Globally Coupled Finite Geometry and Finite Field LDPC Coding Schemes

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***Abstract*—This paper presents two types of concatenated LDPC coding schemes which are viewed as *generalized globally coupled* (*GC*) *LDPC* coding schemes in which outer codes serve as the *local codes* for correcting local errors and inner codes serve as *global coupling codes* to correct global errors. The first type of concatenated LDPC coding scheme globally couples a finite geometry (FG) LDPC code as the local code and a finite field (FF) LDPC code as the global coupling code. This type of global coupling, called *GC-FG/FF-LDPC coupling*, combines the distinct features of both FG- and FF-LDPC codes to achieve low error rates at a rapid decoding convergence and an error performance close to the Shannon limit. Decoding of a GC-FG/FF-LDPC code is carried out in *two iterative phases*, global/local or local/global. In the second type of concatenated LDPC coding scheme, both local and global coupling codes are FF-LDPC codes. If both local and global coupling codes are constructed from the same finite field and have the same graphical structures, a GC-FF/FF-LDPC code can be decoded in one phase or two phases iteratively, otherwise, it can be decoded in two phases. Construction of GC-FF/FF-LDPC codes is very flexible in lengths and rates. The proposed two-phase iterative decoding is practically implementable.**

***Index Terms*—Concatenated coding, finite geometry LDPC code, finite field LDPC code, global coupling, iterative decoding.**

I. Introduction

LDPC codes [1] perform amazingly well with iterative decoding algorithms based on belief propagation, such as the *sum-product algorithm* (SPA) [2] or the *min-sum algorithm* (MSA) [3]. However, with iterative decoding, most LDPC codes have a common severe weakness, known as the *error-floor* [4]. For an additive white Gaussian noise (AWGN) channel, the error-floor of an LDPC code is mostly caused by an undesirable structure, known as a *trapping set* [5] in the Tanner graph of the code based on which the decoding is carried out. Let G be the Tanner graph of a binary LDPC code C given by the null space of an $m \times n$ matrix $\mathbf{H}$ over GF(2). For $1 \leq \kappa \leq n$ and $0 \leq \tau \leq m$, a (κ, τ) trapping set is a set $T(\kappa, \tau)$ of κ variable nodes (VNs) in G which induces a subgraph of G with exactly τ *odd-degree* check nodes (CNs), and an arbitrary number of *even-degree* CNs. The parameter κ is called the *size* of the trapping set $T(\kappa, \tau)$. A trapping set simply corresponds to an error pattern with κ errors which prevents an iterative LDPC-decoder to converge. For an AWGN channel, error patterns with small numbers of errors are more probable than error patterns with larger numbers of errors. Consequently, with iterative decoding, the most *harmful* trapping sets are the trapping sets of *small sizes* which generally result in *high* error-floors. If an LDPC code has a reasonably large minimum distance and its Tanner graph contains no harmful trapping sets with sizes smaller than its minimum distance, the code can achieve a very low error rate without a visible error-floor.

Among all the known classes of LDPC codes, the only known class of LDPC codes with large minimum distances whose Tanner graphs contain no trapping sets with sizes smaller than their minimum distances are LDPC codes constructed based on *finite geometries* such as *projective* and *Euclidean* [6-10]. These codes are referred to as finite-geometry (*FG*) LDPC codes and they can achieve very low error rates without error-floors. Iterative decoding of these codes converges *rapidly* with a small number of iterations, say 5 to 10. Furthermore, these codes are cyclic codes and hence their encoding can be implemented with simple feedback registers in the *systematic form* [7].

Besides the FG-LDPC codes, there are many other classes of *structured* LDPC codes constructed based on finite fields. These codes, called *finite field* (*FF*) LDPC codes, have a *quasi-cyclic* (QC) structure [8-12]. The QC-structure of these codes simplifies both encoding and decoding implementations. The construction of FF-LDPC codes is *very flexible* in lengths and rates. They, *in general*, perform very well over an AWGN

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channel in the waterfall region and can achieve *relatively* low error-floors. However, their minimum distances are, in general, much smaller than that of FG-LDPC codes of the same lengths and rates, and their Tanner graphs do contain small trapping sets if the column weights of their parity-check matrices are too small.

This paper presents two types of concatenated LDPC coding schemes which are viewed as *globally coupled* (GC) LDPC coding schemes in which outer codes serve as the local codes for correcting *local errors* and inner codes serve as global coupling codes to correct *global errors*.

The first type of concatenated LDPC coding scheme globally couples an FG-LDPC code as the *local code* and an FF-LDPC code as the *global coupling code*. This type of concatenation is referred to as globally coupled (GC) FG/FF-LDPC coding scheme. The GC-FG/FF-LDPC coding scheme is devised to combine the distinct features of both FG- and FF-LDPC codes to achieve low error rates with a rapid decoding convergence and an error performance close to the Shannon limit. Decoding of a GC-FG/FF-LDPC code is carried out in two iterative phases, *global/local* or *local/global*. With the two-phase iterative decoding, trapping sets that trap a local decoder may be un-trapped by the global decoder, and conversely, the trapped global decoder can be un-trapped by the local decoder(s). As a result, a GC-FG/FF-LDPC code can achieve a very low error rate without a visible error-floor. Furthermore, during the global iterative decoding process, the reliability information of each decoded local codeword is shared by the others to enhance the overall reliability of all decoded local codewords. This *information sharing* among the decoded local codewords enhances the overall decoding performance of a GC-FG/FF LDPC code. Even though GC-FG/FF-LDPC codes have large minimum distances and good trapping set structure, their construction is limited in lengths and rates.

In the second type of concatenated LDPC coding scheme, both local and global coupling codes are FF-LDPC codes which are constructed from finite fields. Such a concatenated LDPC code is referred to as a GC-FF/FF-LDPC code. If both local and global coupling codes of a GC-FF/FF-LDPC code are constructed from the *same finite field and have the same graphical structures*, it can be decoded either in one phase or two phases iteratively, otherwise, it can be decoded in two phases. Construction of GC-FF/FF-LDPC codes is very flexible in lengths and rates. Two-phase iterative decoding of a GC-FG/FF-LDPC code or a GC-FF/FF-LDPC code can be practically implemented. Examples of concatenated and globally coupled codes were reported in the literature [13-15].

The rest of the paper is organized as follows. Sections II and III present two types of GC-FG/FF-LDPC codes and characterize their random error- and erasure-correction features. Iterative methods for decoding these GC-FG/FF-LDPC codes in two phases are presented. Section IV presents a class of GC-FF/FF-LDPC codes in which both local and global coupling codes are constructed from the finite fields. Section V concludes this paper with some remarks on possible applications of GC-FG/FF-LDPC codes.

II. A GC-FG/FF-LDPC CODING SCHEME

In this section, we present a GC-FG/FF-LDPC coding scheme that combines FG-LDPC codes as local codes and FF-LDPC codes as global coupling codes to form a class of structured LDPC codes that possess the distinct features of both the FG- and the FF-LDPC codes. The codes in this class are called GC-FG/FF-LDPC codes. Here, we give a brief description of the FG- and FF-LDPC codes and characterize their distinct features.

An FG-LDPC code has a large minimum distance and its Tanner graph contains no small harmful trapping sets and has large connectivity, it can achieve a very low error rate without a visible error-floor. Consider a 2-D PG over the field GF(2^s), denoted by PG(2, 2^s), where s is a positive integer. The null space over GF(2) of $\mathbf{H}_{PG}$ gives 2-D PG-LDPC code, denoted by C_{PG}, with a minimum distance of *at least* $2^s + 2$ which is one greater than the column weight $2^s + 1$ of $\mathbf{H}_{PG}$ [6-9]. The Tanner graph G_{PG} of C_{PG} has a girth of at least 6, and contains no harmful trapping sets with sizes smaller than $2^s + 1$ [9] and [10].

To construct QC-FF-LDPC code based on finite fields with a girth of at least 6, consider a finite field GF(q) with q elements where q is a prime or a power of a prime. With $1 \le m, n \le q$ and primitive element α, let $\boldsymbol{S}_0 = \{\alpha^{i_0}, \alpha^{i_1}, \ldots, \alpha^{i_{m-1}}\}$ and $\boldsymbol{S}_1 = \{\alpha^{j_0}, \alpha^{j_1}, \ldots, \alpha^{j_{n-1}}\}$ be two *arbitrary subsets* of elements in GF(q) of size m and n, respectively. We form the following $m \times n$ matrix over GF(q) where η is a nonzero element in GF(q):

$$\mathbf{B}(m,n) = [\alpha^{i_k} + \eta\alpha^{j_l}]_{0 \le k < m, 0 \le l < n} \qquad (1)$$

In the matrix $\mathbf{B}(m, n)$, all the entries in a row (or a column) are distinct elements in GF(q); each row (or each column) contains at most one zero elements; no two rows (or two columns) have identical entries in any position; and each 2×2 submatrix of $\mathbf{B}(m, n)$ is *non-singular* (NS) [11], which is referred to as the 2×2 *submatrix (SM) constraint* [8, 11, 17, 18] and is the *key structure* for constructing QC-FF-LDPC code whose Tanner graph has a girth of at least 6.

For $0 \le i < q - 1$, we represent the nonzero field element α^i in GF(q) by a *circulant permutation matrix* (CPM) of size $(q - 1) \times (q - 1)$ whose *generator* (or the top row) has a single 1-component at the position i. For the zero element $0 = \alpha^{-\infty}$, we represent it by a zero matrix (ZM) of size $(q - 1) \times (q - 1)$. This matrix representation of a field element is called the *CPM-dispersion* [8-12], [18].

The construction of a 2 × 2 SM-constrained base matrix using two subsets $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$ can be put in a *product form* as follows:

$$\mathbf{B}^*(m,n) = [\alpha^{i_k}\alpha^{i_l} - \eta]_{0 \le k < m, 0 \le l < n} \qquad (2)$$

If we set $m = n = q - 1$ and choose $\boldsymbol{S}_0 = \boldsymbol{S}_1 = \{1, \alpha, \alpha^2, \ldots, \alpha^{q-2}\}$, then $\mathbf{B}^*(q - 1, q - 1)$ can be arranged (by row permutation) as a $(q - 1) \times (q - 1)$ *circulant* $\mathbf{B}_{cyc}^*(q - 1, q - 1)$ over GF(q), in which each row is the cyclic-shift of the row above it one place to the right and the top row is the cyclic-shift of the last row one place to the right. Any submatrix of $\mathbf{B}_{cyc}^*(q - 1, q - 1)$

can be used as a base matrix to construct a QC-LDPC code using CPM-dispersion. The resultant code may have a *doubly quasi-cyclic structure* [9].

A. Encoding and Code Construction

Let C_{FG} be an (n_0, k_0) binary FG-LDPC code of length n_0 with dimension k_0 given by the null space of an $n_0 \times n_0$ circulant $\mathbf{H}_{FG}$ over GF(2) which is the line-point incidence matrix of the 2-D finite geometry FG(2, 2^s) over GF(2^s). Let d_0 be the minimum distance of C_{FG}. The Tanner graph G_{FG} of C_{FG} contains no harmful trapping set with a size smaller than d_0 - 1.

Let c be a positive integer and C_{FF} be a $(cn_0 + r, cn_0)$ binary FF-QC-LDPC code of length $cn_0 + r$ and dimension cn_0, where r is the number of parity-check symbols of C_{FF}. The code C_{FF} is given by the null space of an RC-constrained $\lambda \times (cn_0 + r)$ parity-check matrix $\mathbf{H}_{FF}$ constructed by the CPM-dispersion of a base matrix $\mathbf{B}$ over a finite field GF(2^s) that satisfies the 2 × 2 SM-constraint. Note that λ may be greater than r if $\mathbf{H}_{FF}$ contains redundant rows. In a concatenation of these two codes, both codes are put in systematic form.

Let $\mathbf{u}$ be a sequence of ck_0 binary information symbols. Encoding of this information sequence consists of two stages, local and global coupling encodings. First, we divide $\mathbf{u}$ into c subsequences, denoted by $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{c-1}$, each consisting of k_0 information symbols and called a *message*. For $0 \leq i < c$, the message $\mathbf{u}_i$ is encoded into a codeword $\mathbf{v}_i$ of n_0 code symbols in the local FG-LDPC code C_{FG}. Encoding results in c codewords, $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{c-1}$, in C_{FG}, called local codewords. Cascading these c local codewords, we obtain a sequence $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{c-1})$ of cn_0 code symbols, called a cascaded codeword. This completes the first stage of encoding, referred to as the *local encoding*. There are 2^{ck_0} such cascaded codewords which form a (cn_0, ck_0) linear code of length cn_0, denoted by $C_{FG,casc}(c)$. The parity-check matrix of $C_{FG,casc}(c)$ is a $c \times c$ *diagonal array*, denoted by diag($\mathbf{H}_{FG}, \mathbf{H}_{FG}, \ldots, \mathbf{H}_{FG}$), with c copies of $\mathbf{H}_{FG}$ lying on its main diagonal and zeros elsewhere. The code $C_{FG,casc}(c)$ is called a *cascaded code* of C_{FG} and the integer c is called the cascading degree. The minimum distance and rate of $C_{FG,casc}(c)$ are the same as C_{FG}.

In the second stage of encoding, a cascaded codeword $\mathbf{v}$ in $C_{FG,casc}(c)$ is encoded into a codeword $\mathbf{w} = (\mathbf{p}, \mathbf{v})$ of $cn_0 + r$ code symbols in the global coupling FF-LDPC code C_{FF}. The codeword $\mathbf{w}$ consists of two parts $\mathbf{v}$ and $\mathbf{p}$. The first part $\mathbf{v}$ is a cascaded codeword in $C_{FG,casc}(c)$ and the second part $\mathbf{p}$ consists of r parity-check symbols, which are formed based on the parity-check matrix $\mathbf{H}_{FF}$ of C_{FF}. These r parity symbols connect the c local codewords $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{c-1}$ of C_{FG} in $\mathbf{v}$.

The encoding performed at the second stage is referred to as *global coupling encoding*. Local encoding, cascading, and global coupling encoding result in 2^{ck_0} codewords in C_{FF} in the form of $(\mathbf{p}, \mathbf{v})$. These codewords form a $(cn_0 + r, ck_0)$ linear code which is referred to as a GC-FG/FF-LDPC code, denoted by $C_{GC,FG,FF}$. We see that the two-stage encoding of a GC-FG/FF-LDPC code is straightforward and can be easily implemented.

Based on the formation of $C_{GC,FG,FF}$, we readily see that a parity-check matrix of $C_{GC,FG,FF}$ is of the following *global* form:

$$\mathbf{H}_{GC,FG,FF} = \left[\begin{array}{c|c} \mathbf{O} & diag(\mathbf{H}_{FG}, \mathbf{H}_{FG}, \ldots, \mathbf{H}_{FG}) \\ \hline \mathbf{H}_{FF,left} & \mathbf{H}_{FF,right} \end{array}\right] \quad (3)$$

The parity-check matrix $\mathbf{H}_{GC,FG,FF}$ of $C_{GC,FG,FF}$ consists of two submatrices, the upper one and the lower one. The upper submatrix of $\mathbf{H}_{GC,FG,FF}$ consists of two parts. The first part is a zero-matrix $\mathbf{O}$ of size $cn_0 \times r$ and the second part is the parity-check matrix of the cascaded FG-LDPC code $C_{FG,casc}(c)$. The lower submatrix of $\mathbf{H}_{GC,FG,FF}$ is the parity-check matrix $\mathbf{H}_{FF}$ of the global coupling FF-LDPC code C_{FF} which consists of two parts, denoted by $\mathbf{H}_{FF,left}$ and $\mathbf{H}_{FF,right}$, called left and right parts of $\mathbf{H}_{FF}$, respectively. The left part $\mathbf{H}_{FF,left}$ of $\mathbf{H}_{FF}$ consists of the first (leftmost) r columns of $\mathbf{H}_{FF}$ and the right part $\mathbf{H}_{FF,right}$ of $\mathbf{H}_{FF}$ consists of the rightmost cn_0 columns of $\mathbf{H}_{FF}$. Note that the *global* matrix $\mathbf{H}_{GC,FG,FF}$ consists of c disjoint copies of the parity-check matrix $\mathbf{H}_{FG}$ of the local code C_{FG}, which are *globally connected* by the parity-check matrix $\mathbf{H}_{FF}$ of the global coupling FF-LDPC code C_{FF}. It can be easily checked that a codeword $\mathbf{w} = (\mathbf{p}, \mathbf{v})$ in $C_{GC,FG,FF}$ is in the null space of $\mathbf{H}_{GC,FG,FF}$, i.e., $\mathbf{w} \cdot (\mathbf{H}_{GC,FG,FF})^{\mathrm{T}} = \mathbf{0}$.

B. Graphical Structure

Let G_{FG}, G_{FF}, and $G_{GC,FG,FF}$ be the Tanner graphs of C_{FG}, C_{FF}, and $C_{GC,FG,FF}$, respectively. From (3) and the diagonal structure of diag($\mathbf{H}_{FG}, \mathbf{H}_{FG}, \ldots, \mathbf{H}_{FG}$), we readily see that the Tanner graph $G_{GC,FG,FF}$ of the GC-FG/FF-LDPC code $C_{GC,FG,FF}$, consists of c disjoint copies of the Tanner graph G_{FG} of the local FG-LDPC code C_{FG}, which are globally connected by a group of *global CNs* that correspond to rows of the parity-check matrix $\mathbf{H}_{FF}$ of the global coupling FF-LDPC code C_{FF}. Hence, $G_{GC,FG,FF}$ is a CN-based globally coupled graph with c identical and disjoint local graphs G_{FG}, which are connected by the λ CNs of the global coupling graph G_{FF} as shown in Fig. 1.

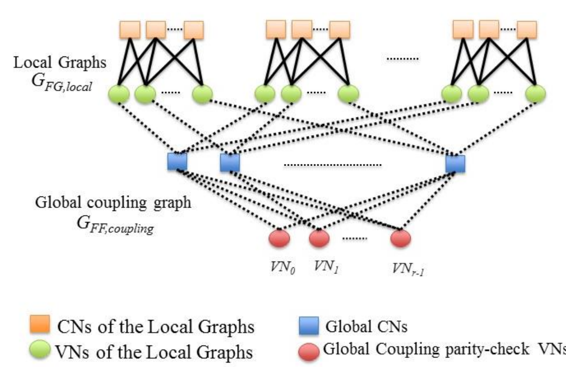

Fig. 1. The Global structure of the Tanner graph of a GC-FG/FF-LDPC code.

Since each local graph G_{FG} in $G_{GC,FG,FF}$ is the Tanner graph of the local FG-LDPC code with minimum distance d_0, it contains no harmful trapping sets of size smaller than $d_0 - 1$. For $\kappa < d_0 - 1$, if the global coupling graph G_{FF} contains a harmful trapping set T of size κ, then the κ VNs in T either reside in one local graph or distribute among the c local graphs. Note some of the VNs in T may reside in the subgraph associated with the left part $\mathbf{H}_{FF,left}$ of $\mathbf{H}_{FF}$. For $0 \le i < c$, let κ_i be the number of VNs in T that reside in the i-th local graph $G_{FG,i}$. Regardless of the distribution of the κ VNs in T, κ_i is less than $d_0 - 1$. Hence, the κ_i VNs will not create harmful trapping set in the i-th local graph $G_{FG,i}$.

C. Two-Phase Iterative Decoding

Based on the structure of the GC-FG/FF-LDPC code $C_{GC,FG,FF}$, two iterative decoding methods can be devised to decode the code. Let $\mathbf{r}$ be the received vector of $cn_0 + r$ symbols. We decode $\mathbf{r}$ in two phases. The two-phase decoding can be carried out in two different manners iteratively: 1) global/local iterative decoding, and 2) local/global iterative decoding.

With the global/local iterative decoding, we first decode $\mathbf{r}$ based on the parity-check matrix $\mathbf{H}_{FF}$ of the global coupling FF-LDPC code C_{FF} using a chosen iterative decoding algorithm. Set the maximum number of global coupling decoding iterations to $I_{gc,max}$. Decode $\mathbf{r}$ based on the chosen iterative decoding algorithm. At the end of each global coupling decoding iteration, we check the syndrome $\mathbf{S}_{FF}$ of the decoded vector $\mathbf{x}$ based on $\mathbf{H}_{FF}$. If $\mathbf{S}_{FF} = \mathbf{0}$, then $\mathbf{x}$ is a codeword in C_{FF}. If $\mathbf{S}_{FF} \neq \mathbf{0}$, we continue the global coupling decoding until either the decoded vector $\mathbf{x}$ is a codeword in C_{FF} or the preset maximum number $I_{gc,max}$ of decoding iterations is reached.

If a codeword $\mathbf{w}$ in C_{FF} is obtained during the global coupling decoding phase, we remove all the r parity-check symbols from $\mathbf{w}$. This gives c decoded vectors $\mathbf{v}^*_0, \mathbf{v}^*_1, \ldots, \mathbf{v}^*_{c-1}$ for the c transmitted local codewords in C_{FG}. For each decoded vector $\mathbf{v}^*_i$, $0 \le i < c$, we compute its syndrome $\mathbf{S}_{FG,i}$ based on the parity-check matrix $\mathbf{H}_{FG}$ of the local FG-LDPC code C_{FG}. If the syndromes of the c decoded vectors are all zeros, then $\mathbf{v}^*_0, \mathbf{v}^*_1, \ldots, \mathbf{v}^*_{c-1}$ are codewords in C_{FG}. In this case, we stop the entire decoding process. Then, we remove all the parity-check symbols from the c decoded local codewords and deliver the ck_0 decoded information symbols to the user (or users in multi-user communications). If the syndrome of any of the c decoded vectors is not zero, the local code decoder is activated to perform decoding on the decoded vectors whose syndromes are not zero. These vectors are referred to as *failed local vectors*. The decoding algorithm used for decoding the local code can be the same or different from the one used for decoding the global coupling FF-LDPC code C_{FF}.

For decoding of each failed local vector at the output of the global coupling decoder, we set the maximum number of local decoding iterations to $I_{local,max}$. If the decoding of all the failed local vectors is successful, we remove all the parity-check symbols from all the decoded local codewords and deliver the ck_0 decoded information symbols to the user(s). If the decoding of any failed local vector is unsuccessful after $I_{local,max}$ iterations, we switch back to the global coupling code decoding with the decoded information and the channel information as input to decode the received vector $\mathbf{r}$ again.

We perform the global/local decoding process iteratively until either the entire decoding is successful, or a preset maximum number I_{max} of global/local decoding iterations is reached. With the global/local decoding of the GC-FG/FF-LDPC code, the local decoder is used to correct the *local errors* that the global coupling decoder fails to correct.

With the above two-phase decoding, no trapping set of size smaller than $d_0 - 1$, regardless of the distribution of its VNs, will trap the two-phase decoder of the GC-FG-FF LDPC code $C_{GC,FG,FF}$. If a trapping set of size $d_0 - 2$ or smaller traps the global coupling decoder, it will be un-trapped by the local decoder.

To achieve a good error performance, both inner and outer decoding algorithms are either the SPA or a properly scaled MSA. Decoding with the SPA requires real number multiplications, additions, and comparisons. However, the MSA requires mainly real additions and comparisons. Since an optimized scaled MSA performs just as well as the SPA and requires much less computational complexity, a scaled MSA is normally preferred for decoding an LDPC code. To reduce the overall decoding complexity, we may use the reduced-complexity *revolving iterative decoding* scheme (RID/MSA) [16] for decoding the local FG-LDPC. The *RID* scheme reduces the decoder complexity without (or with an ignorable) performance degradation. The decoding is based on *a small submatrix* of $\mathbf{H}_{FG}$ that consists of any *l consecutive rows* of $\mathbf{H}_{FG}$, even $l = 1$.

Note that if the global coupling decoding is successful in an iteration loop, we can stop the entire decoding process without checking the syndromes of the decoded local codewords to determine whether to carry out the outer decoding. This reduces computational complexity and decoding delay.

Opposite to global/local decoding presented above, we can decode the GC-FG/FF-LDPC code $C_{GC,FG,FF,}$ in a reverse order, i.e., perform local decoding before global coupling decoding in each global decoding loop. Recall that each globally coupled codeword in systematic form consists of c local codewords in cascade followed by r parity-check symbols of the global coupling FF-LDPC code C_{FF}. Local decoding is carried out as soon as a local vector of n_0 symbols corresponding to a transmitted local codeword is received. The decoding of each received local vector is based on the local parity-check matrix $\mathbf{H}_{FG}$ using a chosen iterative decoding algorithm, say the RID/MSA. If errors spread out among the c received local vectors and errors in each received local vector do not create harmful trapping set with a size greater than $d_0 - 2$ (very rare), each local decoding will be successful.

If the local decoding fails to give c correctly decoded local codewords, the global coupling iterative decoding is then activated. The rest of the decoding process is the same as the global/local iterative decoding. The local/global iterative decoding is effective for correcting local random errors.

Since every codeword in $C_{GC,GF,FF}$ consists of c outer codewords in C_{FG}, the global coupling decoding is actually *performed on a collection of c received local codewords jointly. During the decoding process, the reliability information of each decoded local codeword is shared by the others to enhance the overall reliability of all decoded local codewords. This joint-decoding and information sharing at the global coupling decoding phase reduce the probability of performing the local decoding phase and hence make the decoding converge more quickly. The joint-decoding and information-sharing is the key-feature of two-phase decoding.*

The two-phase iterative decoding can be practically implemented with local FG-LDPC code C_{FG} implemented based on the submatrix $\mathbf{H}_{FG}$ of $\mathbf{H}_{GC,FG,FF}$ and the global coupling code C_{FF} implemented based on the submatrix $\mathbf{H}_{FF}$ of $\mathbf{H}_{GC,FG,FF}$.

Example 1: In this example, we construct a GC-PG/FF-LDPC code with the (1057, 813) PG-LDPC code C_{PG} as the local code and the (10000, 9600) FF-LDPC code C_{FF} as the global coupling code.

Consider PG(2, 2^5) over GF(2^5) which contains 1057 points and 1057 lines. The line-point incidence matrix of PG(2, 2^5) is a 1057 × 1057 circulant matrix $\mathbf{H}_{PG}$ over GF(2) of rank 244 with both column and row weights 33 which has 813 redundant rows. The null space over GF(2) of $\mathbf{H}_{PG}$ gives a (33, 33)-regular (1057, 813) cyclic PG-LDPC code C_{PG} of length 1057 with a rate of 0.7691 and minimum distance at least 34, a very large minimum distance for an LDPC code of such length and rate. The Tanner graph G_{PG} of C_{PG} has girth 6 and contains no harmful trapping sets of sizes smaller than 33. Each VN in G_{PG} is connected to other 1056 VNs in G_{PG} by paths of length 2.

To construct FF-LDPC code, assume α is a primitive element in the prime field GF(101). We form a 4 × 100 base matrix $\mathbf{B}$(4, 100) over GF(101) in the form of (1) [8, 11, 17, 18] using two subsets $\boldsymbol{S}_0 = \{0, 1, \alpha, \alpha^2\}$ and $\boldsymbol{S}_1 = \{1, \alpha, \alpha^2, \ldots, \alpha^{99}\}$. The CPM-dispersion of $\mathbf{B}$(4, 100) gives a 4 × 100 array $\mathbf{H}_{FF}$(4, 100) of CPMs of size 100 × 100. The array $\mathbf{H}_{FF}$(4, 100) is a 400 × 10000 matrix over GF(2) with column and row weights of maximum 4 and 100, respectively, which satisfies the RC-constraint. The null space of $\mathbf{H}_{FF}$(4, 100) over GF(2) gives a binary (4, 100)-regular (10000, 9600) QC-FF-LDPC code C_{FF}(4, 100) with rate 0.96.

To construct GC-PG/FF-LDPC set the cascading degree c = 9. In construction, an information sequence of 7317 symbols is divided into 9 messages, each consisting of 813 information symbols. We first encode these 9 messages into 9 codewords in C_{PG}. Next, these 9 local codewords are cascaded to form a cascaded codeword $\mathbf{v}$ of 9513 code symbols. Then, the cascaded codeword $\mathbf{v}$ is extended by adding 87 *fill-in zeros* to form a word $\mathbf{v}_{ext}$ of 9600 symbols.

At the second encoding stage, the extended cascaded codeword $\mathbf{v}_{ext}$ is encoded into a codeword $\mathbf{w}$ in the global coupling code C_{FF}. The codeword $\mathbf{w}$ consists of 7317 information symbols, 2196 local parity-check symbols, 400 global coupling parity-check symbols, and 87 fill-in zeros. The above local/global coupling encoding results in a (10000, 7317) GC-PG/FF-LDPC code $C_{GC,PG,FF}$ with a rate of 0.7317.

To decode $C_{GC,PG,FF}$, we use global/local two-phase iterative decoding. The BER performance of the code over an AWGN channel decoded using the MSA for the global coupling code and the local code with $I_{gc,max}$, $I_{local,max}$ and I_{max} set to 5, 5, and 5, respectively, is shown in Fig. 2. The scaling factors for decoding the global coupling and local codes are 0.625 and 0.275. When decoding is switched from a global coupling decoding phase to a local decoding phase, the 87 fill-in zeros are removed, conversely, when decoding is switched from a local decoding phase to a global coupling decoding phase, the 87 fill-in zeros are added back.

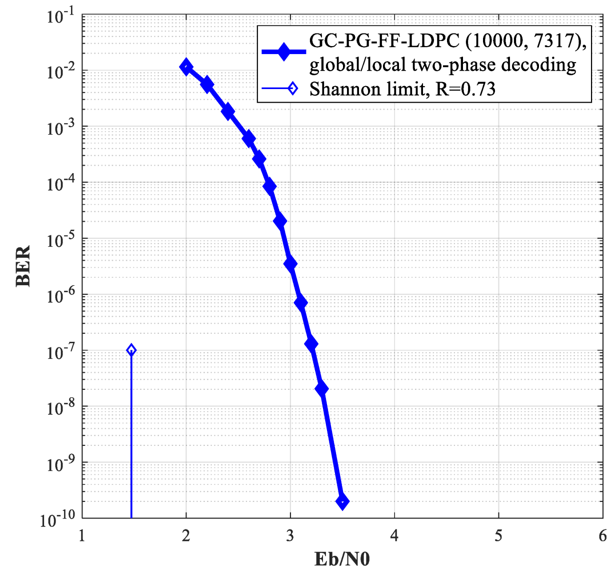

Fig. 2. The BER performance of the GC-PG/FF-LDPC code given in Example 1.

D. *Punctured GC-FG/FF-LDPC Codes*

If we remove the submatrices $\mathbf{O}$ and $\mathbf{H}_{F,left}$ from the global parity-check matrix $\mathbf{H}_{GC,FG,FF}$ given by (3), we obtain a global matrix $\mathbf{H}^*_{GC,FG,FF}$. The null space of $\mathbf{H}^*_{GC,FG,FF}$ gives a *punctuated* GC-FG/FF-LDPC code. Punctuated codes can also be constructed by removing some local parity-check matrices. By puncturing a GC-FG/FF-LDPC code, we obtain globally coupled LDPC codes of various lengths and rates.

E. *GC-FG/FG-LDPC Codes*

The main objective of this section is to combine an FG-LDPC code and an FF-LDPC code to form a GC-FG/FF-LDPC code that possesses the distinct features of both FG- and FF-LDPC codes, large minimum distances, no small trapping sets, fast decoding convergence, and flexibility.

A GC-LDPC code can be constructed using FG-LDPC codes as both local and global coupling codes. However, with this construction, we are limited in choice of length and rate of the global coupling code. For the desired length n_0 of a local FG-LDPC code, a desired cascading (or interleaving) degree c, a desired global coupling degree r, and a desired global code rate, there may not exist for a global coupling FG-LDPC code of length $cn_0 + r$. Furthermore, if the length n_0 of the local FG-LDPC code is not short and the cascading degree c is not small, the length $cn_0 + r$ of the global coupling FG-code may be very long. Iterative decoding of a long FG-LDPC based on a large circulant will be very complex in both hardware and computations even with the complexity-reduced RID/MSA. Construction of global coupling EG-LDPC codes can be made more flexible if we use parallel bundles of lines in the sub-geometry EG*(2, 2^s) [7, 8]. This will not be discussed in this paper.

III. INTERLEAVED GC-FG/FF CODING SCHECME FOR CORRECTING ERASURES

A 2-D FG-LDPC code C_{FG} with minimum distance d_0 constructed based on the 2-D finite geometry FG(2, 2^s) over GF(2^s) is very effective in correcting random erasures over a binary erasure channel (BEC). If the number of erased code symbols, called *erasures,* in a received vector is $d_0 - 1$ or less, there is *at least one check-sum* formed by taking the inner product of the received vector and a row in the parity-check matrix $\mathbf{H}_{FG}$ of C_{FG}, which contains *one and only one* erased code symbol. By using this check-sum, we can recover the erased code symbol.

In decoding a received word $\mathbf{r}$, for each erased code symbol x in $\mathbf{r}$, we find a row $\mathbf{h}$ in $\mathbf{H}_{FG}$ that checks only on this erased symbol and no other erased symbol in $\mathbf{r}$. Form the check-sum $\Sigma = \langle \mathbf{r}, \mathbf{h} \rangle$ which is the inner product of $\mathbf{r}$ and $\mathbf{h}$. Set $\Sigma = 0$. The equation $\Sigma = 0$ contains x as the only unknown. Then, the erased code symbol x is equal to the modulo-2 sum of the known code symbols contained in Σ. The erased symbols in $\mathbf{r}$ can be recovered *one at a time* using the above process, called the *erasure peeling* (*EPL*) *process*. Using the EPL process, C_{FG} is capable of correcting $d_0 - 1$ or fewer random erasures.

Let c be a positive integer and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{c-1}$ be c codewords in C_{FG}. Suppose we interleave these c codewords in C_{FG} *symbol by symbol*, we obtain an interleaved word $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{n_0-1})$ which consists of n_0 sections, each containing c code symbols. For $0 \le j < n_0$, the j-th section $\mathbf{y}_j$ of $\mathbf{y}$ consists of the j-th code symbols from the c codewords $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{c-1}$ in C_{FG}. By interleaving each group of c codewords C_{FG}, we obtain an interleaved FG-LDPC code, denoted by $C_{FG,interl}(c)$, where c is referred to as *interleaving degree*. Let $\mathbf{H}_{FG} = [\mathbf{h}_0\ \mathbf{h}_1 \ldots \mathbf{h}_{n_0-1}]$ be the $n_0 \times n_0$ parity-check matrix of C_{FG} where for $0 \le j < n_0$, $\mathbf{h}_j$ is the j-th column of $\mathbf{H}_{FG}$. For $0 \le j < n_0$, we form a $c \times c$ diagonal array, denoted by $\mathbf{H}_j = \mathrm{diag}(\mathbf{h}_j, \mathbf{h}_j, \ldots, \mathbf{h}_j)$, with c copies of j-th column $\mathbf{h}_j$ of $\mathbf{H}_{FG}$ lying on its main diagonal and zeros elsewhere. Then, the parity-check matrix of the interleaved FG-LDPC code $C_{FG,interl}(c)$ is the following $cn_0 \times cn_0$ matrix:

$$\mathbf{H}_{FG,interl}(c) = [\mathbf{H}_0 \mathbf{H}_1 \ldots \mathbf{H}_{n_0-1}] \qquad (4)$$

Suppose an interleaved codeword $\mathbf{y}$ in $C_{FG,interl}(c)$ is transmitted over a BEC. Let $\mathbf{y}^* = (\mathbf{y}^*_0, \mathbf{y}^*_1, \ldots, \mathbf{y}^*_{n_0-1})$ be the received word with erasures confined in $d_0 - 1$ or fewer interleaved sections in $\mathbf{y}^*$. To recover the erased code symbols in $\mathbf{y}^*$, we first de-interleave $\mathbf{y}^*$ into c received words $\mathbf{v}^*_0, \mathbf{v}^*_1, \ldots, \mathbf{v}^*_{c-1}$, which correspond to the c transmitted codewords contained in $\mathbf{y}$. De-interleaving of $\mathbf{y}^*$ distributes the erased symbols in $\mathbf{y}^*$ into the c received local words $\mathbf{v}^*_0, \mathbf{v}^*_1, \ldots, \mathbf{v}^*_{c-1}$, each containing *no more* than d_0 -1 erasures. Using the EPL-algorithm, the erased symbols in each received word $\mathbf{v}^*_i$, $0 \le i < c$, can be recovered. The maximum number of erasures confined in $d_0 - 1$ interleaved sections, which can be corrected is $c(d_0 - 1)$.

Consider the (33, 33)-regular (1057, 813) PG-LDPC code C_{PG} of Example 1. This code has a minimum distance of 34 and its parity-check matrix has an orthogonal structure. With EPL-decoding, the code can correct 33 or fewer random erasures. If we interleave this code by a degree $c = 100$, then the interleaved code $C_{PG,interl}(100)$ is capable of correcting erasures confined in 33 random interleaved sections, each of length 100, with a maximum of 3300 erasures.

The GC-FG/FF-LDPC coding scheme presented in the section II can be modified by interleaving every sequence of c local codewords at the first stage of encoding. This results in an interleaved GC-FG/FF-LDPC coding scheme which can be used for error control over an AWGN channel as well as a BEC channel.

In the first stage of encoding, the c outer codewords $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{c-1}$ at the output of the local encoder are interleaved into codeword $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{n_0-1})$ in $C_{FG,interl}(c)$. In the second stage of encoding, $\mathbf{y}$ is encoded into a codeword $\mathbf{w} = (\mathbf{p}, \mathbf{y})$ in the global coupling FF-LDPC code for transmission. The local encoding, interleaving, and global coupling encoding result in an interleaved GC-FG/FF-LDPC code, denoted by $C_{GC,FG,FF,interl}$, with the following parity-check matrix:

$$\mathbf{H}_{GC,FG,FF,interl} = \left[\begin{array}{c|c} \mathbf{O} & \mathbf{H}_{FG,interl}(c) \\ \hline \mathbf{H}_{FF,left} & \mathbf{H}_{FF,right} \end{array}\right] \qquad (5)$$

Suppose an interleaved global coupling codeword $\mathbf{w} = (\mathbf{p}, \mathbf{y})$ is transmitted. Let $\mathbf{w}^* = (\mathbf{p}^*, \mathbf{y}^*)$ be the received word. In decoding $\mathbf{w}^*$, the decoder first checks whether $\mathbf{w}^*$ contains erasures. If it does, the decoder removes the received parity-check part $\mathbf{p}^*$ of $\mathbf{w}^*$ and de-interleaves $\mathbf{y}^*$ into c received local words $\mathbf{v}^*_0, \mathbf{v}^*_1, \ldots, \mathbf{v}^*_{c-1}$. Then, the local decoder applies the EPL process to recover all erased symbols in each received local word $\mathbf{v}^*_i$, $0 \le i < c$. If all the erased code symbols are recovered, then decoding stops and the decoder delivers the ck_0 decoded information symbols to the user(s). If the number of erased code symbols in any received local word $\mathbf{v}^*_i$, exceeds the erasure correction capability $d_0 - 1$, the decoder stops the entire decoding process and declares a decoding failure. The above decoding of a received word is referred to as *erasure-correcting mode.*

If no erasure is found in the received word $\mathbf{w}^* = (\mathbf{p}^*, \mathbf{y}^*)$, the two-phase global/local (or local/global) iterative decoding as described in the last section is activated. In decoding, de-interleaving and interleaving of the decoded local words at the outputs of the global coupling and local decoders must be performed before switching from one decoding phase to the other. Each time at the end of the global coupling decoding phase, the decoded interleaved word must be de-interleaved into c decoded local words before the local decoding phase begins, and each time at the end of the local decoding phase, the c decoded local words must be interleaved before the global coupling decoding phase begins. The global coupling decoding/de-interleaving and the local decoding/interleaving processes continue iteratively until either all the local transmitted codewords are successfully decoded or a preset maximum number of global/local (or local/global) iterations is reached.

Since the GC-FG/FF- and the interleaved GC-FG/FF-LDPC codes are *combinatorially equivalent*, they perform the same over an AWGN channel if the same local and global decoding schemes are used. The two types of codes have the same trapping set structure.

Consider the GC-PG/FF-LDPC code given in Example 1. If the 9 local codewords at output of the local encoder are interleaved before adding 87 fill-in zeros and global coupling encoding, then the resultant interleaved GC-PG/FF-LDPC code is capable of correcting erasures confined in 33 random interleaved sections, each of length 9, with a maximum 297 erasures.

The interleaved GC-FG/FF-LDPC coding scheme may find application for error and erasure control in a compound channel over which both random errors and erasures may occur.

IV. GLOBAL COUPLED FF/FF-LDPC CODING SCHEME

In the construction of GC-FG/FF-LDPC codes, we use FG-LDPC codes as local codes and FF-LDPC codes as global coupling codes. Even though FG-LDPC codes are powerful and have a good trapping set structure, however, their choices are limited. Furthermore, since the two types of codes used in the construction have different structures, the global parity-check $\mathbf{H}_{GC,FG,FF}$ of a GC-FG/FF code, in general, does not satisfy the RC-constraint. Hence, a GC-FG/FF-LDPC code must be decoded separately using a two-phase iterative decoding, global/local or local/global.

In contrast to FG-LDPC codes, the construction of FF-LDPC codes is more flexible in lengths and rates. In the construction of a GC-FF/FF-LDPC code $C_{GC,FF,FF}$, we can either use the same field to construct both local and global coupling codes or use two different fields to construct local and global code, separately. Construction of a GC-FF/FF-LDPC code $C_{GC,FF,FF}$ based on a single field, makes it possible to construct a global parity-check matrix $\mathbf{H}_{GC,FF,FF}$ that satisfies the RC-constraint. Hence, $C_{GC,FF,FF}$ can be decoded globally in one phase based on the global parity parity-check matrix $\mathbf{H}_{GC,FF,FF}$. Of course, $C_{GC,FF,FF}$ can also be decoded in two phases similar to decoding of a GC-FG/FF-LDPC code as described in Section II. If local and global coupling codes are constructed from two different fields, the global parity-check matrix $\mathbf{H}_{GC,FF,FF}$ of the resultant GC-FF/FF-LDPC code, in general, does not satisfy the RC-constraint. In this case, the code must be decoded in two phases.

Constructing a GC-FF/FF-LDPC code using two different fields can be achieved by replacing the local parity-check matrix $\mathbf{H}_{FG}$ given in (3) with a parity-check matrix $\mathbf{H}_{local,FF}$ constructed based on a field different from the field used in constructing the global coupling matrix $\mathbf{H}_{FF}$.

In the following, we present a construction of GC-FF/FF-LDPC codes in which both the local and global coupling LDPC codes are constructed using the same field. The global parity-check matrices of these codes satisfy the RC-constraint.

Let m_0, m_1, n_0, c and r be 5 positive integers with $m_0 < n_0$. Construct a $(cm_0 + m_1) \times (cn_0 + r)$ base matrix $\mathbf{B}(cm_0 + m_1, cn_0 + r)$ over GF(q) in the form of (1) using two subsets $\boldsymbol{S}_0 = \{\alpha^{i_0}, \alpha^{i_1}, \ldots, \alpha^{i_{m-1}}\}$ and $\boldsymbol{S}_1 = \{\alpha^{j_0}, \alpha^{j_1}, \ldots, \alpha^{j_{n-1}}\}$ over GF(q) with sizes of $m = cm_0 + m_1$ and $n = cn_0 + r$, respectively, and $n \leq q$. The matrix $\mathbf{B}(cm_0 + m_1, cn_0 + r)$ satisfies the 2 × 2 SM-constraint [12, 17]. Label the rows and columns of $\mathbf{B}(cm_0 + m_1, cn_0 + r)$ from 0 to $cm_0 + m_1 - 1$ and 0 to $cn_0 + r - 1$, respectively. Divide the rows of $\mathbf{B}(cm_0 + m_1, cn_0 + r)$ in to $c + 1$ disjoint groups, denoted by $\mathbf{B}_0(m_0, cn_0 + r)$, $\mathbf{B}_1(m_0, cn_0 + r)$, … , $\mathbf{B}_{c-1}(m_0, cn_0 + r)$ and $\mathbf{B}_{gc}(m_1, cn_0 + r)$. For $0 \leq l < c$, $\mathbf{B}_l(m_0, cn_0 + r)$ consists of m_0 *consecutive rows* of $\mathbf{B}(cm_0 + m_1, cn_0 + r)$, labeled from lm_0 to $(l+1)m_0 - 1$. The submatrix $\mathbf{B}_{gc}(m_1, cn_0 + r)$ consists of the last m_1 rows of $\mathbf{B}(cm_0 + m_1, cn_0 + r)$. For $0 \leq l < c$, let $\mathbf{B}_{local,l}(m_0, n_0)$ be an $m_0 \times n_0$ submatrices of $\mathbf{B}_l(m_0, cn_0 + r)$, which consists of n_0 consecutive columns of $\mathbf{B}_l(m_0, cn_0 + r)$, labeled from ln_0 to $(l + 1)n_0 - 1$.

Form the following $(cm_0 + m_1) \times (cn_0 + r)$ matrix over GF(q):

$$\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, \mathrm{c}, r) = \left[\frac{\mathbf{0}|\mathrm{diag}(\mathbf{B}_{local,0}(m_0,n_0),\mathbf{B}_{local,1}(m_0,n_0),\ldots,\mathbf{B}_{local,c-1}(m_0,n_0))}{\mathbf{B}_{gc}(m_1,cn_0+r)}\right] \tag{6}$$

The matrix $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ consists of two submatrices, the upper and the lower submatrices. The upper submatrix is a $cm_0 \times (cn_0 + r)$ matrix, which consists of two parts, where the first part $\mathbf{O}$ is a $cm_0 \times r$ zero matrix and the second part diag($\mathbf{B}_{local,0,}(m_0, n_0)$, … , $\mathbf{B}_{local,c-1}(m_0, n_0)$) is a $c \times c$ diagonal array with the c submatrices $\mathbf{B}_{local,0,}(m_0, n_0)$, … , $\mathbf{B}_{local,c-1}(m_0, n_0)$ of size $m_0 \times n_0$ lying on its main diagonal and zeros elsewhere. The lower matrix $\mathbf{B}_{gc}(m_1, cn_0 + r)$ of $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ is $m_1 \times (cn_0 + r)$ which globally connects the c disjoint local matrices, $\mathbf{B}_{local,0,}(m_0, n_0)$, …, $\mathbf{B}_{local,c-1}(m_0, n_0)$. Hence, $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ is a globally coupled matrix over GF(q).

The matrix $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ is a submatrix of the base matrix $\mathbf{B}(cm_0 + m_1, cn_0 + r)$. Dispersing each nonzero entry in $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ into a $(q - 1) \times (q - 1)$ CPM and each zero entry into a $(q - 1) \times (q - 1)$ ZM, we obtain the following $(cm_0 + m_1) \times (cn_0 + r)$ array of CPMs and ZMs of size $(q - 1) \times (q - 1)$:

$$\mathbf{H}_{GC,FF,FF}(m_0, m_1, n_0, \mathrm{c}, r) = \left[\frac{\mathbf{O}_{cpm}\,|\,\mathrm{diag}(\mathbf{H}_{local,0}(m_0,n_0),\mathbf{H}_{local,1}(m_0,n_0),\ldots,\mathbf{H}_{local,c-1}(m_0,n_0))}{\mathbf{H}_{gc}(m_1,cn_0+r)}\right] \tag{7}$$

where for $0 \leq l < c$, $\mathbf{H}_{local,l}(m_0, n_0)$= CPM($\mathbf{B}_{loca,l}(m_0, n_0)$), $\mathbf{O}_{cpm}$ is a $cm_0(q-1) \times r(q-1)$ zero matrix, and $\mathbf{H}_{gc}(m_1, cn_0+r)$ = CPM($\mathbf{B}_{gc}(m_1, cn_0+r)$). The array $\mathbf{H}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ is a $(cm_0+m_1)(q-1) \times (cn_0+r)(q-1)$ matrix, in which the submatrix $\mathbf{H}_{gc}(m_1, cn_0+r)$ globally couples c disjoint local submatrices $\mathbf{H}_{local,l}(m_0, n_0)$, $0 \leq l < c$.

The null space over GF(2) of $\mathbf{H}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ gives a quasi-cyclic (QC) GC-FF/FF-LDPC code $C_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ of length $(cn_0+r)(q-1)$, which is composed of c local QC-FF-LDPC codes $C_{local,l}(m_0, n_0)$, $0 \leq l < c$, of length cn_0 and a global coupling QC-FF-LDPC code $C_{gc}(m_1, cn_0+r)$ of length $(cn_0+r)(q-1)$. The l-th local QC-FF-LDPC code $C_{local,l}(m_0, n_0)$ is given by the null space of the l-th local parity-check matrix $\mathbf{H}_{local,l}(m_0, n_0)$ and the global coupling code $C_{gc}(m_1, cn_0+r)$ is given by the null space over GF(2) of the global coupling matrix $\mathbf{H}_{gc}(m_1, cn_0+r)$.

Encoding of $C_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ consists of two stages. In the first stage, c messages of equal length (*maybe from c different senders*) are encoded into c local codewords of length $n_0(q-1)$ by c local encoders for the c local codes, respectively. In the second stage of encoding, the c local codewords are cascaded into a word $\mathbf{v}$ of length $cn_0(q-1)$ and then $\mathbf{v}$ is encoded into a global coupling codeword $\mathbf{w}$ in $C_{gc}(m_1, cn_0+r)$ for transmission.

Since the base matrix $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ satisfies the 2×2 SM-constraint, the global parity-check matrix $\mathbf{H}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ satisfies the RC-constraint. Hence, the Tanner graph $G_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ of $C_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ has a girth at least 6. $C_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ can be decoded in *one phase* based on the entire global matrix $\mathbf{H}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ using an iterative algorithm based on belief propagation, say the MSA. Decoding of $C_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ can also be carried out in two phases similar to decoding a GC-FG/FF-LDPC code to reduce decoding complexity.

Example 2: In the following, we use the prime field GF(101) to construct a GC-FF/FF-LDPC code. First, we set $m_0 = m_1 = 4$, $n_0 = 32$, $c = 3$ and $r = 4$. Next, we form a 16×100 matrix $\mathbf{B}_{FF}(16, 100)$ over GF(101) in the form of (1) based on two subsets $\boldsymbol{S}_0 = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{15}\}$ and $\boldsymbol{S}_1 = \{1, \alpha, \alpha^2, \ldots, \alpha^{99}\}$ of GF(101) with $\eta = 1$. Label the rows and columns from 0 to 15 and 0 to 99, respectively. Divide the rows of $\mathbf{B}_{FF}(16, 100)$ in to 4 disjoint groups, $\mathbf{B}_0(4, 100)$, $\mathbf{B}_1(4, 100)$, $\mathbf{B}_2(4, 100)$ and $\mathbf{B}_{gc}(4, 100)$. For $0 \leq l < 3$, $\mathbf{B}_l(4, 100)$ consists of 4 *consecutive rows* of $\mathbf{B}_{FF}(16, 100)$, labeled from $4l$ to $4(l+1)-1$. The submatrix $\mathbf{B}_{gc}(4, 100)$ consists of the last 4 rows of $\mathbf{B}_{FF}(16, 100)$. For $0 \leq l < 3$, let $\mathbf{B}_{local,l}(4, 32)$ be a 4×32 submatrix of $\mathbf{B}_l(4, 100)$ which consists of 32 consecutive columns of $\mathbf{B}_l(4, 100)$, labeled from $32l$ to $32(l+1)-1$. Form the following 16×100 matrix over GF(101):

$$\mathbf{B}_{GC,FF,FF}(4,4,32,3,4)$$

$$= [\frac{\mathbf{O}|\text{diag}(\mathbf{B}_{local,0}(4,32), \mathbf{B}_{local,1}(4,32), \mathbf{B}_{local,2}(4,32))}{\mathbf{B}_{gc}(4,100)}].$$

The 100×100 CPM-dispersion of $\mathbf{B}_{GC,FF,FF}(4, 4, 32, 3, 4)$ gives the following 16×100 array of CPMs and ZMs of size 100×100:

$$\mathbf{H}_{GC,FF,FF}(4,4,32,3,4)$$

$$= [\frac{\mathbf{O}_{cpm}|\text{diag}(\mathbf{H}_{local,0}(4,32), \mathbf{H}_{local,1}(4,32), \mathbf{H}_{local,2}(4,32))}{\mathbf{H}_{gc}(4,100)}]$$

which is a 1600×10000 matrix over GF(2) with column weights 4 and 8 and two row weights 32 and 100.

The null space of $\mathbf{H}_{GC,FF,FF}(4, 4, 32, 3, 4)$ gives a QC-GC-FF/FF-LDPC code $C_{GC,FF,FF}(4, 4, 32, 3, 4)$ of length 10000 and rate 0.84, which is composed of 3 local QC-LDPC codes of length 3200, and a global coupling QC-LDPC code of length 10000. The code can be decoded either in one phase based on the global parity-check matrix $\mathbf{H}_{GC,FF,FF}(4, 4, 32, 3, 4)$ or in two phases based on local and global coupling parity-check matrices, respectively.

The BER performance of $C_{GC,FF,FF}(4, 4, 32, 3, 4)$ over an AWGN channel using BPSK signaling decoded in one phase based on the global matrix $\mathbf{H}_{GC,FF,FF}(4, 4, 32, 3, 4)$ with 20 and 100 iterations of the MSA, scaled by a factor 0.275 is shown in Fig. 3. We see the performance gap between 20 and 100 iterations is very small, about 0.1 dB.

The BER performance of $C_{GC,FF,FF}(4, 4, 32, 3, 4)$ decoded using the global/local two-phase decoding of the MSA with $\boldsymbol{I}_{out,max} = \boldsymbol{I}_{in,max} = \boldsymbol{I}_{\max} = 5$ is also shown in Fig.3.

We see that one phase decoding and two-phase decoding of the code give almost the same error performance in the simulation range. Simulation results show that one-phase decoding converges slightly faster than two-phase decoding. However, the implementation of two-phase decoding is simpler.

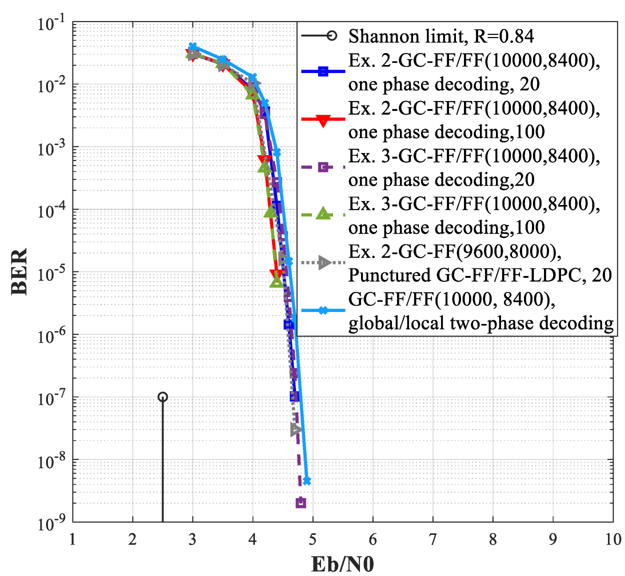

Fig. 3: The BER performance of the GC-FF/FF-LDPC code given in Examples 2 and 3.

Remarks: If we remove the zero matrix $\mathbf{O}$ and the leftmost r column of $\mathbf{B}_{gc}(m_1, cn_0 + r)$ from the global base matrix $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c, r)$ given by (6), we obtain a base matrix $\mathbf{B}_{GC,FF,FF}(m_0, m_1, n_0, c)$ in the form for constructing a CN-based GC-FF-LDPC code proposed in [12]. Hence, the construction of GC-FF/FF-LDPC codes presented above in *concatenated form* is a generalization of the construction of CN-based GC-FF-LDPC codes proposed in [12]. We can also regard a CN-based GC-FF-LDPC code as a punctured GC-FF/FF-LDPC code. The performance of the CN-based GC-FF-LDPC code (punctured GC-FF/FF-LDPC) is shown in Fig. 3

Example 3: In this example, we use the field GF(101) to construct a 2×2 SM-constrained 100×100 circulant $\mathbf{B}_{cyc}*(100, 100)$ in the form given by (2).

Suppose we take the top 16 rows of $\mathbf{B}_{cyc}*(100, 100)$ to form a 16×100 submatrix $\mathbf{B}_{cyc}*(16, 100)$. If we replace the base matrix $\mathbf{B}(16, 100)$ in Example 2 by $\mathbf{B}_{cyc}*(16, 100)$ and follow the same global base matrix construction process and parameters, we obtain the following 16×100 global coupled base matrix over GF(101):

$$\mathbf{B}^*_{cyc,GC,FF,FF}(4,4,32,3,4)$$

$$= [\frac{\mathbf{O}|\mathrm{diag}(\mathbf{B}^*_{cyc,local,0}(4,32),\mathbf{B}^*_{cyc,local,1}(4,32),\mathbf{B}^*_{cyc,local,2}(4,32))}{\mathbf{B}^*_{cyc,gc}(4,100)}].$$

Following the cyclic structure of $\mathbf{B}_{cyc}*(100, 100)$, we can readily see that the three local base matrices $\mathbf{B}*_{cyc,local,0}(4, 32)$, $\mathbf{B}*_{cyc,local,1}(4, 32)$, and $\mathbf{B}*_{cyc,local,2}(4, 32)$ are identical. Hence, the null space over GF(2) of the 100×100 CPM dispersion $\mathbf{H}*_{cyc,GC,FF,FF}(4, 4, 32, 3, 4)$ gives a GC-FF/FF-LDPC code $C*_{cyc,GC,FF,FF}(4, 4, 32, 3, 4)$ which consists of three copies of a local code of length 3200 connected by a global coupling code of length 10000.

The BER performance of the GC-FF/FF-LDPC $C*_{cyc,GC,FF,FF}(4, 4, 32, 3, 4)$ over an AWGN channel using BPSK signaling decoded in one phase based on the entire global matrix $\mathbf{H}*_{cyc,GC,FF,FF}(4, 4, 32, 3, 4)$ with 20 and 100 iterations of the MSA, scaled by a factor 0.275 is also shown in Fig. 3.

V. CONCLUSION AND REMARKS

In this paper, we presented two types of concatenated LDPC codes viewed as generalized globally coupled LDPC codes in which the outer and inner codes serve as local and global coupling codes, respectively. In the construction of a first-type concatenated LDPC code, a cyclic FG-LDPC code is used as the local code and a QC-FF-LDPC code is used as the global coupling code. Such a concatenated LDPC code is referred to as a GC-FG/FF-LDPC code. In the construction of a second-type concatenated LDPC code, both local and global coupling codes are QC-FF-LDPC codes. Such a concatenated LDPC code is referred to as a GC-FF/FF-LDPC code. A GC-FG/FF-LDPC code possesses the distinct features of both FG- and FF-LDPC codes, i.e., large minimum distances, no small trapping sets, fast decoding convergence, capable of correcting both random errors and bursts of erasures. Two two-phase iterative decoding schemes were devised for decoding a GC-FG/FF-LDPC code, one for correcting random errors over an AWGN channel, and the other for correcting random bursts of erasures over a BEC in addition to correcting random errors over an AWGN channel. A GC-FF/FF-LDPC code constructed based on a single field can be decoded either in a single phase or in two phases. The two-phase decoding allows information-sharing in the decoding of local codewords. In one example, we also showed that the two-phase decoding performs almost the same as the one-phase decoding.

The GC-FG/FF-LDPC coding scheme is quite *adaptive for hybrid repeat-request-retransmission*, called HARQ [7]. Various possible retransmission schemes can be devised depending on the communication environments, the requirements of reliability, throughput efficiency, and system complexity. One such possible retransmission scheme is a retransmission of failed decoded local codewords until all the transmitted local codewords are successfully decoded or a preset maximum number of retransmissions is reached.

Since the local FG code is powerful, the frequency of retransmission-requests is relatively low. Hence, a HARQ system based on the GC-FG/FF-LDPC coding scheme should be able to provide high reliability and high throughput performance.

This direction may deserve further research effort. Finally, the authors would like to point out that a concatenated FG-FF-LDPC coding scheme was developed for possible application in a high-speed underwater fiber optical communication system [19].

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