Reachability-based Covariance Control for Pursuit-Evasion in Stochastic Flow Fields

Venkata Ramana Makkapati*, Jack Ridderhof†, and Panagiotis Tsiotras‡
Georgia Institute of Technology, Atlanta, GA 30332-0150

In this paper, pursuit-evasion scenarios in a stochastic flow field involving one pursuer and one evader are analyzed. Using a forward reachability set based approach and the associated level set equations, nominal solutions of the players are generated. The dynamical system is linearized along the nominal solution to formulate a chance-constrained, linear-quadratic stochastic dynamic game. Assuming an affine disturbance feedback structure, the proposed game is solved using the standard Gauss-Seidel iterative scheme. Numerical simulations demonstrate the proposed approach for realistic flow-fields.

I. Introduction

Coordination strategies for autonomous vehicles that are obtained under the framework of pursuit-evasion (PE) games address many of the challenges involving multi-agent systems such as of collision avoidance, surveillance and target acquisition [1–4]. Planning under environmental disturbances, such as wind fields and uncertain currents, is a necessity for technological solutions employing many aerial and underwater autonomous vehicles. Traditionally, such disturbances are assumed to be stochastic, and there is a vast amount of work available for planning under stochastic uncertainties, both endogenous and exogenous. To this end, stochastic pursuit-evasion games have received a great deal of attention by many researchers over the years, who have proposed various formulations and numerical techniques [5–8].

One of the most promising approaches for solving deterministic pursuit-evasion (PE) games involves reachability and level set based analysis [9, 10]. These have been applied in aerospace applications such as for the construction of safety envelope [11]. Sun et al. derived capture conditions and open-loop strategies for agents in multi-player PE problems with dynamic flow fields using a reachability based approach [10]. The work by Sun et al. employs forward reachable sets by solving the level set equations. Under the assumption that the evader is slower than the pursuer in the one-pursuer-one-evader game, the approach led to a simplified capture condition, stating that the optimal capture time is the minimum time taken by the pursuer's reachability set to contain the evader's reachability set. In the stochastic realm, forward reachability based analysis is a relatively new idea with limited previous work, and the system dynamics was mostly assumed to be linear [12–15]. This work attempts to extend forward reachability set based approaches to address pursuit-evasion under general stochastic flow fields, while employing techniques from covariance control theory [16].

The problem of steering the state of a stochastic dynamic system from a given initial Gaussian distribution to a desired one is referred to as the covariance steering problem. The idea of covariance control has its genesis in the 1980s [17]. The problem of finite-horizon covariance control in continuous time was however analyzed only recently by Chen et al. [18–20]. In Ref. [21], state chance constraints were introduced to the covariance control problem in the context of path planning with static obstacles and system uncertainties. The approach was subsequently modified to deal with general nonlinear dynamics [22], and was applied to spacecraft control [16, 23, 24]. Finally, a game-theoretic version of the discrete-time covariance steering problem was analyzed in Ref. [25].

In this work, we consider two-agent PE problems with both agents traversing a stochastic flow field. It is assumed that both agents have speed constraints, and the pursuer is superior to the evader in terms of its speed capabilities. Initially, a forward reachability analysis is performed while considering only the drift term in the flow field to obtain the nominal trajectories for the agents. Assuming a linear feedback control architecture, we then formulate a discrete-time chance-constrained covariance game about the players' nominal trajectories, which is solved using the standard Gauss-Seidel method, to obtain closed-loop controls for both players.

*PhD Candidate, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta. GA 30332-0150. USA. Email: mvramana@gatech.edu

†PhD Candidate, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta. GA 30332-0150. USA. Email: jridderhof3@gatech.edu

‡Professor and David and Andrew Lewis Chair, Institute for Robotics and Intelligent Machines, School of Aerospace Engineering, Georgia Institute of Technology, Atlanta. GA 30332-0150. USA. Email: tsiotras@gatech.edu

The rest of the paper is organized as follows. Section II presents a mathematical formulation of the stochastic pursuit-evasion problem, and the linearized dynamical model that is used to analyze the problem. Section III discusses the reachability approach and the level set methods that are used to obtain the nominal trajectories. The chance-constrained covariance steering game, formulated using the nominal solution from Section III, is presented in Section IV. Numerical simulations demonstrating the proposed approach in the cases of linear and nonlinear flow fields are presented in Section V. Section VI concludes the paper.

II. Problem Formulation

Consider a two agent pursuit-evasion scenario in an external stochastic flow field. The dynamics of each agent are given by

$$\mathrm{d}x^i(t) = u^i(t)\mathrm{d}t + D(x^i(t), \mathrm{d}t, \mathrm{d}w^i), \quad x^i(0) = x_0^i, \tag{1}$$

where $x^i \in \mathbb{R}^2$, for $i \in \{p, e\}$, denotes the position of an agent (p - pursuer, e - evader) with x_0^i being the i^{th} agent's fixed initial position, known to both players. Here, u^i is the i^{th} agent's control input (velocity) such that $u^i(t) \in \mathbb{R}^2$, and

$$\|u^i(t)\|_2 \leq u^i_{\max}. \tag{2}$$

It is assumed that the pursuer is strictly superior in terms of its speed capabilities compared to the evader, i.e., $u^p_{\max} > u^e_{\max}$. The instantaneous dynamic flow field $D(x, \mathrm{d}t, \mathrm{d}w)$ is assumed to have the form

$$D(x, \mathrm{d}t, \mathrm{d}w) = f(x)\mathrm{d}t + g(x)\mathrm{d}w, \tag{3}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a position-dependent function, and $g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$. Here, $w = [w_1 \; w_2]^\intercal$ where w_1 and w_2 are two independent standard Wiener processes. Also, w^p and w^e are assumed to be independent.

In a general pursuit-evasion scenario, the aim of the pursuer is to capture the evader in the shortest time possible, while the evader tries to postpone capture indefinitely. In a deterministic pursuit-evasion game, capture occurs when the Euclidean distance between the agents is less than the capture radius $\varepsilon > 0$. In the proposed formulation, and since the positions of the agents are driven by stochastic processes, capture can only be defined in probabilistic terms. The capture probability at time t is given by

$$C(t) = \mathbb{P}\{\|x^p(t) - x^e(t)\| \leq \varepsilon\}. \tag{4}$$

The goal of this work is to arrive at the control inputs u^i, $i \in \{p, e\}$, that achieve the players' objectives: the pursuer wants to capture the evader in the shortest time possible with high certainty (capture probability); and the evader wants to ensure that the capture probability is as low as possible for all times. To this end, we first obtain the players' nominal trajectories using reachability set analysis while ignoring the disturbance term in the players' dynamics. The system is subsequently linearized along this nominal solution. Then, using the theory of discrete-time linear-quadratic stochastic games, feedback control inputs are constructed that track the trajectory under flow uncertainties while optimizing for the capture probability.

Let $(\hat{x}^i(t), \hat{u}^i(t))$, $i \in \{p, e\}$, for $t \in [0, \hat{T}]$ be the i^{th} player's nominal solution. Here, $\hat{T}$ is the final time of the nominal solution when capture occurs, and $\hat{T} = \infty$ indicate that capture is not possible. The linearized dynamics along the nominal trajectory is given in an augmented fashion as

$$\mathrm{d}x(t) \approx (u(t) + r(t) + A(t)x(t))\,\mathrm{d}t + G(t)\mathrm{d}w, \tag{5}$$

where $x(t) = [x^{p\intercal}(t), x^{e\intercal}(t)]^\intercal$, $u(t) = [u^{p\intercal}(t), u^{e\intercal}(t)]^\intercal$, $w = [w^{p\intercal}, w^{e\intercal}]^\intercal$. Here, $G(t) = \mathrm{blkdiag}(g(x^p(t)), g(x^e(t)))$, $A(t) = \mathrm{blkdiag}(A^p(t), A^e(t))$, where

$$A^i(t) = \frac{\partial f}{\partial x}(\hat{x}^i(t)), \tag{6}$$

and $r(t) = [r^{p\intercal}(t), r^{e\intercal}(t)]^\intercal$, where $r^i(t) = f(\hat{x}^i(t)) - A^i(t)\hat{x}^i(t)$.

A discrete-time representation of the dynamics in (5) can be expressed as

$$x_{k+1} = A_k x_k + B_k u_k + r_k + G_k w_k, \tag{7}$$

where $x_k = x(\tau_k)$, $u_k = u(\tau_k)$, for all $\tau_k = k\hat{T}/N$, $k = \{0, 1, \ldots, N\}$. Assuming a zero-order-hold discretization, we obtain

$$A_k = \Phi(\tau_{k+1}, \tau_k), \tag{8a}$$

$$B_k = \int_{\tau_k}^{\tau_{k+1}} \Phi(\tau_{k+1}, \tau)\mathrm{d}\tau, \tag{8b}$$

$$r_k = \int_{\tau_k}^{\tau_{k+1}} \Phi(\tau_{k+1}, \tau) r(\tau)\mathrm{d}\tau, \tag{8c}$$

$$G_k G_k^\intercal = \int_{\tau_k}^{\tau_{k+1}} \Phi(\tau_{k+1}, \tau) G(\tau) G^\intercal(\tau) \Phi^\intercal(\tau_{k+1}, \tau)\mathrm{d}\tau, \tag{8d}$$

where $\Phi(\tau, s)$ is the state transition matrix for the system in (7). Using the notation introduced in [22, 25], the system dynamics in (7) can be alternatively expressed as

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{R} + \mathcal{G}W, \tag{9}$$

where $X = [x_1^\intercal,\ x_2^\intercal, \ldots, x_N^\intercal]^\intercal$, $U = [u_0^\intercal,\ u_1^\intercal, \ldots, u_{N-1}^\intercal]^\intercal$, $W = [w_0^\intercal,\ w_1^\intercal, \ldots, w_{N-1}^\intercal]^\intercal$, and

$$\mathcal{A} = \begin{bmatrix} A_0 \\ A_1 A_0 \\ \vdots \\ A_{N-1} \ldots A_1 A_0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_0 & 0 & \ldots & 0 \\ A_1 B_0 & B_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{N-1} \ldots A_1 B_0 & A_{N-1} \ldots A_2 B_1 & \ldots & B_{N-1} \end{bmatrix}, \tag{10a}$$

$$\mathcal{R} = \begin{bmatrix} r_0 \\ A_1 r_0 + r_1 \\ \vdots \\ A_{N-1} \ldots A_1 r_0 + \cdots + r_{N-1} \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} G_0 & 0 & \ldots & 0 \\ A_1 G_0 & G_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{N-1} \ldots A_1 G_0 & A_{N-1} \ldots A_2 G_1 & \ldots & G_{N-1} \end{bmatrix}. \tag{10b}$$

The mean and error terms of the augmented position vector are defined as $\bar{X} = \mathbb{E}[X]$ and $\tilde{X} = X - \bar{X}$, respectively.

In order to retrieve the coordinates, and the controls of the agents at each time-step individually from the augmented vectors (X, U), the matrices $E_k = [0_{2\times 2(k-1)}, I_2, 0_{2\times 2(N-k)}]$,

$$E^p = \begin{bmatrix} I_2 & 0_2 & 0_2 & \ldots & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 & \ldots & 0_2 & 0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_2 & 0_2 & 0_2 & \ldots & I_2 & 0_2 \end{bmatrix}_{2N\times 4N}, \tag{11a}$$

$$E^e = \begin{bmatrix} 0_2 & I_2 & 0_2 & 0_2 & \ldots & 0_2 \\ 0_2 & 0_2 & 0_2 & I_2 & \ldots & 0_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & 0_2 & 0_2 & \ldots & I_2 \end{bmatrix}_{2N\times 4N}, \tag{11b}$$

are introduced. Note that $E_k^i = E_k E^i$, for $i = \{p, e\}$, $1 \le k \le N$. Consequently, $E^i X = [x_1^i, \ldots,\ x_N^i]$, and $E_k^i X = x_k^i$.

The relative position vector at time-step k is defined as $x_k^r = x_k^p - x_k^e = E_k^p X - E_k^e X$. The mean and error terms of the relative position vector can be obtained as

$$\begin{aligned} \mathbb{E}[x_k^r] &= \mathbb{E}[E_k^p X - E_k^e X] = (E_k^p - E_k^e)\mathbb{E}[X] \\ &= (E_k^p - E_k^e)\bar{X}, \end{aligned} \tag{12}$$

$$\begin{aligned} x_k^r - \mathbb{E}[x_k^r] &= (E_k^p - E_k^e)X - (E_k^p - E_k^e)\bar{X} = (E_k^p - E_k^e)(X - \bar{X}) \\ &= (E_k^p - E_k^e)\tilde{X}. \end{aligned} \tag{13}$$

As a result, the covariance of the relative position vector is given by

$$\Sigma_k^r = \mathbb{E}[(E_k^p - E_k^e)\tilde{X}\tilde{X}^\top(E_k^p - E_k^e)^\top]. \tag{14}$$

The capture probability at the k^{th} time-step is given by $\mathbb{P}\{\|x_k^r\| \leq \varepsilon\}$. The capture probability can be considered as the payoff function at each time-step, which the pursuer tries to minimize while the evader tries to maximize. An alternative formulation involves the players optimizing over the minimum value within which the relative distance of the players lie with high probability, given by

$$\bar{\epsilon}_k = \inf\{\epsilon > 0 : \mathbb{P}\{\|x_k^r\| \leq \epsilon\} \geq 1 - \beta\}, \tag{15}$$

$1 > \beta > 0$, $1 \leq k \leq N$. The following result provides a lower bound for such an $\bar{\epsilon}_k$ in terms of the mean and the covariance of the relative distance vector x_k^r at the k^{th} time-step.

Theorem II.1. *([24]) Let $z \in \mathcal{N}(\mu, \Sigma)$ be an m-dimensional random vector, where $m = 1$ or $m = 2$, let $\sigma = \sqrt{\lambda_{\max}(\Sigma)}$, let $\rho > 0$, and let $1 > \beta > 0$. Then,*

$$\|\mu\| + \sigma\sqrt{2\log\frac{1}{\beta}} \leq \rho \implies \mathbb{P}(\|z\| \leq \rho) \geq 1 - \beta. \tag{16}$$

Note that the above result provides a lower bound for any $\epsilon > 0$ that satisfies the condition $\mathbb{P}\{\|x_k^r\| \leq \epsilon\} \geq 1 - \beta$. In this paper, we consider the lower bound of $\bar{\epsilon}_k$, as per (16), to be the payoff function that the players try to optimize. To this end, using the result in Theorem II.1, we choose to optimize over the mean and covariance of the relative distance vector. From (16), it can be observed that by increasing the norm of the mean and/or covariance, the lower bound of $\bar{\epsilon}_k$ can be increased. As a result, the maximizing player can establish guarantees on the minimum relative distance that can be achieved with high probability at every time-step. However, the minimizing player can only hope to minimize $\bar{\epsilon}_k$ by reducing its lower bound, and in this case guarantees on $\bar{\epsilon}_k$ cannot be established.

In the proposed formulation, the mean trajectory is assumed to be essentially driven by the players' controls obtained from the reachability analysis, i.e., the nominal control inputs. Therefore, for the chance-constrained covariance game, the players optimize primarily over the covariance of the relative position vector alone. To this end, we consider the payoff function, which the pursuer tries to minimize while the evader tries to maximize,

$$J(U) = \|\Sigma^r\|_F^2 = \|\mathbb{E}[(E^p - E^e)\tilde{X}\tilde{X}^\top(E^p - E^e)^\top]\|_F^2, \tag{17}$$

subject to the deterministic constraints

$$\|E_k^i(\bar{X} - \hat{X})\|_2 \leq \delta_x^i, \tag{18a}$$

$$\|E_k^i(\bar{U} - \hat{U})\|_2 \leq \delta_u^i, \tag{18b}$$

for $i = \{p, e\}$. Here, $\hat{X}$ and $\hat{U}$ are concatenated vectors for the nominal solution that are obtained similar to X and U, respectively. In (17), it is understood that in the case of the pursuer, by minimizing the norm of the covariance of the augmented relative distance vector, it is minimizing the uncertainty in the relative position at every time-step $1 \leq k \leq N$, and vice versa for the evader. Note that the norm in (17) is the Forbenius norm that captures the sum of the squares of the eigenvalues, as opposed to the maximum eigenvalue, suggested by Theorem II.1. This is done for the sake of numerical implementation. The constraints in (18) ensure that the linearized dynamics in (5) remains valid. To account for the control bounds in (2), and since a feedback control structure is considered, we also enforce chance constraints at each time-step k of the form

$$\mathbb{P}\{\|E_k^i U\|_2 \leq u_{\max}^i\} \geq 1 - \beta^i, \quad i = \{p, e\}. \tag{19}$$

A reachable set based approach to obtain the nominal trajectories for the players is presented in the next section.

III. Reachability Analysis

In this section, the concept of a reachable set is first introduced. To this end, we present some definitions and discuss existing results in the area of reachability set based pursuit-evasion under deterministic flow fields. Finally, a scheme to obtain nominal control inputs of the players using level set methods is presented. In order to obtain the nominal trajectories using reachability analysis, we ignore the disturbance term in (3) so that the flow field is deterministic.

The following definitions hold for the case where the agents' dynamics are deterministic.

Definition III.1. *An agent's reachable set at time t with the initial state at x_0^i, $\mathcal{R}^i(x_0^i, t)$, $t \geq 0$, is the set of all points that can be reached in time t. The boundary of the reachable set is the reachability front, denoted by $\partial\mathcal{R}^i(x_0^i, t)$.*

Definition III.2. *The usable reachable set of the evader $\mathcal{R}_*^e(x_0^e, t)$ is the set of all terminal points of the evader at time t, for which the trajectories do not pass through the reachable set of the pursuer at any time in the interval $[0, t]$. Formally,*

$$\mathcal{R}_*^e(x_0^e, t) = \{x \in \mathbb{R}^2 : x = x^e(t) \text{ and } x^e(\tau) \notin \mathcal{R}^p(x_0^p, \tau),\ \forall\, \tau \in [0, t]\}. \tag{20}$$

From Definition III.2, it can be observed that the usable reachable set of the evader contains the set of terminal points of the evader's trajectories that are deemed *safe*. If for some time $t_c > 0$, $\mathcal{R}^e(x_0^e, t_c) \subseteq \mathcal{R}^p(x_0^p, t_c)$, then it follows that for every u^e, there exists u^p such that $x^p(t_c) = x^e(t_c)$. In other words, in the deterministic pursuit-evasion scenario, if $\mathcal{R}_*^e(X_0^e, t_c) = \varnothing$, then the capture of the evader is guaranteed at time t_c and vice versa. Consequently, the optimal capture time for pursuit-evasion problems with deterministic flow fields can be established from the following result.

Theorem III.3. *([10]) Let $T = \inf\{t \geq 0 : \mathcal{R}_*^e(x_0^e, t) = \varnothing\}$. If $T < \infty$, then capture is guaranteed for any time greater than T, whereas the evader can always escape within a time smaller than T. Hence, T is the time to capture if both players play optimally. Furthermore, let x_f denote the location where the evader is captured. Then, we have that $x_f \in \mathcal{X} = \{x \in \mathbb{R}^2 : x = x^e(T) \text{ and } x^e(t) \notin \mathcal{R}^p(x_0^p, \tau),\ \forall\, \tau \in [0, T)\}$*

While the above result provides a criterion for the evader's capture based on its usable reachable set, an instantaneous condition that is easier to implement can be stated as follows. For $u_{\max}^p > u_{\max}^e$, and assuming the magnitude of the flow field is bounded from above by some suitable constant, we have $\mathcal{R}_*^e(x_0^e, t) = \mathcal{R}^e(x_0^e, t) \setminus \mathcal{R}^p(x_0^p, t)$, for all $t \geq 0$. In such cases, the condition $\mathcal{R}_*^e(x_0^e, t) = \varnothing$ is equivalent to the condition $\mathcal{R}^e(x_0^e, t) \subseteq \mathcal{R}^p(x_0^p, t)$ [10].

The above definitions and results form the crux of the theory related to the reachable set based pursuit-evasion with deterministic equations of motion. The evolution of the reachability front can be traced using, for instance, the level set methods [26]. The reachability front is embedded as a hypersurface in a higher dimension with time as the additional dimension. An implicit representation of the front using the signed distance function is considered in this paper. The signed distance function $\varphi(x)$ with respect to a set $\mathcal{S}$ is defined as

$$\varphi(x) = \begin{cases} \min\limits_{y \in \partial\mathcal{S}} |x - y|, & \text{if } x \notin \mathcal{S}, \\ -\min\limits_{y \in \partial\mathcal{S}} |x - y|, & \text{if } x \in \mathcal{S}. \end{cases} \tag{21}$$

For any $c \in \mathbb{R}$, the c-level set of a φ is the set $\{x : \varphi(x) = c\}$. The zero-level set of the signed distance function with respect to an agent's reachable set is expressed as the corresponding reachability front.

Given the signed distance function of an agent's reachable set $\mathcal{R}^i(x_0^i, t)$ as $\phi^i(x, t)$, $i \in \{p, e\}$, the evolution of the corresponding reachability front is governed by the viscosity solution of the Hamilton-Jacobi equation

$$\frac{\partial \phi^i(x, t)}{\partial t} + \bar{u}^i |\nabla \phi^i(x, t)| + \nabla \phi^i(x, t) f(x) = 0, \tag{22}$$

with initial condition $\phi^i(x, 0) = |x - x_0^i|$ [27]. Note that $\mathcal{R}^i(x_0^i, t) = \{x \in \mathbb{R}^2 : \phi^i(x, t) \leq 0\}$, and $\partial\mathcal{R}^i(x_0^i, t) = \{x \in \mathbb{R}^2 : \phi^i(x, t) = 0\}$. Once the reachable set of the evader is contained in the pursuer's reachable set at time $\hat{T}$, the nominal trajectories of the players can be obtained in the following manner [10, 27].

From Theorem III.3, it can be observed that the terminal point of the evader's nominal trajectory $\hat{x}_f$ is the point in its reachable set that is not covered by the pursuer's reachable set until time $\hat{T}$. Consequently, $\hat{x}_f$ is the point to which the pursuer can drive its nominal trajectory at $\hat{T}$ using its control input $\hat{u}^p$. This point resides on the pursuer's reachability front. When ϕ^i is differentiable, the nominal trajectory of the pursuer can be obtained from the differential equation

$$\frac{\mathrm{d}\hat{x}^p}{\mathrm{d}t} = \hat{u}_{\max}^p \frac{\nabla \phi^p}{|\nabla \phi^p|} + f(\hat{x}^p), \tag{23}$$

and the corresponding optimal control is given by

$$\hat{u}^p = \hat{u}_{\max}^p \frac{\nabla \phi^p}{|\nabla \phi^p|}. \tag{24}$$

Here, $\hat{u}_{\max}^p \leq u_{\max}^p$ is the bound on the nominal control input of the pursuer.

Similarly, the evader's nominal trajectory is obtained from the differential equation

$$\frac{\mathrm{d}\hat{x}^e}{\mathrm{d}t} = \hat{u}^e_{\max}\frac{\nabla\phi^e}{|\nabla\phi^e|} + f(\hat{x}^e), \tag{25}$$

and the corresponding optimal control is given by

$$\hat{u}^e = \hat{u}^e_{\max}\frac{\nabla\phi^e}{|\nabla\phi^e|}, \tag{26}$$

where $\hat{u}^e_{\max} \leq u^e_{\max}$ is the bound on the nominal control input of the evader. The chance-constrained covariance control problem, introduced in Section II, is analyzed and solved using an iterative numerical technique in the following section.

IV. Covariance Control Game

A linear feedback control structure for the players' control inputs is considered in order to solve the covariance control problem, formulated in Section II. Subsequently, the players' inputs are assumed to take the form

$$u_k = v_k + K_k y_k, \tag{27}$$

where $K_k \in \mathbb{R}^{4\times 4}$, and $y_k \in \mathbb{R}^4$ is obtained from the difference equation

$$y_{k+1} = A_k y_k + G_k w_k, \quad y_0 = 0. \tag{28}$$

In order to calculate y_k, $1 \leq k \leq N-1$, as per (28), we assume that each player can observe the instantaneous positions of the players and the control input of their adversary at the previous time-step to evaluate the noise term $G_k w_k$ experienced by the players at an earlier time-step in (28). Note that $u_k = [u_k^{p\intercal}, u_k^{e\intercal}]^\intercal$ contains the control inputs of both the pursuer and the evader at time-step k.

Using the matrices introduced in Section II, we obtain

$$Y = \mathcal{G}W, \tag{29}$$

where $Y = [y_0^\intercal, \ldots, y_{N-1}^\intercal]^\intercal \in \mathbb{R}^{4N}$. Therefore,

$$U = V + KY, \tag{30}$$

where $V = [v_0^\intercal, v_1^\intercal, \ldots, v_{N-1}^\intercal]^\intercal$ and $K = \text{blkdiag}\,(K_0, K_1, \ldots, K_{N-1})$. Substituting (30) in (9), the mean and the error terms of the augmented state vector X can be obtained as

$$\bar{X} = \mathcal{A}x_0 + \mathcal{B}V + \mathcal{R}, \tag{31a}$$

$$\begin{aligned}\tilde{X} &= X - \bar{X} = \mathcal{B}K\mathcal{G}W + \mathcal{G}W \\ &= (I + \mathcal{B}K)\mathcal{G}W,\end{aligned} \tag{31b}$$

and for the augmented control vector U, we obtain

$$\bar{U} = \mathbb{E}[U] = V, \quad \tilde{U} = U - \bar{U} = K\mathcal{G}W. \tag{32}$$

Subsequently, the covariance matrices are given by

$$\Sigma^y = \mathbb{E}[YY^\intercal] = \mathcal{G}\mathcal{G}^\intercal \tag{33a}$$

$$\Sigma^x = \mathbb{E}[\tilde{X}\tilde{X}^\intercal] = (I + \mathcal{B}K)\Sigma^y(I + \mathcal{B}K)^\intercal \tag{33b}$$

$$\begin{aligned}\Sigma^r &= \mathbb{E}[(E^p\tilde{X} - E^e\tilde{X})(E^p\tilde{X} - E^e\tilde{X})^\intercal] \\ &= (E^p - E^e)\Sigma^x(E^p - E^e)^\intercal\end{aligned} \tag{33c}$$

$$\Sigma^u = \mathbb{E}[\tilde{U}\tilde{U}^\intercal] = K\Sigma^y K^\intercal \tag{33d}$$

The payoff function in (17) can be rewritten in the form

$$\mathcal{J}(V, K) = \|\Sigma^r\|_F^2 = \text{tr}\{(E^p - E^e)^\intercal(E^p - E^e)\Sigma^x\}. \tag{34}$$

The deterministic constraints in (18) can be expressed as

$$\|E_k^i(\mathcal{A}x_0 + \mathcal{B}V + \mathcal{R} - \hat{X})\|_2 \leq \delta_x^i, \tag{35a}$$

$$\|E_k^i(V - \hat{U})\|_2 \leq \delta_u^i, \tag{35b}$$

for $i = \{p, e\}$. Finally, consider the chance constraints on the control inputs of the players given in (19). Using Theorem II.1, the control chance constraint in (19) can be captured using the expression

$$\|E_k^i V\| + \|\Sigma^{y\,1/2} K^\intercal E_k^{i\intercal}\| \sqrt{2 \log \frac{1}{\beta^i}} \leq u_{\max}^i, \quad i = \{p, e\}. \tag{36}$$

The augmented vector V and the matrix K contains the control inputs of both the pursuer and the evader. The pursuer tries to minimize the payoff function in (34) by choosing its control inputs $(E^p V, E^p K) \in \mathcal{P}$, while the evader tries to maximize the payoff function by choosing its control inputs $(E^e V, E^e K) \in \mathcal{E}$. the set $\mathcal{P}$ contains all possible tuples $(E^p V, E^p K) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N \times 4N}$ such that the constraints (35) and (36) are satisfied for $i = p$ and for all possible $k \in \{0, \dots, N-1\}$. Similarly, the set $\mathcal{E}$ contains all possible tuples $(E^e V, E^e K) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N \times 4N}$ such that the constraints (35) and (36) are satisfied for $i = e$ and for all possible $k \in \{0, \dots, N-1\}$. Therefore, the proposed stochastic game in Section II, given by (17)-(19), is transformed by considering an equivalent payoff function in (34), and constraints in (35) and (36). Note that the constraints in (35) and (36) are orthogonal constraints [28], which are player-specific and are not coupled.

In order to arrive at a saddle point equilibrium for the aforementioned game, assuming one exists, a simple Gauss-Seidel procedure given in Algorithm 1 is considered. For Algorithm 1 to converge to an equilibrium solution for any $V^{(0)}$, $K^{(0)}$, the solution must be stable [29]. The necessary conditions for the existence of a stable equilibrium can be found in Ref. [29].

Algorithm 1 Gauss–Seidel procedure to obtain saddle points

procedure G-S($V^{(0)}$,$K^{(0)}$)
    for i = 0,1,2,. . . **do**
        $[E^p V^{(i+1)}, E^p K^{(i+1)}] \leftarrow \underset{(E^p V, E^p K) \in \mathcal{P}}{\arg\min} \ \mathcal{J}(V, K)$ such that $E^e V = E^e V^{(i)}$, $E^e K = E^e K^{(i)}$
        $[E^e V^{(i+1)}, E^e K^{(i+1)}] \leftarrow \underset{(E^e V, E^e K) \in \mathcal{E}}{\arg\max} \ \mathcal{J}(V, K)$ such that $E^p V = E^p V^{(i+1)}$, $E^p K = E^p K^{(i+1)}$
    end for
    return $V^{(i+1)}$, $K^{(i+1)}$
end procedure

V. Simulations

In this section, we present simulation results for pursuit-evasion scenarios in realistic flow fields with disturbances. The chance constrained covariance steering game is implemented in MATLAB using its in-built function `fmincon` in conjunction with YALMIP [30]. The convergence criterion for the iterative method is $\|V^{(i+1)} - V^{(i)}\| \leq \epsilon_v$, and $\|K^{(i+1)} - K^{(i)}\| \leq \epsilon_k$. First, a generalized Rankine vortex model is used to generate state-dependent wind field approximations to analyze the performance of the proposed approach on linear flow fields [31]. We then consider a third-order nonlinear flow field that is constructed using orthogonal polynomials [32].

For linear flow fields, the drift part of the wind field is assumed to be of the form

$$f(x) = A(x - x_s), \tag{37}$$

and the diffusion part is a constant matrix $g(x) = G$. The simulation parameters are

$$A = \begin{bmatrix} 0.2 & 0.3 \\ -0.15 & 0.1 \end{bmatrix}, \quad x_s = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \tag{38}$$

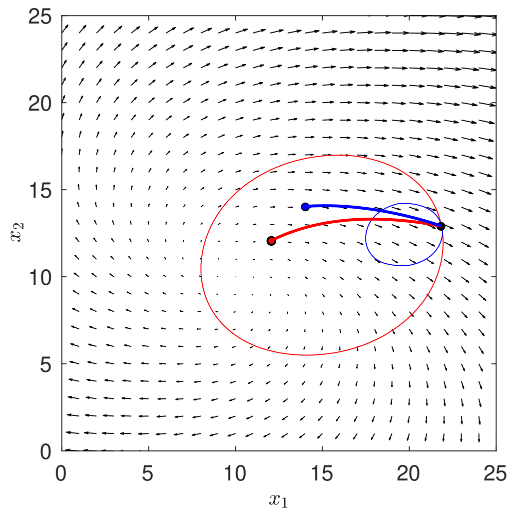


Fig. 1 Nominal trajectories in the case of a linear flow field

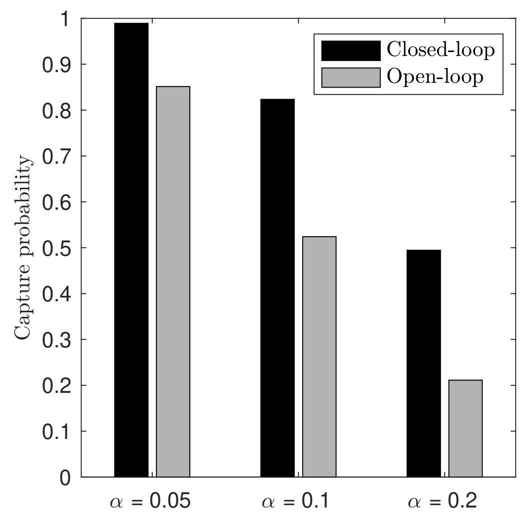
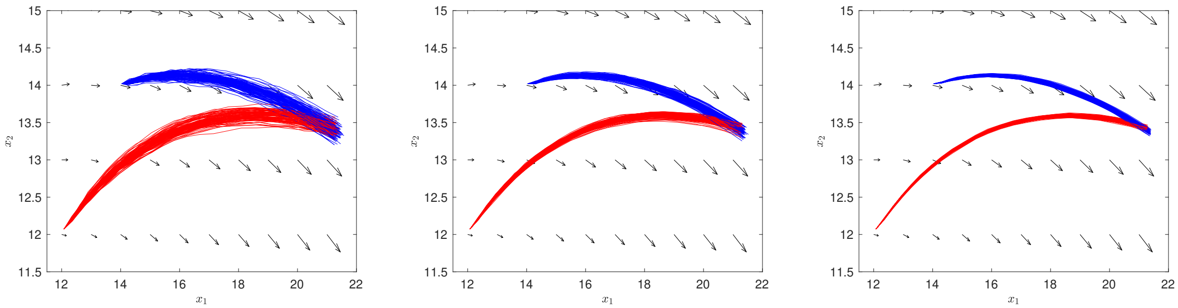


Fig. 2 Capture probabilities at $\hat{T}$ for different values of α under open-loop and closed-loop control inputs

$x_0^p = [12\ 12]^\top$, $x_0^e = [14\ 14]^\top$, $\hat{T}/N = 0.1$, $\varepsilon = 0.09$, $u_{\max}^p = 3$, $u_{\max}^e = 1$, $\hat{u}_{\max}^i = 0.8u_{\max}^i$, $\delta_u^i = 0.2u_{\max}^i$, $\delta_x^i = 0.1$, $\beta^i = 0.01$, $i \in \{p, e\}$, and finally $\epsilon_v = \epsilon_k = 5 \times 10^{-3}$. The diffusion matrix of the flow field is assumed to be of the form $G = \alpha 0.25 I_2$.

As explained earlier, the forward reachable sets of the players are first propagated until the pursuer's reachable set fully engulfs the evader's reachable set. The closed curves in the Fig. 1 are the reachable sets of the players at the final time $\hat{T}$ for the aforementioned simulation parameters. The differential equations (23), (26) are solved backwards in time from the capture point to obtain the nominal trajectories in Fig. 1 (red - pursuer, blue - evader). Subsequently, the closed-loop trajectories (solid line) are obtained by solving the associated covariance steering game along the nominal (or open-loop) trajectories.



(a) $\alpha = 0.2$ (b) $\alpha = 0.1$ (c) $\alpha = 0.05$

Fig. 3 Trajectory dispersion of the players in the linear flow field for different α values: Red - pursuer; Blue - evader.

Along the trajectory, the capture probability at each time-step can be obtained by numerically evaluating the integral

$$C_k = \int_{\|x\| \le \varepsilon} \frac{1}{2\pi\sqrt{|\Sigma_k^r|}} \exp\left(-\frac{(x-\mu_k^r)^\top \Sigma_k^{-1}(x-\mu_k^r)}{2}\right) \mathrm{d}x, \tag{39}$$

where $\mu_k^r = E_k(E^p - E^e)\bar{X}$. Figure 2 presents the capture probabilities at the final time ($\hat{T}$) under nominal and optimized control inputs for $\alpha = \{0.05, 0.1, 0.2\}$. Figure 3 presents the trajectory dispersion experienced by the players under the closed-loop control for the three α values. It can be observed as α becomes lower, the trajectory dispersion reduces, leading to higher capture probability.

The drift part of the nonlinear flow field, for $x = [x_1,\ x_2]^\top \in \mathbb{R}^2$ is assumed to be of the form

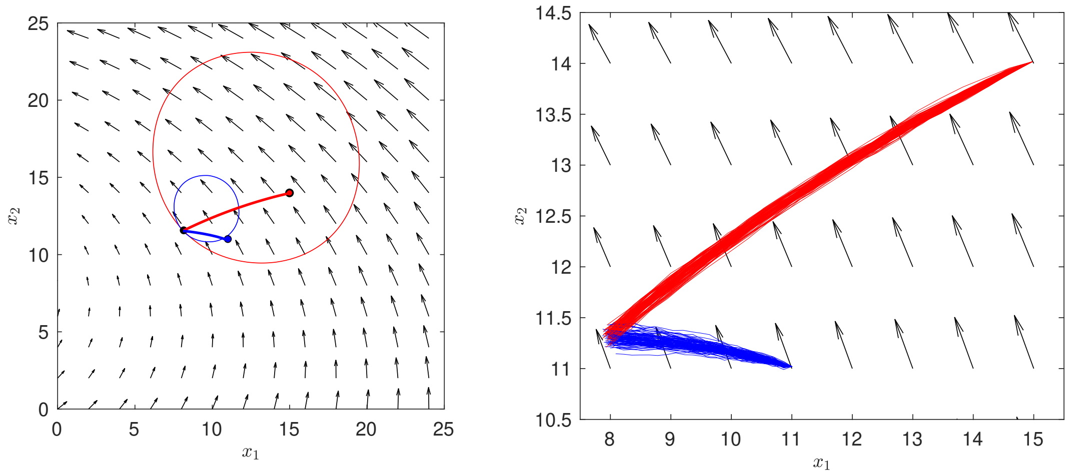
$$f(x) = \begin{bmatrix} a^\top \phi(x_1, x_2) \\ b^\top \phi(x_1, x_2) \end{bmatrix}, \tag{40}$$

where $\phi(y,z) = [1,\ y,\ z,\ y^2,\ yz,\ z^2,\ y^3,\ y^2 z,\ yz^2,\ z^3]^\top$ is the basis of the third-order polynomial vector space. The coefficients are set to $a^\top = (1/25) \times [10.8, -0.421, -1.46, -1.78\times 10^{-3}, 2.42\times 10^{-3}, 1.07\times 10^{-4}, -8.61\times 10^{-7}, 1.17\times 10^{-7}, -3.03\times 10^{-5}, -3.32\times 10^{-8}]$, and $b^\top = (1/25)\times[8.67, 0.689, -3.88\times 10^{-2}, 2.41\times 10^{-4}, 2.26\times 10^{-3}, 9.96\times 10^{-4}, 1.26\times 10^{-6}, -2.23\times 10^{-5}, -3.55\times 10^{-5}, -4.29\times 10^{-5}]$. The above nonlinear function represents a single critical point model that was employed to approximate the wind field during Hurricane Hugo [32]. The diffusion part of the nonlinear flow field is consider to be a constant matrix $G = 0.033 I_2$. The initial points of the players are chosen to be $x_0^p = [15\ 14]^\top$, $x_0^e = [11\ 11]^\top$, and $\hat{T}/N = 0.15$. The rest of the simulation parameters are same as the ones used for the case of linear flow fields.

The nominal trajectories of the players along with the reachable sets at the final time $\hat{T}$ are shown in Fig. 4(a). The trajectory dispersion of the players under closed loop control can be seen in Fig. 4(b). For this simulation, the capture probability at the final time is found to be 46.29% under open-loop control, while it is 82.78% under the closed-loop control. From the simulation results, it can be observed that the proposed closed-loop control strategy for the pursuer is effective in maximizing the capture probability in stochastic flow fields.

VI. Conclusion

A novel approach to addressing pursuit-evasion problems under external stochastic flow fields is presented. The players' nominal trajectories are obtained using forward reachability analysis while ignoring the diffusion part of the flow field. The nominal solution thus obtained is time-optimal for the players under deterministic conditions. Assuming a linear feedback control strategy, a chance-constrained covariance game is constructed around the nominal solution. The proposed covariance steering game involves optimizing over the value of the smallest relative distance that can be achieved with high probability. The pursuer tries to minimize this value while the evader tries to maximize it by equivalently optimizing over the covariance of the relative distance. The proposed approach is tested on realistic linear and nonlinear flow fields. Numerical simulations suggest that the pursuer can effectively steer the game towards capture while controlling the covariance of the relative distance.



(a) Nominal solution

(b) Trajectory dispersion

Fig. 4 Simulation results in the case of a nonlinear flow field: Red - pursuer; Blue - evader.

Acknowledgment

This work has been supported by NSF award CMMI-1662542. The work of the second author is supported by a NASA Space Technology Research Fellowship.

References

[1] Mylvaganam, T., Sassano, M., and Astolfi, A., "A constructive differential game approach to collision avoidance in multi-agent systems," *American Control Conference*, IEEE, Portland, OR, 2014, pp. 311–316.

[2] Isler, V., Sun, D., and Sastry, S., "Roadmap Based Pursuit-Evasion and Collision Avoidance," *Robotics: Science and Systems*, Vol. 1, Cambridge, United States, 2005, pp. 257–264.

[3] Schenato, L., Oh, S., Sastry, S., and Bose, P., "Swarm coordination for pursuit evasion games using sensor networks," *IEEE International Conference on Robotics and Automation*, Barcelona, Spain, 2005, pp. 2493–2498.

[4] Pan, S., Huang, H., Ding, J., Zhang, W., Tomlin, C. J., et al., "Pursuit, evasion and defense in the plane," *American Control Conference*, IEEE, Montreal, Canada, 2012, pp. 4167–4173.

[5] Yavin, Y., "Stochastic pursuit-evasion differential games in the plane," *Journal of Optimization Theory and Applications*, Vol. 50, No. 3, 1986, pp. 495–523.

[6] Vidal, R., Shakernia, O., Kim, H. J., Shim, D. H., and Sastry, S., "Probabilistic pursuit-evasion games: Theory, implementation, and experimental evaluation," *IEEE Transactions on Robotics and Automation*, Vol. 18, No. 5, 2002, pp. 662–669.

[7] Kachroo, P., Shedied, S. A., and Vanlandingham, H., "Pursuit evasion: The herding noncooperative dynamic game-The stochastic model," *IEEE Transactions on Systems, Man, and Cybernetics, Part C (Applications and Reviews)*, Vol. 32, No. 1, 2002, pp. 37–42.

[8] Li, D., Cruz Jr, J. B., and Schumacher, C. J., "Stochastic multi-player pursuit–evasion differential games," *International Journal of Robust and Nonlinear Control*, Vol. 18, No. 2, 2008, pp. 218–247.

[9] Mitchell, I. M., Bayen, A. M., and Tomlin, C. J., "A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games," *IEEE Transactions on Automatic Control*, Vol. 50, No. 7, 2005, pp. 947–957.

[10] Sun, W., Tsiotras, P., Lolla, T., Subramani, D. N., and Lermusiaux, P. F., "Multiple-pursuer/one-evader pursuit–evasion game in dynamic flowfields," *Journal of Guidance, Control, and Dynamics*, Vol. 40, No. 7, 2017, pp. 1627–1637.

[11] Nabi, H., Lombaerts, T., Zhang, Y., van Kampen, E., Chu, Q., and de Visser, C. C., "Effects of structural failure on the safe flight envelope of aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 41, No. 6, 2018, pp. 1257–1275.

[12] Vinod, A. P., HomChaudhuri, B., and Oishi, M. M., "Forward stochastic reachability analysis for uncontrolled linear systems using fourier transforms," *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*, ACM, Pittsburgh, PA, 2017, pp. 35–44.

[13] Vinod, A. P., HomChaudhuri, B., Hintz, C., Parikh, A., Buerger, S. P., Oishi, M. M., Brunson, G., Ahmad, S., and Fierro, R., "Multiple Pursuer-Based Intercept via Forward Stochastic Reachability," *American Control Conference*, IEEE, Milwaukee, WI, 2018, pp. 1559–1566.

[14] Bertram, J., and Wei, P., "Distributed Computational Guidance for High-Density Urban Air Mobility with Cooperative and Non-Cooperative Collision Avoidance," *AIAA Scitech Forum*, Orlando, FL, 2020. AIAA Paper 2020-1371.

[15] Jain, A., Gueho, D., and Singla, P., "A Computationally Efficient Approach for Stochastic Reachability Set Analysis," *AIAA Scitech Forum*, Orlando, FL, 2020.

[16] Ridderhof, J., and Tsiotras, P., "Minimum-Fuel Closed-Loop Powered Descent Guidance with Stochastically Derived Throttle Margins," *Journal of Guidance, Control, and Dynamics*, Vol. 44, No. 3, 2020, pp. 537–547.

[17] Hotz, A. F., and Skelton, R. E., "A covariance control theory," *24th IEEE Conference on Decision and Control*, Vol. 24, Fort Lauderdale, FL, 1985, pp. 552–557.

[18] Chen, Y., Georgiou, T. T., and Pavon, M., "Optimal steering of a linear stochastic system to a final probability distribution, Part I," *IEEE Transactions on Automatic Control*, Vol. 61, No. 5, 2016, pp. 1158–1169.

[19] Chen, Y., Georgiou, T. T., and Pavon, M., "Optimal steering of a linear stochastic system to a final probability distribution, Part II," *IEEE Transactions on Automatic Control*, Vol. 61, No. 5, 2016, pp. 1170–1180.

[20] Chen, Y., Georgiou, T. T., and Pavon, M., "Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part III," *IEEE Transactions on Automatic Control*, Vol. 63, No. 9, 2018, pp. 3112 – 3118.

[21] Okamoto, K., and Tsiotras, P., "Optimal Stochastic Vehicle Path Planning Using Covariance Steering," *IEEE Robotics and Automation Letters*, Vol. 4, No. 3, 2019, pp. 2276–2281.

[22] Ridderhof, J., Okamoto, K., and Tsiotras, P., "Nonlinear Uncertainty Control with Iterative Covariance Steering," *58th IEEE Conference on Decision and Control (CDC)*, Nice, France, 2019, pp. 3484–3490.

[23] Ridderhof, J., and Tsiotras, P., "Minimum-fuel powered descent in the presence of random disturbances," *AIAA Scitech Forum*, San Diego, CA, 2019. AIAA Paper 2019-0646.

[24] Ridderhof, J., Pilipovsky, J., and Tsiotras, P., "Chance-Constrained Covariance Control for Low-thrust Minimum-Fuel Trajectory Optimization," *AAS/AIAA Astrodynamics Specialist Conference*, South Lake Tahoe, CA, 2020.

[25] Makkapati, V. R., Rajpurohit, T., Okamoto, K., and Tsiotras, P., "Covariance Steering for Discrete-Time Linear-Quadratic Stochastic Dynamic Games," *59th IEEE Conference on Decision and Control*, Jeju Island, Republic of Korea, 2020, pp. 1771–1776.

[26] Osher, S., Fedkiw, R., and Piechor, K., "Level set methods and dynamic implicit surfaces," *Appl. Mech. Rev.*, Vol. 57, No. 3, 2004, pp. B15–B15.

[27] Lolla, T., Ueckermann, M. P., Yiğit, K., Haley, P. J., and Lermusiaux, P. F., "Path planning in time dependent flow fields using level set methods," *IEEE International Conference on Robotics and Automation*, Piscataway, NJ, 2012, pp. 166–173.

[28] Altman, E., and Solan, E., "Constrained Games: The Impact of the Attitude to Adversary's Constraints," *IEEE Transactions on Automatic Control*, Vol. 54, No. 10, 2009, pp. 2435–2440.

[29] Li, S., and Başar, T., "Distributed algorithms for the computation of noncooperative equilibria," *Automatica*, Vol. 23, No. 4, 1987, pp. 523–533.

[30] Löfberg, J., "YALMIP : A Toolbox for Modeling and Optimization in MATLAB," *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[31] Corpetti, T., Mémin, E., and Pérez, P., "Extraction of singular points from dense motion fields: an analytic approach," *Journal of Mathematical Imaging and Vision*, Vol. 19, No. 3, 2003, pp. 175–198.

[32] Ford, R. M., and Strickland, R. N., "Representing and visualizing fluid flow images and velocimetry data by nonlinear dynamical systems," *Graphical Models and Image Processing*, Vol. 57, No. 6, 1995, pp. 462–482.