

Cohomological Blowups of Graded Artinian Gorenstein Algebras along Surjective Maps

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We introduce the cohomological blowup of a graded Artinian Gorenstein algebra along a surjective map, which we term BUG (blowup Gorenstein) for short. This is intended to translate to an algebraic context the cohomology ring of a blowup of a projective manifold along a projective submanifold. We show, among other things, that a BUG is a connected sum, that it is the general fiber in a flat family of algebras, and that it preserves the strong Lefschetz property. We also show that standard graded compressed algebras are rarely BUGs, and we classify those BUGs that are complete intersections. We have included many examples throughout this manuscript.

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1 Introduction

Given two graded Artinian Gorenstein (AG) algebras A and T over a field $\mathbb{F}$, of socle degrees $d > k$, respectively, and a surjective algebra map between them $\pi: A \rightarrow T$, we construct a new graded AG algebra $\tilde{A}$ of socle degree d called the cohomological blowup of A along π ; we shall sometimes refer to $\tilde{A}$ as BUG, short for blowup Gorenstein. As the name suggests, our construction is based on the blow-up operation in complex geometry, and particularly, its effect on the (singular) cohomology rings of the spaces involved. The purpose of the present paper is to extend this blow-up operation on cohomology rings to the more general class of graded AG algebras, to study its interactions with other familiar algebraic constructs, and to draw parallels to corresponding geometric ones. Our cohomological blowups are very different from the much-studied blow-up algebras (Rees algebras and related rings), which correspond to the coordinate ring (not the cohomology ring) of the blow-up variety [45].

Generally speaking, graded AG algebras are algebraic analogues of cohomology algebras (in even degrees) of smooth, compact, even-dimensional manifolds, for instance complex manifolds or symplectic manifolds. For compact complex (or symplectic) manifolds $\pi: Y \hookrightarrow X$ of (complex) dimensions $k < d$, respectively, the blowup of X along π is another compact complex d -dimensional manifold $\tilde{X}$ obtained from X by removing Y and gluing in its place a codimension one submanifold $\tilde{Y}$ called the exceptional divisor, which can be realized as the projectivization of the normal bundle $\mathcal{N}_{Y/X}$. The cohomology algebras (over $\mathbb{Q}$) of these spaces Y , X , $\tilde{Y}$, and $\tilde{X}$ satisfy the following three algebraic properties. For notational convenience, let $A = H^{2\bullet}(X)$, $T = H^{2\bullet}(Y)$, $\tilde{A} = H^{2\bullet}(\tilde{X})$, and $\tilde{T} = H^{2\bullet}(\tilde{Y})$, graded so that $A_i = H^{2i}(X)$, and so on.

First, if $\beta: \tilde{X} \rightarrow X$ is the blow-down map, $\beta_0: \tilde{Y} \rightarrow Y$ its restriction to the exceptional divisor, and $\hat{\pi}: \tilde{Y} \hookrightarrow \tilde{X}$ the natural inclusion map, then the obvious commutative diagram of spaces induces a commutative diagram of cohomology algebras:

$$\begin{array}{ccc}
 X & \xleftarrow{\beta} & \tilde{X} \\
 \uparrow \pi & & \uparrow \hat{\pi} \\
 Y & \xleftarrow{\beta_0} & \tilde{Y}
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ccc}
 A & \xrightarrow{\beta^*} & \tilde{A} \\
 \pi^* \downarrow & & \downarrow \hat{\pi}^* \\
 T & \xrightarrow{\beta_0^*} & \tilde{T}
 \end{array}
 \tag{1}$$

Second, as a projective space bundle over Y , the cohomology algebra of the exceptional divisor $\tilde{Y}$ is a free extension over the cohomology of Y generated by the

Euler class $\xi = e(\mathcal{N}_{\tilde{Y}/\tilde{X}}) \in H^2(\tilde{Y})$ of the normal bundle $\mathcal{N}_{\tilde{Y}/\tilde{X}}$, in symbols

$$\tilde{T} \cong \frac{T[\xi]}{(\xi^n + t_\xi^{n-1} + \cdots + t_{n-1}\xi + t_n)}, \quad (2)$$

where $t_i = (-1)^i \cdot c_i(Y) \in T_i$ are the Chern classes of the normal bundle $\mathcal{N}_{Y/X}$, the top one of which is the Euler class $c_n(Y) = e(\mathcal{N}_{Y/X}) \in T_n$, ($n = d - k = \text{codimension of } Y \text{ in } X$).

Third, the induced map β^* is injective and fits into a short exact sequence of A -modules:

$$0 \longrightarrow A \xrightarrow{\beta^*} \tilde{A} \xrightarrow{\tilde{\pi}^*} \tilde{T}/\beta_0^*(T) \longrightarrow 0. \quad (3)$$

Equations (2) and (3) imply that the Hilbert function of the cohomology of the blowup is

$$H(\tilde{A}) = H(A) + H(T)[1] + \cdots + H(T)[n-1]. \quad (4)$$

In fact, one can show that Equations (1)–(3) uniquely determine the algebra structure of $\tilde{A}$. Moreover, if the restriction map $\pi^*: A \rightarrow T$ is surjective, one can show that the cohomology algebra of the blowup is given by

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, \xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi + a_n)}, \quad (5)$$

where $K \subset A$ is the kernel of π^* , $\pi^*(a_i) = t_i$, and $a_n = (-1)^n \cdot \tau(Y) \in A_n$ where $\tau(Y)$ is the Thom class of the normal bundle $\mathcal{N}_{Y/X}$. See [14, 17, 21, 23, 32, 37] for further details on geometric and topological aspects of blowups; specifically, [17, Proposition 6.4], [37, Proposition 2.4], and [14, Theorem 3.11]. See also [41] for further details on characteristic classes.

The novelty of this paper is to show that Equations (1)–(3) and (5) can be extended to arbitrary AG algebras over any field $\mathbb{F}$ to define a new construction on these rings called the cohomological blowup. Specifically, given any surjective degree preserving map of graded AG algebras $\pi: A \rightarrow T$, we define (Definition 3.10) a *cohomological blowup of A along π* as in Equation (5), leaving $a_1, \dots, a_{n-1}$ as free parameters and setting a_n to be the algebraic analogue of the Thom class.

The algebraic analogue of the Thom class for a map of AG algebras $\pi: A \rightarrow T$ is defined as a certain annihilator of the kernel of π , determined by choices of socle

generators, called orientations, of A and T (Definition 2.1). Assuming that π is surjective, and $A = R/I$ where $I \subset R$ is a homogeneous ideal in a graded polynomial ring, an alternative characterization of the Thom class is an element $\tau \in R$ for which $T = R/(I : \tau)$. In terms of Macaulay duality, if R is acting on its dual divided power algebra Q by contraction and if $F \in Q$ and $G \in Q$ are dual generators of $A = R/\text{Ann}(F)$ and $T/\text{Ann}(G)$, then $\tau \in R$ is a contraction operator satisfying $\tau \circ F = G$. Each of these interpretations of the Thom class leads to distinct, but equivalent descriptions of the cohomological blowup. We frame these in terms of three general constructions on AG algebras, starting from the given data consisting of two oriented AG algebras A and T of socle degrees d and k with $n = d - k \geq 2$ and a surjective degree preserving algebra map between them $\pi: A \rightarrow T$.

Our 1st construction (Construction 3.1) fixes an indeterminate ξ , then chooses any homogeneous monic polynomial $f_A(\xi) \in A[\xi]$ (we use the subscript to indicate the coefficient algebra) and uses Equation (5) to construct a new algebra

$$f_A(\xi) = \xi^n + a_1\xi^{n-1} + \cdots + a_n \quad \Rightarrow \quad \hat{A} = \frac{A[\xi]}{(\xi \cdot K, \hat{f}(\xi))},$$

where K is the kernel of π and $a_i \in A_i$ are any homogenous elements. We show that $\hat{A}$ is an AG algebra if and only if $a_n = \lambda \cdot \tau$ is a nonzero scalar multiple of the Thom class $\tau \in A_n$ of π (Theorem 3.6). In this case, we replace the “hat” with a “tilde” and call the resulting algebra $\tilde{A}$ the *cohomological blowup* of A along π with parameters $(\pi(a_1), \dots, \pi(a_{n-1}), \lambda)$, and we call A the *cohomological blowdown* of $\tilde{A}$ (Definition 3.10). We show that $\tilde{A}$ is essentially characterized by analogues of Equations (1)–(3) above (Theorem 3.14). We then use a standard result in commutative algebra to show that $\tilde{A}$ is isomorphic to the general fiber in a flat family of algebras (Theorem 8.3). Using this flat family, we further show that if both A and T have the strong Lefschetz property (SLP) and $\mathbb{F}$ is infinite of characteristic either zero or sufficiently large, then $\tilde{A}$ must also have SLP (Theorem 8.5). This result has parallels in complex geometry; if $Y \subset X$ are projective manifolds, then the blowup $\tilde{Y} \subset \tilde{X}$ is also projective, and projective manifolds always satisfy the hard Lefschetz theorem; see, for example, [17, 21, 23]. We remark that the essential result of Theorem 8.5, that blowups preserve SLP, seems to be well known in various other contexts and guises, for example, [2, 9, 10, 31, 38, 44]; on the other hand, to our knowledge, the generality of Theorem 8.5 is new, and we hope it will be a welcome reference for this useful result. We further show that over an infinite field, if A and T both have the weak Lefschetz property (WLP) and the difference between their socle

degrees is two, then $\tilde{A}$ also has WLP (Theorem 8.9). We give examples that show general cohomological blowups may not preserve WLP (Example 8.8) and blowdowns may not preserve SLP or WLP in general (Example 8.6). In a sequel [34], we plan to investigate the behavior of the related Hodge–Riemann bilinear relations (HRR) in relation with cohomological blowups and blowdowns.

Our 2nd construction (Construction 4.2) starts with Macaulay dual generators $F \in Q_d$ and $G \in Q_k$ for $A = R/\text{Ann}(F)$ and $T = R/\text{Ann}(G)$, and a contraction operator $\tau \in R_n$ such that $\tau \circ F = G$. Then one again fixes an indeterminate ξ and chooses a monic homogeneous polynomial $f_R(\xi) \in R[\xi]$ and, from this, constructs a new dual generator $\hat{F} \in Q[\Xi]_d$ and its associated AG algebra

$$f_R(\xi) = \xi^n + r_1 \xi^{n-1} + \cdots + r_n \quad \Rightarrow \quad \hat{A}_{MD} = \frac{R[\xi]}{\text{Ann}(\hat{F})},$$

where the $r_i \in R_i$ are homogeneous of degree i , $1 \leq i \leq n$. Applying the projection map $R[\xi] \rightarrow A[\xi]$ to $f_R(\xi) \mapsto f_A(\xi) = \overline{f_R(\xi)}$, one can also construct $\hat{A}$ from above, but in general, the algebras $\hat{A}$ and $\hat{A}_{MD}$ will be non-isomorphic. However, we show that they are isomorphic exactly when they are equal to some cohomological blowup of A along π , and we give necessary and sufficient conditions for this to occur (Theorem 4.6); in this case, we shall replace the “hat” with “tilde”, drop the subscript, and simply write $\tilde{A}$. We use this result to show that every cohomological blowup along a surjective map can be realized as a connected sum in the sense of [30] (Theorem 5.4). We show that one can also obtain the cohomological blowdown as a connected sum (Theorem 5.6). This latter result implies that every AG algebra A can be realized as a connected sum (over some $\tilde{T} \neq \mathbb{F}$) by blowing up then blowing down.

Our 3rd construction (Construction 6.2) starts with presentations $A = R/I$ and $T = R/(I : \tau)$. One then fixes an indeterminate ξ and chooses a monic homogeneous polynomial $f_R(\xi) \in R[\xi]$ and constructs a new ideal $\hat{I} \subset R[\xi]$ by

$$f_R(\xi) = \xi^n + r_1 \xi^{n-1} + \cdots + r_n \quad \Rightarrow \quad \hat{I} = I + \xi \cdot (I : \tau) + (f_R(\xi)),$$

where the $r_i \in R_i$ are homogeneous of degree i , $1 \leq i \leq n$. It is clear from the definition that $R[\xi]/\hat{I} \cong \hat{A}$ from Construction 3.1 above, and hence we deduce that $\hat{I}$ is a Gorenstein ideal (i.e., irreducible and primary to the maximal ideal of $R[\xi]$) if and only if $a_n = \lambda \cdot \tau$, a nonzero scalar multiple of the Thom class of π . In this case, we replace the “hat” with “tilde” and call $\tilde{I}$ a *cohomological blow-up ideal* because the quotient algebra $R[\xi]/\tilde{I}$ is isomorphic to the cohomological blow-up algebra $\tilde{A}$ defined above.

We compute a minimal generating set of $\tilde{I}$ in terms of minimal generating sets of I and $(I : \tau)$ (Theorem 6.5). Using this result in conjunction with the notion of exact pairs of zero divisors borrowed from [22, 33], we classify BUGs that are complete intersections (CIs) (Theorem 6.12). We also use cohomological blowups to prove a special case of Watanabe’s bold conjecture (WBC) [47], namely that any standard graded Artinian CI cut out by products of linear and/or quadratic forms can be embedded as a subalgebra of another standard graded Artinian CI cut out by quadratic forms and of the same socle degree (Theorem 6.16); see [35] for a different proof of a related result, and see [36] for a proof of other cases of WBC.

It is interesting to note that our cohomological blowup along a surjective map preserves the standard grading, that is, if A and T are standard graded, then the cohomological blowup $\tilde{A}$ is also standard graded (Remark 5.5). However, we show by examples that neither the cohomological blowdown nor, what one might call, the cohomological blowup along a non-surjective map necessarily preserve standard grading at all (Example 5.8, Remark 9.4). We say that a standard graded AG algebra $\tilde{A}$ has a *standard BUG structure* if it is a cohomological blowup of a standard graded AG algebra A along some surjective map π . Using Equation (4) above, we deduce that some Hilbert functions cannot occur with standard BUG structures, and we call such Hilbert functions *inaccessible*. Specifically, we show that the compressed Hilbert functions of [24] with embedding dimension at least three and socle degree at least six are always inaccessible (Theorem 7.2). We plan for a sequel [29] with further results in this direction, including a complete description of cohomological blow-up algebras of codimension two—where all AG Hilbert functions are accessible—and a study of inaccessible AG Hilbert functions in codimension at least three.

This paper is organized as follows. In Section 2, we give a description of the algebraic analogue of the Thom class of a map of graded AG algebras. In Section 3, we give Construction 3.1 and necessary and sufficient conditions for it to yield a Gorenstein algebra. Then we define the cohomological blow-up algebra and give a description of its Hilbert function. In Section 4, we discuss Macaulay duality, we give Construction 4.2, and we give necessary and sufficient conditions for it to yield the cohomological blow-up algebra. In Section 5, we show that every cohomological blow-up algebra is a connected sum. We also describe the blowdown in terms of connected sums and show that every AG algebra has a connected sum decomposition, obtained by blowing up then blowing down. In Section 6, we give Construction 6.2, introduce the cohomological blow-up ideal, and compute its minimal generating set. In Section 7, we define inaccessible Hilbert functions and show that almost all compressed AG algebras

are inaccessible. In Section 8, we show that cohomological blowing up preserves SLP. In Section 9, we discuss further the connection of our work with geometry and other areas, give some geometrically motivated examples, and suggest problems for future work. In this paper, we have made a special effort to include a wide array of examples, many of which were found as counter-examples to conjectures or questions that arose in our discussions and preparations of this manuscript, and we hope the reader will find them useful. In the appendix, we have included a list of all examples and brief descriptions of them for the readers convenience.

2 AG Algebras, Orientations, and Thom Classes

Let $\mathbb{F}$ be any field. A *graded AG algebra* means a commutative $\mathbb{Z}_{\geq 0}$ -graded connected AG $\mathbb{F}$ -algebra; in particular, if A is a graded AG algebra of socle degree d , then $A = \bigoplus_{i=0}^d A_i$ with $A_0 = \mathbb{F}$ and $\text{socle}(A) = A_d \cong \mathbb{F}$. Since most algebraic objects in this paper are graded, we may sometimes drop the adjective and refer simply to an AG algebra. Unless explicitly stated otherwise (as Remark 3.11), we will not restrict the ground field $\mathbb{F}$ except in our study of Lefschetz properties (Section 8). We say that A is *standard graded* if it is generated as an algebra by its linear forms, that is, $\mathbb{F}[A_1] = A$. Although many of our examples are standard graded, we will not assume this, except in our discussion of compressed algebras (Section 7). An *orientation* on the AG algebra A is a choice of linear isomorphism $\int_A : A_d \xrightarrow{\cong} \mathbb{F}$ or, equivalently, a choice of socle generator, which we shall denote by $0 \neq a_{\text{soc}} \in A_d$. An *oriented AG algebra* is a pair $(A, \int_A)$ consisting of a graded AG algebra A and an orientation $\int_A$; when the orientation is understood we shall drop the $\int_A$ and just speak of an oriented AG algebra A , and its distinguished socle generator a_{soc} .

Suppose that A and T are two oriented AG algebras of socle degrees d and k , respectively, with $d > k$, and suppose that $\pi : A \rightarrow T$ is a degree-preserving algebra map between them; we may occasionally drop the adjectives and simply refer to the map π . Since A is Gorenstein, multiplication defines a perfect pairing for all $0 \leq i \leq d$:

$$\begin{aligned} A_i \times A_{d-i} &\longrightarrow \mathbb{F} \\ (a, a') &\longmapsto \int_A aa' \end{aligned}$$

In particular, for any $\phi \in \text{Hom}_{\mathbb{F}}(A_i, \mathbb{F})$, there exists a unique $a \in A_{d-i}$ such that $\phi(b) = \int_A a \cdot b$ for every $b \in A_i$. Pulling back the orientation on T by π defines a homomorphism $\int_T \circ \pi : A_k \rightarrow \mathbb{F}$ and hence, as shown in [30, Lemma 2.1], there exists a unique element

$\tau \in A_{d-k}$ for which

$$\int_T \pi(a) = \int_A \tau \cdot a, \quad \forall a \in A. \quad (6)$$

Definition 2.1 (Thom class, Euler class). The unique element $\tau \in A_{d-k}$ defined by Equation (6) is called the *Thom class of π* . The image of the Thom class $\pi(\tau) \in T_{d-k}$ is called the *Euler class of π* .

Topologically, if $A = H^{2\bullet}(X)$ and $T = H^{2\bullet}(Y)$ are the cohomology rings of a manifold X and a submanifold $\pi: Y \hookrightarrow X$ and $\pi^*: H^{2\bullet}(X) \rightarrow H^{2\bullet}(Y)$ is the induced restriction map on cohomology algebras, then the Thom class of π^* is exactly the Thom class of the normal bundle of Y in X , also known as the *Poincaré dual class of $Y \subset X$* , and its image under the restriction map is the Euler class of the normal bundle, e.g., [41]. Note that the Thom class is zero if and only if the socle of T does not belong to the image of π . We call $\pi: A \rightarrow T$ a *restriction map* if its Thom class is nonzero. In particular, if π is surjective, then it is a restriction map, but not every restriction map is surjective. In complex geometry, if X is a Kähler manifold and $\pi: Y \subset X$ is a codimension n Kähler submanifold, then the induced map on cohomology $\pi^*: H^{2\bullet}(X) \rightarrow H^{2\bullet}(Y)$ is a restriction map [23, Exercise 3.3.9], though it need not be surjective, for example Example 9.3. Here is a useful characterization of the Thom class of a restriction map.

Lemma 2.2. Let $\pi: A \rightarrow T$ be a restriction map between two oriented AG algebras of socle degrees $d > k$, respectively, and let $K = \ker(\pi) \subset A$ be its kernel. Let $a \in A$ be any homogeneous element of degree $n = d - k$. Then $a \cdot K = 0$ if and only if $a = \lambda \cdot \tau$ is a multiple of the Thom class of π .

Proof. Let $\int_A: A_d \rightarrow \mathbb{F}$ and $\int_T: T_k \rightarrow \mathbb{F}$ be orientations on A and T , respectively. Assume first that $a \in A_n$ and $a \cdot K = 0$. Consider the short exact sequence of vector spaces $0 \rightarrow K_k \rightarrow A_k \rightarrow T_k \rightarrow 0$. Since T is Gorenstein of socle degree k , T_k is one-dimensional, and hence $K_k \subset A_k$ is a codimension-one subspace. Therefore, the set of homomorphisms $\phi \in \text{Hom}_{\mathbb{F}}(A_k, \mathbb{F})$ that vanish on K_k is one-dimensional. Since $\phi_1(u) = \int_A \tau \cdot u = \int_T \pi(u) = 0$ for every $u \in K$, and also $\phi_2(u) = \int_A a \cdot u = 0$ for every $u \in K$, we must have $\phi_1 = \lambda \cdot \phi_2$, which implies that $\tau = \lambda \cdot a$. The converse is clear. ■

If $\pi: A \rightarrow T$ is surjective, which will be a standing assumption throughout this paper, then the Thom class has some nice alternative descriptions. First, let

$R = \mathbb{F}[x_1, \dots, x_r]$ be a graded polynomial ring with homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_r) \subset R$, and let $I \subset R$ be a homogeneous $\mathfrak{m}$ -primary ideal. Then R/I is graded Artinian and it is Gorenstein if and only if I is irreducible, meaning that it cannot be written as an intersection of two strictly larger ideals [39, Lemma I.1.3]. Suppose that $J \subset R$ is another homogeneous $\mathfrak{m}$ -primary irreducible ideal with $I \subset J$, let $T = R/J$, and let $\pi: A \rightarrow T$ be the natural projection map. The following fact is well known; see [46, Lemma 4] and [39, Theorem I.2.1].

Lemma 2.3. With the above assumptions, let $\bar{\tau} \in A = R/I$ be the Thom class of π and let $\tau \in R$ any homogeneous lift. Then we have

$$J = (I : \tau) \text{ and } (I : J) = (\tau) + I.$$

In [39], the authors refer to the homogeneous lift of the Thom class $\tau \in R$ in Lemma 2.3 as a transition element for $I \subseteq J$.

Next, let $Q = \mathbb{F}[X_1, \dots, X_r]$ be a divided power algebra on which R acts by contraction, for example [25, Appendix A] or [12, Appendix A.2.4]:

$$x_i \circ X_1^{a_1} \cdots X_i^{a_i} \cdots X_r^{a_r} = \begin{cases} X_1^{a_1} \cdots X_i^{a_i-1} \cdots X_r^{a_r} & \text{if } a_i > 0 \\ 0 & \text{if } a_i = 0 \end{cases}.$$

Then it is well known that a homogeneous ideal $I \subset R$ is $\mathfrak{m}$ -primary irreducible of socle degree d (meaning the socle degree of $A = R/I$) if and only if there exists a homogeneous form $F \in Q_d$ for which $I = \text{Ann}(F) = \{r \in R \mid r \circ F = 0\}$. In this case, F is called a Macaulay dual generator of A . Suppose that $G \in Q_k$ ($k < d$) is another homogeneous form for which $I = \text{Ann}(F) \subset J = \text{Ann}(G)$, let $T = R/J$, and let $\pi: A = R/\text{Ann}(F) \rightarrow R/\text{Ann}(G)$ be the natural projection map. Note that A and T have natural orientations coming from F and G given by $\int_A a = (a \circ F)(0)$ and $\int_T t = (t \circ G)(0)$.

Lemma 2.4. Under the assumptions of Lemma 2.3, if $\tau \in R$ is any homogeneous lift of the Thom class of π , then

$$\tau \circ F = G.$$

Proof. Indeed, the Thom class condition (6) translates to the condition

$$(a \circ G)(0) = (\tau \cdot a \circ F)(0) = (a \circ (\tau \circ F))(0), \quad \forall a \in R$$

or, equivalently,

$$0 = a \circ (G - \tau \circ F)(0), \quad \forall a \in R, \quad (7)$$

but because the pairing $R_i \times Q_i \rightarrow \mathbb{F}$, $(a, H) = (a \circ H)(0)$ is nondegenerate, we see that Condition (9) is equivalent to the claimed condition that $G = \tau \circ F$. ■

Remark 2.5. If τ is the Thom class of a map of AG algebras $\pi: A \rightarrow T$, and if $\lambda \in \mathbb{F}^\times$ is a nonzero constant, we can get a new Thom class $\tau' = \lambda \cdot \tau$ by either scaling the distinguished socle generator of A by λ , that is, $a_{\text{soc}} \mapsto \lambda \cdot a_{\text{soc}} = a'_{\text{soc}}$, or by scaling the distinguished socle generator of T by λ^{-1} , that is, $t_{\text{soc}} \mapsto \lambda^{-1} \cdot t_{\text{soc}} = t'_{\text{soc}}$.

In our proofs, we make frequent use of the following result: it is valid for an arbitrary AG algebra A (not necessarily graded) and an arbitrary Artinian algebra B .

Lemma 2.6. Let $\varphi: A \rightarrow B$ be a homomorphism of Artinian algebras so that A is Gorenstein and φ restricted to the socle of A is injective. Then φ is injective.

In particular, if $\varphi: A \rightarrow B$ is a surjective homomorphism of graded AG rings of the same socle degree, then φ is an isomorphism.

Proof. Let $\varphi: A \rightarrow B$ be a ring homomorphism with A AG. Let a be an element of the kernel of φ . If $a \neq 0$, then there exists $0 \neq a' \in A$ such that $0 \neq aa' \in \text{soc}(A)$. However, $\varphi(aa') = \varphi(a)\varphi(a') = 0$ since $\varphi(a) = 0$ by assumption. This contradicts the restriction of φ to $\text{soc}(A)$ being injective. Thus, it must be the case that $a = 0$ and consequently φ is injective.

Now, assume $\varphi: A \rightarrow B$ is a surjective homomorphism of graded AG rings of the same socle degree. Then $\varphi: \text{soc}(A) \rightarrow \text{soc}(B)$ is a vector space isomorphism. Applying the 1st assertion, φ is injective, hence bijective. ■

3 Cohomological Blowups

Let A and T be oriented AG algebras of socle degrees d and k respectively with $d > k$ and $\pi: A \rightarrow T$ a surjective degree preserving algebra map with Thom class $\tau \in A_{d-k}$.

Construction 3.1. Set $n = d - k$, and let ξ be an indeterminate of degree one. Choose any homogeneous elements $a_i \in A$ for $1 \leq i \leq n$, and define the monic homogeneous polynomial of degree n , $f_A(\xi) \in A[\xi]$, by

$$f_A(\xi) = \xi^n + a_1 \xi^{n-1} + \cdots + a_{n-1} \xi + a_n. \quad (8)$$

Then, let $K \subset A$ be the kernel of π , and construct the algebra $\hat{A}$ by

$$\hat{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))}. \quad (9)$$

We further set $t_i = \pi(a_i)$ for $1 \leq i \leq n$, define the monic homogeneous polynomial $f_T(\xi) \in T[\xi]$

$$f_T(\xi) = \xi^n + t_1 \xi^{n-1} + \cdots + t_n, \quad (10)$$

and construct the algebra $\tilde{T}$ by

$$\tilde{T} = \frac{T[\xi]}{(f_T(\xi))}. \quad (11)$$

Before giving the formal definition of the cohomological blowup (Definition 3.10), we shall discuss some of the fundamental properties of the algebras $\hat{A}$ and $\tilde{T}$ from Construction 3.1. First, note that since $\xi \cdot K = 0$ in $\hat{A}$, the algebra structure of $\hat{A}$ depends only on the images $t_1, \dots, t_{n-1}$, as well as a_n . There is a natural degree preserving algebra map $\beta: A \rightarrow \hat{A}$ induced from the natural inclusions $A \hookrightarrow A[\xi]$ that makes $\hat{A}$ into an A -algebra. Also, the surjective tensor product map $\pi \otimes 1: A[\xi] \rightarrow T[\xi]$ passes to a surjective degree preserving map on quotients $\hat{\pi}: \hat{A} \rightarrow \tilde{T}$. Moreover, these maps $\pi: A \rightarrow T$, $\beta: A \rightarrow \hat{A}$ and $\hat{\pi}: \hat{A} \rightarrow \tilde{T}$ together with the natural inclusion $\beta_0: T \rightarrow \tilde{T}$ fit together in a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & \hat{A} \\ \pi \downarrow & & \downarrow \hat{\pi} \\ T & \xrightarrow{\beta_0} & \tilde{T}. \end{array} \quad (12)$$

The following elementary observation will be useful and we record it as a lemma. We offer our own proof here for completeness, but it can also be deduced from [27, Lemma 1.8] and [43, Lemma 2.1]. Recall that for graded Artinian algebras A, B , and C , we say that C is a *free extension of A with fiber B* if there are maps $\iota: A \rightarrow C$ making C into a free A -module and $\pi: C \rightarrow B$ with kernel $\ker(\pi) = \mathfrak{m}_A \cdot C$ where $\mathfrak{m}_A$ is the maximal ideal of A .

Lemma 3.2. The algebra $\tilde{T}$ from Construction 3.1 in Equation (11) is a free extension of T with fiber $F \cong \mathbb{F}[\xi]/(\xi^n)$. In particular, since T and F are Gorenstein, $\tilde{T}$ is also Gorenstein. (A more general result established in a 2008 unpublished note “Coexact Sequences of Poincaré Duality Algebras” by L. Smith and R.E. Stong states that if $\tilde{T}$,

T and F are graded Artinian algebras in which $\tilde{T}$ is a free extension of T with fiber F , and any two are Gorenstein, then so is the 3rd.)

Proof. We first claim that the map $\beta_0: T \rightarrow \tilde{T}$ makes $\tilde{T}$ into a free T -module with basis $\{1, \xi, \dots, \xi^{n-1}\}$. Clearly, this set generates $\tilde{T}$ as a T -module, since powers of ξ generate $T[\xi]$ as a T -module, and all powers greater than $n-1$ can be eliminated with the relation $\xi^n \equiv -(t_1 \xi^{n-1} + \dots + t_n)$. A T -dependence relation in $\tilde{T}$ lifts to a T -relation in $T[\xi]$ of the form $c_0 \cdot 1 + c_1 \xi + \dots + c_{n-1} \xi^{n-1} = t \cdot (\xi^n + t_1 \xi^{n-1} + \dots + t_n)$ for some $c_i, r \in T$. Comparing ξ -coefficients, we conclude that $t = 0$, and hence $c_i = 0$ for all i since $\{1, \xi, \dots, \xi^{n-1}\}$ are T -linearly independent in $T[\xi]$. Finally, note that the natural projection map $\phi: T \rightarrow \mathbb{F}$ extends to a projection map $\tilde{\phi}: \tilde{T} \rightarrow \mathbb{F}[\xi]/(\xi^n)$ with kernel $\ker(\tilde{\phi}) = \mathfrak{m}_T \cdot \tilde{T}$. Therefore, $\tilde{T}$ is a free extension of T with fiber $F = \mathbb{F}[\xi]/(\xi^n)$ as claimed.

For the last statement, it suffices to show that the socle of $\tilde{T}$ satisfies $\text{soc}(\tilde{T}) = \text{soc}(T) \cdot \xi^{n-1}$. One containment is obvious, and for the other, assume that $\hat{t} \in \text{soc}(\tilde{T})$ is a homogeneous socle element. From our arguments above, we may decompose it as $\hat{t} = s_0 + s_1 \xi + \dots + s_{n-1} \xi^{n-1}$ for some unique $s_i \in T$. Since $\hat{t}$ is in the socle, we must have $t \cdot \hat{t} = 0$ for any positive degree element $t \in T_+$, which implies that $s_i \in \text{soc}(T)$ for all i , by linear independence of $\{1, \dots, \xi^{n-1}\}$. On the other hand, since $\hat{t}$ is homogeneous, and $\deg(s_i) = \deg(\hat{t}) - i$ it follows that there is only one nonzero s_i , and since $\xi \cdot \hat{t} = 0$, it follows that $i = n-1$. Therefore, we have shown that $\hat{t} = s_{n-1} \cdot \xi^{n-1}$ where $s_{n-1} \in \text{soc}(T)$, which implies that $\text{soc}(\tilde{T}) = \text{soc}(T) \cdot \xi^{n-1}$, as desired. ■

Lemma 3.2 implies that $\tilde{T}$ is always Gorenstein. On the other hand, the algebra $\hat{A}$ may not always be Gorenstein, as the following example shows.

Example 3.3. Let

$$A = \frac{\mathbb{F}[x, y]}{(x^3, y^3)} \xrightarrow{\pi} T = \frac{\mathbb{F}[x, y]}{(x^2, y)},$$

where $\pi(x) = x$ and $\pi(y) = 0$ (here $d = 4$, $k = 1$, and $n = d - k = 3$). Note $K = \ker(\pi) = (x^2, y)$. Orient A and T with socle generators $a_{\text{soc}} = x^2 y^2$ and $t_{\text{soc}} = x$; then, the Thom class of π is $\tau = xy^2 \in A_3$. Set $f_T(\xi) = \xi^3 + x\xi^2 \in T[\xi]$, so that $t_1 = x$ and $t_2 = t_3 = 0$, and let $\tilde{T}$ be the associated free extension:

$$\tilde{T} = \frac{T[\xi]}{(f_T(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^2, y, \xi^3 + x\xi^2)}.$$

Below are different algebras $\hat{A}$ for different choices of π -lifts $f_A(\xi) = \xi^3 + a_1\xi^2 + a_2\xi + a_3$ of $f_T(\xi)$, according to Construction 3.1.

1. $a_1 = x$, $a_2 = 0$, and $a_3 = (x^2y + xy^2)$. Then $f_A(\xi) \in A[\xi]$ is $f_A(\xi) = \xi^3 + x\xi^2 + (x^2y + xy^2)$, and

$$\hat{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, x^2\xi, y\xi, \xi^3 + x\xi^2 + x^2y + xy^2)}.$$

Then an $\mathbb{F}$ -basis for $\hat{A}$ is

$$\{1, x, y, \xi, x^2, xy, y^2, x\xi, \xi^2, x^2y, xy^2, x\xi^2\}$$

from which it follows that the Hilbert function for $\hat{A}$ is

$$H(\hat{A}) = (1, 3, 5, 3)$$

and hence $\hat{A}$ is not Gorenstein. Note that in this case, the socle generator $a_{soc} = x^2y^2$ is actually in the ideal $(\xi \cdot K, \hat{f}(\xi))$, hence $\beta(a_{soc}) = 0$, and thus β is not injective.

2. $a_1 = x$, $a_2 = 0$, and $a_3 = xy^2 = \tau$. Then, $f_A(\xi) = \xi^3 + x\xi^2 + xy^2 \in A[\xi]$ and

$$\hat{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, x^2\xi, y\xi, \xi^3 + x\xi^2 + xy^2)}$$

has basis

$$\{1, x, y, \xi, x^2, xy, y^2, x\xi, \xi^2, x^2y, xy^2, x\xi^2, x^2y^2\}$$

and Hilbert function

$$H(\hat{A}) = (1, 3, 5, 3, 1).$$

Here, the socle of $\hat{A}$ is generated by $\hat{a}_{soc} = a_{soc} = x^2y^2$; hence, $\hat{A}$ is Gorenstein.

3. $a_1 = x + y$ and $a_2 = a_3 = 0$. Here, we chose a different π -lift of $t_1 = x$, but the reader will see it does not affect $\hat{A}$; the important choice is $a_3 = 0$. Then, $f_A(\xi) = \xi^3 + x\xi^2 \in A[\xi]$ and

$$\hat{A} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, x^2\xi, y\xi, \xi^3 + (x + y)\xi^2)} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, x^2\xi, y\xi, \xi^3 + x\xi^2)}$$

with basis

$$\{1, x, y, \xi, x^2, xy, y^2, x\xi, \xi^2, x^2y, xy^2, x\xi^2, x^2y^2\}$$

and Hilbert function

$$H(\hat{A}) = (1, 3, 5, 3, 1).$$

However, note that in this case the socle of $\hat{A}$ is the two dimensional $\mathbb{F}$ -vector space generated by $x\xi^2$ and x^2y^2 ; therefore, $\hat{A}$ is not Gorenstein. In contrast to case (1), in this case, the image of the socle of A in $\hat{A}$ via β is nonzero and in fact the map $\beta: A \rightarrow \hat{A}$ is injective.

The example suggests a strong dependence of the Gorenstein property on the choice of a_n from Equation (8). Before we give our main result in this direction, we need some lemmas. The following lemma computes the Hilbert function of $\hat{A}$ in terms of those of A and T .

Lemma 3.4. For each $\hat{a} \in \hat{A}$, there exist unique elements $b_0 \in A$ and elements $s_1, \dots, s_{n-1} \in T$ for which

$$\hat{a} = \beta(b_0) + \beta(b_1)\xi^1 + \dots + \beta(b_{n-1})\xi^{n-1}, \text{ for some } b_i \in A, \text{ where } \pi(b_i) = s_i.$$

In particular, the Hilbert function of $\hat{A}$ satisfies

$$H(\hat{A}) = H(\beta(A)) + H(T)[1] + \dots + H(T)[n-1]. \quad (13)$$

Proof. Existence of such a decomposition is easy: every element $\hat{a} \in \hat{A}$ has a representative in the polynomial ring $A[\xi]$, and high powers of ξ can be reduced via the relation

$$f_A(\xi) = \xi^n + a_1\xi^{n-1} + \dots + a_n \equiv 0.$$

As for uniqueness, suppose that there are some other elements $c_0 \in A$ and other $s'_1, \dots, s'_{n-1} \in T$ with π -lifts $c_1, \dots, c_{n-1} \in A$ such that

$$\hat{a} = \beta(c_0) + \beta(c_1)\xi + \dots + \beta(c_{n-1})\xi^{n-1}.$$

Comparing their decompositions in $\tilde{T}$ via the projection $\hat{\pi}$, we find that

$$0 = \beta_0(\pi(b_0 - c_0)) + \beta_0(\pi(b_1 - c_1))\xi + \cdots + \beta_0(\pi(b_{n-1} - c_{n-1}))\xi^{n-1}.$$

It follows from Lemma 3.2 that $\tilde{T}$ is a free T -module with basis $\{1, \xi, \dots, \xi^{n-1}\}$ via the injective map $\beta_0: T \rightarrow \tilde{T}$, and hence we conclude that $s_i = \pi(b_i) = \pi(c_i) = s'_i$ for all $i = 1, \dots, n-1$ and that $\pi(a_0) = \pi(c_0)$; hence, $b_0 - c_0 \in K$. But we also have the relation in $\hat{A}$

$$0 = \beta(b_0 - c_0) + \beta(b_1 - c_1)\xi + \cdots + \beta(b_{n-1} - c_{n-1})\xi^{n-1}.$$

Since $\beta(b_i - c_i)\xi^i \in (\xi \cdot K)$, the identity displayed above simplifies to

$$0 = \beta(b_0 - c_0)$$

and hence $\beta(b_0) = \beta(c_0)$ in $\hat{A}$. Thus, the decomposition is unique in the desired sense. Equation (13) follows immediately from this decomposition. ■

Lemma 3.5. With A, T, π, K , and $f_A(\xi) = \xi^n + a_1\xi^{n-1} + \cdots + a_n \in A[\xi]$ as in Construction 3.1, we have in $A[\xi]$

$$(\xi \cdot K, f_A(\xi)) \cap \beta(A) = a_n \cdot K.$$

Proof. Let $b \in (\xi \cdot K, f_A(\xi)) \cap \beta(A)$. Then there is a polynomial $g(\xi) \in A[\xi]$ for which $b - g(\xi)f_A(\xi) \in (\xi \cdot K)$ in $A[\xi]$. Writing $g(\xi) = g_m\xi^m + \cdots + g_1\xi + g_0$ for $g_i \in A$, since $f_A(\xi)$ is monic, we must therefore have in $A[\xi]$

$$b - g_m\xi^{m+n} + (\text{lower order terms}) \in (\xi \cdot K). \quad (14)$$

Since $m + n \geq n \geq 1$ we can compare coefficients on the left-hand side and right-hand side of (14) to deduce that $g_m \in K$. If $m \geq 1$, we can combine the term $g_m\xi^m \cdot \hat{f}(\xi)$ with the other $(\xi \cdot K)$ terms and lower the ξ -degree of $g(\xi)$. Repeating this procedure, we may assume that the ξ -degree is $m = 0$, that $g_m = g_0 \in K$, and hence that $b - g_0 \cdot f_A(\xi) \in (\xi \cdot K)$. Since $g_0 \in K$, it follows that $g_0(f_A(\xi) - a_n) \in (\xi \cdot K)$ as well and therefore that $b - g_0 \cdot a_n \in (\xi \cdot K)$, which implies that $b - g_0 \cdot a_n = 0$ and hence that $b \in a_n \cdot K$, as desired. The reverse containment is obvious. ■

Theorem 3.6. Let $A, T, \pi, \tau, f_A(\xi), \hat{A}$, and β be as above. Then,

1. the algebra map $\beta: A \rightarrow \hat{A}$ is injective if and only if the constant coefficient of $f_A(\xi)$ satisfies $a_n = \lambda \cdot \tau$ for some $\lambda \in \mathbb{F}$ (possibly $\lambda = 0$) and
2. the algebra $\hat{A}$ is Gorenstein if and only if $a_n = \lambda \cdot \tau$ for some nonzero $\lambda \in \mathbb{F}^\times$.

Proof. For (1.), observe that $\beta: A \rightarrow \hat{A}$ is injective if and only if $(\xi \cdot K, f_A(\xi)) \cap \beta(A) = 0$, which by Lemma 3.5 is equivalent to $a_n \cdot K = 0$, which is in turn, equivalent to $a_n = \lambda \cdot \tau$ for some $\lambda \in \mathbb{F}$ by Lemma 2.2.

For (2.), assume first that $\hat{A}$ is Gorenstein. Then $\text{soc}(\hat{A}) = \hat{A}_e$ where e is the largest integer for which $\hat{A}_e \neq 0$. It follows from Lemma 3.4 that $e \leq d$ and that $e < d$ if and only if $\beta(A_d) = 0$. But if $e < d$, then by the surjectivity of the algebra map $\hat{\pi}: \hat{A} \rightarrow \tilde{T}$ we must have $\hat{\pi}(\hat{A}_{d-1}) = \tilde{T}_{d-1} \neq 0$, and hence $0 \neq \hat{A}_{d-1} = \text{soc}(\hat{A})$ and $e = d - 1$. By Lemma 2.6, $\hat{\pi}$ must be an isomorphism, that is,

$$\hat{\pi}: \hat{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} \xrightarrow{\cong} \frac{A[\xi]}{(K, f_A(\xi))} = \tilde{T}.$$

In particular, we see that $K \subseteq (\xi \cdot K, f_A(\xi))$ in $A[\xi]$, which by Lemma 3.5 implies that $K \subseteq a_n \cdot K$, which is impossible for degree reasons (since we are assuming $d > k$). Therefore, we must have $\text{soc}(\hat{A}) = \hat{A}_d \neq 0$, which by Lemma 3.4 must be the image of A_d , and hence $\beta(a_{\text{soc}}) \neq 0$. By (1), this implies that $a_n = \lambda \cdot \tau$ for some $\lambda \in \mathbb{F}$. Next, we claim that $\lambda \neq 0$. Indeed, assume that $\lambda = 0$ so that $a_n = 0$. Let $b_0 \in A$ be any π -lift of $t_{\text{soc}} \in T_k$. Then note that $\xi^{n-1} \beta(b_0) \in \text{soc}(\hat{A})$. Indeed, in $A[\xi]$, we have $\xi \cdot (\xi^{n-1} b_0) - b_0 f_A(\xi) \in (\xi \cdot K)$, hence $\xi \cdot (\xi^{n-1} b_0) \in (\xi \cdot K, f_A(\xi))$, but also for any $b \in A$ of positive degree $b_0 \cdot b \in K$ and hence $b \cdot (\xi^{n-1} \cdot b_0) \in (\xi \cdot K)$ as well. We further claim that $\{\xi^{n-1} \cdot \beta(b_0), \beta(a_{\text{soc}})\}$ are linearly independent in $\hat{A}$. Indeed, $\hat{\pi}(\xi^{n-1} \cdot \beta(b_0)) = \xi^{n-1} t_{\text{soc}}$ is a socle generator of $\tilde{T}$ (hence is nonzero), whereas $\hat{\pi}(\beta(a_{\text{soc}})) = 0$. Since $\beta(a_{\text{soc}}) \neq 0$, this shows that the socle of $\hat{A}$ has dimension at least two, contradicting our assumption that $\hat{A}$ is Gorenstein.

Conversely assume that $a_n = \lambda \cdot \tau$ for some nonzero $\lambda \in \mathbb{F}^\times$. Then, by (1.) $\beta: A \rightarrow \hat{A}$ is injective, and hence $\beta(a_{\text{soc}}) \in \text{soc}(\hat{A})$. We want to show that $\beta(a_{\text{soc}})$ generates the socle. To that end, suppose that $\hat{a} \in \hat{A}$ is any other socle element, and as in Lemma 3.4, write

$$\hat{a} = \beta(b_0) + \beta(b_1)\xi + \cdots + \beta(b_{n-1})\xi^{n-1}. \quad (15)$$

Note that since $\hat{\pi}$ is surjective, it must map socle elements of $\hat{A}$ to socle elements of $\tilde{T}$. Thus, applying $\hat{\pi}$ to (15), we find that there is a constant c (possibly $c = 0$) such that

$$\hat{\pi}(\hat{a}) = \pi(b_0) + \pi(b_1)\xi + \cdots + \pi(b_{n-1})\xi^{n-1} = c \cdot t_{soc} \cdot \xi^{n-1}.$$

Since $\tilde{T}$ is a free T -module with basis $\{1, \xi, \dots, \xi^{n-1}\}$, it follows that $\pi(b_i) = 0$ in T for $0 \leq i \leq n-2$ and $\pi(b_{n-1}) = c \cdot t_{soc}$. This implies that $b_0, \dots, b_{n-2} \in K$, and therefore that $\beta(b_i)\xi^i = 0$ in $\hat{A}$ for $i = 1, \dots, n-2$. Thus, $\hat{a}$ reduces to

$$\hat{a} = \beta(b_0) + \beta(b_{n-1})\xi^{n-1}.$$

Since $\hat{a} \in \text{soc}(\hat{A})$, we have $\xi \cdot \hat{a} = 0$ in $\hat{A}$, and therefore,

$$\begin{aligned} 0 = \xi \cdot \hat{a} &= \xi \cdot \beta(b_0) + \beta(b_{n-1})\xi^n \\ & \quad (\text{since } b_0 \in K, \text{ and } \xi^n \equiv -(a_1\xi^{n-1} + \cdots + a_n)) \\ &= -\beta(b_{n-1}) \cdot (a_1\xi^{n-1} + \cdots + a_{n-1}\xi + a_n) \\ & \quad (\text{since } \pi(b_{n-1}) = c \cdot t_{soc}, \text{ and } a_i \in A_i \text{ for } 1 \leq i \leq n-1) \\ &= \beta(b_{n-1}) \cdot a_n \\ & \quad (\text{since } b_{n-1} = c \cdot t_{soc} \text{ and } a_n = \lambda \cdot \tau) \\ &= \lambda \cdot c \cdot \beta(a_{soc}). \end{aligned}$$

Since $\lambda \neq 0$ and $\beta(a_{soc}) \neq 0$ we must conclude that $c = 0$ and hence $b_{n-1} \in K$ as well and therefore that $\hat{a} = \beta(b_0)$. But since $\beta(b_0) = \hat{a} \in \text{soc}(\hat{A})$ and β is injective, it follows that $b_0 \in \text{soc}(A)$, as desired. Therefore, $\text{soc}(\hat{A}) \subseteq \beta(\text{soc}(A))$, from which it follows that $\hat{A}$ must be Gorenstein. ■

Corollary 3.7. If $a_n = \lambda \cdot \tau$ with $\lambda \in \mathbb{F}$ then

$$H(\hat{A}) = H(A) + H(T)[1] + \cdots + H(T)[n-1]$$

and there is a split exact sequence of A -modules

$$0 \longrightarrow A \xrightarrow{\beta} \tilde{A} \xrightarrow{\tilde{\pi}} \tilde{T}/\beta_0(T) \longrightarrow 0.$$

Proof. Injectivity for β follows from Theorem 3.6 part (1) and surjectivity of $\hat{\pi}$ is by definition. That the displayed sequence is a complex follows from the commutative diagram (12). Exactness of the sequence above viewed as a sequence of vector spaces follows from Lemma 3.4 and the identity

$$H(\hat{A}) = H(\beta(A)) + H(T)[1] + \cdots + H(T)[n-1] = H(A) + H(\tilde{T}/\beta_0(T)).$$

Finally, since A is a Gorenstein ring, A is injective as an A -module; thus, the above sequence splits. $\blacksquare$

Remark 3.8. It follows from Theorem 3.6 and Corollary 3.7 that if $a_n = 0$, then $\beta: A \rightarrow \hat{A}$ is injective and

$$H(\hat{A}) = H(A) + H(T)[1] + \cdots + H(T)[n-1],$$

but $\hat{A}$ is not Gorenstein. We call such algebras *boundary Gorenstein algebras*, indicating that they are in the closure of the Gorenstein locus of the Hilbert scheme; the algebra $\hat{A}$ in Example 3.3(3.) is a boundary Gorenstein algebra of this type. More precisely, one can show that if $\mathbb{F}$ is algebraically closed, then the algebra

$$\tilde{A}[\lambda] = \frac{A[\xi, \lambda]}{(\xi \cdot K, \xi^n + a_1 \xi^{n-1} + \cdots + \lambda \cdot \tau)}$$

is flat as a module over $\mathbb{F}[\lambda]$, where the fibers $\tilde{A}[c]/(\lambda - c) \cdot \tilde{A}[\lambda]$ are Gorenstein if $c \neq 0$ and not Gorenstein, but boundary Gorenstein, if $c = 0$. We shall give another flat family in which the Gorenstein algebra $\hat{A}$ is a general fiber in Section 8.

Definition 3.9 (Preferred orientations). By Lemma 3.2, $\tilde{T}$ from Construction 3.1 is always Gorenstein, and we define its preferred orientation as the one corresponding to the socle generator, $\hat{t}_{soc} = \xi^{n-1} \cdot t_{soc}$. If $\hat{A}$ from Construction 3.1 is Gorenstein, then we define its preferred orientation as the one corresponding to the socle generator $\hat{a}_{soc} = \beta(a_{soc})$; hence, the preferred orientation on $\hat{A}$ is the one inherited from A via β .

Definition 3.10 (Cohomological blowup, exceptional divisor, cohomological blowdown). Given oriented AG algebras A and T of socle degrees $d > k$, respectively, and surjective degree-preserving algebra map $\pi: A \rightarrow T$ with Thom class $\tau \in A_n$ where $n = d - k$, and given a homogeneous monic polynomial $f_A(\xi) = \xi^n + a_1 \xi^{n-1} + \cdots + a_n \in A[\xi]$ of degree n

with homogeneous elements $a_i \in A_i$ for $1 \leq i \leq n$ and with $a_n = \lambda \cdot \tau$ for some nonzero constant λ , and setting $t_i = \pi(a_i)$ for $1 \leq i \leq n-1$, we call the corresponding oriented AG algebra from Construction 3.1, Equation (9) the *cohomological blowup of A along π with parameters $(t_1, \dots, t_{n-1}, \lambda)$* , or BUG for short, and write

$$\tilde{A} = (\hat{A} =) \frac{A[\xi]}{(\xi \cdot K, \underbrace{\xi^n + a_1 \xi^{n-1} + \dots + \lambda \cdot \tau}_{f_A(\xi)})}.$$

with its preferred orientation $\tilde{a}_{soc} = a_{soc}$. The oriented AG algebra from Construction 3.1, Equation (11)

$$\tilde{T} = (\tilde{T} =) \frac{T[\xi]}{(\xi^n + t_1 \xi^{n-1} + \dots + \lambda \cdot \pi(\tau))_{f_T(\xi)}}$$

with its preferred orientation $\tilde{t}_{soc} = \xi^{n-1} \cdot t_{soc}$ is called the *exceptional divisor of T with parameters $(t_1, \dots, t_{n-1}, \lambda)$* . In this case, we refer to A as the *cohomological blowdown of $\tilde{A}$ along $\hat{\pi}$* .

Remark 3.11. One can force $\lambda = 1$ in Definition 3.10 by scaling orientations on either A or T . Specifically, given $\pi: A \rightarrow T$ with Thom class τ , take new distinguished socle generator either $a'_{soc} = \lambda \cdot a_{soc}$ or $t'_{soc} = \lambda^{-1} \cdot t_{soc}$ so that the same map with one of these scaled orientations $\pi': A' \rightarrow T'$ will have Thom class $\tau' = \lambda \cdot \tau$, and hence $a_n = \lambda \cdot \tau = \tau'$.

If $\lambda \in \mathbb{F}$ has an n^{th} -root, say μ (e.g., if $\mathbb{F}$ is algebraically closed), then one can also force $\lambda = 1$ by rescaling the parameters in the cohomological blowup. Specifically, if $\tilde{A}$ is the cohomological blowup of A along $\pi: A \rightarrow T$ with parameters $(t_1, \dots, t_{n-1}, \lambda)$, then $\tilde{A}$ is isomorphic to the cohomological blowup of A along π with parameters $(t'_1, \dots, t'_{n-1}, 1)$ where $t'_i = \mu^i \cdot t_i$ via the map $\xi \mapsto \mu \cdot \xi$.

The following example shows that the hypothesis on $\mathbb{F}$ in Remark 3.11 is necessary.

Example 3.12. Let $A = \mathbb{Q}[x]/(x^3)$ and $T = \mathbb{Q} = \mathbb{F}$ with $\pi: A \rightarrow T$ the natural projection having $\ker(\pi) = (x)$ and Thom class $\tau = x^2$. Taking $f_A(\xi) = \xi^2 + \lambda x^2$ with $\lambda \in \mathbb{Q}^\times$, the

cohomological blow-up algebra of A along π with parameters $(0, \lambda)$ is

$$\tilde{A}(\lambda) = \frac{\mathbb{Q}[x, \xi]}{(x\xi, \xi^2 + \lambda x^2)}.$$

We claim that any pair of integers p, q such that p is a prime that does not divide q yields non-isomorphic algebras $\tilde{A}(p)$ and $\tilde{A}(q)$. Indeed, assume that there is a $\mathbb{Q}$ -algebra isomorphism $\psi : \tilde{A}(q) \rightarrow \tilde{A}(p)$. Then, for some $a, b, c, d \in \mathbb{Q}$, we must have $\psi(x) = ax + b\xi$ and $\psi(\xi) = cx + d\xi$. Moreover, we may assume that a, b, c, d are integers such that $\gcd(a, b, c, d) = 1$ because ψ is an isomorphism if and only if $\alpha \cdot \psi$ is an isomorphism for any $\alpha \in \mathbb{Q}^\times$. We have

$$\psi((x\xi, \xi^2 + qx^2)) = (acx^2 + (ad + bc)x\xi + bd\xi^2, (c^2 + qa^2)x^2 + 2(cd + qab)x\xi + (d^2 + qb^2)\xi^2).$$

In order for ψ to be a $\mathbb{Q}$ -algebra isomorphism, the ideal above must be equal to $(x\xi, \xi^2 + px^2)$ and equating the two ideals in $\mathbb{Q}[x, \xi]/(x\xi)$ yields the following equations:

$$ac = pbd \tag{16}$$

$$c^2 + qa^2 = p(d^2 + qb^2). \tag{17}$$

It follows from (16) that p divides a or c , and hence from (17) that p divides both a and c . Returning to (16), we now deduce that p divides bd and hence p divides b or d . Similarly, we must also have $p \mid (d^2 + qb^2)$, which implies that p must divide b and d . Therefore, we must have p divides a, b, c , and d , contradicting our assumption that $\gcd(a, b, c, d) = 1$.

In particular, we have shown that every prime p gives a cohomological blowup of A along π with parameters $(0, p)$, which is not isomorphic to the cohomological blow-up algebra $\hat{A}(1)$ with parameters $(0, 1)$. In fact, this shows that there are infinitely many distinct isomorphism classes of cohomological blow-up algebras $\tilde{A}(p)$ of A along π with parameters $(0, p)$, one for each prime p .

On the other hand, we shall see in Theorem 8.3 that the algebras $\tilde{A}(\lambda^2)$ and $\tilde{A}(1)$ are always isomorphic for any rational number λ .

The next lemma gives the Thom class of the restriction map from the cohomological blowup to its exceptional divisor.

Lemma 3.13. With notations as in Definition 3.10, if $\tilde{A}$ is the cohomological blowup of A along $\pi : A \rightarrow T$ with parameters $(t_1, \dots, t_{n-1}, \lambda)$, and $\tilde{T}$ is its exceptional divisor then

the Thom class of the projection map $\tilde{\pi}: \tilde{A} \rightarrow \tilde{T}$ is

$$\tilde{\tau} = -\lambda^{-1}\xi.$$

Proof. The socle degree of $\tilde{T}$ is $d-1$, and it follows from Lemma 3.4 that the graded component $\tilde{A}_{d-1}$ has a $\mathbb{F}$ -vector space decomposition as

$$\mathbb{F} \cdot a_T \xi^{n-1} \oplus A_{d-1}, \text{ where } \pi(a_T) = t_{soc} \in T_k.$$

We may assume that $k < d-1$, since if $k = d-1$, we clearly have $\tilde{A} = A$ and $\tilde{T} = T$. Then, for each $a \in A_{d-1}$, we have $a \in K$; hence $\xi \cdot a \equiv 0$ in $\tilde{A}$. Also, we have

$$\begin{aligned} \xi \cdot a_T \xi^{n-1} &\equiv a_T \xi^n \\ &\equiv a_T \left(- \left(a_1 \xi^{n-1} + \cdots + a_n \right) \right) \\ &\text{since } a_i \in A_+ \Rightarrow a_T \cdot a_i \in K \\ &\equiv a_T \cdot (-a_n) \equiv a_T \cdot (-\lambda \cdot \tau) = -\lambda \cdot a_{soc} = -\lambda \cdot \tilde{a}_{soc}, \end{aligned}$$

from which it follows that $\xi = -\lambda \cdot \tilde{\tau}$, as desired. ■

The following theorem gives a useful characterization of the blow-up algebra in terms of some universal properties, analogous to Equations (1)–(3) from the Introduction.

Theorem 3.14. Suppose that we are given oriented AG algebras $(A, \int_A)$, $(T, \int_T)$, $(\tilde{A}, \int_{\tilde{A}})$, and $(\tilde{T}, \int_{\tilde{T}})$ with socle degrees d, k, d , and $d-1$, respectively, with $d > k$, and surjective degree preserving algebra maps $\pi: A \rightarrow T$ and $\tilde{\pi}: \tilde{A} \rightarrow \tilde{T}$. Then $\tilde{A}$ is a cohomological blowup of A along π for some parameters $(t_1, \dots, t_{n-1}, \lambda)$, and $\tilde{T}$ is its exceptional divisor if and only if the following conditions are satisfied.

1. There are degree preserving algebra maps $\beta: A \rightarrow \hat{A}$ and $\beta_0: T \rightarrow \tilde{T}$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\beta} & \hat{A} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ T & \xrightarrow{\beta_0} & \tilde{T} \end{array}$$

2. The Euler class $\epsilon = \tilde{\pi}(\tilde{\tau}) \in \tilde{T}_1$ generates $\tilde{T}$ as a T -algebra (via β_0) with a single homogeneous relation in degree $n = d - k$:

$$\epsilon^n + \beta_0(t'_1)\epsilon^{n-1} + \cdots + \beta_0(t'_n)\epsilon \equiv 0$$

for some homogeneous elements $t'_i \in T_i$ for $1 \leq i \leq n$.

3. There is a short exact sequence of A -modules

$$0 \longrightarrow A \xrightarrow{\beta} \tilde{A} \xrightarrow{\tilde{\pi}} \tilde{T}/\beta_0(T) \longrightarrow 0.$$

Proof. Assume that $(\tilde{A}, \int_{\tilde{A}})$ is the cohomological blowup of $(A, \int_A)$ along π and that $(\tilde{T}, \int_{\tilde{T}})$ is the exceptional divisor, with parameters $(t_1, \dots, t_{n-1}, \lambda)$ from Definition 3.10, and their preferred orientations from Definition 3.9. Then our discussion following Construction 3.1 shows condition (1.) is satisfied. Also, Lemma 3.13 shows that the Thom class of $\tilde{\pi}$ is $\tilde{\tau} = -\lambda^{-1}\xi$, and its Euler class is $\epsilon = \tilde{\pi}(-\lambda^{-1}\xi)$. From the presentation of $\tilde{T}$ in (11), we have $f_T(\xi) = \xi^n + t_1\xi^{n-1} + \cdots + \lambda \cdot \pi(\tau) \equiv 0$ in $\tilde{T}$. Therefore, setting $t'_i = (-1)^i \lambda^{-i} t_i$ for $1 \leq i \leq n-1$ yields $(-1)^n \lambda^n (\epsilon^n + t'_1 \epsilon^{n-1} + \cdots + t'_{n-1} \epsilon + \lambda^{1-n} \cdot \tau) \equiv 0$ in $\tilde{T}$, which is (2.). Finally, Corollary 3.7 implies condition (3.).

Conversely, assume that Conditions (1.)–(3.) hold. Define an algebra map

$$\phi: A[\xi] \longrightarrow \tilde{A} \tag{18}$$

$$a \longmapsto \beta(a)$$

$$\xi \longmapsto \tilde{\tau}$$

where $\beta: A \rightarrow \tilde{A}$ is the map given by (1.), and $\tilde{\tau}$ is the Thom class of $\tilde{\pi}: \tilde{A} \rightarrow \tilde{T}$. Then Conditions (2.) and (3.) guarantee that ϕ is surjective. Indeed, by (2.), the quotient $\tilde{T}/\beta_0(T)$ is generated as an A module by nonzero powers of the Euler class $\epsilon = \tilde{\pi}(\tilde{\tau})$, and hence by (3.), $\tilde{A}$ is generated as an A module by 1 and nonzero powers of the Thom class $\tilde{\tau}$. Furthermore, note that the ideal generated by $\xi \cdot u$ for $u \in K = \ker(\pi)$ is contained in $\ker(\phi)$. Indeed, $\phi(u \cdot \xi) = \beta(u) \cdot \tilde{\tau}$ and for any $\tilde{a} \in \tilde{A}$, we have

$$\begin{aligned} \int_{\tilde{A}} \tilde{\tau} \cdot \beta(u) \cdot \tilde{a} a &= \int_{\tilde{T}} \tilde{\pi}(\beta(u) \cdot \tilde{a}) \\ &= \int_{\tilde{T}} \beta_0(\pi(u)) \cdot \tilde{\pi}(\tilde{a}) = 0, \end{aligned}$$

which implies that $\beta(u)\tilde{\tau} = \phi(u \cdot \xi) = 0$ since $\tilde{A}$ is Gorenstein.

Consider the relation on the Euler class from (2.):

$$\epsilon^n + \beta_0(t'_1)\epsilon^{n-1} + \cdots + \beta_0(t'_{n-1})\epsilon + \beta_0(t'_n) \equiv 0.$$

For each $1 \leq i \leq n-1$, let a_i be any π -lift of t'_i (which exists since π is surjective), and set $g_A(\xi) = \xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi \in A[\xi]$. Note that $\tilde{\pi}(\phi(g_A(\xi))) = -\beta_0(t'_n) \in \beta_0(T)$, and hence according to condition (3.), there exists $a_n \in A$ such that $\beta(a_n) = \phi(g_A(\xi))$ in $\tilde{A}$. Then, setting $f_A(\xi) = g_A(\xi) + a_n$, we see that $f_A(\xi) \in \ker(\phi)$ as well. Thus, we have shown the containment of ideals $(\xi \cdot K, f_A(\xi)) \subseteq \ker(\phi)$, and in particular, the map ϕ induces a surjective map on the quotient

$$\bar{\phi}: \tilde{A}' = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} \rightarrow \tilde{A}.$$

Since $\tilde{A}'$ follows Construction 3.1, Lemma 3.4 implies its Hilbert function is as in (13):

$$H(\tilde{A}') = H(\beta'(A)) + H(T)[1] + \cdots + H(T)[n-1], \quad (19)$$

where $\beta': A \rightarrow \tilde{A}'$ is the natural map described after Construction 3.1. By condition (3.), we know that the Hilbert function of $\tilde{A}$ is

$$H(\tilde{A}) = H(A) + H(T)[1] + \cdots + H(T)[n-1]. \quad (20)$$

Since $H(\beta(A))_i \leq H(A)_i$ for all i , but also $H(\tilde{A}')_i \geq H(A)_i$ for all i by surjectivity of $\bar{\phi}$, we deduce that the Hilbert functions (19) and (20) must be equal and therefore that $\bar{\phi}: \tilde{A}' \rightarrow \tilde{A}$ must be an isomorphism. Finally, since $\tilde{A}$ is Gorenstein, it follows that $\tilde{A}'$ is Gorenstein and hence must be a cohomological blowup, and the result follows. ■

Remark 3.15. As discussed in the Introduction, the conditions of Theorem 3.14 are satisfied by cohomology algebras. More precisely, if $Y \subset X$ are compact complex manifolds of dimension $k < d$, with cohomology algebras $A = H^{2\bullet}(X)$ and $T = H^{2\bullet}(Y)$ and $\pi: A \rightarrow T$ the induced restriction map, surjective or not, then the cohomology algebras of the blown up manifolds $\tilde{T} = H^{2\bullet}(\tilde{Y})$ and $\hat{A} = H^{2\bullet}(\tilde{X})$ with (possibly non-surjective) restriction map $\hat{\pi}: \hat{A} \rightarrow \tilde{T}$ satisfies Conditions (1.)–(3.) of Theorem 3.14. Therefore, Theorem 3.14 seems to offer a way to define the cohomological blowup of an oriented AG algebra A along any, possibly non-surjective, map $\pi: A \rightarrow T$. On the other hand, without the surjectivity assumption on π , one must sacrifice, among other

things, the nice presentation given by Construction 3.1; see Example 9.3 and Remark 9.4 in Section 9.

4 Macaulay Dual Generators

Our reference for this section is [25, Appendix A], but see also [12, Appendix A.2.4]. Let $R = \mathbb{F}[x_1, \dots, x_r]$ be a polynomial ring, and let $Q = \mathbb{F}[X_1, \dots, X_r]$ be the dual divided power algebra. Let $F \in Q_d$ and $G \in Q_k$ be homogeneous forms of degrees $d > k$, respectively, and suppose that $\tau \in R_{d-k}$ is a polynomial for which $G = \tau \circ F$. Then, if $A = R/\text{Ann}(F)$ and $T = R/\text{Ann}(G)$ are the corresponding oriented AG algebras, the identity map on R induces a surjective map on the quotients

$$\pi: A = \frac{R}{\text{Ann}(F)} \rightarrow \frac{R}{\text{Ann}(G)} = T$$

for which the Thom class is τ as shown in Lemma 2.4.

Next, we give another construction, similar to Construction 3.1, that will lead to yet another characterization of the cohomological blowup (Theorem 4.6). First, we need some notation. Throughout this section, we use the notation $\bar{r}$ for the coset of $r \in R$ in the quotient algebra T .

Definition 4.1 (*G-dual polynomial*). Given G and T as above, let ξ be an indeterminate, let n be any positive integer, and let $f_R(\xi) = \xi^n + a_1\xi^{n-1} + \dots + a_n \in R[\xi]$ be a homogeneous monic polynomial with $a_i \in R_i$ for $1 \leq i \leq n$. Evaluation at $\xi = 1$ gives a non-homogeneous element in $f_R(1) \in R$ and projection to the local ring T gives a (non-homogeneous) unit $\mu_f = \bar{1} + \bar{a}_1 + \dots + \bar{a}_n \in T$. Let $\mu_h = \bar{1} + \bar{u}_1 + \dots + \bar{u}_k \in T$ be its T -inverse, that is, $\mu_f \cdot \mu_h = \bar{1}$ in T , with homogeneous components $\bar{u}_i \in T_i$. A homogeneous monic polynomial $h_R(\xi) = \xi^k + u_1\xi^{k-1} + \dots + u_k$ with homogeneous coefficients $u_i \in R_i$ that project to $\bar{u}_i \in T_i$ for every $1 \leq i \leq n$ is called a *G-dual polynomial for $f_R(\xi)$* .

Since T is graded and $\mu_f \cdot \mu_h = \bar{1}$ in T , it follows that the polynomials $f_R(\xi)$ and its G -dual polynomial $h_R(\xi)$ satisfy

$$f_R(\xi) \cdot h_R(\xi) \equiv \xi^{n+k} \pmod{\text{Ann}_R(G) \cdot R[\xi]}.$$

Construction 4.2. Set $n = d - k$, let ξ be an indeterminate, and let Ξ be its dual divided power. Choose homogeneous elements $a_i \in R_i$ for $1 \leq i \leq n$, and define a homogeneous

monic polynomial $f_R(\xi) \in R[\xi]$ by

$$f_R(\xi) = \xi^n + a_1 \xi^{n-1} + \cdots + a_n. \quad (21)$$

Let $h_R(\xi) \in R[\xi]$ be any G -dual polynomial of $f_R(\xi)$:

$$h_R(\xi) = \xi^k + u_1 \xi^{k-1} + \cdots + u_k. \quad (22)$$

Construct the $(d-1)$ -form $\tilde{G} \in Q[\Xi]$ by

$$\tilde{G} = h_R(\xi) \circ (\Xi^{d-1} \cdot G) = \Xi^{n-1} G + \Xi^n (u_1 \circ G) + \cdots + \Xi^{d-1} (u_k \circ G), \quad (23)$$

and construct the oriented AG algebra

$$\tilde{T}_{MD} = \frac{R[\xi]}{\text{Ann}(\tilde{G})}. \quad (24)$$

Next, fix a parameter $\lambda \in \mathbb{F}^\times$, and construct the d -form $\hat{F} \in Q[\Xi]$

$$\hat{F} = F - \lambda \cdot \Xi \tilde{G} = F - \lambda \left(\Xi^n G + \Xi^{n+1} (u_1 \circ G) + \cdots + \Xi^{n+k} (u_k \circ G) \right) \quad (25)$$

and construct the oriented AG algebra

$$\hat{A}_{MD} = \frac{R[\xi]}{\text{Ann}(\hat{F})}. \quad (26)$$

In Construction 4.2, one can easily check that we have

$$\text{Ann}_{R[\xi]}(\hat{F}) \cap R = \text{Ann}_R(F), \text{ and } \text{Ann}_{R[\xi]}(\tilde{G}) \cap R = \text{Ann}_R(G),$$

which implies that the inclusion map $R \hookrightarrow R[\xi]$ induces *injective* maps

$$\beta: A = \frac{R}{\text{Ann}_R(F)} \rightarrow \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\hat{F})} = \hat{A}_{MD} \quad \text{and} \quad \beta_0: T = \frac{R}{\text{Ann}_R(G)} \rightarrow \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\tilde{G})} = \tilde{T}_{MD},$$

Next, we observe that $\xi \circ \tilde{F} = \lambda \cdot \tilde{G}$, and hence the identity map $R[\xi] \rightarrow R[\xi]$ passes to a surjective map on the quotient algebras

$$\hat{\pi}: \hat{A}_{MD} = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\hat{F})} \rightarrow \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\lambda^{-1} \xi \circ \hat{F} = -\tilde{G})} = \tilde{T}_{MD}$$

with Thom class $\tilde{\tau} = -\lambda^{-1}\xi$ (compare with Lemma 3.13). Since the identity and inclusion maps on R and $R[\xi]$ form a commutative square, it follows that the maps on quotients do too, resulting in a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & \hat{A}_{MD} \\ \pi \downarrow & & \downarrow \hat{\pi} \\ T & \xrightarrow{\beta_0} & \tilde{T}_{MD} \end{array} \quad (27)$$

compare with condition (1.) of Theorem 3.14. The following result is related to condition (2.) of Theorem 3.14, namely that $\hat{T}_{MD}$ is a free extension of T ; compare with Lemma 3.2.

Lemma 4.3. Let $G \in Q_k$ be any homogeneous form, let $f_R(\xi) = \xi^n + a_1\xi^{n-1} + \dots + a_n$ any monic homogeneous polynomial with coefficients $a_i \in R_i$, and let $h_R(\xi) = \xi^k + u_1\xi^{k-1} + \dots + u_k \in R[\xi]$ be a G -dual of $f_R(\xi)$, as in Construction 4.2. Then we have the following equality of ideals:

$$\text{Ann} \left(h_R(\xi) \circ \left(\Xi^{d-1} \cdot G \right) \right) = \text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi)).$$

In particular, the oriented AG algebra $\tilde{T}_{MD}$ from Construction 4.2, Equation (26) is a free extension over T and in fact is equal to the algebra $\tilde{T}$ from Construction 3.1 Equation (11), that is,

$$\tilde{T}_{MD} = \frac{R[\xi]}{\text{Ann}(\tilde{G} = h_R(\xi) \circ (\Xi^{d-1} \cdot G))} = \frac{R[\xi]}{\text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi))} = \frac{T[\xi]}{(f_T(\xi) = \overline{f_R(\xi)})} = \tilde{T}.$$

Proof. The containment

$$\text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi)) \subseteq \text{Ann}_{R[\xi]} \left(\tilde{G} = h_R(\xi) \circ (\Xi^{d-1} \cdot G) \right)$$

follows from the relation

$$f_R(\xi) \cdot h_R(\xi) \equiv \xi^d \pmod{\text{Ann}_R(G) \cdot R[\xi]}.$$

Therefore, the identity map $R[\xi] \rightarrow R[\xi]$ passes to a surjective map of algebras

$$\phi: \tilde{T} = \frac{T[\xi]}{(\tilde{f}(\xi))} = \frac{R[\xi]}{\text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi))} \rightarrow \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\tilde{G})} = \tilde{T}_{MD}.$$

Let $t_{soc} \in R_k$ be any homogeneous polynomial that projects onto the socle generator of T , so that the distinguished socle generator of $\tilde{T}$ is $\overline{t_{soc} \cdot \xi^{n-1}}$. Since

$$t_{soc} \cdot \xi^{n-1} \cdot h_R(\xi) \circ (\Xi^{d-1} \cdot G) = 1,$$

it follows from Lemma 2.6 that ϕ must also be injective and hence an isomorphism, and the result follows. $\blacksquare$

Remark 4.4. In [43], the authors refer to the monic polynomial $f_R(\xi)$ as the *homogenizing polynomial* for T and to the G -dual $h_R(\xi)$ as the *dual homogenizing polynomial*. We refer also to [8] for more on de-homogenization. We also remark monic polynomial $f_R(\xi)$ and its G -dual polynomial $h_R(\xi)$ also satisfy

$$f_R(\xi) \cdot h_R(\xi) \circ (\Xi^d \cdot G) = f_R(\xi) \circ (\Xi \cdot \hat{G}) = G; \quad (28)$$

in particular, $f_R(\xi)$ is the Thom class of the projection map

$$\pi_B: B := \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\Xi \cdot \tilde{G})} \rightarrow \frac{R[\xi]}{\text{Ann}_{R[\xi]}(G)} = T.$$

The algebra B defined above will show up again when we discuss connected sums.

Lest the reader think that the AG algebra

$$\tilde{T} = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(h_R(\xi) \circ (\Xi^{d-1} \cdot G))}$$

is always a free extension over $T = R/\text{Ann}_R(G)$; the following example shows otherwise.

Example 4.5. Let $R = \mathbb{F}[x, y, z]$, $Q = \mathbb{F}[X, Y, Z]$, and let $G = XYZ$ with $k = \deg(G) = 3$. Suppose that we choose $n = 2$, and $h_R(\xi) = \xi^3 + (xy + xz + yz)\xi + (xyz)$ and define $\tilde{G}$ as in Construction 4.2 Equation (23):

$$\tilde{G} = \Xi XYZ + \Xi^3(X + Y + Z) + \Xi^4 = h(\xi) \circ (\Xi^4 G).$$

Note that there is no monic polynomial $f_R(\xi) \in \text{Ann}_{R[\xi]}(\tilde{G})$ of degree $n = 2$, and in fact we have

$$\tilde{T} = \frac{R[\xi]}{\text{Ann}(\tilde{G})} = \frac{\mathbb{F}[x, y, z, \xi]}{(x^2, y^2, z^2, \xi xyz - \xi^3(x + y + z), \xi^3(x + y + z) - \xi^4, \xi^5)},$$

which is not a free extension over $T = R/\text{Ann}(G)$.

Note, however, that the corresponding unit $\overline{h(1)} \in T$ is $\mu_h = \bar{1} + \overline{(xy + xz + yz)} + \overline{(xyz)}$ and its T -inverse is $(\mu_h)^{-1} = \bar{1} - \overline{(xy + xz + yz)} - \overline{(xyz)}$, indicating that the “correct choice” is $f_R(\xi) = \xi^3 - (xy + xz + yz)\xi - xyz$ of degree $n = 3$ (not $n = 2$). Indeed, we see that if we define the homogeneous form of degree 5

$$\tilde{G}' = \Xi^2 XYZ + \Xi^4(X + Y + Z) + \Xi^5 = h_R(\xi) \circ (\Xi^5 G),$$

then

$$\tilde{T}' = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\tilde{G}')} = \frac{R[\xi]}{\text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi))} = \frac{T[\xi]}{(f_T(\xi) = \xi^3 - \overline{(xy + xz + yz)}\xi - \overline{(xyz)})},$$

which is a free extension over $T = R/\text{Ann}(G)$ with fiber $F = \mathbb{F}[\xi]/(\xi^3)$. The moral of the story here is that given a k -form $G \in Q_k$ and $T = R/\text{Ann}(G)$, then in order to construct the dual generator of a free extension over T by the formula $h_R(\xi) \circ (\Xi^{n+k-1} \cdot G) \in Q_{d-1}$, we can either choose the integer n and the monic polynomial $f_R(\xi)$ of degree n and then take $h_R(\xi)$ as its G -dual, or we can choose the monic polynomial $h_R(\xi)$ of degree k , then take its G -dual $f_R(\xi)$ and take $n = \deg(f_R(\xi))$, but we cannot necessarily choose the integer n and the polynomial $h_R(\xi)$ simultaneously. Of course, the $\tilde{G}$ of Construction 4.2 Equation (23) follows the former procedure and, by Lemma 4.3, is always the dual generator of a free extension of T .

The following result gives necessary and sufficient conditions for $\hat{A}_{MD}$ to be a cohomological blow-up algebra of A along $\pi: A \rightarrow T$ in the sense of Definition 3.10.

Theorem 4.6. Let $f_R(\xi)$, $h_R(\xi)$, $\tilde{G}$, $\tilde{T}_{MD}$, λ , $\hat{F}$, and $\hat{A}_{MD}$ be as in Construction 4.2. Then the following statements are equivalent.

1. The algebra $\hat{A}_{MD}$ is isomorphic to a cohomological blowup of A along π with some parameters $(t_1, \dots, t_{n-1}, \lambda)$.

2. The Hilbert function of $\hat{A}_{MD}$ satisfies

$$H(\hat{A}_{MD}) = H(A) + H(T)[1] + \cdots + H(T)[n-1].$$

3. There exists an element $r \in R$ for which $f_R(\xi) - r \in \text{Ann}_{R[\xi]}(\tilde{F})$.
 4. The constant coefficient r_n of $f_R(\xi)$ satisfies

$$(r_n - r) \circ F = \lambda \cdot G$$

for some $r \in \text{Ann}_R(G)$.

Proof. (1.) $\Rightarrow$ (2.). Assume that $\hat{A}_{MD}$ is isomorphic to a cohomological blowup of $\pi: A \rightarrow T$. Then Corollary 3.7 gives (2.).

(2.) $\Rightarrow$ (3.). Assume that $H(\hat{A}_{MD}) = H(A) + H(T)[1] + \cdots + H(T)[n-1]$. Then the sequence of maps

$$0 \longrightarrow A \xrightarrow{\beta} \hat{A}_{MD} \xrightarrow{\hat{\pi}} \tilde{T}/\beta_0(T) \longrightarrow 0$$

is exact by the argument in the proof of Corollary 3.7. Let $f_A(\xi)$ be the equivalence class of $f_R(\xi) \in R[\xi]$ in $\hat{A}_{MD}$. Note that by Lemma 4.3, $f_A(\xi)$ must be in the kernel of $\hat{\pi}$, and hence by exactness, there exists $\bar{r} \in A = R/\text{Ann}(F)$ for which $\beta(\bar{r}) = f_A(\xi)$. Then if $r \in R$ be any homogeneous lift of $\bar{r}$, we have $f_R(\xi) - r \in \text{Ann}(\hat{F})$, which is (3.).

(3.) $\Rightarrow$ (4.). Assume that there exists $r \in R$ for which $f_R(\xi) - r \in \text{Ann}(\hat{F})$. Then we have

$$\begin{aligned} (f_R(\xi) - r) \circ \hat{F} &= (f_R(\xi) - r) \circ \left(F - \lambda \cdot h_R(\xi) \circ \left(\Xi^{n+k} \cdot G \right) \right) \\ &= f_R(\xi) \circ F - r \circ F - \lambda \cdot (f_R(\xi) \cdot h_R(\xi)) \circ (\Xi^{n+k} \cdot G) + \lambda \cdot \Xi \cdot r \circ \tilde{G} \\ &= r_n \circ F - r \circ F - \lambda \cdot G + \lambda \cdot \Xi \cdot r \circ \tilde{G} \\ &= (r_n - r) \circ F - \lambda \cdot G + \lambda \cdot \Xi \cdot r \circ \tilde{G} = 0, \end{aligned} \tag{29}$$

which implies, by comparing Ξ -coefficients, that

$$(r_n - r) \circ F - \lambda \cdot G = 0.$$

Moreover, since $f_R(\xi) - r \in \text{Ann}(\hat{F}) \subseteq \text{Ann}(\tilde{G} = \xi \circ \hat{F})$, and $f_R(\xi) \in \text{Ann}(\tilde{G})$, we see that $r \in \text{Ann}(\tilde{G})$ too, which implies that $r \in \text{Ann}_{R[\xi]}(\tilde{G}) \cap R = \text{Ann}_R(G)$, and (4.) follows.

(4.) $\Rightarrow$ (1.). Assume that there exists $r \in \text{Ann}(G)$ such that the constant coefficient of $f_R(\xi) - r \in R[\xi]$ satisfies $(r_n - r) \circ F = \lambda \cdot G$. Then we claim that $f_R(\xi) - r \in \text{Ann}(\hat{F})$. Indeed, as in (31), we have

$$(f_R(\xi) - r) \circ \hat{F} = (r_n - r) \circ F - \lambda \cdot G - \lambda \cdot \Xi \cdot r \circ \tilde{G} = -\Xi(\lambda r \circ \tilde{G}) = 0$$

since $r \circ \tilde{G} = h_R(\xi) \circ (\Xi^d \cdot (r \circ G)) = 0$.

Next, observe that for any $r' \in \text{Ann}_R(G) = \text{Ann}_{R[\xi]}(\tilde{G}) \cap R$ we have $\xi \cdot r' \in \text{Ann}_{R[\xi]}(\hat{F})$. In particular then, the kernel of the surjective map

$$\Phi: R[\xi] \rightarrow \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\hat{F})} = \hat{A}_{MD}$$

contains $\text{Ann}_R(G) \cdot \xi R[\xi]$, and $f_R(\xi) - r$, not to mention the ideal $\text{Ann}_R(F) \cdot R[\xi]$. Set $I \subset R[\xi]$ to be the sum of these components, that is, $I = \text{Ann}_R(F) \cdot R[\xi] + \text{Ann}_R(G) \cdot \xi R[\xi] + (f_R(\xi) - r)$. Then, clearly, we have $R[\xi]/I \cong A[\xi]/(\xi \cdot K, f_A(\xi) = \overline{f_R(\xi) - r})$ where $K = \ker(\pi) = \text{Ann}_R(G) \cdot R[\xi]/\text{Ann}_R(F) \cdot R[\xi]$. Moreover, if $t_i \in T_i$ is the equivalence class of $r_i \in R_i$, then since the equivalence class of $r_n - r \in R_n$ in A is $\lambda \cdot \tau \in A_n$, it follows that $R[\xi]/I$ is the cohomological blowup of A along π with parameters $(t_1, \dots, t_{n-1}, \lambda)$. Therefore, we have a surjective map of Gorenstein algebras

$$\bar{\Phi}: \hat{A} = R[\xi]/I = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} \rightarrow \hat{A}_{MD}.$$

Since $\hat{A}$ and $\hat{A}_{MD}$ have the same socle degree, $\bar{\Phi}$ must be an isomorphism by Lemma 2.6, which is (1.). ■

The preceding Theorem 4.6 implies that the algebras $\hat{A}$ from Construction 3.1 Equation (9) and $\hat{A}_{MD}$ from Construction 4.2 Equation (26) are equal precisely when they are cohomological blowups; in this case, we shall replace “hat” with “tilde” on all symbols and write

$$\tilde{A} = \hat{A}_{MD} = \frac{R[\xi]}{\text{Ann}\left(\tilde{F} = F - \Xi \cdot \left(\tilde{G} = h_R(\xi) \circ (\Xi^{d-1} \cdot G)\right)\right)} = \frac{A[\xi]}{(\xi \cdot K, \overline{f_R(\xi)})} = \hat{A}.$$

The following gives an example in which $\hat{A} \neq \hat{A}_{MD}$.

Example 4.7. Let $F = X^2Y^2$ and $G = XY$ so that

$$\pi: A = \frac{\mathbb{F}[x, y]}{\text{Ann}(X^2Y^2)} \rightarrow \frac{\mathbb{F}[x, y]}{\text{Ann}(XY)} = T, \quad \pi(x) = x, \quad \pi(y) = y$$

is the natural projection map of corresponding AG algebras with Thom class $\tau = xy$; here, $d = 4$, $k = 2$, and $n = 2$. Taking $f_R = \xi^2$, we see that its G -dual is itself, that is, $h_R(\xi) = \xi^2$, and hence setting $\tilde{G} = h_R(\xi) \circ (\Xi^{d-1}G) = \xi^2 \circ (\Xi^3 \cdot XY) = \Xi XY$, we get

$$\tilde{T} = \frac{\mathbb{F}[x, y, \xi]}{\text{Ann}(\Xi XY)} = \frac{\mathbb{F}[x, y, \xi]}{(x^2, y^2, \xi^2)}.$$

Then taking $\lambda = 1$, Construction 4.2 yields $\hat{F} = F - \Xi \tilde{G} = X^2Y^2 - \Xi^2XY$, and hence

$$\hat{A}_{MD} = \frac{\mathbb{F}[x, y, \xi]}{\text{Ann}(X^2Y^2 - \Xi^2XY)} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, x^2\xi, y^2\xi, x(\xi^2 + xy), y(\xi^2 + xy), \xi^3)}.$$

Note in this case, there is no $r \in \text{Ann}(G = XY)$ for which $f_R(\xi) - r = \xi^2 - r \in \text{Ann}(\hat{F})$. We can also compute the Hilbert function $H(\hat{A}_{MD}) = (1, 3, 6, 3, 1)$, whereas $H(A) + H(T)[1] = (1, 3, 5, 3, 1)$, and hence $H(\hat{A}_{MD}) \neq H(A) + H(T)[1]$, and therefore $\hat{A}_{MD}$ is not a cohomological blowup of A along π , by Theorem 4.6 (or by Theorem 3.14). Moreover, taking $f_A(\xi) = \overline{f_R(\xi)} = \xi^2$, Construction 3.1 yields

$$\hat{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi) = \xi^2)} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, \xi x^2, \xi y^2, \xi^2)},$$

which is not even Gorenstein; in particular, $\hat{A} \neq \hat{A}_{MD}$.

Alternatively, a “correct choice” is $f_R(\xi) = \xi^2 - xy$ for which a G -dual polynomial is $h_R(\xi) = \xi^2 + xy$ yielding $\tilde{G} = (\xi^2 + xy) \circ (\Xi^3 XY) = \Xi XY + \Xi^3$ and

$$\tilde{T} = \frac{\mathbb{F}[x, y, \xi]}{\text{Ann}(\Xi XY + \Xi^3)} = \frac{\mathbb{F}[x, y, \xi]}{(x^2, y^2, \xi^2 - xy)}.$$

In this case, taking $\lambda = -1$ (Theorem 4.6 requires it!), Construction 4.2 yields $\tilde{F} = X^2Y^2 + \Xi(\Xi XY + \Xi^3)$ with

$$\tilde{A} = (\hat{A}_{MD}) = \frac{\mathbb{F}[x, y, \xi]}{\text{Ann}(\Xi^2XY + \Xi^4 + X^2Y^2)} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, \xi x^2, \xi y^2, \xi^2 - xy)}.$$

Here, we can verify that the conditions of Theorem 4.6 are satisfied, for example the Hilbert function is $H(\tilde{A}) = (1, 3, 5, 3, 1) = H(A) + H(T)[1]$. Hence, in this case, $\tilde{A} (= \hat{A}_{MD})$

is the cohomological blowup of A along π with parameters $(t_1, \lambda) = (0, -1)$. In this case, taking $f_A(\xi) = \overline{f_R(\xi)} = \xi^2 - xy$, then Construction 3.1 yields the same algebra:

$$\tilde{A} = (\hat{A} =) \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, \xi x^2, \xi y^2, \xi^2 - xy)}.$$

5 Relation with Connected Sums

5.1 The cohomological blow-up algebra as a connected sum

In this section, we relate cohomological blow-up algebras to a different algebraic construction termed connected sum, an algebraic analogue of the better known topological construction by the same name. In the topological construction, a connected sum is obtained by gluing two $2d$ -dimensional manifolds M_1 and M_2 along diffeomorphic tubular neighborhoods of a common $2k$ -dimensional submanifold N . In complex geometry, it is well known that the connected sum of an n -dimensional complex manifold M with a projective space $\mathbb{P}^n$ is diffeomorphic to the blowup of M at a point [23, p.101]. Our contribution here is to recognize that *any* cohomological blow-up algebra along a surjective map is a connected sum in the algebraic sense; see Theorem 5.4. We now recall the algebraic connected sum construction as defined in [30], based on the original construction defined in [3].

Definition 5.1 (Fibered product, connected sum). Given oriented AG algebras $(A, \int_A)$, $(B, \int_B)$, another AG algebra $(T, \int_T)$, and algebra maps $\pi_A: A \rightarrow T$ and $\pi_B: B \rightarrow T$ one forms the *fibered product algebra* as the sub algebra $A \times_T B \subseteq A \times B$ of the direct product algebra given by

$$A \times_T B = \{(a, b) \in A \times B \mid \pi_A(a) = \pi_B(b)\}.$$

If A, B have the same socle degree d , π_A and π_B have Thom classes τ_A and τ_B respectively, and the Euler classes $\pi_B(\tau_B) = \pi_A(\tau_A)$ are equal, then the *total Thom class* (τ_A, τ_B) is in the fibered product algebra $A \times_T B$, and we define the *connected sum algebra* as the quotient of the fibered product by the principal ideal generated by the total Thom class, that is,

$$A \#_T B = \frac{A \times_T B}{\langle (\tau_A, \tau_B) \rangle}.$$

If the projection maps π_A and π_B are both surjective, then the connected sum algebra defined above is an AG algebra of the same socle degree d as A and B ; see for example [30, Lemma 3.8].

In terms of Macaulay duality, the following result in [30, Theorem 4.6] gives a useful criterion to recognize an AG algebra as a connected sum.

Proposition 5.2. Let $R = \mathbb{F}[x_1, \dots, x_r]$, and let $F, H \in R_d$ be two linearly independent homogeneous forms of degree d . Suppose that there exists $\sigma \in R_{d-k}$ (for some $k < d$) satisfying

1. $\sigma \circ F = \sigma \circ H \neq 0$, and
2. $\text{Ann}(\sigma \circ F = \sigma \circ H) = \text{Ann}(F) + \text{Ann}(H)$.

In this case, set

$$A = \frac{Q}{\text{Ann } F}, \quad B = \frac{Q}{\text{Ann}(H)}, \quad T = \frac{Q}{\text{Ann}(\sigma \circ F = \sigma \circ H)},$$

and let $\pi_A: A \rightarrow T$ and $\pi_B: B \rightarrow T$ be the natural projection maps. Then the Thom classes of π_A and π_B are given by $\tau_A = \sigma + \text{Ann}(F)$ and $\tau_B = \sigma + \text{Ann}(H)$, and we have algebra isomorphisms

$$A \times_T B \cong \frac{Q}{\text{Ann}(F) \cap \text{Ann}(H)}, \quad A \#_T B \cong \frac{Q}{\text{Ann}(F - H)}.$$

And, conversely, every connected sum $A \#_T B$ of graded AG algebras of the same socle degree over graded AG algebra T arises in this way.

We utilize this result to realize a cohomological blow-up algebra defined as in Construction 4.2 as a connected sum.

Example 5.3. Let $F = X^2Y^2$ and $G = Y$ and π the projection between their corresponding AG algebras

$$\pi: A = \frac{\mathbb{F}[x, y]}{\text{Ann}(X^2Y^2)} \rightarrow \frac{\mathbb{F}[x, y]}{\text{Ann}(Y)} = T,$$

with Thom class $\tau = x^2y$. Here, $H(A) = (1, 2, 3, 2, 1)$, $H(T) = (1, 1)$, $d = 4$, $k = 1$, and $n = 3$. Take $h_R(\xi) = \xi + y$ with corresponding unit $\mu_h = 1 + y \in T$, and inverse $(\mu_h)^{-1} = 1 - y$, so that $f_R(\xi) = \xi^3 - \xi^2y$. Then,

$$\tilde{G} = (\xi + y) \circ (\Xi^3 Y) = \Xi^2 Y + \Xi^3.$$

Choosing $\lambda = -2$, we have

$$\tilde{F} = X^2 Y^2 - 2\Xi \left(\Xi^2 Y + \Xi^3 \right) = X^2 Y^2 - 2\Xi^3 Y - 2\Xi^4.$$

Setting $r = -2x^2 y \in \text{Ann}_R(G)$, we have $\hat{f}_R(\xi) = f_R(\xi) - r = \xi^3 - \xi^2 y + 2x^2 y \in \text{Ann}_{R[\xi]}(\tilde{F})$, and hence

$$\tilde{A} = \frac{\mathbb{F}[x, y, \xi]}{\text{Ann}(\tilde{F})} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, \xi x, \xi y^2, \xi^3 - \xi^2 y + 2x^2 y)}$$

is according to Theorem 4.6 the cohomological blowup of A along π with parameters $(-\bar{y}, 0, , 0, 2)$, with Hilbert function $H(\tilde{A}) = H(A) + H(T)[1] + H(T)[2] = (1, 3, 5, 3, 1)$.

Setting $H = 2\Xi^3 Y + 2\Xi^4$, we have $\tilde{F} = F - H$. Moreover, the element $\tau = \hat{f}_R(\xi)$ satisfies $\tau \circ H = \tau \circ F = Y$ and $\text{Ann}_{R[\xi]}(Y) = \text{Ann}_{R[\xi]}(F) + \text{Ann}_{R[\xi]}(H)$. Hence, it follows from Proposition 5.2 that we can also realize $\tilde{A}$ as a connected sum of $A = R[\xi]/\text{Ann}(F)$ and $B = R[\xi]/\text{Ann}(H)$ over $T = R/\text{Ann}(Y)$, that is, $\tilde{A} \cong A \#_T B$.

In the following, we show that Example 5.3 is an instance of a general phenomenon: any cohomological blow-up algebra is a connected sum. For the remainder of this section, we work under the following set up. Fix a d -form $F \in Q_d$ and a k -form $G \in Q_k$ with $d > k$, and let $A = R/\text{Ann}(F)$ and $T = R/\text{Ann}(G)$ be the associated oriented AG algebras. Assume there exists homogeneous polynomial $\tau \in R_{d-k}$ for which $\tau \circ F = G$, so that the natural projection map $\pi_A: A \rightarrow T$ has Thom class $\tau_A = \bar{\tau}$. Let $n = d - k$, let ξ be an indeterminate, Ξ a divided power variable dual to ξ , and fix a monic homogeneous polynomial $f_R(\xi) = \xi^n + r_1 \xi^{n-1} + \dots + r_n \in R[\xi]$ with coefficients $r_i \in R_i$ such that $r_n = \lambda \cdot \tau$ for some nonzero constant $\lambda \in \mathbb{F}^\times$. Let $h_R(\xi) = \xi^k + u_1 \xi^{k-1} + \dots + u_k \in R[\xi]$ be a G -dual polynomial of $f_R(\xi)$. Define the $(d-1)$ -form $\tilde{G}$ and its associated oriented AG algebra $\tilde{T}$ by

$$\tilde{G} = h_R(\xi) \circ \left(\Xi^{d-1} \cdot G \right), \quad \tilde{T} = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\tilde{G})}.$$

Define the d -form $\tilde{F}$ and its associated oriented AG algebra $\tilde{A}$

$$\tilde{F} = F - \lambda \cdot \Xi \cdot \tilde{G}, \quad \tilde{A} = \hat{A}_{MD} = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\tilde{F})}$$

as in construction 4.2; note that Theorem 4.6 guarantees that $\hat{A}_{MD} = \tilde{A}$ here. Define the new d -form H and its associated AG algebra B

$$H = \lambda \cdot \Xi \cdot \tilde{G} \in Q[\Xi]_d, \quad B = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(H)}$$

and recall that

$$H = \lambda \cdot \Xi \cdot \tilde{G} = \lambda \cdot h_R(\xi) \circ (\Xi^d \cdot G).$$

It follows that $h_R(\xi)$ is also a G -dual polynomial for $\xi \cdot f_R(\xi)$ of degree $n + 1$, and hence by Lemma 4.3, we have

$$B = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(H)} = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\text{Ann}_R(G) \cdot R[\xi] + (\xi \cdot f_R(\xi)))} = \frac{T[\xi]}{(\xi \cdot f_R(\xi))}.$$

Thus, the distinguished socle generator of B as $b_{soc} = \lambda^{-1} \cdot \xi^n \cdot t_{soc}$. The evaluation $\xi = 0$ passes to a map on quotients $\pi_B: B \rightarrow T$, and from the identity (28), it follows that its Thom class is

$$\tau_B = \lambda^{-1} \cdot f_R(\xi).$$

Note that the Euler class of π_B is $\pi_B(\lambda^{-1} \cdot f_R(\xi)) = \tau$, which is equal to the Euler class of π_A , and hence it makes sense to form the connected sum $A \#_T B$.

Theorem 5.4. The connected sum of A and B over T is equal to the cohomological blowup of A along π with parameters $(\overline{a_1}, \dots, \overline{a_{n-1}}, \lambda)$, that is,

$$\tilde{A} = A \#_T B.$$

Proof. With our setup above, Theorem 4.6 implies that $\tilde{A}$ is the cohomological blowup of A along π with parameters $(\overline{a_1}, \dots, \overline{a_{n-1}}, \lambda)$, that is,

$$\hat{A}_{MD} = \tilde{A} = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(\tilde{F} = F - H)}.$$

Setting $\sigma = f_R(\xi)$ in Proposition 5.2, it then suffice to check that Conditions (1.) and (2.) hold. Condition (1.) holds since we have

$$f_R(\xi) \circ F = (\xi^n + r_1 \xi^{n-1} + \dots + \lambda \cdot \tau) \circ F = \lambda \cdot G = f_R(\xi) \circ (H).$$

It remains to see that

$$\mathrm{Ann}_{R[\xi]}(G) = \mathrm{Ann}_{R[\xi]}(H) + \mathrm{Ann}_{R[\xi]}(F). \quad (30)$$

By Lemma 4.3, we have

$$\mathrm{Ann}_{R[\xi]}(H) = \mathrm{Ann}_R(G) \cdot R[\xi] + (\xi \cdot f_R(\xi)).$$

Also, since F is independent of ξ , we have

$$\mathrm{Ann}_{R[\xi]}(F) = \mathrm{Ann}_R(F) \cdot R[\xi] + (\xi).$$

Since $\mathrm{Ann}_R(F) \subset \mathrm{Ann}_R(G)$ and

$$\mathrm{Ann}_{R[\xi]}(G) = \mathrm{Ann}_R(G) \cdot R[\xi] + (\xi),$$

(30) follows, and the desired conclusion follows by Proposition 5.2. ■

Remark 5.5. In general, fibered products and connected sums of standard graded AG algebras need not be standard graded, even in the simplest cases. Examples illustrating this appear for the fibered product in [30, Example 4.5], and for the connected sum in [30, Proposition 5.22]. Theorem 5.4 distinguishes cohomological blow-up algebras of surjective maps as a class of connected sums, which preserves the standard grading. However, there are examples from geometry where A and T are standard graded, but the restriction map $\pi: A \rightarrow T$ is not surjective, and the cohomological blowup has a nonstandard grading; see Remark 9.4.

5.2 The blowdown as a connected sum

Continuing with our setup above, let $\tilde{\pi}_A: \tilde{A} \rightarrow \tilde{T}$ be the projection map with Thom class $\tilde{\tau}_A = -\lambda^{-1}\xi$. Set $H = \lambda \cdot \Xi \cdot \tilde{G}$, and consider the surjective map of AG algebras

$$\tilde{\pi}_B: \tilde{B} := \frac{R[\xi]}{\mathrm{Ann}(-H = -\lambda \cdot \Xi \cdot \tilde{G})} \rightarrow \frac{R[\xi]}{\mathrm{Ann}(\tilde{G})} = \tilde{T}.$$

The algebra $\tilde{B}$ is the AG algebra B from the previous subsection, but with orientation reversed, that is, $\tilde{b}_{soc} = -b_{soc}$. Thus, the Thom class of the map $\tilde{\pi}_B$ is $\tilde{\tau}_B = -\lambda^{-1}\xi$, which is equal to the Thom class of $\tilde{\pi}_A$. Then it makes sense to form the connected sum $\tilde{A} \#_{\tilde{T}} \tilde{B}$.

Theorem 5.6. The connected sum of $\tilde{A}$ and $\tilde{B}$ over $\tilde{T}$ is equal to A , that is,

$$A = \tilde{A} \#_{\tilde{T}} \tilde{B}.$$

Proof. Write

$$A = \frac{R[\xi]}{\text{Ann}_{R[\xi]}(F = \tilde{F} - (-H))} = \frac{R[\xi]}{\text{Ann}_R(F) \cdot R[\xi] + (\xi)}.$$

Setting $\sigma = \xi$ in Proposition 5.2, we will show that Conditions (1.) and (2.) hold. Condition (1.) holds because

$$\xi \circ \tilde{F} = -\lambda \cdot \tilde{G} = \xi \circ (-H).$$

For Condition (2.), note that it follows from Theorem 4.6 and Construction 3.1 that

$$\tilde{A} \cong \frac{A[\xi]}{(\xi \cdot K, \tilde{f}_R(\xi))} = \frac{R[\xi]}{\text{Ann}_R(F) \cdot R[\xi] + \xi \cdot \text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi))}$$

and hence that

$$\text{Ann}_{R[\xi]}(\tilde{F}) = \text{Ann}_R(F) \cdot R[\xi] + \xi \cdot \text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi)).$$

Also, from Lemma 4.3, we have

$$\text{Ann}_{R[\xi]}(H) = \text{Ann}_R(G) \cdot R[\xi] + (\xi \cdot f_R(\xi)),$$

and since $\text{Ann}_R(F) \subseteq \text{Ann}_R(G = \tau \circ F)$, their sum satisfies

$$\text{Ann}_{R[\xi]}(\tilde{F}) + \text{Ann}_{R[\xi]}(H) = \text{Ann}_R(G) \cdot R[\xi] + (f_R(\xi)) = \text{Ann}_{R[\xi]}(\tilde{G}),$$

which is condition (2). Hence, the result follows from Proposition 5.2. ■

One interesting consequence of Theorem 5.6 is that every AG algebra has a nontrivial connected sum decomposition over some algebra $\tilde{T}$. This stands in direct contrast to connected sums over the ground field $T = \mathbb{F}$, where $\#_{\mathbb{F}}$ -indecomposable AG algebras exist; see [3, Theorem 8.3] and also [42, Proposition 3.1]. On the other

hand, Theorem 5.6 shows that over general T , there may be no $\#_T$ -indecomposable AG algebras.

Example 5.7. Set $F = Z^2XY - X^2Y^2$ so that the corresponding AG algebra is

$$A = \frac{\mathbb{F}[x, y, z]}{\text{Ann}(Z^2XY - X^2Y^2)} = \frac{\mathbb{F}[x, y, z]}{(x^3, y^3, x^2z, y^2z, x(z^2 + xy), y(z^2 + xy), z^3)}$$

with Hilbert function $H(A) = (1, 3, 6, 3, 1)$. Set $G = 1$ so that $T = \mathbb{F}$ and the Thom class for the projection $\pi: A \rightarrow \mathbb{F}$ is just the socle generator $\tau = z^2xy$. Then we can set $f_R(\xi) = \xi^4 - z^2xy$, which has G -dual $h_R(\xi) = 1$. Therefore, we have $\tilde{G} = \Xi^3$, $\tilde{F} = (Z^2XY - X^2Y^2) - \Xi^4$, and $H = \Xi^4$. Then,

$$\tilde{T} = \frac{\mathbb{F}[x, y, z, \xi]}{\text{Ann}(\Xi^3)} = \frac{\mathbb{F}[x, y, z]}{(x, y, z, \xi^4)}$$

and

$$\begin{aligned} \tilde{A} &= \frac{\mathbb{F}[x, y, z, \xi]}{\text{Ann}(Z^2XY - X^2Y^2 - \Xi^4)} \\ &= \frac{\mathbb{F}[x, y, z, \xi]}{(x^3, y^3, x^2z, y^2z, x(z^2 + xy), y(z^2 + xy), z^3, \xi x, \xi y, \xi z, \xi^4 - z^2xy)} \end{aligned}$$

and

$$\tilde{B} = \frac{\mathbb{F}[x, y, z, \xi]}{\text{Ann}(-\Xi^4)} = \frac{\mathbb{F}[x, y, z, \xi]}{(x, y, z, \xi^5)}.$$

Since $\xi \circ (Z^2XY - X^2Y^2 - \Xi^4) = -\Xi^3 = \xi \circ (-\Xi^4)$, and also $\text{Ann}(Z^2XY - X^2Y^2 - \Xi^4) + \text{Ann}(\Xi^4) = \text{Ann}(\Xi^3)$, it follows that A is a connected sum of $\tilde{A}$ and $\tilde{B}$ along $\tilde{T}$, as guaranteed by Theorem 5.6.

We conclude this section with an example that shows that unlike the blow-up operation, the blow-down operation may not always preserve the standard grading.

Example 5.8. Define the AG algebra

$$A = \frac{\mathbb{F}[x, y, u]}{(x^2, u^2, xy, xu - yu, xu - y^3)} = \frac{\mathbb{F}[x, y, u]}{\text{Ann}(XU + YU + Y^3)}$$

with the nonstandard grading $\deg(x) = \deg(y) = 1$ and $\deg(u) = 2$. We can blow up along the projection

$$\pi: A \rightarrow T = \frac{\mathbb{F}[x]}{\text{Ann}(X)} = \frac{F[x]}{(x^2)}$$

with Thom class $\tau = u - y^2$ and kernel $K = (y, u)$. Then setting $f_R(\xi) = \xi^2 - (u - y^2)$ (which has $G = X$ -dual polynomial $h_R(\xi) = \xi^k$), the cohomological blowup of A along π is

$$\tilde{A} = \frac{\mathbb{F}[x, y, u, \xi]}{(x^2, u^2, xy, xu - yu, xu - y^3, \xi y, \xi u, \xi^2 - (u - y^2))} = \frac{\mathbb{F}[x, y, u, \xi]}{\text{Ann}(\Xi^2 X - XU - YU - Y^3)}.$$

Since $u \equiv \xi^2 + y^2$ in $\tilde{A}$, we can eliminate u and get

$$\begin{aligned} \tilde{A} &= \frac{\mathbb{F}[x, y, \xi]}{(x^2, (\xi^2 + y^2)^2, xy, (x - y)(\xi^2 + y^2), x(\xi^2 + y^2) - y^3, \xi y, \xi(\xi^2 + y^2))} \\ &\cong \frac{\mathbb{F}[x, y, \xi]}{(x^2, \xi^3, xy, x\xi^2 - y^3, y\xi)}, \end{aligned}$$

which has a standard grading.

Incidentally, $\tilde{A}$ is also the cohomological blow-up algebra of a *standard graded* AG algebra A' along a different T' , namely

$$A' = \frac{\mathbb{F}[x, \xi]}{(x^2, \xi^3)} \rightarrow T' = \mathbb{F}.$$

Here, the kernel is $K' = (x, \xi)$ and the Thom class is the socle generator $\tau = x\xi^2$, and taking y as the “blow-up variable” with $f_R(y) = y^3 - x\xi^2$, we find that the blowup satisfies

$$\tilde{A}' = \frac{A[y]}{(yK', \overline{f(y)})} = \frac{\mathbb{F}[x, \xi, y]}{(x^2, \xi^3, yx, y\xi, y^3 - x\xi^2)} \cong \tilde{A}.$$

6 Minimal Generating Sets and CIs

In this paper, a complete intersection (CI) is a quotient of a polynomial ring by an ideal generated by a regular sequence of maximal length. At a cursory glance, the presentation in Construction 3.1 may lead one to believe that cohomological blowups cannot be CIs, except in embedding dimension two where all AG algebras are CIs, but the following example shows otherwise.

Example 6.1. Define the AG algebras and the surjective map between them

$$A = \frac{\mathbb{F}[x, y]}{(x^3, y^3)} \xrightarrow{\pi} T = \frac{\mathbb{F}[x, y]}{(x^3, y)}, \quad \pi(x) = x, \quad \pi(y) = 0.$$

Then the kernel of π is $K = (y)$ and the Thom class is $\tau = y^2$. Here, $d = 4$, $k = 2$ and $n = 2$. Letting ξ be the blow-up variable and $f_A(\xi) = \xi^2 - y^2$, define the associated cohomological blow-up algebra as in Construction 3.1 as

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, \xi y, \xi^2 - y^2)}.$$

In this presentation, the generator y^3 is redundant, and we see that $\tilde{A}$ is indeed a CI of Hilbert function $H(\tilde{A}) = (1, 3, 4, 3, 1) = H(A) + H(T)[1]$.

In this section, we will show that Example 6.1 is prototypical of the class of BUGs, which are CIs. First, we introduce yet another description of the cohomological blowup in terms of its defining ideal.

As usual, let $R = \mathbb{F}[x_1, \dots, x_r]$ be a graded polynomial ring with homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_r)$. Recall from Section 2 a homogeneous ideal $I \subseteq R$ is $\mathfrak{m}$ -primary and irreducible if and only if the quotient R/I is a graded AG algebra. We abuse notation slightly and call the socle degree of such an ideal the socle degree of the corresponding quotient.

Construction 6.2. Fix an $\mathfrak{m}$ -primary irreducible homogeneous ideal $I \subset R$ of socle degree d , and fix a homogeneous polynomial $\tau \in R$ of degree n where $2 \leq n < d - 1$ and such that $I \subsetneq (I : \tau) \subsetneq R$. Then the ideal $(I : \tau)$ is also homogeneous, $\mathfrak{m}$ -primary, and irreducible of socle degree $k = d - n$; see Lemma 2.3.

Let ξ be an indeterminate, and fix a homogeneous monic polynomial $f_R(\xi) = \xi^n + r_1 \xi^{n-1} + \dots + r_n$ where $r_i \in R_i$.

Define the ideal $\hat{I} \subset R[\xi] = \hat{R}$ by

$$\hat{I} = I \cdot R[\xi] + \xi \cdot (I : \tau) \cdot R[\xi] + f_R(\xi) \cdot R[\xi]. \quad (31)$$

We now describe some properties of the ideal (31) of Construction 6.2. Since I and $(I : \tau)$ are both $\mathfrak{m}$ -primary and irreducible it follows that $A = R/I$ and $T = R/(I : \tau)$ are AG algebras, and since $I \subseteq (I : \tau)$, the identity map on R passes to a surjective map of quotient algebras $\pi : A \rightarrow T$. Moreover, it follows from Lemma 2.2 that one can choose

orientations on A and T such that $\bar{\tau} \in A_n$ is the Thom class of π . Note that the kernel of π is $K = (I : \tau)/I \subset R/I = A$. It follows that if $f_A(\xi) = \overline{f_R(\xi)} \in A[\xi] = R[\xi]/I$, then $\hat{A}$ from Construction 3.1 satisfies

$$\hat{A} = \frac{A[\xi]}{(\xi \cdot K, \overline{f_R(\xi)})} = \frac{R[\xi]}{I \cdot R[\xi] + \xi \cdot (I : \tau) \cdot R[\xi] + (f_R(\xi)) \cdot R[\xi]} = \frac{\hat{R}}{\hat{I}}. \quad (32)$$

It follows therefore from Lemma 3.6 that the ideal $\hat{I}$ from Construction 6.2 is $\hat{m} = (x_1, \dots, x_r, \xi)$ -primary irreducible if and only if the constant coefficient $r_n \in R_n$ of $f_R(\xi)$ satisfies $r_n - \lambda \cdot \tau \in I$ for some $\lambda \in \mathbb{F}^\times$. In this case, we shall replace the “hat” with “tilde” and call $\tilde{I} = \hat{I}$ the *cohomological blow-up ideal of I and τ* , as it is the defining ideal of the cohomological blow-up algebra $\tilde{A}$ of A along π with parameters $(\overline{r_1}, \dots, \overline{r_{n-1}}, \lambda)$.

It is also clear from Construction 6.2, and Lemma 3.13, that the colon ideal $(\hat{I} : \xi) \subset \hat{R}$ satisfies

$$(\hat{I} : \xi) = (I : \tau) + (f_T(\xi) = \overline{f_R(\xi)}) \quad (33)$$

and we have

$$\tilde{T} = \frac{T[\xi]}{(f_T(\xi))} = \frac{R[\xi]}{(I : \tau) \cdot R[\xi] + f_R(\xi) \cdot R[\xi]} = \frac{\hat{R}}{(\hat{I} : \xi)}, \quad (34)$$

which is the algebra from Construction 3.1 Equation (11).

6.1 Minimal generating sets

Next, we would like to know how the minimal generators of the cohomological blow-up ideal $\tilde{I}$ compare with those of I and of $(I : \tau)$. We also determine the relations among these minimal generators. We start by providing a lemma that helps explain some of these relations.

Lemma 6.3. Let $B(\xi) = b_p \xi^p + \dots + b_0$ and $C(\xi) = c_q \xi^q + \dots + c_0$ be any homogeneous polynomials in $R[\xi]$ with homogeneous coefficients $b_i, c_i \in R$. Let $J \subset R$ is any homogeneous ideal in R and assume that $C(\xi)$ is monic, that is, $c_q = 1$. If the product of B and C is in the ideal in $R[\xi]$ generated by J , that is,

$$B(\xi) \cdot C(\xi) \in J \cdot R[\xi],$$

then every coefficient of $B(\xi)$ must lie in J , that is, $b_i \in J \cdot R$ for all $0 \leq i \leq p$.

Proof. The key observation here is that if $D(\xi) = d_r \xi^r + \cdots + d_0$ is any polynomial in the ideal $J \cdot R[\xi]$, then its coefficients $d_i \in J \cdot R$. Indeed, if $D(\xi) \in J \cdot R[\xi]$, plug in $\xi = 0$ to see that $D(0) = d_0 \in J \cdot R$. Then $D(\xi) - d_0 = D_1(\xi) = \xi (d_r \xi^{r-1} + \cdots + d_1) \in J \cdot R[\xi]$ and hence $d_r \xi^{r-1} + \cdots + d_1 \in (J \cdot R[\xi] : \xi) = J \cdot R[\xi]$, since ξ is a nonzero divisor of $J \cdot R[\xi]$. Then plug in $\xi = 0$ to see that $d_1 \in J \cdot R$, and so on.

Therefore, if the product

$$B(\xi) \cdot C(\xi) = \sum_{i=0}^{p+q} (b_0 c_i + b_1 c_{i-1} + \cdots + b_i c_0) \xi^i$$

is in the ideal $J \cdot R[\xi]$, then each of its coefficients must be in $J \cdot R$, that is,

$$b_0 c_i + \cdots + b_i c_0 \in J \cdot R \text{ for each } 0 \leq i \leq p+q.$$

Taking first $i = p+q$, we find that $b_p \cdot c_q \in J \cdot R$ and since $c_q = 1$ it follows that $b_p \in J \cdot R$. Inductively, assume that $b_{p-j+1}, \dots, b_p \in J \cdot R$. Then taking $i = p+q-j$ we find that

$$b_p c_{q-j} + b_{p-1} c_{q-j+1} + \cdots + b_{p-j+1} c_{q-1} + b_{p-j} c_q \in J \cdot R$$

from which it follows that $b_{p-j} c_q = b_{p-j} \in J \cdot R$ as well. Therefore, by induction, all coefficients b_j are in $J \cdot R$. ■

We can now provide a short exact sequence that determines the relations among the obvious (not necessarily minimal) set of generators of the cohomological blow-up ideal $\tilde{I}$.

Proposition 6.4. Let $\tilde{I} = I \cdot R[\xi] + \xi \cdot (I : \tau) \cdot R[\xi] + (f_R(\xi))$ be the ideal described in Construction 6.2, let $I' = I \cdot R[\xi]$, $K' = (I : \tau) \cdot R[\xi]$, and let $g = (f_R(\xi) - \lambda \tau)/\xi$. There is a short exact sequence of graded $R[\xi]$ -modules

$$0 \rightarrow I'(-1) \oplus K'(-n) \xrightarrow{\begin{bmatrix} \xi & \lambda \tau \\ -1 & g \\ 0 & -1 \end{bmatrix}} I' \oplus K'(-1) \oplus R[\xi] \xrightarrow{\begin{bmatrix} 1 & \xi & f_R \end{bmatrix}} \tilde{I} \rightarrow 0. \quad (35)$$

Proof. The definition of $\tilde{I}$ yields surjectivity of the rightmost nonzero map and the injectivity of the leftmost nonzero map is clear from its description (note there is a

unit entry in each column). That (35) is a complex is seen by matrix multiplication and utilization of $f = g\xi + \lambda\tau$.

It remains to prove exactness in the middle of (35). For this, consider a triple $(\alpha(\xi), \beta(\xi), \gamma(\xi)) \in I' \oplus K'(-1) \oplus R[\xi]$ so that

$$\alpha(\xi) + \beta(\xi)\xi + \gamma(\xi)f_R(\xi) = 0. \quad (36)$$

Since $I' \subseteq K'$ we have $\alpha(\xi) + \beta(\xi)\xi \in K'$, so that also $\gamma(\xi)f_R(\xi) \in K'$, and by Lemma 6.3, it follows that $\gamma(\xi) \in K'$ since $f_R(\xi)$ is monic. Adding the relation

$$\lambda\tau\gamma(\xi) + g(\xi)\gamma(\xi) - f_R(\xi)\gamma(\xi) = 0,$$

to (36) yields $(\alpha(\xi) - \lambda\tau\gamma(\xi)) + (\beta(\xi) - g(\xi)\gamma(\xi))\xi = 0$. Since $\gamma(\xi) \in K' = (I : \tau) \cdot R[\xi]$, we have $\lambda\tau\gamma(\xi) \in I \cdot R[\xi] = I'$ and thus $\beta'(\xi) := (\beta(\xi) - g(\xi)\gamma(\xi)) \in I'$. We have thus obtained the identity

$$\begin{bmatrix} \alpha(\xi) \\ \beta(\xi) \\ \gamma(\xi) \end{bmatrix} = -\beta'(\xi) \begin{bmatrix} \xi \\ -1 \\ 0 \end{bmatrix} - \gamma(\xi) \begin{bmatrix} \lambda\tau \\ g \\ -1 \end{bmatrix},$$

where $\beta'(\xi) \in I'$ and $\gamma(\xi) \in K'$, establishing the desired exactness. ■

Based on the presentation in Proposition 6.4, one can infer a minimal generating set for $\tilde{I}$. Setting $\hat{\mathfrak{m}}$ to be the homogeneous maximal ideal of $R[\xi]$, tensoring the short exact sequence (35) with $R[\xi]/\hat{\mathfrak{m}}$, and observing that there are isomorphisms $I'/\hat{\mathfrak{m}}I' \cong I/\mathfrak{m}I$ and $K'/\hat{\mathfrak{m}}K' \cong (I : \tau)/\mathfrak{m}(I : \tau)$, we obtain a new exact sequence of $\mathbb{F}$ -vector spaces

$$\frac{I}{\mathfrak{m}I}(-1) \oplus \frac{(I : \tau)}{\mathfrak{m}(I : \tau)}(-n) \xrightarrow[\psi_1]{\begin{bmatrix} 0 & \lambda\phi_2 \\ -\phi_1 & 0 \\ 0 & 0 \end{bmatrix}} \frac{I}{\mathfrak{m}I} \oplus \frac{(I : \tau)}{\mathfrak{m}(I : \tau)}(-1) \oplus \mathbb{F} \xrightarrow[\psi_2]{\begin{bmatrix} 1 & \xi & f_R \end{bmatrix}} \frac{\tilde{I}}{\hat{\mathfrak{m}}\tilde{I}} \rightarrow 0. \quad (37)$$

The zero entries in the 1st matrix are due to the containments $\tau I' \subseteq \hat{\mathfrak{m}}I'$, $gI' \subseteq \hat{\mathfrak{m}}K'$, and $\xi I' \subseteq \hat{\mathfrak{m}}I'$. We discuss the remaining maps ϕ_1 and ϕ_2 —the former is induced by inclusion and the other by multiplication by τ . These maps fit into the sequence

$$(I : \tau)/\mathfrak{m}(I : \tau) \xrightarrow[\phi_2]{\times \tau} I/\mathfrak{m}I \xrightarrow[\phi_1]{} (I : \tau)/\mathfrak{m}(I : \tau) \xrightarrow[\phi_2]{\times \tau} I/\mathfrak{m}I. \quad (38)$$

Note this sequence of maps (38) forms a complex, that is, $\phi_1 \circ \phi_2 = 0$ and $\phi_2 \circ \phi_1 = 0$; hence, there are homology groups

$$H = \frac{\ker(\phi_1)}{\operatorname{im}(\phi_2)} \quad \text{and} \quad H' = \frac{\ker(\phi_2)}{\operatorname{im}(\phi_1)}. \quad (39)$$

We will see that these homology groups H and H' measure the difference between minimal generating sets of $\tilde{I}$ and those of I or $(I : \tau)$. In fact, we will see that these homology groups are obstructions to $\tilde{I}$ being generated by a regular sequence.

Using (37), we can express a minimal generating set of $\tilde{I}$ in terms of the minimal generating sets of I and $(I : \tau)$ and the homology groups H and H' .

Theorem 6.5. Let ϕ_1, ϕ_2 be defined as in (39), and consider the vector space decompositions

$$\frac{I}{\mathfrak{m}I} = U \oplus \underbrace{\operatorname{im}(\phi_2) \oplus H}_{\ker(\phi_1)} \quad \frac{(I : \tau)}{\mathfrak{m}(I : \tau)} = W \oplus \underbrace{\operatorname{im}(\phi_1) \oplus H'}_{\ker(\phi_2)}, \quad (40)$$

where U and W are some (noncanonical) complements for $\ker(\phi_1)$ and $\ker(\phi_2)$, respectively. Then the vector space spanned by the minimal generators for $\tilde{I}$ decomposes as

$$\frac{\tilde{I}}{\hat{\mathfrak{m}}\tilde{I}} \cong U \oplus H \oplus \xi W \oplus \xi H' \oplus \langle f_R(\xi) \rangle. \quad (41)$$

Proof. This follows almost immediately from exactness of Sequence (37): since $\ker(\psi_2) = \operatorname{im}(\psi_1) = \operatorname{im}(\phi_2) \oplus \operatorname{im}(\phi_1) \subset I/\mathfrak{m}I \oplus (I : \tau)/\mathfrak{m}(I : \tau) \oplus \mathbb{F}$ is a direct summand, and since ψ_2 is surjective, it passes to an isomorphism

$$\frac{\tilde{I}}{\hat{\mathfrak{m}}\tilde{I}} \cong \frac{\frac{I}{\mathfrak{m}I} \oplus \frac{(I:\tau)}{\mathfrak{m}(I:\tau)}(-1) \oplus \mathbb{F}}{\ker(\psi_2)} \cong U \oplus H \oplus W \oplus H' \oplus \mathbb{F}$$

and the result follows. ■

For a homogeneous ideal $J \subset R$, we denote by $\mu(J) = \dim_{\mathbb{F}}(J/\mathfrak{m}J)$ the number of minimal generators of J . The following corollary is an immediate consequence of Theorem 6.5, and we omit the proof.

Corollary 6.6. With I , τ , $(I : \tau)$ and $\tilde{I}$ as above, we have

$$\mu(\tilde{I}) = \mu(I) + \dim(H') + 1 = \mu(I : \tau) + \dim(H) + 1.$$

Note that in our set up of Construction 6.2, $\mu(\tilde{I}) \geq r + 1 = \dim(\hat{R})$ with equality if and only if $\tilde{I}$ is generated by a regular sequence. In particular, Corollary 6.6 shows that the vanishing of homology groups H and H' from (39) is necessary for the cohomological blow-up ideal to be generated by a regular sequence, and hence for the BUG quotient $\tilde{A} = \hat{R}/\tilde{I}$ to be a CI. However, the following example shows that the vanishing of H and H' is not quite sufficient for that purpose.

Example 6.7. Let $I = (x^4, y^4, zx^2, zy^2, z^4 - x^2y^2)$ and $\tau = z^2 - xy$ so that $(I : \tau) = (x^4, y^4, zx^2, zy^2, z^2 + xy)$. Then taking $\hat{R} = R[\xi] = \mathbb{F}[x, y, z][\xi]$ and $f_R(\xi) = \xi^2 - \tau$ the cohomological blow-up ideal is

$$\tilde{I} = (x^4, y^4, zx^2, zy^2, \xi(z^2 + xy), \xi^4 - (z^2 - xy)).$$

Hence, the cohomological blowup $\tilde{A} = \hat{R}/\tilde{I}$ is not a CI. Note that in this case, the homology groups H and H' are both zero.

Note in Example 6.7 that the colon ideal $(\tilde{I} : \xi) = \tilde{I} + (z^2 + xy)$ is principal over $\tilde{I}$. This is related to exact pairs of zero divisors, which allow a complete characterization of BUGs that are CI.

6.2 CI blow-up algebras and exact zero divisors

Below, we give a necessary and sufficient condition for a cohomological blow-up algebra to be a CI based on exact zero divisors, a notion introduced by Henriques and Şega [22], (see also [4, Section 3]), which is defined as follows.

Definition 6.8. A pair of non-unit elements a, b of a ring A is an *exact pair of zero divisors* if $(0 :_A a) = b \cdot A$ and $(0 :_A b) = a \cdot A$.

Example 6.9.

1. In Example 6.1 $a = \overline{y^2}$ and $b = \overline{y}$ form an exact pair of zero divisors on

$$A = \frac{\mathbb{F}[x, y]}{(x^3, y^3)}.$$

2. In Example 6.7, $a = \overline{z^2 - xy}$ and $b = \overline{z^2 + xy}$ form an exact pair of zero divisors on

$$A = \frac{\mathbb{F}[x, y, z]}{(x^4, y^4, zx^2, zy^2, z^4 - x^2y^2)}.$$

If A is Artinian, then it suffices to check only one of the conditions in Definition 6.8; see also [4, Lemma 3.2].

Lemma 6.10. If A is a graded Artinian $\mathbb{F}$ -algebra, then homogeneous elements $a, b \in A$ of positive degree form an exact pair of zero divisors of A provided that $(0 :_A a) = b \cdot A$.

Proof. From $(0 :_A a) = b \cdot A$, we deduce there is an A -module isomorphism $A/(b) = A/(0 :_A a) \cong a \cdot A$ by means of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A/(0 :_A a) & \xrightarrow{\times a} & A & \longrightarrow & A/(a) \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow = & & \downarrow = \\ 0 & \longrightarrow & a \cdot A & \longrightarrow & A & \longrightarrow & A/(a) \longrightarrow 0 \end{array}$$

Similarly, there is an isomorphism $A/(0 :_A b) \cong b \cdot A$. Moreover, the hypothesis yields $ab = 0$ and thus $a \cdot A \subseteq (0 :_A b)$. To see that this containment is in fact an equality, we compute

$$\dim_{\mathbb{F}}(A) = \dim_{\mathbb{F}}(A/(b)) + \dim_{\mathbb{F}}(b \cdot A) = \dim_{\mathbb{F}}(a \cdot A) + \dim_{\mathbb{F}}(A) - \dim_{\mathbb{F}}(0 :_A b);$$

whence $\dim_{\mathbb{F}}(a) = \dim_{\mathbb{F}}(0 :_A b)$, and hence $a \cdot A = (0 :_A b)$ as desired. ■

The next lemma relates exact zero divisors to the homology group H in (39).

Lemma 6.11. Let $A = R/I$ and $T = R/(I : \tau)$ be AG algebras, and suppose that $\bar{\tau} \in A_n$ is part of a pair of exact zero divisors on A . Then the homology groups H and H' from (39) vanish.

Proof. Assume that there exists a homogeneous element (of positive degree) $\sigma \in R$ for which $(I : \tau) = I + (\sigma)$ and $(I : \sigma) = I + (\tau)$, so that $\bar{\tau}$ and $\bar{\sigma}$ are an exact pair of zero divisors in A .

Suppose that $s \in I$ is any minimal generator of I for which $s + \mathfrak{m}I \in \ker(\phi_1)$. Then, $s \in \mathfrak{m}(I : \tau) = \mathfrak{m}(I + (\sigma))$, and hence there exists elements $g_1, \dots, g_t \in I$ and $a_1, \dots, a_t, b \in \mathfrak{m}$ such that

$$s = \sum_{i=1}^t a_i g_i + b \cdot \sigma.$$

It follows that $b \in (I : \sigma) = I + (\tau)$ and hence $b = q + r \cdot \tau$ for some $q \in I$ and $r \in R$. Therefore, we see that

$$s - r \cdot \tau \cdot \sigma \in \mathfrak{m}I.$$

Then the equivalence class of s in H is

$$[s] \equiv [r\tau\sigma] \equiv 0,$$

which shows that $H = 0$, as claimed.

Moreover, by [33, Proposition 1.9], we have $\tau \cdot \sigma \in I \setminus \mathfrak{m}I$, which yields that $(I : \tau)/\mathfrak{m}(I : \tau) = \langle \sigma \rangle$. Consequently, the 2nd equation in (40) yields $W = \langle \sigma \rangle$, $\text{im}(\phi_1) = 0$, and $H' = 0$, also as claimed. ■

The following is a characterization of CI cohomological blow-up algebras in terms of exact pairs of zero divisors.

Theorem 6.12. Fix oriented AG algebras $A = R/I$ and $T = R/(I : \tau)$ of socle degrees $d > k$, and let $\pi : A \rightarrow T$ be the natural surjective algebra map between them with Thom class $\bar{\tau} \in A_n$ and kernel $K \subset A$. Let $f_R(\xi) = \xi^n + r_1 \xi^{n-1} + \dots + \lambda \cdot \tau \in R[\xi] = \hat{R}$ for some homogeneous elements $r_i \in R$, and let $\tilde{A} = \hat{R}/\tilde{I}$ be the associated cohomological blowup of A along π with parameters $(\bar{r}_1, \dots, \bar{r}_{n-1}, \lambda)$. Then the following are equivalent.

1. $\tilde{A}$ is a CI.
2. A is a CI and $\bar{\tau} \in A_n$ is part of an exact pair of zero divisors on A .
3. T is a CI and $\bar{\tau} \in A_n$ is part of an exact pair of zero divisors on A .

Proof. Assume that (1) holds. Then $\tilde{I}$ must be generated by a $\hat{R}$ -sequence, and hence it follows from Corollary 6.6 that both I and $(I : \tau)$ must also be generated by R -sequences and hence that A and T must be CIs and that the homology groups H and H' must vanish.

Thus, according to Equations (40) and (41), we have

$$\frac{I}{\mathfrak{m}I} = U \oplus \phi_2(W), \quad \frac{(I: \tau)}{\mathfrak{m}(I: \tau)} = \phi_1(U) \oplus W, \quad \text{and} \quad \frac{\tilde{I}}{\hat{\mathfrak{m}}\tilde{I}} = U \oplus \xi \cdot W \oplus \langle f_R(\xi) \rangle.$$

It follows that $\dim_{\mathbb{F}}(W) = 1$, since no two elements of a minimal generating set of I (or $\tilde{I}$) can have a common divisor. Let $\sigma \in (I: \tau)$ be a minimal generator for which $\sigma + \mathfrak{m}(I: \tau) \in W$. Then we can write

$$I = (u_1, \dots, u_{r-1}, \tau \cdot \sigma) \text{ and } (I: \tau) = (u_1, \dots, u_{r-1}, \sigma) = I + (\sigma),$$

which implies by Lemma 6.10 that $a = \bar{\tau}$ and $b = \bar{\sigma}$ form an exact pair of zero divisors for A . This shows that (1.) implies (2.) and (3.).

Next, assume that (2.) holds: A is a CI and $\bar{\tau} \in A_n$ is an exact zero divisor for A . It follows from Lemma 6.11 that $H = H' = 0$, and hence by Corollary 6.6, it follows that $\tilde{I}$ is generated by a $\hat{R}$ -sequence and hence that the cohomological blowup $\tilde{A}$ is a CI, which is (1.).

Finally, assume that (3.) holds: T is a CI and $\bar{\tau} \in A_n$ is an exact zero divisor for A . Again, it follows from Lemma 6.11 that $H = 0$, and hence it follows from Corollary 6.6 that the cohomological blowup $\tilde{A}$ is a CI, and thus (1.) holds. ■

Remark 6.13. It follows from the description of the minimal generators of $\tilde{I}$ in Equation (41) together with Theorem 6.12 and Lemma 6.11 that when $\tilde{A}$ is a CI and thus $\bar{\sigma}, \bar{\tau}$ is an exact pair of zero divisors on A , a minimal generating set for $\tilde{I}$ can be described as

$$\frac{\tilde{I}}{\hat{\mathfrak{m}}\tilde{I}} \cong U \oplus \langle \xi \sigma \rangle \oplus \langle f_R(\xi) \rangle.$$

In particular, since $\tilde{I}$ is generated by a regular sequence and $\xi \sigma$ is a minimal generator for $\tilde{I}$ it follows from [33, Proposition 1.9] that ξ is an exact zero divisor of $\tilde{A}$. The following example shows that this condition is not quite sufficient to identify a CI as a cohomological blow-up algebra.

Example 6.14. Consider the CI

$$\tilde{A} = \frac{\mathbb{F}[x, y]}{(x^4 + y^4, x^2 y^2)}.$$

Then $\xi = \bar{x} \in \tilde{A}_1$ is an exact zero divisor for $\tilde{A}$, and its Hilbert function is $H(\tilde{A}) = (1, 2, 3, 4, 3, 2, 1)$. This particular CI algebra (or any isomorphic to it) cannot be a cohomological blow-up algebra by a result of [29] concerning codimension two.

6.3 Application: Watanabe's Bold Conjecture (WBC)

The following “rather bold” conjecture was put forth by the 5th author [47] after noticing that many CIs arising as invariant rings could be realized as subrings of CIs cut out by quadrics and of the same socle degree.

Conjecture 6.15 (WBC). For any standard graded Artinian CI A of socle degree d , there is another standard graded Artinian CI B of the same socle degree d cut out by quadrics and an injective algebra map from A into B , in symbols:

$$\phi: A = \frac{\mathbb{F}[x_1, \dots, x_r]}{(f_1, \dots, f_r)} \hookrightarrow \frac{\mathbb{F}[X_1, \dots, X_N]}{(F_1, \dots, F_N)} = B, \quad \deg(F_i) = 2, \quad \forall 1 \leq i \leq N.$$

In 2016, the 3rd author proved [35] WBC in the special case where A is cut out by polynomials that factor into a product of linear forms (see also [36, Theorem 1] for another case in which WBC holds). Here, we give another, much shorter proof of this result (in fact something slightly stronger) using cohomological blowups. First, some notation. Let $A = R/I$ is a CI where $R = \mathbb{F}[x_1, \dots, x_r]$ has the standard grading and I is minimally generated by some regular sequence $(f_1, \dots, f_r)$. If the minimal generators $f_1, \dots, f_n$ can be chosen such that each f_i is a product of linear and/or quadratic forms, that is, $f_i = L_1 \cdots L_k$ where $\deg(L_j) = 1$ or 2 , then we shall say A is of class $\mathcal{W}$. If $d_i = \deg(f_i) > 1$ for all i , then the degree sequence is $(d_1, \dots, d_n)$ and we define its *defect* to be

$$\text{def}(A) = d_1 + \cdots + d_n - 2n = \text{soc. deg}(A) - n.$$

Let $\mathcal{W}(m)$ denote the subclass consisting of CIs of defect m , so that $\mathcal{W} = \bigsqcup_{m=0}^{\infty} \mathcal{W}(m)$. CIs of class $\mathcal{W}(0)$ are called *quadratic CIs*.

Theorem 6.16. For every CI A of class $\mathcal{W}$, there is a CI B of class $\mathcal{W}(0)$ of the same socle degree as A and an injective map of algebras $\phi: A \hookrightarrow B$.

Proof. We use induction on m to show that every CI $A \in \mathcal{W}(m)$ embeds into a CI $B \in \mathcal{W}(0)$ of the same socle degree. For the base case $m = 0$ there is nothing to show.

For the inductive step, assume that $m > 0$ and that the statement holds for every $0 \leq m' < m$. Fix $A \in \mathcal{W}(m)$ with presentation as in (20) with minimal generators $f_1, \dots, f_n$ that factor into products of linear and/or quadratic forms. Since $\text{def}(A) = m > 0$, we may assume without loss of generality that $\deg(f_n) \geq 3$. Write $f_n = \tau \cdot g$ where $\deg(\tau) = 2$, which is possible since we are assuming that f_n factors as a product of linear and quadratic forms. Then let ξ be an indeterminate, and define the monic quadratic polynomial $f_R(\xi) = \xi^2 - \tau \in R[\xi]$. Then the cohomological blow-up ideal of I with respect to τ and $f_R(\xi)$ is by (31)

$$\tilde{I} = (f_1, \dots, f_{n-1}, \xi g, f_R(\xi)) \subset R[\xi]$$

and the quotient $\tilde{A} = R[\xi]/\tilde{I}$ is the cohomological blowup of A along the map

$$\pi: A = \frac{\mathbb{F}[x_1, \dots, x_n]}{(f_1, \dots, f_{n-1}, \tau g)} \rightarrow \frac{\mathbb{F}[x_1, \dots, x_n]}{(f_1, \dots, f_{n-1}, g)} = T.$$

Note that the degree sequence of $\tilde{A}$ is $(d_1, \dots, d_{n-1}, d_n - 1, 2)$ and its defect is $d_1 + \dots + d_n - 1 + 2 - 2(n+1) = (d_1 + \dots + d_n) - 2n - 1 = m - 1$. Finally, it is clear that the minimal generators of $\tilde{I}$ are either quadratic (like $f_R(\xi)$) or factor into a product of linear and/or quadratic forms (since the minimal generators of I do). Therefore, $\tilde{A} \in \mathcal{W}(m-1)$. For the injection, note that the blow-down map will do the trick:

$$\beta: A = \frac{\mathbb{F}[x_1, \dots, x_n]}{(f_1, \dots, f_{n-1}, \tau g)} \hookrightarrow \frac{\mathbb{F}[x_1, \dots, x_n][\xi]}{(f_1, \dots, f_{n-1}, \xi g, f(\xi))} = \tilde{A}.$$

To complete the proof note that by induction, there is an embedding of $\tilde{A} \in \mathcal{W}(m-1)$ into some CI $B \in \mathcal{W}(0)$ of the same socle degree as $\tilde{A}$ (and hence also A), say

$$\iota: \tilde{A} \hookrightarrow B.$$

Composing β and ι then gives an embedding $\iota \circ \beta: A \hookrightarrow B$, as desired. ■

7 Restrictions on Hilbert Functions

In this section, we show that standard graded cohomological blow-up algebras cannot have arbitrary Hilbert functions. In fact, in the parameter space of AG algebras of fixed embedding dimension ≥ 3 and socle degree 4 or ≥ 6 , cohomological blow-up algebras are quite rare. To justify this assertion, we recall the notion of compressed AG algebra [24]. We use the notation of Section 4: $R = \mathbb{F}[x_1, \dots, x_r]$ is a standard graded polynomial

ring, $Q = \mathbb{F}[X_1, \dots, X_r]$ its dual divided power algebra. For a d -form H , its associated AG algebra $C = R/\text{Ann}(H)$ has embedding dimension r if $\text{Ann}(H)$ contains no linear forms.

Definition 7.1. A standard graded AG algebra C of embedding dimension r and socle degree d is *compressed* provided

$$\dim_k C_i = \begin{cases} \dim_{\mathbb{F}} R_i = \binom{r-1+i}{r-1} & \text{if } i \leq \left\lfloor \frac{d}{2} \right\rfloor \\ \dim_{\mathbb{F}} C_{d-i} & \text{if } i > \left\lfloor \frac{d}{2} \right\rfloor \end{cases}.$$

We can parametrize graded AG algebras of fixed socle degree d and codimension r by their Macaulay dual generators in Q_d , essentially by elements of the projective space $\mathbb{P}(Q_d)$. The compressed algebras form a dense Zariski open set in this parameter space [25, Proposition 3.12]. (This depends on $R_j/(I^2)_j$ being the tangent space to $\text{Gor}(T)$ at $A = R/I$ and gives perhaps the shortest proof. Original references are [13, Theorem 3.31] and [16, Theorem 5.1]). We now show that for sufficiently large parameters the cohomological blow-up algebras do not belong to this set.

Theorem 7.2. Compressed algebras of embedding dimension $r \geq 3$ whose socle degree d satisfies $d = 4$ or $d \geq 6$ are not cohomological blow-up algebras.

Proof. Let $C = R/\text{Ann}(H)$ be a compressed algebra of socle degree d and embedding dimension r , and assume that C is the blowup along a surjective morphism $\pi : A \rightarrow T$ of some AG algebras of socle degrees d and k , respectively. Then since C is compressed $\tilde{I} = \text{Ann}(H)$ is a homogeneous ideal with initial degree

$$\min \{i \mid \tilde{I}_i \neq 0\} = \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

By [7, Proposition 3.2] the minimal generators of $\tilde{I}$ have degrees $\left\lfloor \frac{d}{2} \right\rfloor + 1$ and possibly $\left\lfloor \frac{d}{2} \right\rfloor + 2$. Recall that the polynomial $f_R(\xi)$ of degree $n = d - k$ is a minimal generator of $\tilde{I}$ by Theorem 6.5 and thus

$$n = d - k \in \left\{ \left\lfloor \frac{d}{2} \right\rfloor + 1, \left\lfloor \frac{d}{2} \right\rfloor + 2 \right\}, \quad \text{thus} \quad k \in \left\{ \left\lceil \frac{d}{2} \right\rceil - 1, \left\lceil \frac{d}{2} \right\rceil - 2 \right\}. \quad (42)$$

By the definition of blowup, when $n > 1$, the embedding dimension of A is $r - 1$ (one less than the embedding dimension of C) and by the surjectivity of π embedding dimension of T is at most $r - 1$. This yields the following upper bounds on the Hilbert

functions of A and T :

$$\dim_{\mathbb{F}} A_i \leq \min \left\{ \binom{r-2+i}{r-2}, \dim_{\mathbb{F}} A_{d-i} \right\} \quad (43)$$

$$\dim_{\mathbb{F}} T_i \leq \min \left\{ \binom{r-2+i}{r-2}, \dim_{\mathbb{F}} T_{k-i} \right\}. \quad (44)$$

By Theorem 4.6, we have

$$H(C) = H(A) + H(T)[1] + \cdots + H(T)[n-1]. \quad (45)$$

Evaluating the above identity in degrees $i \leq \left\lfloor \frac{k}{2} \right\rfloor + 1$ combined with the above inequalities yields

$$\binom{r+i-1}{r-1} = H(A)_i + \sum_{j=1}^i H(T)_{i-j} \leq \sum_{j=0}^i \binom{r+j-2}{r-2} = \binom{r+i-1}{r-1},$$

where the last equality is a well-known combinatorial identity. This implies that equality must hold both in (43) and in (44) for $i \leq \left\lfloor \frac{k}{2} \right\rfloor + 1$. In degree $i = \left\lfloor \frac{k}{2} \right\rfloor + 2$ since $r \geq 3$ the inequality (44) yields

$$H(T)_{i-1} = H(T)_{k-i+1} = H(T)_{\left\lceil \frac{k}{2} \right\rceil - 1} \leq \binom{r-3 + \left\lceil \frac{k}{2} \right\rceil}{r-2} < \binom{r+i-2}{r-2}.$$

Evaluating Equation (45) in degree $i = \left\lfloor \frac{k}{2} \right\rfloor + 2$ and combining the result with the inequality (43) and the inequality displayed above gives

$$H(C)_i = H(A)_i + \sum_{j=1}^i H(T)_{i-j} < \sum_{j=0}^i \binom{r+j-2}{r-2} = \binom{r+i-1}{r-1}.$$

This contradicts our assumption that C is compressed, provided that

$$\left\lfloor \frac{k}{2} \right\rfloor + 2 \leq \left\lfloor \frac{d}{2} \right\rfloor, \text{ for } k \in \left\{ \left\lceil \frac{d}{2} \right\rceil - 1, \left\lceil \frac{d}{2} \right\rceil - 2 \right\}.$$

The above inequality is satisfied if and only if $d = 4$ or $d \geq 6$. ■

The following example shows that there exists compressed algebras of socle degree 5, which are cohomological blowups.

Example 7.3. Let

$$\pi: A = \frac{\mathbb{F}[x, y]}{(x^4, y^3)} \rightarrow T = \frac{\mathbb{F}[x, y]}{(x^2 - xy, y^2)}$$

be the map defined by $\pi(x) = x$ and $\pi(y) = y$ and distinguished socle generators $\sigma_A = x^3y$ and $\sigma_T = x^2$. Then $\tau = xy$ and $K = (x^2 - xy, y^2)$, and hence taking parameters $t_1 = t_2 = 0$ and $\lambda = 1$, we get the cohomological blowup

$$\tilde{A} = \frac{\mathbb{F}[x, y, \xi]}{(x^4, y^3, \xi(x^2 - xy), \xi y^2, \xi^2 - xy)},$$

which has Hilbert function $H(\tilde{A}) = (1, 3, 6, 6, 3, 1)$, and hence $\tilde{A}$ is a compressed BUG of socle degree 5.

8 The Lefschetz Property

The SLP for graded AG algebras is an algebraic version of a property of cohomology rings of smooth complex projective varieties stemming from the Hard Lefschetz theorem in algebraic geometry.

Definition 8.1. A graded Artinian $\mathbb{F}$ -algebra $A = \bigoplus_{i=0}^d A_i$ is said to satisfy the strong Lefschetz property (SLP) if there is a linear form $\ell \in A_1$ for which the multiplication maps $\times \ell^j: A_i \rightarrow A_{i+j}$ have full rank $\text{rank}(\times \ell^j) = \min\{\dim_{\mathbb{F}} A_i, \dim_{\mathbb{F}} A_{i+j}\}$ for each degree i and each exponent j . A linear form ℓ with this property is called a strong Lefschetz element for A .

If the multiplication maps $\times \ell: A_i \rightarrow A_{i+1}$ have full rank for each degree i , then A is said to satisfy the weak Lefschetz property (WLP).

More generally, given any graded Artinian algebra A , and any linear form $\ell \in A_1$, we can define its *Jordan type* P_ℓ to be the partition corresponding to the block decomposition of the Jordan canonical form for the nilpotent linear operator $\times \ell: A \rightarrow A$. It is well known that for a standard-graded AG algebra with unimodal Hilbert function then ℓ is strong Lefschetz if and only if the Jordan type P_ℓ is equal to $H^\vee$, the conjugate of the Hilbert function regarded as a partition (switch rows and columns in the Ferrers graph) [26, Proposition 2.10] and ℓ is weak Lefschetz if P_ℓ has number of parts equal to the Sperner number of H [19, Proposition 3.5].

In this section, we study the SLP for cohomological blow-up algebras. To attain this goal, we observe the behavior of these rings in families. Our strategy is to consider

every cohomological blow-up algebra as the general fiber in a certain flat family. Interestingly, all fibers in these families will be AG algebras with the exception of the special fiber. Our proof of the SLP for the cohomological blowup then employs the following well-known fact: in a flat family, if SLP holds on the special fiber, then it must also hold on a sufficiently general fiber by semicontinuity of Jordan type, for example [26, Corollary 2.44].

We continue with the notation established in previous sections, namely $\pi : A \rightarrow T$ is a surjective map of oriented AG algebras with Thom class τ and kernel K , $f_A(\xi) \in A[\xi]$ is a homogeneous monic polynomial yielding the cohomological blow-up algebra

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{A[\xi]}{I},$$

as in Construction 3.1.

Consider a weighted monomial order $<$ on $A[\xi]$ obtained by assigning weight 1 to the variable ξ and weight 0 to each element of A . Then the weight of a monomial μ is $w(\mu) = \max\{n : \xi^n \mid \mu\}$ and the weight of a polynomial $g = \sum c_i \mu_i \in A[\xi]$ with $c_i \in F^\times$ and μ_i monomials is $w(g) = \max\{w(\mu_i)\}$. Two monomials μ, μ' satisfy $\mu < \mu'$ if and only if $w(\mu) < w(\mu')$. The initial form of a polynomial $g = \sum c_i \mu_i \in A[\xi]$ is $\text{in}_<(g) = \sum_{w(\mu_j)=w(g)} c_j \mu_j$ and the initial ideal of an ideal $J \subseteq A[\xi]$ is

$$\text{in}_<(J) = (\text{in}_<(g) \mid g \in J).$$

While it is not usually the case that the generators of the initial ideal of an ideal J are the initial forms of the generators of J , this is nevertheless the case for I since, as we show below, the generators of I form a Gröbner basis with respect to $<$. We obtain the following description for the initial ideal of I .

Lemma 8.2. The set $\{\xi \cdot K, f_A(\xi)\}$ is a Gröbner basis for I with respect to $<$, that is, the initial ideal of I is $\text{in}_<(I) = (\xi \cdot K, \xi^n)$.

Proof. To show that the set $\{\xi \cdot K, f_A(\xi)\}$ is a Gröbner basis for I one utilizes Buchsberger's criterion; see [12, Theorem 15.8]. Since any element in $\xi \cdot K$ is equal to its initial form, the S-polynomial of any two such elements is 0. It remains to compute the S-polynomial of $u\xi$ (with $u \in K$) and $f_A(\xi)$, which is

$$S(f_A(\xi), u\xi) = u(\xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi + \lambda\tau) - u\xi \cdot \xi^{n-1} = ua_1\xi^{n-1} + \cdots + ua_{n-1}\xi$$

since $K \cdot \lambda\tau = 0$ by Lemma 2.2. Since $S(f_A(\xi), u\xi) \in (\xi \cdot K)$, this polynomial reduces to 0 modulo the set $\{\xi \cdot K, f_A(\xi)\}$, concluding the proof that this is a Gröbner basis. It follows that $\text{in}_<(I) = (\text{in}_<(\xi \cdot K), \text{in}_<(f_A(\xi))) = (\xi \cdot K, \xi^n)$. ■

The relationship between $\tilde{A} = A[\xi]/I$ and its initial algebra $\text{in}_<(\tilde{A}) := A[\xi]/\text{in}_<(I)$ is established by means of the following well-known construction; see [12, p. 343]. Given a parameter z , one considers an ideal $\mathcal{I}$ of $A[\xi, z]$ given by

$$\mathcal{I} = \left(\xi \cdot K, \xi^n + a_1 z \xi^{n-1} + \cdots + a_{n-1} \xi z^{n-1} + \lambda \tau z^n \right).$$

Note that setting $z = 1$ in $\mathcal{I}$ recovers I while setting $z = 0$ gives $\text{in}_<(I)$. We recall some key properties of this construction; cf. [12, Theorem 15.17].

Theorem 8.3. The following ring $\mathcal{A}$ is free and hence flat as a $\mathbb{F}[z]$ -algebra:

$$\mathcal{A} = \frac{A[\xi, z]}{\mathcal{I}} = \frac{A[\xi, z]}{(\xi \cdot K, \xi^n + a_1 \xi^{n-1} z + \cdots + a_{n-1} \xi z^{n-1} + a_n z^n)}. \quad (46)$$

Thus, $\mathcal{A}$ can be viewed as a flat family of algebras with fibers $\mathcal{A}_c = \mathcal{A} \otimes_{\mathbb{F}[z]} \mathbb{F}[z]/(z - c)$ given by

$$\mathcal{A}_c \cong \begin{cases} A[\xi]/I = A[\xi]/(\xi \cdot K, f_A(\xi)) = \tilde{A} & \text{if } c \in \mathbb{F}^\times \\ A[\xi]/\text{in}_<(I) = A[\xi]/(\xi \cdot K, \xi^n) = \text{in}_<(\tilde{A}) & \text{if } c = 0 \end{cases}$$

In the literature, flat families in which the general fibers are isomorphic are sometimes called *jump deformations*. We remark that Theorem 8.3 remains true if we replace $\tilde{A}$ with any $\hat{A}$ from Construction 3.1 but recall that $\hat{A}$ is Gorenstein if and only if $\hat{A} = \tilde{A}$ by Theorem 3.6.

Since $\tilde{A}$ is Gorenstein, Theorem 8.3 implies that the general fibers of the family $\mathcal{A}$ are Gorenstein. However, Theorem 3.6 implies that the special fiber $\mathcal{A}_0$ is not Gorenstein but is boundary-Gorenstein (Remark 3.8). Moreover, Corollary 3.7 implies that all the fibers have the same Hilbert function given by

$$H(\mathcal{A}_c) = H(A) + H(T)[1] + \cdots + H(T)[n-1] \quad \text{for all } c \in \mathbb{F}.$$

We now consider the SLP for the special fiber. The ring $in_{<}(\tilde{A})$ can thus be recognized as a fibered product

$$in_{<}(\tilde{A}) = A \times_T B \quad \text{with} \quad B = \frac{T[\xi]}{(\xi^n)}$$

with respect to the projections $\pi_A = \pi : A \rightarrow T$ and $\pi_B : B \rightarrow T = B/(\xi)$, where π_B is the canonical projection. Our proof of Lemma 8.4 below closely resembles the proof of [30, Theorem 5.12], where the SLP is established for fibered products of certain AG algebras of the same socle degree. This is not the case here as $\text{soc. deg}(A) = \text{soc. deg}(B) + 1$.

Lemma 8.4. Let $\pi : A \rightarrow T$ be a surjective homomorphism of graded AG $\mathbb{F}$ -algebras of socle degrees $d > k$, respectively, such that both A and T have SLP. Assume that $\mathbb{F}$ has characteristic zero, or characteristic $p > d > k$. Then the initial algebra of $\tilde{A}, A[\xi]/in_{<}(I)$ has SLP as well.

Proof. Recall from Lemma 8.2 that

$$in_{<}(\tilde{A}) = \frac{A[\xi]}{in_{<}(I)} = \frac{A[\xi]}{(\xi \cdot K, \xi^n)}.$$

This algebra admits a decomposition

$$in_{<}(\tilde{A}) = A \oplus J, \text{ where } J = T\xi \oplus \cdots \oplus T\xi^{n-1} = \xi \cdot \frac{T[\xi]}{(\xi^{n-2})}.$$

The important point here is that in the algebra $in_{<}(\tilde{A})$, the vector space J is actually an ideal, and thus multiplication by a linear form $\ell = \ell_A + \xi$ with $\ell_A \in A_1$ is represented with respect to the above decomposition by a block matrix

$$\begin{pmatrix} \ell_A & 0 \\ * & \ell|_J \end{pmatrix}, \quad (47)$$

where ℓ_A stands for the map given by multiplication by ℓ_A on A and $\ell|_J$ denotes the restriction of multiplication by ℓ to J . Since A and T have SLP, one can pick ℓ_A so that both ℓ_A and its image $\ell_T = \overline{\ell_A}$ in T are strong Lefschetz elements. Then, by [20, Theorem 6.1] if characteristic $\mathbb{F}$ is zero, or [28, Theorem 2.6] in general, $\ell = \ell_T + \xi$ is also a strong Lefschetz element on the free extension $T[\xi]/(\xi^{n-2})$ of T and hence on J . Since the Hilbert functions of A and J are both symmetric around $d/2$ (due to both being AG) and unimodal (due to both having SLP) we conclude that $\ell_A^j : A_i \rightarrow A_{i+j}$ and $\ell|_J^j : J_i \rightarrow J_{i+j}$

are simultaneously injective or simultaneously surjective. It follows from (47) that ℓ is a strong Lefschetz element on $in_{<}(\tilde{A})$. ■

We are now ready to prove that cohomological blow-up algebras have SLP.

Theorem 8.5. Let $\mathbb{F}$ be an infinite field, and let $\pi : A \rightarrow T$ be a surjective homomorphism of graded AG $\mathbb{F}$ -algebras of socle degrees $d > k$, respectively, such that both A and T have SLP. Assume that characteristic $\mathbb{F}$ is zero or characteristic F is $p > d$. Then every cohomological blow-up algebra of A along T satisfies SLP.

Proof. We have seen that $\mathcal{A}$ is a flat family. By Lemma 8.4, the special fiber $\mathcal{A}_0$ has SLP. Therefore, by semicontinuity of Jordan type [26, Corollary 2.44], and since $\mathbb{F}$ is infinite, we deduce that there is some $c \neq 0$ for which the general fiber $\mathcal{A}_c$ has SLP. By Theorem 8.3, it follows that the cohomological blowup $\tilde{A} = \mathcal{A}_1$ has SLP as well. ■

The following example shows that our assumptions on the characteristic of the field in Theorem 8.5 are necessary.

Example 8.6. Let d and k be integers satisfying $2k < d$, and, following the usual notation, write $n = d - k$. Let $\mathbb{F}$ be a field of characteristic p , and suppose there is an integer $m \geq 1$ such that $n < p^m \leq d - 2$. Consider the AG algebras

$$A = \frac{\mathbb{F}[x, y]}{\text{Ann}(X^d + Y^d)} = \frac{\mathbb{F}[x, y]}{(xy, x^d - y^d)} \quad \text{and} \quad T = \frac{\mathbb{F}[x, y]}{\text{Ann}(X^k)} = \frac{\mathbb{F}[x, y]}{(y, x^{k+1})} \cong \frac{\mathbb{F}[x]}{(x^{k+1})}.$$

The Thom class of the natural surjection $\pi : A \rightarrow T$ is $\tau = x^n$, and choosing $f_R(\xi) = \xi^n - x^n$ and $h_R(\xi) = 1$ as in Construction 4.2, we get a cohomological blow-up algebra

$$\tilde{A} = \frac{\mathbb{F}[x, y, \xi]}{\text{Ann}(X^d + Y^d + X^k \Xi^n)} = \frac{\mathbb{F}[x, y, \xi]}{(xy, x^d - y^d, y\xi, x^{k+1}\xi, \xi^n - x^n)}.$$

We can easily check that $\ell_A = x + y$ and $\ell_T = x$ are Lefschetz elements in A and T , respectively. We also know that the Hilbert function of $\tilde{A}$ satisfies $H(\tilde{A})_1 = H(\tilde{A})_{d-1} = 3$. This means that if $\tilde{A}$ satisfies the SLP, multiplication by ℓ^{d-2} , for a general $\ell \in \tilde{A}_1$ must have rank 3. However, writing $\ell = ax + by + c\xi$, we have $\ell^{p^m} = a^{p^m}x^{p^m} + b^{p^m}y^{p^m} + c^{p^m}\xi^{p^m} = a^{p^m}x^{p^m} + b^{p^m}y^{p^m}$, because ξ^{p^m} is zero in $\tilde{A}$. Therefore, $\ell^{p^m}\xi = 0$ (note that $p^m > k$),

meaning that multiplication by ℓ^{p^m} has rank less than 3; hence so does multiplication by ℓ^{d-2} . Therefore, $\tilde{A}$ does not satisfy the SLP.

Theorem 8.5 once again singles out cohomological blow-up algebras among connected sums. Connected sums $A\#_{\mathbb{F}}B$ over a field $\mathbb{F}$ of two strong or weak Lefschetz algebras A, B are strong or weak Lefschetz, respectively; however, taking connected sum $A\#_TB$ over an arbitrary AG algebra T may not in general preserve SLP or WLP [30, Section 5.2].

The following example shows that the converse of Theorem 8.5 is not true: if the cohomological blowup $\tilde{A}$ has SLP, it does not follow that A has SLP. In other words, while the process of blowing up preserves SLP, the process of blowing down does not preserve SLP, or even WLP.

Example 8.7. The following example, originally due to U. Perazzo but re-examined more recently by Gondim and Russo [15], is an AG algebra with unimodal Hilbert function that does not have SLP or WLP:

$$\begin{aligned} A &= \frac{\mathbb{F}[x, y, z, u, v]}{\text{Ann}(XU^2 + YUV + ZV^2)} \\ &= \frac{\mathbb{F}[x, y, z, u, v]}{(x^2, xy, y^2, xz, yz, z^2, u^3, u^2v, uv^2, v^3, xv, zu, xu - yv, zv - yu)}. \end{aligned}$$

Taking the quotient T of A given by the Thom class $\tau = u^2$ yields

$$T = \frac{\mathbb{F}[x, y, z, u, v]}{\text{Ann}(X)} = \frac{\mathbb{F}[x, y, z, u, v]}{(x^2, y, z, u, v)} \cong \frac{\mathbb{F}[x]}{(x^2)}.$$

Fix a parameter $\lambda \in \mathbb{F}$, and define polynomials $f_T(\xi) \in T[\xi]$ and $f_A(\xi) \in A[\xi]$ by

$$f_T(\xi) = \xi^2 - \lambda x\xi \quad \text{and} \quad f_A(\xi) = \xi^2 - \lambda x\xi + u^2.$$

Denoting the ideal of relations of A by I , we obtain the cohomological blowup

$$\tilde{A} = \frac{\mathbb{F}[x, y, z, u, v, \xi]}{I + \xi(y, z, u, v) + (f_A(\xi))},$$

which has Hilbert function $H(\tilde{A}) = H(A) + H(T)[1] = (1, 6, 6, 1)$. Fix $\mathbb{F}$ -bases

$$\tilde{A}_1 = \text{span}_{\mathbb{F}}\{x, y, z, u, v, \xi\}, \text{ and } \tilde{A}_2 = \text{span}_{\mathbb{F}}\{u^2, uv, v^2, yv, yu, -x\xi\},$$

and let $\ell \in \hat{A}_1$ be a general linear form

$$\ell = ax + by + cz + du + ev + f\xi.$$

Then the matrix for the Lefschetz map $\times \ell: \tilde{A}_1 \rightarrow \tilde{A}_2$ and its determinant are given by

$$M = \begin{pmatrix} 0 & 0 & 0 & d & 0 & -f \\ 0 & 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ d & e & 0 & a & b & 0 \\ 0 & d & e & b & c & 0 \\ -f & 0 & 0 & 0 & 0 & -(a + \lambda f) \end{pmatrix} \Rightarrow \det(M) = f^2 e^4.$$

Thus, ℓ is a strong Lefschetz element for $\tilde{A}$ if and only if $e \cdot f \neq 0$. In particular, $\tilde{A}$ satisfies SLP and also WLP.

Surprisingly, the analogous result to Theorem 8.5 does not hold for the WLP. The obstruction to establishing such a result via an analogue of Lemma 8.4 is that the tensor product of two weak Lefschetz algebras need not be weak Lefschetz. Examples of non-AG quadratic algebras that demonstrate this are given in [40, Section 4.1]. Here, we point out an AG example. The AG algebra T in Example 8.8 below is an example of R. Gondim, quoted as [26, Example 3.4]. Using the Clebsch–Gordan theorem [19, Theorem 3.29], one can see that if $\text{char } \mathbb{F} = 0$ or $\text{char } \mathbb{F} \geq 7$, then the tensor product $T \otimes B$, where $B = \mathbb{F}[\xi]/(\xi^2)$, has Jordan type (for generic linear form) the tensor product of $P_T = (5, 3, 3, 3, 2, 2)$ and $P_B = (2)$, which is $(6, 4^4, 3^2, 2^3, 1^2)$ with 12 parts; on the other hand, $H(T \otimes B) = (1, 6, 11, 11, 6, 1)$; hence, $H(T \otimes B)^\vee = (6, 4^5, 2^5)$ has 11 parts, so $T \otimes B$ is not weak Lefschetz.

Building upon this, we give an example illustrating that blowing up does not preserve WLP.

Example 8.8. Consider the following algebra:

$$\begin{aligned} A &= \frac{\mathbb{F}[x, y, z, u, v]}{\text{Ann}(XU^6 + YU^4V^2 + ZU^5V)} \\ &= \frac{\mathbb{F}[x, y, z, u, v]}{(yz, xz, xy, vy - uz, vx, ux - vz, u^5y, u^5v^2, u^6v, u^7, v^3, x^2, y^2, z^2)} \end{aligned}$$

with $H(A) = (1, 5, 6, 6, 6, 5, 1)$ and its quotient corresponding to the Thom class $\tau = u^3$

$$\begin{aligned} T &= \frac{\mathbb{F}[x, y, z, u, v]}{\text{Ann}(XU^3 + YUV^2 + ZU^2V)} \\ &= \frac{\mathbb{F}[x, y, z, u, v]}{(z^2, yz, xz, y^2, xy, vy - uz, x^2, vx, ux - vz, u^2y, v^3, u^2v^2, u^3v, u^4)} \end{aligned}$$

with $H(T) = (1, 5, 6, 5, 1)$. Both A and T satisfy WLP, but not SLP: for a generic ℓ , the Jordan types are $P_{A,\ell} = (8, 6, 6, 6, 5, 5)$ and $P_{T,\ell} = (5, 3, 3, 3, 2, 2)$. Denoting the ideal of relations of A by I and the ideal of relations of T by K , consider the cohomological blow-up algebra

$$\tilde{A} = \frac{\mathbb{F}[x, y, z, u, v, \xi]}{I + \xi \cdot K + (\xi^3 - u^3)},$$

which has Hilbert function $H(\tilde{A}) = H(A) + H(T)[1] + H(T)[2] = (1, 6, 12, 17, 17, 12, 6, 1)$ and dual Macaulay generator $\tilde{F} = \Xi^3 G + \Xi^6 X + F$, here $\lambda = -1$ in the Construction 4.2. Fix the following bases for the seventeen dimensional vector spaces

$$\begin{aligned} \tilde{A}_3 &= \text{span}_{\mathbb{F}} \left\{ x\xi^2, yu^2, yuv, yu\xi, yv^2, yv\xi, y\xi^2, zv\xi, z\xi^2, u^2v, u^2\xi, uv^2, uv\xi, u\xi^2, v^2\xi, v\xi^2, \xi^3 \right\} \\ \tilde{A}_4 &= \text{span}_{\mathbb{F}} \left\{ x\xi^3, yuv\xi, yu\xi^2, yv^2\xi, yv\xi^2, y\xi^3, zv\xi^2, z\xi^3, u^2v^2, u^2v\xi, u^2\xi^2, uv^2\xi, uv\xi^2, u\xi^3, \right. \\ &\quad \left. v^2\xi^2, v\xi^3, \xi^4 \right\} \end{aligned}$$

Taking a general linear form $\ell \in \tilde{A}_1$ as $\ell = ax + by + cz + du + ev + f\xi$, we compute the matrix for the Lefschetz map $\times \ell: \tilde{A}_3 \rightarrow \tilde{A}_4$ and its determinant, which shows $\tilde{A}$ does not

satisfy WLP

$$M = \begin{pmatrix} f & 0 & e & 0 & d & 0 & 0 & 0 & 0 & c & 0 & b & 0 & 0 & 0 & 0 & a \\ 0 & 0 & f & e & 0 & d & 0 & 0 & 0 & 0 & c & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & e & 0 & d & 0 & 0 & a & 0 & c & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & e & 0 & d & 0 & 0 & 0 & 0 & c & 0 & b & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & e & 0 & 0 & 0 & 0 & a & 0 & c & 0 \\ 0 & e & d & 0 & 0 & 0 & 0 & 0 & 0 & f & b & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & e & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & e & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & e & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & f & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & f \end{pmatrix} \Rightarrow \det(M) = 0.$$

A Macaulay2 [18] calculation gives the generic Jordan type $P_{\tilde{A}} = (8, 6^5, 4^6, 2^4, 1^2)$ with 18 parts, whereas the conjugate partition of the Hilbert function is $H(\tilde{A})^\vee = (8, 6^5, 4^6, 2^5)$ with 17 parts, which implies that $\tilde{A}$ has neither SLP nor WLP.

In Example 8.8, the Thom class of the map $A \rightarrow T$ has degree 3. This is the minimal possible value for such an example based on the following result.

Theorem 8.9. Let $\mathbb{F}$ be an infinite field, and let $\pi : A \rightarrow T$ be a surjective homomorphism of graded AG $\mathbb{F}$ -algebras such that the difference between the socle degrees of A and T is at most 2 and A and T both satisfy WLP. Then every cohomological blow-up algebra of A along π satisfies WLP.

Proof. The hypothesis on the socle degrees of A and T translates into $n \leq 2$. The proofs of Lemma 8.4 in the spacial cases $J = 0$ (for $n = 1$) or $J = T$ (for $n = 2$) and Theorem 8.5 go through upon replacing SLP by WLP throughout. ■

9 Geometric View and Examples

As in the Introduction, our motivation for studying the cohomological blow-up algebras stems from the blow-up construction in algebraic geometry. That the cohomology ring of the blowup of a compact complex manifold along a closed complex submanifold satisfies the conditions of Theorem 3.14 can be pieced together from results in the

book of Griffiths and Harris [17, Chapter 4, Section 6]; see also the paper of McDuff [37, Proposition 2.4] for the statements in the symplectic category. The cohomology of the blowup was also studied by S. Gitler who has obtained a presentation, as in Equation (5), of the cohomology of the blowup of a complex manifold along a complex submanifold in the case of a surjective restriction map as in Equation (5) [14, Theorem 3.11]. An analogous presentation for the Chow ring of the blowup of an algebraic variety along a regularly embedded subvariety with a surjective restriction map also appears in the paper of Keel [32, Appendix, Theorem 1].

Blowing up and blowing down are fundamental building blocks in birational geometry. For example, in the theory of algebraic surfaces, a classical result states that every birational map between algebraic surfaces (smooth complex projective variety of dimension two) admits a strong factorization, meaning it factors as a sequence of blowups followed by a sequence of blowdowns [6, Corollary II.12]. More recently, Abramovich *et al.* have proved the weak factorization conjecture, a higher-dimensional analogue that states that every birational map between complete non-singular algebraic varieties over an algebraically closed field of characteristic zero factors as a product of blowups and blowdowns (in no specific order) [1, Theorem 0.1.1]. In the theory of (smooth, projective) toric varieties, one can be quite explicit with these factorizations by working with the associated (simplicial, polytopal) fan, where blowing up along a toric subvariety corresponds to subdividing a cone of the fan [11, Proposition 3.3.15]. As we have seen in Section 8 concerning the SLP and will show in a sequel [34] concerning the Hodge-Riemann bilinear relations (HRR), factorizations in terms of blowups and blowdowns can be useful in establishing these properties.

McMullen [38] in his proof of SLP and HRR for the polytope algebra, an AG algebra that he shows is isomorphic to a certain Artinian reduction of the Stanley-Reisner ring of the corresponding simplicial polytopal fan, has given an explicit formula for a (weak) factorization of the birational map $\mathbb{P}^n \supset (\mathbb{C}^*)^n \rightarrow X$ where X is a smooth projective toric variety of dimension n ; see also Timorin [44] for an exposition from the point of view of Macaulay duality. Karu [31] used similar arguments in his proof of SLP and HRR for non-simplicial polytopal fans; see also [5, 9]. More recently Ardila *et al.* [2] have exploited blow-up factorizations to prove SLP and HRR for the Chow ring of the Bergman fan associated with a matroid. Especially important in their work is the special case of blowups that correspond to edge subdivisions in these Bergman fans, and in fact, they factor any blowup as a sequence of successive edge subdivision blowups and blowdowns. Geometrically, these edge subdivisions correspond to cohomological blowups with $n = 2$. Theorem 8.9 indicates that this $n = 2$ scenario is particularly

favorable in terms of ascent of the WLP to the cohomological blow-up algebra, but it does not necessarily guarantee the descent of WLP or SLP to the blowdown; see Example 8.7.

Algebraically, one might say that two oriented graded AG algebras A and A' are *birationally equivalent* if there is a sequence of oriented graded AG algebras $(A_0, A_1, \dots, A_m)$ of some fixed socle degree d , where $A_0 = A$ and $A_m = A'$ and for each i , A_i is either a cohomological blowup or a cohomological blowdown of A_{i-1} . We give an example below of several birationally equivalent AG algebras using fans corresponding to smooth projective toric surfaces; in fact, we derive a (strong) factorization of the birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. First, we recall a few fundamental facts on toric varieties and their associated fans; for further details, we refer the reader to the book [11]. In these geometric examples, as in the Introduction, we take cohomology with coefficients over the rationals $\mathbb{F} = \mathbb{Q}$.

A presentation for the cohomology ring of a complete simplicial toric variety can be obtained from the associated fan in the following explicit manner. Let Σ be a complete simplicial fan, and let X be the corresponding toric variety. Let $\rho_1, \dots, \rho_r$ be the rays of Σ , each ρ_i having minimal generator u_i . Introduce a variable x_i for each ρ_i . In the ring $\mathbb{Q}[x_1, \dots, x_r]$, let I be the square-free monomial ideal

$$I = (x_{i_1} \cdots x_{i_s} \mid i_1 < \cdots < i_s \text{ and } \rho_{i_1}, \dots, \rho_{i_s} \text{ are not part of the same cone of } \Sigma).$$

We call I the Stanley–Reisner ideal of Σ . Let J be the ideal generated by the linear forms

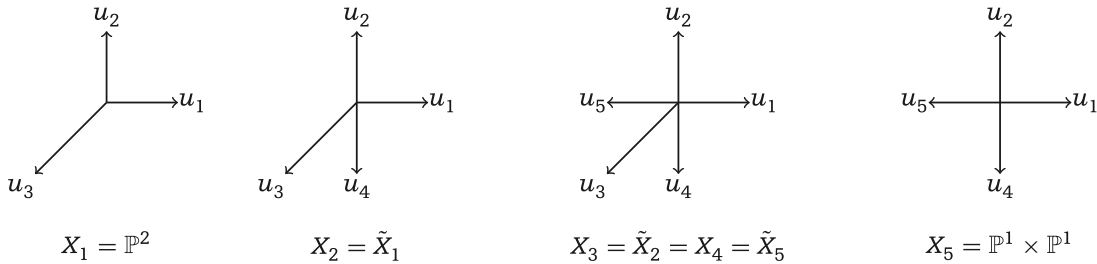
$$J = \left(\sum_{i=1}^r \langle m, u_i \rangle x_i \mid m \in \sum_{i=1}^r u_i \mathbb{Z} \right).$$

By [11, Theorem 12.4.1], the singular cohomology ring of X can be presented as

$$H^{2\bullet}(X, \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \dots, x_r]}{I + J}. \quad (48)$$

The following example shows that the oriented graded AG algebras $A_1 = \mathbb{Q}[x]/(x^3)$, $A_2 = \mathbb{Q}[y, z]/(y^2, z^2 - yz)$, $A_3 = \mathbb{Q}[r, s, t]/(rt, st, r^2, s^2, t^2 + rs) = A_4$, and $A_5 = \mathbb{Q}[u, v]/(u^2, v^2)$ are all birationally equivalent to one another. This corresponds to the well-known fact from algebraic geometry that if $\mathbb{P}^2$ is blown up at two points and the proper transform of the line joining the two points is blown down, the resulting surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We work through the details of this example from the perspective of cohomological blowups.

Example 9.1. The algebraic varieties featured in this example correspond to the following fans



As a toric variety, $X_1 = \mathbb{P}^2$ is defined by the fan Σ with ray generators $u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis vectors; see [11, Example 12.4.2]. According to the formula above, its cohomology algebra is

$$A_1 = H^{2\bullet}(X_1) \cong \frac{\mathbb{Q}[x_1, x_2, x_3]}{(x_1 x_2 x_3, x_1 - x_3, x_2 - x_3)} \cong \frac{\mathbb{Q}[x]}{(x^3)}.$$

The blowup $X_2 = \tilde{X}_1$ of $X_1 = \mathbb{P}^2$ at a point is obtained by subdividing a cone of Σ by adding the ray generated by $u_4 = -e_2$. Then,

$$A_2 = H^{2\bullet}(X_2) \cong \frac{\mathbb{Q}[x_1, x_2, x_3, x_4]}{(x_1 x_3, x_2 x_4, x_1 - x_3, x_2 - x_3 - x_4)} = \frac{\mathbb{Q}[x_1, x_2]}{(x_1^2, x_2^2 - x_2 x_1)}. \quad (49)$$

In terms of cohomological blowups, take $T_1 = \mathbb{Q}$, and let $\pi: A_1 \rightarrow T_1$ be the natural projection, with Thom class $\tau_1 = x^2$ and kernel $K_1 = (x)$. The normal bundle of a point in $\mathbb{P}^2$ has total Chern class $c = 1$, and hence $f_{T_1}(\xi) = \xi^2$ and $f_{A_1}(\xi) = \xi^2 + x^2$, and hence the cohomological blowup of A along π_1 is

$$\tilde{A}_1 = \frac{\mathbb{Q}[x, \xi]}{(x^3, x\xi, \xi^2 + x^2)} = \frac{\mathbb{Q}[x, \xi]}{(x\xi, \xi^2 + x^2)} \cong \frac{\mathbb{Q}[x_1, x_2]}{(x_1^2, x_2^2 - x_1 x_2)} = A_2, \quad (50)$$

where the last isomorphism sends $x \mapsto x_2$ and $\xi \mapsto x_1 - x_2$.

Next, define $X_3 = \tilde{X}_2$ as the blowup of X_2 obtained by adding the ray generated by $u_5 = -e_1$. Then its cohomology algebra is given by

$$\begin{aligned}
 A_3 = H^{2\bullet}(X_3) &= \frac{\mathbb{Q}[x_1, x_2, x_3, x_4, x_5]}{(x_1 x_3, x_1 x_5, x_2 x_3, x_2 x_4, x_4 x_5, x_1 - x_3 - x_5, x_2 - x_3 - x_4)} \\
 &= \frac{\mathbb{Q}[x_1, x_2, x_3]}{(x_1 x_3, x_1^2, x_2 x_3, x_2^2, x_3^2 + x_1 x_2)}.
 \end{aligned}$$

Again, let $T_2 = \mathbb{Q}$ and $\pi_2: A_2 \rightarrow T_2$ be the canonical projection, with Thom class $\tau_2 = x_1 x_2$ and kernel $K = (x_1, x_2)$. Then we have $f_{A_2}(\xi) = \xi^2 + x_1 x_2$ and the cohomological blowup of A_2 along π_2 is

$$\tilde{A}_2 = \frac{\mathbb{Q}[x_1, x_2, \xi]}{(x_1^2, x_2^2 - x_2 x_1, \xi x_1, \xi x_2, \xi^2 + x_1 x_2)} \cong \frac{\mathbb{Q}[x_1, x_2, x_3]}{(x_1 x_3, x_1^2, x_2 x_3, x_2^2, x_3^2 + x_1 x_2)} = A_3,$$

where the last isomorphism sends $x_1 \mapsto x_1$, $x_2 \mapsto x_2 - x_3$ and $\xi \mapsto x_3$.

Finally, let $X_5 = \mathbb{P}^1 \times \mathbb{P}^1$, defined by the fan Σ' with ray generators $u_1 = e_1$, $u_2 = e_2$, $u_4 = -e_2$, and $u_5 = -e_1$, and let $X_4 = \tilde{X}_5$ be the blowup at a point obtained by adding the ray to Σ' $u_3 = -e_1 - e_2$. Since the fans for X_4 and X_3 are identical, it follows that the toric varieties coincide as well. Hence, their cohomology algebras are

$$A_5 = H^{2\bullet}(\mathbb{P}^1 \times \mathbb{P}^1) = \frac{\mathbb{Q}[x_1, x_2, x_4, x_5]}{(x_1 x_5, x_2 x_4, x_1 - x_5, x_2 - x_4)} = \frac{\mathbb{Q}[x_1, x_2]}{(x_1^2, x_2^2)}$$

and

$$\begin{aligned} A_4 = A_3 = H^{2\bullet}(X_4) = H^{2\bullet}(X_3) &= \frac{\mathbb{Q}[x_1, x_2, x_3]}{(x_1 x_3, x_1^2, x_2 x_3, x_2^2, x_3^2 + x_1 x_2)} \\ &\cong \frac{\mathbb{Q}[x_1, x_2, \xi]}{(x_1^2, x_2^2, x_1 \xi, x_2 \xi, \xi^2 + x_1 x_2)} = \tilde{A}_5, \end{aligned}$$

where the last isomorphism is the obvious $x_1 \mapsto x_1$, $x_2 \mapsto x_2$, and $x_3 \mapsto \xi$.

It follows that the oriented graded AG algebras A_1 , A_2 , $A_3 = A_4$, and A_5 are all birationally equivalent, corresponding to the (strong) factorization of the birational map

$$\begin{array}{ccccc} & & X_3 = X_4 & & \\ & \swarrow & & \searrow & \\ & X_2 & & & \\ \swarrow & & & & \searrow \\ X_1 = \mathbb{P}^2 & \dashrightarrow & & & \mathbb{P}^1 \times \mathbb{P}^1 = X_5 \end{array}$$

Remark 9.2. The cohomology algebra of the blowup of a smooth toric variety X along a smooth torus invariant subvariety $Y \subset X$ will always agree with the Constructions 3.1, 4.2, or 6.2 of this paper since in that case the restriction map $\pi^*: H^{2\bullet}(X) \rightarrow H^{2\bullet}(Y)$ is always surjective. Indeed, in that case, the associated fan of Y corresponds to a subfan of X , and the surjectivity follows from the combinatorial presentation of cohomology algebras as in (48). The following examples show what can happen in cases where that restriction map is not surjective.

Example 9.3. Let $\pi: Y = \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 = X$ be the Segre embedding. Then we have a short exact sequence of vector bundles on Y :

$$0 \longrightarrow A \xrightarrow{\beta} \hat{A}_{MD} \xrightarrow{\hat{\pi}} \tilde{T}/\beta_0(T) \longrightarrow 0$$

where $\mathcal{T}_Y$ is the tangent bundle of Y , $\pi^*\mathcal{T}_X$ is the restriction of the tangent bundle of X to $\pi(Y)$ and $\mathcal{N}_{Y/X}$ is the normal bundle to $\pi(Y) \subset X$. If we identify the cohomology algebras as the oriented graded AG algebras

$$A = H^{2\bullet}(X) \cong \frac{\mathbb{Q}[x]}{(x^6)}, \quad T = H^{2\bullet}(\mathbb{P}^1 \times \mathbb{P}^2) \cong \frac{\mathbb{Q}[y, z]}{(y^2, z^3)},$$

where x , y , and z are the classes of a hyperplane in $H^2(\mathbb{P}^5)$, and the factors $H^2(\mathbb{P}^1)$, and $H^2(\mathbb{P}^2)$ of $H^{2\bullet}(\mathbb{P}^1 \times \mathbb{P}^2) \cong H^{2\bullet}(\mathbb{P}^1) \otimes_{\mathbb{Q}} H^{2\bullet}(\mathbb{P}^2)$, then the induced map $\pi^*: A \rightarrow T$ satisfies $\pi^*(x) = y + z$. Note that π^* is not surjective here. From the Euler sequence, we compute the total Chern classes $c(\pi^*\mathcal{T}_X) = (1 + \pi^*(x))^6$ and $c(\mathcal{T}_Y) = (1 + y)^2 \cdot (1 + z)^3$. It follows from the Whitney product formula that the total Chern class for the normal bundle is

$$\begin{aligned} c(\mathcal{N}_{Y/X}) &= \frac{c(\pi^*\mathcal{T}_X)}{c(\mathcal{T}_Y)} = \frac{(1+y+z)^6}{(1+y)^2 \cdot (1+z)^3} = (1+y+z)^6 \cdot (1-y)^2 \cdot (1-z+z^2)^3 \\ &= (6yz + 3z^2) + (4y + 3z) + 1 \end{aligned}$$

and hence the Chern classes are

$$\begin{cases} c_1(\mathcal{N}_{Y/X}) = 4y + 3z \\ c_2(\mathcal{N}_{Y/X}) = 6yz + 3z^2. \end{cases}$$

Hence, if we blow up X along Y then, according to Equation (2), the cohomology algebra of the exceptional divisor $\tilde{Y}$ is given by

$$\tilde{T} = \frac{\mathbb{Q}[y, z, \xi]}{(y^2, z^3, \xi^2 - (4y + 3z)\xi + (6yz + 3z^2))} \cong H^{2\bullet}(\tilde{Y}).$$

Moreover, using the conditions of Theorem 3.14, we can derive a presentation of the cohomology algebra of the blow-up manifold $\tilde{X}$:

$$\tilde{A} = \frac{\mathbb{Q}[x, \xi]}{(\xi^3 - 6x\xi^2 + 12x^2\xi - 8x^3, 3\xi^4 - 9x\xi^3 + 6x^2\xi^2 + 4x^3\xi)} \cong H^{2\bullet}(\tilde{X}), \quad (51)$$

where the restriction map $\tilde{\pi}: \tilde{A} \rightarrow \tilde{T}$ defined by $\tilde{\pi}(x) = y + z$ and $\tilde{\pi}(\xi) = \xi$ has Thom class $\tilde{\tau} = -\xi$, the blow-up map $\beta: A \rightarrow \tilde{A}$ defined by $\beta(x) = x$ is injective with $\beta(a_{soc}) = \tilde{a}_{soc} = x^5$, and the Hilbert function satisfies $H(\tilde{A}) = H(A) + H(T)[1] = (1, 2, 3, 3, 2, 1)$. Furthermore, a Macaulay2 [18] calculation computes the Macaulay dual generator of $\tilde{A}$ as

$$\tilde{F} = X^5 - 3X^3\Xi^2 - 10X^2\Xi^3 - 24X\Xi^4 - 48\Xi^5.$$

Note that $\tilde{A}$ in (51) does not fit the model described by our Construction 3.1; in particular, the defining ideal of $\tilde{A}$ does not contain any monic polynomial of degree $n = 2$.

Remark 9.4. The algebra $\tilde{A}$ computed in Equation (51) might be termed a cohomological blowup along the *non-surjective* map π . In that case, A and T are both standard graded, and $\tilde{A}$ is, too, but this need not hold in general. For example, if we blow up $X = \mathbb{P}^8$ along the Segre embedding of $Y = \mathbb{P}^2 \times \mathbb{P}^2$, then again we have a non-surjective restriction map

$$\pi: A = H^{2\bullet}(X) = \mathbb{Q}[x]/(x^9) \rightarrow \mathbb{Q}[y, z]/(y^3, z^3) = H^{2\bullet}(Y) = T$$

but the cohomology of the blowup of X along Y , $\tilde{A} \cong H^{2\bullet}(\tilde{X})$ has Hilbert function

$$H(\tilde{A}) = H(A) + H(T)[1] + H(T)[2] + H(T)[3] = (1, 2, 4, 7, 8, 7, 4, 2, 1),$$

which implies that $\tilde{A}$ cannot be standard graded.

Motivated by these examples, we pose some problems for further research.

Problem 9.5. Generalize Example 9.1, and find other algebras that are birationally equivalent to $A_0 = \mathbb{F}[x]/(x^{d+1})$. Can one classify them?

Problem 9.6. Generalize Example 9.3, and find a construction, similar in spirit to Construction 3.1, for a cohomological blowup of an AG algebra A along *any* (i.e., possibly non-surjective) restriction map $\pi: A \rightarrow T$. Does it have similar properties as the cohomological blowup along a surjective map, that is, flat family, strong Lefschetz, connected sum, minimal generators?

1 A Guide to Our Examples

Given below is a list of the examples in this paper together with a brief description of the idea that example is attempting to illustrate.

1. Example 3.3 shows that without further qualifications, Construction 3.1 can produce non-Gorenstein, Gorenstein, or boundary-Gorenstein algebras, that is, non-Gorenstein algebras in the closure of the Gorenstein locus of the Hilbert scheme of that Hilbert function.
2. Example 3.12 shows that over non-algebraically closed fields, distinct parameter values λ can produce non-isomorphic cohomological blow-up algebras.
3. Example 4.5 shows that $\tilde{T} = R[\xi]/\text{Ann}(h_R(\xi) \circ (\Xi^{d-1} \cdot G))$ is not necessarily a free extension over $T = R/\text{Ann}(G)$ for any choice of $h_R(\xi) \in R[\xi]$.
4. Example 4.7 provides an algebra $\hat{A}_{MD}$ from Construction 4.2 that is not a cohomological blowup.
5. Example 5.3 shows a cohomological blowup as a connected sum.
6. Example 5.7 shows a cohomological blowdown of Hilbert function $H(A) = (1, 3, 6, 3, 1)$ as a connected sum.
7. Example 5.8 shows that the cohomological blowup $\tilde{A}$ may be standard graded even if A is not.
8. Example 6.1 gives a cohomological blowup that is a CI.
9. Example 6.7 gives ideals I and $(I : \tau)$ with homology groups H and H' equal to zero, but where the cohomological blow-up ideal $\tilde{I}$ is not generated by a regular sequence.
10. Example 6.9 gives examples of exact pairs of zero divisors.
11. Example 6.14 gives a CI with exact zero divisors, which is not a BUG.
12. Example 7.3 shows a compressed AG algebra of socle degree 5 and embedding dimension 3 can be a cohomological blowup of a standard graded AG algebra.
13. Example 8.6 shows if A and T have SLP over $\mathbb{F}_p$, then $\tilde{A}$ may fail SLP.
14. Example 8.7 gives algebras in which $\tilde{A}$ and $\tilde{T}$ both have SLP, but the cohomological blowdown A does not have SLP.
15. Example 8.8 shows that if A and T both have WLP (but fail SLP), then $\tilde{A}$ may fail WLP.
16. Example 9.1 gives a geometric example of a strong factorization of a birational map between toric varieties, which yields several birationally equiv-

alent AG algebras, that is, a sequence of AG algebras in which each one is either a cohomological blowup or blowdown of the previous.

17. Example 9.3 computes a presentation of the blowup of $\mathbb{P}^5$ along the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ in which case the restriction map on cohomology is not surjective.

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