estimates we actually have

$$(3.4) \quad |K(w, z)| = \frac{1}{d(w, z)^{2n}}.$$

3.2. Boundedness and compactness of the Commutator $[b, \mathcal{C}]$.

Proof of Theorem 1.4. We point out that the proof of Theorem 1.4 follows from the proof of Theorem 1.1, and in fact, it is simpler, since the operator $\mathcal{C}$ is a Calderón–Zygmund operator. We only sketch the proof here.

We first prove (1) of Theorem 1.4.

From Theorem 3.1 and the size and smoothness conditions of the kernel K above, we see that $\mathcal{C}$ is a Calderón–Zygmund operator. According to [17, Theorem 3.1], if $b \in \text{BMO}(bD, d\lambda)$, we can obtain that $[b, \mathcal{C}]$ is bounded on $L^p(bD, d\lambda)$. Thus, it suffices to verify the sufficient condition.

To see this, assume that b is in $L^1(bD, d\lambda)$ and that $\|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)} < \infty$. We now verify that b is in $\text{BMO}(bD, d\lambda)$.

In fact, by using the size and smoothness estimates as in (3.4) and (3.3), following the argument in the proof of the sufficiency part of Theorem 1.1, we see that there exist positive constants γ_1 , A_4 and A_5 such that for every ball $B = B_r(z_0) \subset bD$ with $r < \gamma_1$, there is another ball $\tilde{B} = B_r(w_0) \subset bD$ with $A_4 r \leq d(w_0, z_0) \leq (A_4 + 1)r$ such that at least one of the following properties holds:

c) for every $z \in B$ and $w \in \tilde{B}$, $K_1(w, z)$ does not change sign and

$$|K_1(w, z)| \geq A_5 \frac{1}{d(w, z)^{2n}};$$

d) for every $z \in B$ and $w \in \tilde{B}$, $K_2(w, z)$ does not change sign and

$$|K_2(w, z)| \geq A_5 \frac{1}{d(w, z)^{2n}},$$

where $K_1(w, z)$ and $K_2(w, z)$ are the real and imaginary parts of $K(w, z)$, respectively. Here the constants γ_1 , A_4 and A_5 depend only on the implicit constants in (3.3) and the quasi-distance d .

We test the $\text{BMO}(bD, d\lambda)$ condition on the case of balls with big radius and small radius.

Case 1: In this case we work with balls with a large radius, $r \geq \gamma_1$.

By (2.5) and by the fact that $\lambda(B) \geq \lambda(B_{\gamma_1}(z_0)) \approx \gamma_1^{2n}$, we obtain that

$$\frac{1}{\lambda(B)} \int_B |b(w) - b_B| d\lambda(w) \lesssim \gamma_1^{-2n} \|b\|_{L^1(bD, d\lambda)}.$$

Case 2: In this case we work with balls B with a small radius, $r < \gamma_1$.

Then, by using one of the arguments c) and d), following the approach of the proof of Case 2 in the proof of Theorem 1.1, we obtain that

$$\frac{1}{\lambda(B)} \int_B |b(w) - b_B| d\lambda(w) \lesssim \|[b, \mathcal{C}]\|_{L^p(bD, d\lambda) \rightarrow L^p(bD, d\lambda)}.$$

This finishes the proof of (1) of Theorem 1.4.

Proof of (2): Necessary condition: Since the kernel $K(\cdot, \cdot)$ of $\mathcal{C}$ is a standard kernel, by [18, Theorem 1.1], $[b, \mathcal{C}]$ is compact on $L^p(bD, d\lambda)$. The sufficient condition follows from the proof of (2) of Theorem 1.4. $\square$

We now provide a strict explanation to Remark 1.5.

Proposition 3.2. *Theorem 1.4 holds for the L^p , BMO and VMO spaces with respect to any measure $d\omega$ on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD .*

Proof. The proof of this proposition is similar to that of Proposition 3.2. Suppose $d\omega$ is a measure on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD . Following the proof of Proposition 3.2, we see that the spaces $L^p(bD, d\omega)$, $\text{BMO}(bD, d\omega)$ and $\text{VMO}(bD, d\omega)$ are equivalent to $L^p(bD, d\lambda)$, $\text{BMO}(bD, d\lambda)$ and $\text{VMO}(bD, d\lambda)$.

Next, we recall that the proof of Theorem 1.4 depends only on the $L^p(bD)$ boundedness of the Cauchy–Leray transform $\mathcal{C}$ and the pointwise estimate of its kernel.

Hence, by repeating the proof of Theorem 1.4, we see that it holds for the $L^p(bD, d\omega)$, $\text{BMO}(bD, d\omega)$ and $\text{VMO}(bD, d\omega)$ spaces. $\square$

4. A REMARK ON THE COMMUTATOR OF CAUCHY–SZEGŐ OPERATOR ON A BOUNDED STRICTLY PSEUDOCONVEX DOMAIN IN $\mathbb{C}^n$ WITH SMOOTH BOUNDARY

Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and $\rho(z)$ be a strictly pluri-superharmonic defining function for D . Set

$$\psi(z, w) = \sum_{j=1}^n \frac{\partial \rho}{\partial w_j}(z_j - w_j) + \frac{1}{2} \sum_{j,k} \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k}(z_j - w_j)(z_k - w_k),$$

Then there is a positive number $\tilde{\delta} > 0$ such that the Cauchy–Szegő kernel $S(z, w)$ has the following form

$$(4.1) \quad S(z, w) = F(z, w)\psi(z, w)^{-n} + G(z, w) \log \psi(z, w)$$

for all $(z, w) \in R_{\tilde{\delta}} = \{(z, w) \in bD \times bD : d(z, w) < \tilde{\delta}\}$, where $F, G \in C^\infty(bD \times bD)$ and $F(z, z) > 0$ on bD , d is the usual quasi-distance on bD (see for example [31, p. 33]). According to [10, 28, 1], the Cauchy–Szegő kernel $S(z, w) \in C^\infty(bD \times bD \setminus \{(z, z) : z \in bD\})$ is a standard kernel.

We define the Cauchy–Szegő operator T_S as the singular integral associated with the Cauchy–Szegő kernel $S(z, w)$, i.e.

$$T_S(f)(z) = \int_{bD} S(z, w)f(w)d\mu(w),$$

for suitable f on bD , where $d\mu$ is the usual Lebesgue–Hausdorff surface measure on bD . We still use $B_r(z)$ to denote the ball on bD determined by the quasi-distance d , then $\mu(B_r(z)) \approx r^n$ (c.f. [31, p. 34], [18]).

From (2.1), we can see that if $|w - z| \leq \mu_0/2$ for some fixed small $\mu_0 > 0$, then $|\psi(z, w)| = |g_0(w, z)|$.

We now provide another proof of Theorem A as stated in the introduction.

Proof of Theorem A. We point out that the proof here follows from the proof of Theorem 1.1, and in fact, it is simpler, since the operator S is a Calderón–Zygmund operator with a specific kernel. We only sketch the proof here.

Proof of (i):

It suffices to show the sufficient condition. To see this, assume that b is in $L^1(bD)$ and that $\|[b, T_S]\|_{L^p(bD) \rightarrow L^p(bD)} < \infty$.

Since $S(z, w)$ is a standard kernel, there exist positive constants ϵ and A_6 such that for every $(z, w) \in R_{\tilde{\delta}}$ with $w \neq z$,

$$(4.2) \quad \begin{cases} a)' & |S(z, w)| \leq A_6 \frac{1}{d(w, z)^n}; \\ b)' & |S(z, w) - S(z, w')| \leq A_6 \left(\frac{d(w, w')}{d(w, z)} \right)^\epsilon \frac{1}{d(w, z)^n}, \quad \text{if } d(w, z) \geq cd(w, w'); \\ c)' & |S(z, w) - S(z', w)| \leq A_6 \left(\frac{d(z, z')}{d(w, z)} \right)^\epsilon \frac{1}{d(w, z)^n}, \quad \text{if } d(w, z) \geq cd(z, z') \end{cases}$$

for an appropriate constant $c > 0$.

Then similar to the proof of Theorem 1.1, there exist positive constants γ_2 , A_7 and A_8 such that for every ball $B = B_r(z_0) \subset bD$ with $r < \gamma_2$, there exists another ball $\tilde{B} = B_r(w_0) \subset bD$ with $A_7 r \leq d(w_0, z_0) \leq (A_7 + 1)r$, such that at least one of the following properties holds:

ε) for every $z \in B$ and $w \in \tilde{B}$, $S_1(z, w)$ does not change sign and

$$|S_1(z, w)| \geq A_8 \frac{1}{d(z, w)^n};$$

f) for every $z \in B$ and $w \in \tilde{B}$, $S_2(z, w)$ does not change sign and

$$|S_2(z, w)| \geq A_8 \frac{1}{d(z, w)^n},$$

where $S_1(z, w)$ and $S_2(z, w)$ are the real and imaginary parts of $S(z, w)$, respectively.

We test the $\text{BMO}(bD, d\lambda)$ condition on the case of balls with big radius and small radius.

Case 1: In this case we work with balls with a large radius, $r \geq \gamma_2$.

By the fact that $\mu(B) \geq \mu(B_{\gamma_2}(z_0)) \approx \gamma_2^n$, we obtain that

$$\frac{1}{\mu(B)} \int_B |b(w) - b_B| d\mu(w) \lesssim \gamma_2^{-n} \|b\|_{L^1(bD)}.$$

Case 2: In this case we work with balls B with a small radius, $r < \gamma_2$.

Then, following the approach of the proof of Case 2 in the proof of Theorem 1.1, we obtain that

$$\frac{1}{\mu(B)} \int_B |b(w) - b_B| d\mu(w) \lesssim \|[b, T_S]\|_{L^p(bD) \rightarrow L^p(bD)}.$$

Proof of (ii): the proof follows from the approach in the proof of (2) of Theorem 1.1 together with ε) or f).

This finishes the proof of Theorem A. $\square$

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