Indivisible Mixed Manna: On the Computability of MMS+PO Allocations (Extended Abstract)

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In this paper we study the problem of finding fair and efficient allocations of a *mixed manna*, i.e., a set $\mathcal{M}$ of discrete items that are goods/chores, among a set $\mathcal{N}$ of agents with additive valuations. We note that a mixed manna allows an item to be a good (positively valued) for some agents, and a chore (negatively valued) for others, and thereby strictly generalizes the extensively studied goods (chores) manna. To measure *fairness* and *efficiency* we consider the popular and well studied notions of maximin-share (MMS) and Pareto optimality (PO) respectively.

An MMS *allocation* is one where every agent gets at least her MMS value. However, [6] showed that an MMS allocation may not always exist. This prompted a series of works on the efficient computation of α-MMS allocations, where every agent gets at least α $(1/\alpha)$ times her MMS value for a goods (chores) manna, for progressively better $\alpha \in [0,1]$; the best factor known so far is $\alpha = (3/4 + 1/(12n))$ by Garg and Taki [2] for $n \geq 5$ agents for goods, and 9/11 for chores [5]. No such results are known for the mixed manna. Even for the goods (chores) manna, no work has explored the PO guarantee in addition to MMS, to the best of our knowledge.

In this paper, we first show that, for *any* fixed $\alpha \in (0,1]$, an α-MMS allocation may not always exist; in contrast, non-existence with a goods manna is known for α close to one. This rules out efficient computation for any fixed α, and naturally raises the following problem.

Problem of Interest. *Design an efficient algorithm to find an* α-MMS + PO *allocation for the best possible* α, *i.e., the maximum* $\alpha \in (0,1]$ *for which it exists.*

This *exact* problem is intractable: In the case of identical agents, an $(\alpha = 1)$-MMS allocation exists by definition. However, finding one is known to be NP-hard for a goods manna. On the positive side, a polynomial-time approximation scheme (PTAS) is known for this case; given a *constant* $\epsilon \in (0,1]$, the algorithm finds a $(1-\epsilon)$-MMS allocation in polynomial time. Guaranteeing PO in addition adds to the complexity, since even checking if a given allocation is PO is coNP-hard. In light of these results, we ask,

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Problem Investigated. *Can we design a* PTAS*, namely an efficient algorithm to find an* $(\alpha-\epsilon)$-MMS + γ-PO *allocation, given* $\epsilon, \gamma > 0$*, for the best possible* α*?*

Our Contribution. In this paper we make significant progress towards this question for mixed manna by showing the following dichotomy result: We derive two conditions and show that the problem is tractable under these conditions, while dropping either renders the problem intractable. The two conditions are: (i) number of agents n is a constant, and (ii) for every agent i, her total (absolute) value for all the items ($|v_i(\mathcal{M})|$) is significantly greater than the minimum of her total value of goods and her total (absolute) value for chores.

In particular, first, for instances satisfying (i) and (ii), we design a PTAS (as asked in the above question). Second, we show that if either condition is not satisfied, then finding an α-MMS allocation for *any* $\alpha \in (0, 1]$ is NP-hard, even with *identical agents* where a solution exists for $\alpha = 1$. This hardness is interesting because it shows inapproximability within *any* non-trivial factor when either (i) or (ii) is not satisfied and no other restrictions are imposed on the problem.

Our algorithm, in principle, gives a little more than a PTAS. It runs in time $2^{O(1/\min\{\epsilon^2,\gamma^2\})}poly(\mathrm{m})$ for given ϵ, γ, thus gives polynomial run-time for ϵ, γ as small as $O(1/\sqrt{\log m})$, where $m = |\mathcal{M}|$.

α-MMS + PO *for goods (chores) manna.* As a corollary, we obtain a PTAS for finding α-MMS + PO allocations of a goods manna and a chores manna when the number of agents is a constant. This improves the previous results for these settings in two aspects: (i) provides the best possible approximation factor; best known factors for the goods manna other than the general case are 4/5 for $n = 4$ [3], 8/9 for $n = 3$ [4], and 1 for $n = 2$ [1], and (ii) provides an additional (approximate) PO guarantee.

PTAS *to find* MMS *values.* As the first key step for our main algorithm, we design a PTAS that returns $(1-\epsilon)$ approximate MMS values of agents, which may be of independent interest.

A new technique to prove PO. Since certifying a PO allocation is a known coNP-hard problem, known works maintain a PO allocation with market equilibrium as a certificate . We develop a novel approach to ensure PO with α-MMS through LP rounding. The LP itself is intuitive, however the rounding is involved. It makes use of *envy-graph* and properties of the MMS in a novel way. This approach may be of independent interest.

The full version of this paper can be accessed at: https://arxiv.org/abs/2007.09133.

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