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A note on volume thresholds for random polytopes

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Abstract
We study the expected volume of random polytopes generated by taking the convex hull of independent identically distributed points from a given distribution. We show that, for log-concave distributions supported on convex bodies, we need at least exponentially many (in dimension) samples for the expected volume to be significant, and that super-exponentially many samples suffice for κ-concave measures when their parameter of concavity κ is positive.

Keywords Random polytopes · Convex bodies · Log-concave measures · Volume threshold · High dimensions

Mathematics Subject Classification Primary 52A23 · Secondary 52A22 · 60D05

1 Introduction

Let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) random vectors uniform on a set K in $\mathbb{R}^n$. Let

$$K_N = \mathrm{conv}\{X_1, \ldots, X_N\}. \tag{1}$$

We are interested in bounds on the number N of points needed for the volume $|K_N|$ of K_N to be asymptotic in expectation to the volume $|\mathrm{conv}\, K|$ of the convex hull of K as $n \to \infty$. In the pioneering work [12], Dyer, Füredi and McDiarmid established sharp thresholds for the vertices of the cube $K = \{-1, 1\}^n$, as well as for the solid cube $K = [-1, 1]^n$. More precisely, they showed that for every $\varepsilon > 0$,

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$$\frac{\mathbb{E}|K_N|}{2^n} \xrightarrow[n\to\infty]{} \begin{cases} 0, & \text{if } N \le (\nu-\varepsilon)^n \\ 1, & \text{if } N \ge (\nu+\varepsilon)^n \end{cases}, \tag{2}$$

where for $K = \{-1, 1\}^n$, we have $\nu = 2/\sqrt{e} = 1.213\ldots$ and for $K = [-1, 1]^n$, we have $\nu = 2\pi e^{-\gamma-1/2} = 2.139\ldots$ (see also [13]). For further generalisations establishing sharp exponential thresholds see [16] (in a situation when the X_i are not uniform on a set but have i.i.d. components compactly supported in an interval).

The case of a Euclidean ball is different. Pivovarov showed in [22] (see also [7]) that when

$$K = B_2^n = \{x \in \mathbb{R}^n,\ \sum x_i^2 \le 1\},$$

the threshold is superexponential, that is for every $\varepsilon > 0$,

$$\frac{\mathbb{E}|K_N|}{|K|} \xrightarrow[n\to\infty]{} \begin{cases} 0, & \text{if } N \le e^{(1-\varepsilon)\cdot\frac{1}{2}n\log n} \\ 1, & \text{if } N \ge e^{(1+\varepsilon)\cdot\frac{1}{2}n\log n} \end{cases}. \tag{3}$$

He additionally considered the situation when the X_i are not uniform on a set but are Gaussian.

In recent works [7,8], the authors study the case of the X_i having rotationally invariant densities of the form const$\cdot(1-\sum x_i^2)^\beta \mathbf{1}_{B_2^n}$, $\beta > -1$. This is the so-called Beta model of random polytopes attracting considerable attention in stochastic geometry. In particular, $\beta = 0$ corresponds to the uniform distribution on the unit ball and the limiting case $\beta \to -1$ corresponds to the uniform distribution on the unit sphere. As established in [7], the threshold here is as follows: for every constant $\varepsilon \in (0, 1)$ and sequences $N = N(n)$, $-1 < \beta = \beta(n)$, we have

$$\frac{\mathbb{E}|K_N|}{|B_2^n|} \xrightarrow[n\to\infty]{} \begin{cases} 0, & \text{if } N \le e^{(1-\varepsilon)(\frac{n}{2}+\beta)\log n} \\ 1, & \text{if } N \ge e^{(1+\varepsilon)(\frac{n}{2}+\beta)\log n} \end{cases}, \tag{4}$$

which was further refined in [8]: for every positive constant c, the limit is e^{-c} if N grows like $e^{(\frac{n}{2}+\beta)\log\frac{n}{2c}}$ as $n \to \infty$.

We would like to focus on establishing general bounds for some large natural families of distributions. Specifically, suppose that for each dimension n, we are given a family $\{\mu_{n,i}\}_{i\in I_n}$ of probability measures such that each $\mu_{n,i}$ is supported on a compact set $V_{n,i}$ in $\mathbb{R}^n$. We would like to find the largest number N_0 and the smallest number N_1 (in terms of n and some parameters of the family) such that for every $\mu_{n,i}$ from the family, $\frac{\mathbb{E}|K_N|}{|\operatorname{conv} V_{n,i}|} = o(1)$ for $N \le N_0$ and $\frac{\mathbb{E}|K_N|}{|\operatorname{conv} V_{n,i}|} = 1 - o(1)$ for $N \ge N_1$ as $n \to \infty$ (K_N is a random polytope given by (1) with $X_1, X_2, \ldots$ being i.i.d. drawn from $\mu_{n,i}$).

For instance, the examples of the cube and the ball suggest that for the family of uniform measures on convex bodies, N_0 is exponential and N_1 is super-exponential in n.

In fact, the latter can be quickly deduced from a classical result by Groemer from [17], combined with the thresholds for Euclidean balls established by Pivovarov in [22]. Groemer's theorem says that for every $N > n$, we have

$$\mathbb{E}|\operatorname{conv}\{X_1, \ldots, X_N\}| \ge \mathbb{E}|\operatorname{conv}\{Y_1, \ldots, Y_N\}|,$$

where the X_i are i.i.d. uniform on a convex set K and the Y_i are i.i.d. uniform on a Euclidean ball with the same volume as K. We thus get from (3) that

$$\frac{1}{|K|}\mathbb{E}|\operatorname{conv}\{X_1, \ldots, X_N\}| = 1 - o(1), \tag{5}$$

as long as $N \ge e^{(1+\varepsilon)\frac{n}{2}\log n}$.

In this work, we shall establish an exponential bound on N_0 for the family of log-concave distributions on convex sets and extend (5) to the family of the so-called κ-concave distributions.

2 Results

Recall that a Borel probability measure μ on $\mathbb{R}^n$ is κ-concave, $\kappa \in [-\infty, \frac{1}{n}]$, if for every $\lambda \in [0, 1]$ and every Borel sets A, B in $\mathbb{R}^n$, we have

$$\mu(\lambda A + (1-\lambda)B) \geq \Big(\lambda\mu(A)^\kappa + (1-\lambda)\mu(B)^\kappa\Big)^{1/\kappa}$$

(for background on κ-concave measures see e.g. [9,10] or Sect. 2.1.1 in [11]). We say that a random vector is κ-concave if its law is κ-concave. For example, vectors uniform on convex bodies in $\mathbb{R}^n$ are $1/n$-concave by the Brunn-Minkowski inequality. The right hand side increases with κ, so as κ increases, the class of κ-concave measures becomes smaller. It is a natural extension of the class of log-concave random vectors, corresponding to $\kappa = 0$, with the right hand side in the defining inequality understood as the limit $\kappa \to 0+$. Many results for convex sets have analogues for concave measures (for instance, see [4–6,14,18]).

Consider $\kappa \in (0, 1/n)$. By Borell's theorem from [9], a κ-concave random vector is supported on a convex body, has a density and its density is a $1/\beta$-concave function, that is of the form h^β for a concave function h with $\beta = \kappa^{-1} - n$. The notion of κ-concavity was introduced and studied by Borell in [9,10], which are standard references on this topic.

We also recall that a random vector X in $\mathbb{R}^n$ is isotropic if it is centred, that is $\mathbb{E}X = 0$ and its covariance matrix $\mathrm{Cov}(X) = [\mathbb{E}X_iX_j]_{i,j\leq n}$ is the identity matrix. The isotropic constant L_X of a log-concave random vector X which is isotropic and has density f on $\mathbb{R}^n$ is defined as $L_X = (\operatorname{ess\,sup}_{\mathbb{R}^n} f)^{1/n}$ (see e.g. [11]). By Borell's theorem, every log-concave random vector in $\mathbb{R}^n$ is supported on an affine subspace of $\mathbb{R}^n$ and has a density with respect to Lebesgue measure on that subspace.

Our first main result suggests a necessary condition on N (in the form of a lower bound for N exponential in the dimension n) so that $\mathbb{E}|K_N|$ will be significant in the case of symmetric log-concave distributions supported in convex bodies. We recall that a measure μ on $\mathbb{R}^n$ is symmetric (sometimes also called even) if $\mu(A) = \mu(-A)$ for every μ-measurable set A in $\mathbb{R}^n$.

Theorem 1 *Let μ be a symmetric log-concave probability measure supported on a convex body K in $\mathbb{R}^n$. Let $X_1, X_2, \ldots$ be i.i.d. random vectors distributed according to μ. Let $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$. There are universal positive constants c_1, c_2 such that if $N \leq e^{c_1 n/L_\mu^2}$, then*

$$\frac{\mathbb{E}|K_N|}{|K|} \leq e^{-c_2 n/L_\mu^2},$$

where L_μ is the isotropic constant of μ.

Our second main result provides a sufficient condition on N so that $\mathbb{E}|K_N|$ will be significant in the case of κ-concave distributions.

Theorem 2 *Let μ be a symmetric κ-concave measure on $\mathbb{R}^n$ with $\kappa \in (0, \frac{1}{n})$, supported on a convex body K in $\mathbb{R}^n$. Let $X_1, X_2, \ldots$ be i.i.d. random vectors uniformly distributed* *according to μ. Let $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$. There is a universal constant C such that for every $\omega > C$, if $N \geq e^{\frac{1}{\kappa}(\log n + 2\log\omega)}$, then*

$$\frac{\mathbb{E}|K_N|}{|K|} \geq 1 - \frac{1}{\omega}.$$

3 Floating bodies

It turns out that the following quasi-concave function plays a crucial role in estimates for the expected volume of the convex hull of random points (see [2,3,12]): for a random vector X in $\mathbb{R}^n$ define

$$q_X(x) = \inf\{\mathbb{P}(X \in H),\ H \text{ half-space containing } x\}, \qquad x \in \mathbb{R}^n. \tag{6}$$

It is clear that $q(\lambda x + (1-\lambda)y) \geq \min\{q(x), q(y)\}$, because if a half-space H contains $\lambda x + (1-\lambda)y$, it also contains x or y. Consequently, superlevel sets

$$L_{q_X,\delta} = \{x \in \mathbb{R}^n,\ q_X(x) \geq \delta\} \tag{7}$$

of this function are convex. Another way of looking at these sets is by noting that they are intersections of half-spaces: $L_{q_X,\delta} = \bigcap\{H:\ H \text{ is a half-space},\ \mathbb{P}(X \in H) > 1-\delta\}$. When X is uniform on a convex set K, they are called convex floating bodies ($K \setminus L_{q_X,\delta}$ is called a wet part). The function q_X in statistics is called the Tukey or half-space depth of X. The two notions have been recently surveyed in [21].

A key lemma from [12] relates the volume of random convex hulls of i.i.d. samples of X to the volume of the level sets $L_{q_X,\delta}$. Bounds on the latter are obtained by a combination of elementary convexity arguments and deep results from asymptotic convex geometry (notably, Paouris' reversal of the L_p-affine isoperimetric inequality due to Lutwak, Yang and Zhang). We shall present these and all the necessary background material in Sect. 4. Section 5 is devoted to our proofs.

4 Auxiliary results

4.1 Log-concave and κ-concave measures

Theorem 4.3 from [10] provides in particular the following stability of κ-concavity with respect to taking marginals: if $\kappa \in (0, \frac{1}{n})$ and f is the density of a κ-concave random vector in $\mathbb{R}^n$, then

$$\text{the marginal } x \mapsto \int_{\mathbb{R}^{n-1}} f(x,y)\mathrm{d}y \text{ is a } \frac{\kappa}{1-\kappa}\text{-concave function.} \tag{8}$$

We will also need the following basic estimate: if $g\colon \mathbb{R} \to [0, +\infty)$ is the density of a log-concave random variable X with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$, then

$$\frac{1}{2\sqrt{3}e} \leq g(0) \leq \sqrt{2} \tag{9}$$

(see e.g. Chapter 10.6 in [1]).

4.2 Central lemma

The idea of using floating bodies to estimate volume of random polytopes goes back to [3]. The following is a key lemma from [12] (called by the authors "central") about asymptotically matching upper and lower bounds for the volume of the random convex hull.

Lemma 3 *([12]) Suppose $X_1, X_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^n$. Let $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ and define $q = q_{X_1}$ by (6). Then for every Borel subset A of $\mathbb{R}^n$, we have*

$$\mathbb{E}|K_N| \leq |A| + N \cdot \left(\sup_{A^c} q\right) \cdot |A^c \cap \{x \in \mathbb{R}^n,\ q(x) > 0\}| \tag{10}$$

and, if additionally μ assigns zero mass to every hyperplane in $\mathbb{R}^n$, then

$$\mathbb{E}|K_N| \geq |A| \left(1 - 2\binom{N}{n}\left(1 - \inf_A q\right)^{N-n}\right). \tag{11}$$

(The proof therein concerns only the cube, but their argument repeated verbatim justifies our general situation as well – see also [16].)

4.3 Bounds related to the function *q*

Lemma 3 is applied to level sets $L_{q,\delta}$ of the function q (see (7)). We gather here several remarks concerning bounds for the volume of such sets. For the upper bound, we will need the containment $L_{q,\delta} \subset cZ_\alpha(X)$, where c is a universal constant and Z_α is the centroid body (defined below). This was perhaps first observed in Theorem 2.2 in [28] (with a reverse inclusion as well). We recall an argument below.

Remark 4 Plainly, for the infimum in the definition (6) of $q_X(x)$, it is enough to take half-spaces for which x is on the boundary, that is

$$q_X(x) = \inf_{\theta \in \mathbb{R}^n} \mathbb{P}\left(\langle X - x, \theta\rangle \geq 0\right), \tag{12}$$

where $\langle u, v\rangle = \sum_i u_i v_i$ is the standard scalar product in $\mathbb{R}^n$. Of course, by homogeneity, this infimum can be taken only over unit vectors. We also remark that by Chebyshev's inequality,

$$\mathbb{P}\left(\langle X - x, \theta\rangle \geq 0\right) \leq e^{-\langle\theta, x\rangle}\mathbb{E}e^{\langle\theta, X\rangle}.$$

Consequently,

$$q_X(x) \leq \exp\left(-\sup_{\theta \in \mathbb{R}^n}\left\{\langle\theta, x\rangle - \log \mathbb{E}e^{\langle\theta, X\rangle}\right\}\right)$$

and we have arrived at the Legendre transform $\Lambda_X^\star$ of the log-moment generating function Λ_X of X,

$$\Lambda_X(x) = \log \mathbb{E}e^{\langle x, X\rangle} \qquad \text{and} \qquad \Lambda_X^\star(x) = \sup_{\theta \in \mathbb{R}^n}\left\{\langle\theta, x\rangle - \Lambda_X(\theta)\right\}.$$

Thus, for every $\alpha > 0$, we have

$$\{x \in \mathbb{R}^n,\ q_X(x) > e^{-\alpha}\} \subset \{x \in \mathbb{R}^n,\ \Lambda_X^\star(x) < \alpha\}. \tag{13}$$

Remark 5 The level sets $\{\Lambda_X^\star < \alpha\}$ have appeared in a different context of the so-called optimal concentration inequalities introduced by Latała and Wojtaszczyk in [19]. Modulo universal constants, they turn out to be equivalent to centroid bodies playing a major role in asymptotic convex geometry (see [20,23–26]). Specifically, for a random vector X in $\mathbb{R}^n$ and $\alpha \geq 1$, we define its L_α-centroid body $Z_\alpha(X)$ by

$$Z_\alpha(X) = \{x \in \mathbb{R}^n,\ \sup\{\langle x, \theta\rangle\,,\ \mathbb{E}|\,\langle X, \theta\rangle\,|^\alpha \leq 1\} \leq 1\}$$

(equivalently, the support function of $Z_\alpha(X)$ is $\theta \mapsto (\mathbb{E}|\,\langle X, \theta\rangle\,|^\alpha)^{1/\alpha}$). By Propositions 3.5 and 3.8 from [19], if X is a symmetric log-concave random vector X (in particular, uniform on a symmetric convex body),

$$\{\Lambda_X^* < \alpha\} \subset 4eZ_\alpha(X), \qquad \alpha \geq 2. \tag{14}$$

(A reverse inclusion $Z_\alpha(X) \subset 2^{1/\alpha}e\{\Lambda_X^* < \alpha\}$ holds for any symmetric random vector, see Proposition 3.2 therein.)

We shall need an upper bound for the volume of centroid bodies. This was done by Paouris (see [25]). Specifically, Theorem 5.1.17 from [11] says that if X is an isotropic log-concave random vector in $\mathbb{R}^n$, then

$$|Z_\alpha(X)|^{1/n} \leq C\sqrt{\frac{\alpha}{n}}, \qquad 2 \leq \alpha \leq n, \tag{15}$$

where C is a universal constant.

Remark 6 Significant amount of work in [12] was devoted to showing that, for the cube, inclusion (13) is nearly tight (for *correct* values of α, using exponential tilting of measures typically involved in establishing large deviation principles). We shall take a different route and put a direct lower bound on q_X described in the following lemma. Our argument is based on property (8).

Lemma 7 *Let $\kappa \in (0, \frac{1}{n})$. Let X be a symmetric isotropic κ-concave random vector supported on a convex body K in $\mathbb{R}^n$. Then for every unit vector θ in $\mathbb{R}^n$ and $a > 0$, we have*

$$\mathbb{P}(\langle X, \theta\rangle > a) \geq \frac{1}{16}\kappa\left(1 - \frac{a}{h_K(\theta)}\right)^{1/\kappa}, \tag{16}$$

where $h_K(\theta) = \sup_{y\in K}\langle y, \theta\rangle$ is the support function of K. In particular, denoting the norm given by K as $\|\cdot\|_K$, we have

$$q_X(x) \geq \frac{1}{16}\kappa\,(1 - \|x\|_K)^{1/\kappa}\,, \qquad x \in K. \tag{17}$$

Proof Consider the density g of $\langle X, \theta\rangle$. Let $b = h_K(\theta)$. Note that g is supported in $[-b, b]$. By (8), $g^{\frac{\kappa}{1-\kappa}}$ is concave, thus on $[0, b]$ we can lower-bound it by a linear function whose values agree at the end points,

$$g(t)^{\frac{\kappa}{1-\kappa}} \geq g(0)^{\frac{\kappa}{1-\kappa}}\left(1 - \frac{t}{b}\right), \qquad t \in [0, b].$$

This gives

$$\mathbb{P}(\langle X, \theta\rangle > a) = \int_a^b g(t)\mathrm{d}t \geq g(0)\int_a^b \left(1 - \frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}} \mathrm{d}t = \kappa g(0)b\left(1 - \frac{a}{b}\right)^{1/\kappa}.$$

Since $\langle X, \theta \rangle$ is in particular log-concave, by (9), we have $\frac{1}{2\sqrt{3e}} \leq g(0) \leq \sqrt{2}$. Moreover, by isotropicity,

$$1 = \mathbb{E}\langle X, \theta \rangle^2 = \int_{-b}^{b} t^2 g(t)\mathrm{d}t \leq b^2 \int_{-b}^{b} g(t)\mathrm{d}t = b^2.$$

Thus, say $g(0)b > \frac{1}{16}$ and we get (16). To see (17), first recall (12). By symmetry, $\mathbb{P}(\langle X - x, \theta \rangle \geq 0) = \mathbb{P}(\langle X, \theta \rangle \geq |\langle x, \theta \rangle|)$, so we use (16) with $a = |\langle \theta, x \rangle|$ and note that by the definition of h_K, $\left|\left\langle \frac{x}{\|x\|_K}, \theta \right\rangle\right| \leq h_K(\theta)$, so $\frac{|\langle x, \theta \rangle|}{h_K(\theta)} \leq \|x\|_K$. □

5 Proofs

5.1 Proof of Theorem 1

Since the quantity $\frac{\mathbb{E}|K_N|}{|K|}$ does not change under invertible linear transformations applied to μ, without loss of generality we can assume that μ is isotropic. Let $q = q_{X_1}$ be defined by (6). Fix $\alpha > 0$ and apply (10) to the set $A = \{x,\ q(x) > e^{-\alpha}\}$. We get

$$\frac{\mathbb{E}|K_N|}{|K|} \leq \frac{|A|}{|K|} + Ne^{-\alpha}$$

(we have used $\{x,\ q(x) > 0\} \subset K$ to estimate the last factor in (10) by 1). Combining (13), (14) and (15),

$$|A| \leq |4eZ_\alpha(X)| \leq \left(4eC\sqrt{\frac{\alpha}{n}}\right)^n.$$

Moreover, by the definition of the isotropic constant of μ,

$$1 = \int_K \mathrm{d}\mu \leq L_\mu^n |K|.$$

Thus,

$$\frac{|A|}{|K|} \leq \left(4eCL_\mu\sqrt{\frac{\alpha}{n}}\right)^n.$$

We set α such that $4eCL_\mu\sqrt{\frac{\alpha}{n}} = e^{-1}$ and adjust the constants to finish the proof. □

5.2 Proof of Theorem 2

As in the proof of Theorem 1, we can assume that μ is isotropic. Let $q = q_{X_1}$ be defined by (6). Consider $0 < \beta < 1$ (to be fixed shortly). By (11) which we apply to the set $A = \{x \in K,\ q(x) > \beta^{1/\kappa}\}$, we have

$$\frac{\mathbb{E}|K_N|}{|K|} \geq \frac{|A|}{|K|}\left(1 - 2\binom{N}{n}\left(1 - \beta^{1/\kappa}\right)^{N-n}\right).$$

(The extra assumption needed in (11) is satisfied: by Borell's theorem from [9], μ has a density on its support which by our assumption is n-dimensional, hence $\mu(H) = 0$ for every hyperplane H in $\mathbb{R}^n$.) By the lower bound on q from (17),

$$A \supset \{x \in \mathbb{R}^n,\ \|x\|_K \leq 1 - (16\kappa^{-1})^\kappa \beta\},$$

hence, as long as $(16\kappa^{-1})^\kappa \beta < 1$,

$$\frac{|A|}{|K|} \geq \left(1 - (16\kappa^{-1})^\kappa \beta\right)^n \geq 1 - n(16\kappa^{-1})^\kappa \beta \geq 1 - 32n\beta.$$

We choose β such that $32n\beta = \frac{1}{2\omega}$ and crudely deal with the second term,

$$\binom{N}{n}\left(1 - \beta^{1/\kappa}\right)^{N-n} \leq N^n e^{-\beta^{1/\kappa}(N-n)},$$

which is nonincreasing in N as long as $N \geq n\beta^{-1/\kappa}$. This holds for ω large enough if, say $N \geq n^{1/\kappa}\omega^{2/\kappa}$. Then we easily conclude that the dominant term above is $e^{-\beta^{1/\kappa}N}$ which yields, say

$$\frac{\mathbb{E}|K_N|}{|K|} \geq \left(1 - \frac{1}{2\omega}\right)(1 - 2e^{-\omega n/2}) \geq 1 - \frac{1}{\omega},$$

provided that n and ω are large enough. □

6 Final remarks

Remark 8 Groemer's result used in (5) for uniform distributions has been substantially generalised by Paouris and Pivovarov in [27] to arbitrary distributions with bounded densities. We remark that in contrast to (5), using the extremality result of the ball from [27] does not seem to help obtain bounds from Theorem 2 for two reasons. For one, it concerns bounded densities and rescaling will cost an exponential factor. Moreover, for the example of β-polytopes from [7], we have that they are generated by κ-concave measures with $\kappa = \frac{1}{\beta+n}$ and the sharp threshold for the volume is of the order $n^{(\beta+n/2)}$ (see (3)). The ball would give that $N_1 = n^{(1+\varepsilon)n/2}$ points is enough.

Remark 9 The example of beta polytopes from (3) shows that the bound on N in Theorem 2 has to be at least of the order $n^{\beta+n/2} = n^{\frac{1}{\kappa}-n/2} \geq n^{\frac{1}{2\kappa}}$. Our bound $n^{\frac{1}{\kappa}}$ is perhaps suboptimal. It is not inconceivable that as in the uniform case, the extremal example is supported on a Euclidean ball.

Remark 10 It is reasonable to ask about sharp thresholds like the ones in (2), (3) and (4) for other sequences of convex bodies, say simplices, cross-polytopes, or in general ℓ_p-balls. This is a subject of ongoing work. We refer to [15] for recent results establishing exponential nonsharp thresholds for a simplex (i.e. with a gap between the constants for lower and upper bounds).

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