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ANALYSIS AND EVALUATION OF PIECEWISE LINEAR SYSTEMS WITH COULOMB FRICTION USING A HYBRID SYMBOLIC-NUMERIC COMPUTATIONAL METHOD

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ABSTRACT

A general formulation of piecewise linear systems with discontinuous force elements is provided in this paper. It has been demonstrated that this class of nonlinear systems is of great importance due to their ability to accurately model numerous scientific and engineering phenomena. Additionally, it is shown that this class of nonlinear systems can demonstrate a wide spectrum of nonlinear motions and in fact, the phenomenon of weak chaos is observed in a mechanical assembly for the first time.

Despite such importance, efficient methods for fast and accurate evaluation of piecewise linear systems' responses are lacking and the methods of the literature are either incompatible, very slow, very inaccurate, or bear a combination of the aforementioned deficiencies. To overcome this shortcoming, a novel symbolic-numeric method is presented in this paper that is able to obtain the analytical response of piecewise linear systems with discontinuous elements in an efficient manner. Contrary to other efficient methods that are based on stationary steady state dynamics, this method will not experience failure upon the occurrence of complex motion and is able to capture the entirety of the dynamics.

INTRODUCTION

The use of discontinuous models to emulate the behavior of various natural phenomena has been studied considerably for

a number of physical systems [1, 2]. As a significant subset of discontinuous systems, piecewise linear (PWL) models form an important class with applications in most fields of science and especially engineering [3, 4]. Their strength in capturing the physics of systems with gaps or prestress, owing to the existence of cracks, joints or intermittent contacts, is considerable and further signifies their importance [5, 6]. Alternatively, another class of discontinuous systems are also formed due to the existence of elements that inherently exhibit discontinuous nature. The relative motion between contacting surfaces in many systems can be thought of as a discontinuous element that results in Coulomb friction which has the capability to fundamentally alter the dynamical behavior of these systems [7, 8]. Accurate models for numerous applications in mechanical, civil, marine, and aerospace engineering can lead to piecewise linear systems with discontinuous force elements being observed [9, 10]. It is necessary to state that PWL systems with discontinuous force elements (PWLDfE) are not only of great significance due to their exceptional modeling capabilities, but are also important from a theoretical point of view since a wide range of motions (non-chaotic, weakly chaotic and chaotic) can be generated by them.

Studying the dynamical behavior of the combination of PWL systems with such alternative discontinuous elements is a powerful tool for many applications. Due to the nonlinear nature of such models, the employment of linear techniques is typically inadequate [11] and other approaches are employed. The mathematical foundation of such systems, commonly referred to as

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differential inclusions, is studied in detail by Filippov [12] and no analytical solutions exist for the general case of such systems. The arduous mathematics of differential inclusions, and consequently PWL systems, obligated engineers to employ simplifying assumptions that resulted in inaccurate estimations of the dynamical behavior of these systems. Indirect fundamental mathematical methods were also introduced and adapted to qualitatively capture some dynamical characteristics [13]. Contrary to the general approaches of mathematicians, researchers in various fields of engineering attempted to propose approximate solutions or to only focus on special cases that were of significant importance to a desired application. Nakhla proposed the harmonic balance method to calculate the periodic steady-state solution of nonlinear systems [14]. His method was adapted by Lau and Zhang [15] to be suitable for PWL systems. Many alternative methods have since been proposed that are capable of accurately approximating the periodic steady state response of PWL systems. Saito et al. developed one such method that exploits both the frequency and time domain to estimate the steady state system response [16]. Recent studies that propose novel methods for obtaining the steady state response of PWL systems are mainly focused on their capability in treating higher dimensional systems [17–19] but it should be stated that all the preceding methods are susceptible to failure in case of complex periodic responses. Additionally, these methods are ineffective for non-periodic steady state responses, which are known to occur in PWLDFE systems. Recently, a novel analytical-numerical method was developed [20, 21] that is able to obtain an arbitrary response of PWL systems. The proposed method is computationally efficient and does not limit its scope to systems with periodic steady state responses and serves as the foundation in which this work is built upon.

It is also important to mention that due to the lack of efficient techniques, researchers have been forced to utilize numerical integration (NI) methods to evaluate the response of this class of systems. The employment of NI methods such as Runge-Kutta or Adams-Bashforth is computationally expensive and, in many cases, excessively time consuming. In addition to this shortcoming, due to numerical error, NI methods might fail to capture the entirety of the dynamics, especially in the case of stick-slip scenarios. The problem with NI methods runs deeper when such systems start to demonstrate nonlinear phenomena such as chaotic behavior [20].

To overcome the aforementioned shortcomings in evaluating and analyzing PWLDFE systems, a new method is introduced in this work. The methodology presented in this work is based on mathematically manipulating the PWLDFE system in intervals that are in fact continuous. Using such techniques, it is possible to convert the strongly nonlinear problem to a series of systems that are linear in specified intervals; thereby enabling the use of linear modal transformations and an analytical determination of the response in each time interval. The proposed method can be

considered as an adaptive fusion of numerical methods that are utilized to speed up the process of identifying the switch from one interval to another and analytical methods. Consequently, this hybrid symbolic numerical computational (HSNC) method is more accurate, faster and more reliable than traditional NI methods.

To demonstrate the merits of the proposed method and to verify its validity, a multi-degree of freedom mass-spring-damper system with intermittent contact and friction is studied and the time response is obtained and compared with the results of NI. It is shown that the HSNC method's runtime is hundreds to thousands of times faster than conventional NI methods. It is noteworthy to state that the chaos encountered in this paper does not exhibit conventional chaotic characteristics and is considered to be weakly (or slowly) chaotic by some references [22–24]. The fact that this phenomenon has never been reported in mechanical systems adds to its theoretical significance. It is also important to note that the accuracy and speed superiority of the HSNC method to NI improves as the system studied increases in dimension.

The remainder of this work is structured in the following manner. Section 2 introduces the methodology of the HSNC approach and explains the procedure that one must follow to obtain accurate results. Section 2 also offers a detailed prelude on PWLDFE systems. Section 3 demonstrates the capabilities of the HSNC method by simulating a number of PWLDFE systems. Section 4 presents the conclusions and detailed discussions on the obtained results.

METHODOLOGY

The methodology for computing the response of PWLDFE systems using HSNC is presented in this section. First, a mathematical description of PWLDFE systems is given. Next, the HSNC method [21] originally created for PWL systems is extended for PWLDFE systems. Although the use of HSNC in this work is focused on industrial oriented applications, it also has academic applications due to the scientific value of PWLDFE systems.

PWLDFE standard formulation

HSNC can be used to obtain the total response of PWLDFE systems but, prior to its detailed introduction, it is important to establish a standard formulation of PWLDFE systems. The formulation here is focused on mechanical systems, as illustrated in Fig. 1, however the same idea can be employed for any PWLDFE system that can be solved analytically in the specified intervals.

The standard formulation of PWLDFE systems in this paper consists of a multi-degree of freedom mass-spring-damper system with intermittent contact and Coulomb friction. The intermittent contact can represent cracks or gaps between jointed assemblies in mechanical systems. The only assumption in this

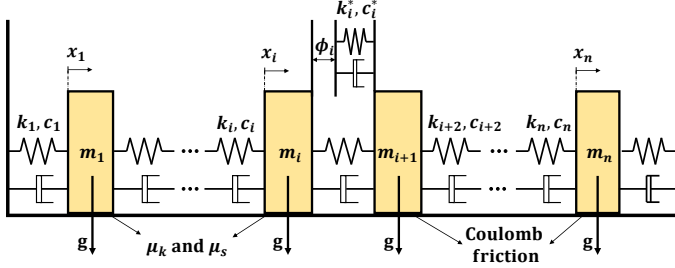


FIGURE 1: CONFIGURATION OF PWLDFE SYSTEMS IN THIS WORK.

work is that the PWLDFE systems are proportionally damped since a disproportionate damping would complicate the process of modal decomposition yet presents no significant scientific novelty. Equation (1) mathematically defines PWLDFE systems in the form of a differential inclusion.

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) - \mathbf{F}\sin(\omega t) \in \mathcal{D}(\dot{\mathbf{x}}(t)), \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ represent the mass, damping, and stiffness matrices, respectively. Vectors $\ddot{\mathbf{x}}(t)$, $\dot{\mathbf{x}}(t)$, $\mathbf{x}(t)$ and $\mathbf{F}$ correspond to the system's acceleration, velocity, position and excitation magnitude. Symbol $\in$ is conventionally used in defining differential inclusions. The right hand side of the differential inclusion, which represents the discontinuity from Coulomb friction, is defined as

$$\mathcal{D}(\dot{\mathbf{x}}(t)) = -\mu_k g V_{mass} \text{sign}(\dot{\mathbf{x}}(t)) = \begin{bmatrix} -\mu_k m_1 g \text{sign}(\dot{x}_1(t)) \\ \vdots \\ -\mu_k m_n g \text{sign}(\dot{x}_n(t)) \end{bmatrix}, \quad (2)$$

where μ_k represents the coefficient of kinetic friction, which in general can vary between from mass to mass even though that is not shown in Eq. (2). V_{mass} is the vector representation of the masses and is identical to the components of the principal diagonal elements of the $\mathbf{M}$ matrix. It is written in the vector form to have dimensional compatibility. It is important to note that matrices $\mathbf{C}$ and $\mathbf{K}$ will change due to intermittent contact. It should be noted that when a subsystem has a zero velocity and becomes stuck, the magnitude of the friction force will be equal to the sum of all the forces acting on the subsystem and its direction will oppose the motion. The static friction force on the subsystem can vary up until it reaches the maximum static friction denoted as $\mu_s m_i g$.

To further elucidate the intermittent contact's dynamics, it can be noted that the subsystems (individual masses) that have intermittent contact pairs can have two distinct configurations. One configuration is where the subsystems' intermittent contact

pair does not experience contact and the subsystems are "open". Alternatively when the intermittent contact pair starts to have contact, then the configuration will be considered "closed". The status of the configuration is contingent upon the position of the two intertwined subsystems and configuration parameters. This is shown in Fig. 2 in detail. Mathematically speaking, when $x_i(t) - x_{i+1}(t) - \phi_i < 0$ the intermittent contact pairs would not experience contact, but for $x_i(t) - x_{i+1}(t) - \phi_i \geq 0$ the contact pairs will be in contact. It is important to note that the method works just as well when there is more than one intermittent contact pair in the model. The parameter ϕ_i represents the gap from the i^{th} mass to its next adjacent intermittent contact pair when the springs are unstretched. To further elaborate on this model and its characteristics, it is assumed that the i^{th} and $i+1^{th}$ subsystems have intermittent contact. Assume that for the case where the intermittent contact's status is open the motion is described by Eq. (1). Upon a change in the positions of the subsystems and an alteration in the configuration's status, from open to close, the equation of motion can be rewritten as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}'\dot{\mathbf{x}}(t) + \mathbf{K}'\mathbf{x}(t) - \mathbf{F}\sin(\omega t) \in \mathcal{D}(\dot{\mathbf{x}}(t)) + \mathcal{F}_{cte}, \quad (3)$$

where $\mathbf{C}'$ and $\mathbf{K}'$ are calculated as

$$\mathbf{C}' = \mathbf{C} + \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}, \quad \mathbf{K}' = \mathbf{K} + \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}, \quad (4)$$

where the indicated rows and columns of Eq.(4) are the i^{th} and $i+1^{th}$. To elaborate on the meaning of the constant vector $\mathcal{F}_{cte}$, it is beneficial to discuss the contact in detail. When the adjacent subsystems experience contact, the spring force of the intermittent contact can be written as

$$\pm k^* \underbrace{(x_i - x_{i+1})}_{\text{inconstant}} - \underbrace{\phi_i}_{\text{constant}}, \quad (5)$$

By updating matrices $\mathbf{C}$ and $\mathbf{K}$ as illustrated in Eq. (4), the inconstant terms are included in the new dynamics but the constant portion of the intermittent contact pair's spring force remains unaccounted for. Therefore, they are stored in the i^{th} and $i+1^{th}$ columns of $\mathcal{F}_{cte}$ as

$$\mathcal{F}_{cte} = [\dots -k^*(-\phi)_i + k^*(-\phi)_{i+1} \dots]^T. \quad (6)$$

It should be noted that the dotted elements in $\mathbf{C}'$, $\mathbf{K}'$ and $\mathcal{F}_{cte}$ are determined in accordance with other intermittent contact pairs.

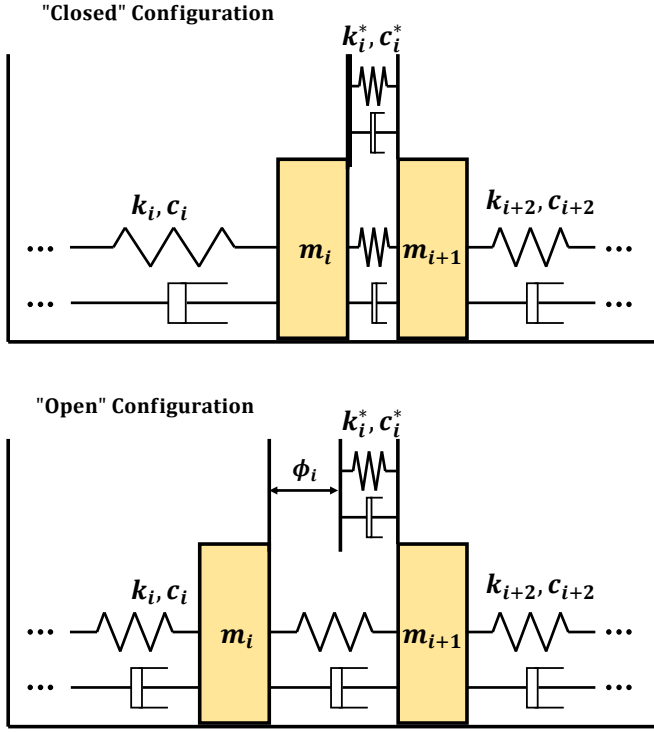


FIGURE 2: SCHEMATIC VIEW OF THE POSSIBLE CONFIGURATIONS FOR PWLDFE SYSTEMS WITH INTERMITTENT CONTACT.

The left hand side of Eq.(3) is compatible with modal analysis but the right hand side does cause certain issues. In addition to the presence of constant forces caused by intermittent contact pairs ($\mathcal{F}_{cte}$), the friction force presents issues of its own. The friction force on each subsystem is contingent upon the direction of its motion and changes sign upon having a change in the direction of motion. Additionally, it is possible and quite probable that some subsystems experience moments of staticity and this can greatly affect the general equations of motion. To elucidate this scenario, assume that the i^{th} subsystem of the studied system represented by Eq. (3) goes into staticity; this results in a change in the equation of motion. Mathematically speaking, the staticity requires that $\ddot{x}_i = 0$, $\dot{x}_i = 0$ and $x_i = cte$ and due to the fact that the i^{th} subsystem is no longer dynamic, the i^{th} row of Eq. (3) must be eliminated. The i^{th} column of the matrices $\mathbf{M}$ and $\mathbf{C}$ are multiplied by $\ddot{x}_i = 0$ and $\dot{x}_i = 0$, respectively and, since their values are zero during staticity, they produce no force element. Contrary to the i^{th} column of $\mathbf{M}$ and $\mathbf{C}$, the i^{th} column of $\mathbf{K}$ is multiplied by the constant term $x_i = cte$ and produces $i - 1$ force elements (where some are zero depending on the problem). These force elements exist for the entire duration of staticity and must be carefully accounted for. To rewrite the problem after the

staticity of the i^{th} subsystem, it is reasonable to eliminate the i^{th} row and column of matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ and obtain the new reduced mass, damping and stiffness matrices $\mathbf{M}_{red}$, $\mathbf{C}_{red}$ and $\mathbf{K}_{red}$ where the i^{th} column of the $\mathbf{K}$ matrix affects the dynamics. The resulting constant force vector $\mathcal{C}$ is introduced as

$$\mathcal{C} = [K_{1i} \cdots K_{(i-1)i} \quad K_{(i+1)i} \cdots K_{ni}]_{(n-1) \times 1}^T \times x_i, \quad (7)$$

Subsequently, the new equation of motion can be represented as

$$\mathbf{M}_{red}\ddot{\mathbf{x}}(t) + \mathbf{C}_{red}\dot{\mathbf{x}}(t) + \mathbf{K}_{red}\mathbf{x}(t) - \mathbf{F}_{red}\sin(\omega t) \in \mathcal{D}_{red}(\dot{\mathbf{x}}(t)) - \mathcal{C}, \quad (8)$$

where $\mathbf{F}_{red}$ and $\mathcal{D}_{red}$ are vectors $\mathbf{F}$ and $\mathcal{D}$ respectively with the i^{th} row eliminated.

It can now be noted that the correct analysis of these systems is dependent upon the treatment of varying forces that are constant in certain intervals ($\mathcal{D}$, $\mathcal{C}$ and $\mathcal{F}_{cte}$). To properly model such systems, it is necessary to identify these constant forces and crucial to fully explore the possible scenarios. The next section will delineate the methodology utilized by HSNC to obtain the response of such strongly nonlinear systems. It is also helpful to state the most general form of this problem that contains all the constant force vectors and can be denoted as

$$\mathbf{M}_{red}\ddot{\mathbf{x}}(t) + \mathbf{C}_{red}\dot{\mathbf{x}}(t) + \mathbf{K}_{red}\mathbf{x}(t) - \mathbf{F}_{red}\sin(\omega t) \in \mathcal{D}_{red}(\dot{\mathbf{x}}(t)) - \mathcal{C} + \mathcal{F}_{cte}, \quad (9)$$

HSNC

As stated previously, the PWLDFE systems benefit from a linear formulation except for constant terms representing the frictional forces, the intermittent contact pairs' spring forces and the constant forces coming from the occurrence of staticity. The first step in HSNC concentrates on finding the switches between states but, prior to this step, it is necessary to eliminate the constant terms via mathematical manipulations. This is done by introducing new temporary coordinate systems.

Temporary coordinates to eliminate constant forces To obtain a fully linear formulation, it is essential to eliminate the constant terms. Temporary coordinate systems for each interval that the force vectors remain constant are used to remove the force vectors.

To elucidate this step in detail, it is beneficial to delve into the mathematical procedure. Assume the system of Eq. (9). A new set of coordinates will be introduced that eliminates all the constant forces when used. The new coordinates can be defined as $\mathbf{x}'(t) = \mathbf{x}(t) - \gamma$ and γ is obtained in accordance with Eq. (10)

below. Based on the different physical scenarios (friction, intermittent contact, staticity or any of their combinations) the corresponding formula of γ changes.

$$\gamma = \begin{cases} +\mathbf{K}^{-1} \times \mathcal{D} & \text{for sole friction} \\ -\mathbf{K}_{\text{red}}^{-1} \times \mathcal{C} & \text{for sole staticity} \\ +\mathbf{K}^{-1} \times \mathcal{F}_{cte} & \text{for sole intermittent contact} \end{cases} \quad (10)$$

After doing so, the governing equation of motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}}'(t) + \mathbf{C}\dot{\mathbf{x}}'(t) + \mathbf{K}\mathbf{x}'(t) - \mathbf{F}\sin(\omega t) = 0, \quad (11)$$

It is now feasible to implement modal transformation to decouple the coupled differential equations and obtain the equations of motion in modal coordinates. To do so, the modal coordinate is defined as $\mathbf{x}'(t) = \boldsymbol{\psi}\mathbf{q}(t)$. It should be noted that the modal matrix $\boldsymbol{\psi}$ should be obtained based on the system matrices (mass and stiffness since the system is assumed to be proportionally damped) corresponding to that interval and, upon having intermittent contact or staticity, it is expected to observe an alteration in $\boldsymbol{\psi}$. After applying the modal transformation, assuming a n degree of freedom *dynamic* system, the decoupled equations of motion in modal coordinates can be obtained as

$$\boldsymbol{\psi}^T \mathbf{M} \boldsymbol{\psi} \ddot{\mathbf{q}}(t) + \boldsymbol{\psi}^T \mathbf{C} \boldsymbol{\psi} \dot{\mathbf{q}}(t) + \boldsymbol{\psi}^T \mathbf{K} \boldsymbol{\psi} \mathbf{q}(t) = \boldsymbol{\psi}^T \mathbf{F} \sin(\omega t), \quad (12)$$

or alternatively

$$\ddot{q}_i(t) + 2\xi_i \omega_{n_i} \dot{q}_i(t) + \omega_{n_i}^2 q_i(t) = f_i \sin(\omega t), \quad i = 1, \dots, n. \quad (13)$$

Now, one can employ fundamental vibrational analysis to acquire the response of the decoupled differential equations. For an underdamped case

$$q_i(t) = e^{-\xi_i \omega_{n_i} t} \left[c_1 \sin(\omega_d t) + c_2 \cos(\omega_d t) \right] + \frac{f_i \sin(\omega t - \theta_i)}{\omega_{n_i}^2 \sqrt{\left(1 - \left(\frac{\omega}{\omega_{n_i}}\right)^2\right)^2 + \left(2\xi_i \frac{\omega}{\omega_{n_i}}\right)^2}}, \quad (14)$$

where f , ω_n , ω_d , ξ and ω represent the modal force, undamped natural frequency, damped natural frequency, viscous damping ratio and excitation frequency, respectively. c_1 and c_2 are constant terms that are identified based on the initial values of the system, and θ_i is calculated as $\theta = \tan^{-1}\left(\frac{2\xi_i \omega \omega_{n_i}}{\omega_{n_i}^2 - \omega^2}\right)$. The analytical response of the Eq. (11) is now obtained; this solution is only

valid for a limited interval where the equations of motion remain continuous. After the alteration of the equations of motion due to the effects of discontinuous force elements, the proposed coordinates systems are no longer correct and a new set of coordinate systems must be calculated and implemented. The coordinate systems used in the structure of HSNC are temporary otherwise the system would simply behave as a linear system.

Switches: Definition, categorization and detection

After reformulating the problem and obtaining the analytical response, the next step consists of finding the interval in which the discontinuous force vectors remain constant. To do so is to find the moments where the governing equations of motion switches from one continuous equation to another.

Definition 1. (Switch) Switch is the transition from one continuous interval of the governing equations to another. The discontinuities in the governing equations that result in a switch are caused by a change in contact status, a change in the direction of friction forcing or a change in the staticity state.

In order to find the switches in PWLDFE systems and to understand their underlying effects, it is essential to study them in detail. Two types of switches may occur in PWLDFE systems that have substantial differences and affect the system in dissimilar manners. Based on their impact on the dynamical behavior of the system, they are categorized as minor or major switches.

Major switch

Definition 2. (Major switch) A switch is labelled as major if it changes the dynamical characteristics of a subsystem.

There are two types of major switches in PWLDFE systems that occur as a result of the activation of intermittent contact pairs or the staticity of (a) subsystem/s.

Type I. Type I major switches are those that are triggered by intermittent contact pairs. When such switches occur, a change in matrices $\mathbf{C}$ or $\mathbf{K}$ or both is unavoidable and subsequently the dynamical characteristics of the system, such as natural frequencies and damping ratios, are affected.

Type II. Type II major switches take place when one subsystem gets stuck in its position. The staticity of a subsystem affects the governing equations of motion by transforming the connected system into two separated systems. This event partitions the mass, damping and stiffness matrices and prompts fundamental changes to the dynamical characteristics of the system. It is important to note that the return of a previously static subsystem to motion is also considered as this type of switch.

Next, consider the triggering condition for each type of major switch.

Triggering Condition (Type I). Assume that two adjacent subsystems have intermittent contact as depicted in Fig. (2). It is evident that the switch happens when the intermittent contact pair starts to experience contact (changing of the status of the intermittent contact pair from open to closed) or starts to lose

one (changing of the status of the intermittent contact pair from closed to open). Each of the scenarios happen when Eq. (15), called the gap function, switches its sign.

$$G_i(t) = x_i(t) - x_{i+1}(t) - \phi_i. \quad (15)$$

To have the formatting compatible with the structure of HSNC, the definition of Eq. (15) is rewritten in discrete time and without loss of generality as

$$G_i(k) = x_i(k) - x_{i+1}(k) - \phi_i, \quad (16)$$

where k represents the evaluation of the function in the k^{th} step. Consequently, the condition that triggers a type I major switch can be mathematically described as

$$G_i(k)G_i(k+1) < 0, \quad (17)$$

Although trivial, it should be noted that the switching condition might be triggered multiple times during the simulation but only the earliest should be considered since it represents the occurrence of a discontinuity in the equation of motion. In fact, the earliest switch of any type in any of the subsystems serves as a *critical* switch since it indicates the invalidity of the obtained solution for further evaluation of the system's response.

Triggering Condition (Type II). The triggering condition corresponding to type II major switches has complexities of its own because contrary to the preceding case, it cannot be summarized using one equation. To clearly discuss the triggering condition, it is best to separate the switch into two scenarios. The first scenario occurs when the i^{th} subsystem is moving and then reaches zero velocity. In this scenario, to check for the occurrence of a major switch requires checking if the i^{th} subsystem gets stuck or not. Mathematically speaking, for this scenario, the triggering condition is

$$|\text{Acting force on the } i^{th} \text{ subsystem}| < \mu_s m_i g, \quad (18)$$

which means that after the subsystem stops, the acting force cannot overcome static friction and the subsystem does not move. The acting force on the i^{th} subsystem is defined as the vector sum of all external forces acting on the mass including the spring, damper and excitation forces, and μ_s represents the coefficient of static friction.

The second scenario occurs when the i^{th} subsystem is static. The forces acting on the i^{th} subsystem are constantly checked to detect the first instance when the acting forces overcome static

friction and begin to move the stuck subsystem. Mathematically speaking this can be expressed as

$$|\text{Acting force on the } i^{th} \text{ subsystem}| > \mu_s m_i g. \quad (19)$$

In fact this condition ensures that the acting forces are greater than static friction.

Definition 3. (Minor switch) A switch is categorized as minor if its occurrence does not affect the dynamical characteristics of the subsystem.

The only type of minor switch for the discontinuities considered in this work is the alteration of the direction of friction where the subsystem does not get stuck (instantaneous staticity). This discontinuity will only result in a change in the sign of the friction force that is being exerted onto the corresponding subsystem. Minor switches are still very important since they do generate a discontinuity in the governing equations of motion that prevent the use of linear techniques.

Triggering Condition. The minor switches occur as a result of a change in the direction of the motion of a subsystem and their triggering condition corresponds to a change in the sign of the velocity of a subsystem. This can be expressed as

$$\begin{aligned} \dot{x}_i(k)\dot{x}_i(k+1) < 0 \quad \text{and} \\ |\text{Acting force on the } i^{th} \text{ subsystem}| > \mu_s m_i g, \end{aligned} \quad (20)$$

It should be noted that the triggering conditions for each type of switch must be checked when evaluating the analytical solution in Eq. (14). To efficiently detect these switches one can use an adaptive numerical scheme.

Adaptive numerical techniques To check for the occurrence of switches, it is required to evaluate the response of the system using the analytical response given in Eq. (14). It is noteworthy to mention that even for a specified system with fixed parameters, consecutive switches may happen in a variety of time scales (e.g. ranging from milliseconds to tens of seconds). If one used a constant time step to evaluate the response, then one would need a tiny time step, which would place an unnecessarily heavy computational burden on the method. To cope with this issue, two time steps are proposed that attempt to create an adaptive structure for faster evaluation. The smaller time step is used to construct a vector that evaluates the response of the system for a small amount of time (usually 200 – 1000 steps) and checks for the occurrence of any early switches. If no switch is detected, the method changes to the larger time increment (about 190 – 990 times larger) and evaluates the response up to a predetermined time (usually thrice the period of the external excitation to ensure the capture of at least one switch). After the detection

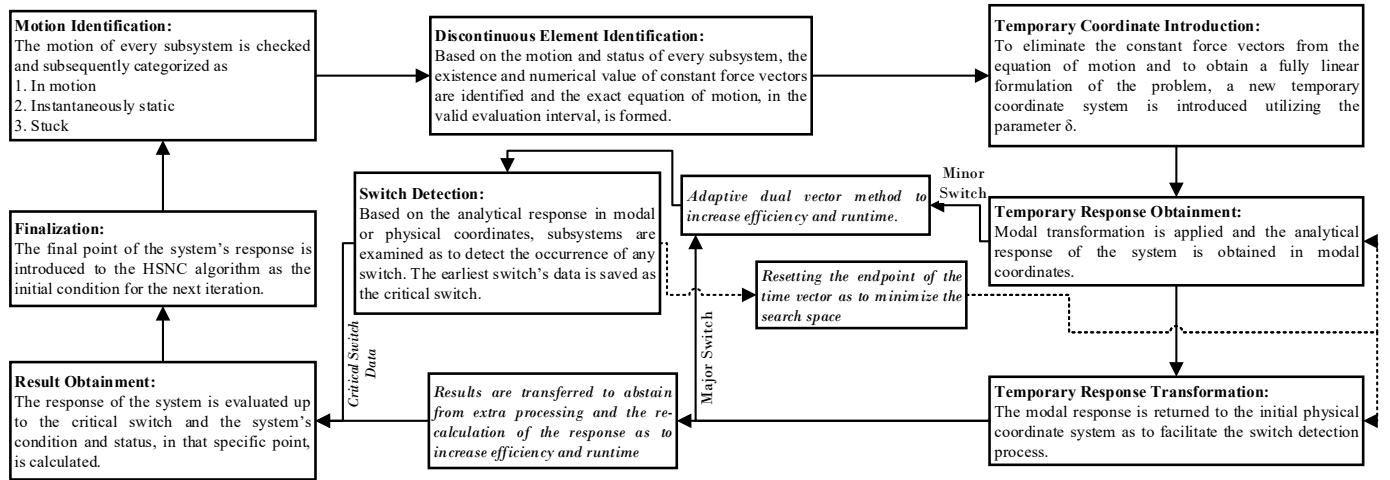


FIGURE 3: DETAILED SCHEMATIC VIEW OF THE HSNC ALGORITHM FOR PWLDFE SYSTEMS.

of one switch (e.g., a minor switch) the time vector's final value is changed from the predetermined time to the time where the switch was found to shorten the search space and the same algorithm goes on to identify all the switches.

As stated previously, the triggering conditions of switches indicate the change in the sign of a known function and therefore the algorithm only identifies the occurrence of a switch in an interval of the size of the time step. To significantly enhance the accuracy of the method, the MATLAB's *fzero* nonlinear solver method is utilized to spot the exact time where the switch condition is triggered. Instead of providing the solver with one initial guess, the two time instances before and after the triggering of the switch condition are provided as the bounds. This makes the solving process fast and accurate.

PWLDFE system response By checking for the occurrence of all the switches (minor and major) in all of the subsystems, a vector is created which contains all the time instances where switches happen. It is known that the obtained responses are only correct until the first switch occurs (where the equation of motion remains continuous). The minimum value of the switch time vector is the final time where the results are valid and the response should be obtained up to that point and that point should serve as the initial conditions for the next iteration of response evaluation. This same process then repeats stitching together the individual linear responses to obtain the complete response of the PWLDFE system. Figure 3 demonstrates a detailed schematic view of the HSNC algorithm used in this work.

RESULTS

To demonstrate the effectiveness of the method the system shown in Fig. (4) is studied. The system under study is a 2-DOF mass-spring-damper system with friction that experiences intermittent contact between the two masses.

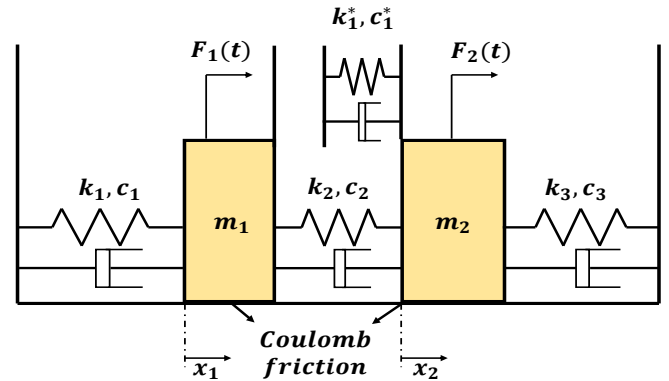


FIGURE 4: TWO DOF PWLDFE SYSTEM STUDIED.

The parameters of the aforementioned system are selected to demonstrate two scenarios that highlight the effectiveness of the HSNC method. The parameters of the system, for both scenarios, are provided in Table 1 and are all in SI units. β represents the proportionality coefficient and is used to calculate the damping matrix as $C = \beta K$.

In the first scenario, the parameters are chosen to demonstrate the effectiveness of HSNC over NI when a very small step

TABLE 1: NUMERICAL VALUES OF THE SYSTEM PARAMETERS FOR SIMULATION.

Scenario	m_1	m_2	k_1	k_2	k_3	β	μ_k	μ_s	F_1	F_2	ω	k^*	ϕ_1
1	1	2	5	10	40	0.1	0.6	0.75	22500	-15000	175.5151	100	0.2
2	1	2	5	10	40	0	0.6	0.75	225	150	17.5515	100	0.2

size is needed. The second scenario highlights a deficiency in NI methods that cannot be fixed by reducing the step size since the motion is weakly chaotic and the small numerical error in NI will cause a systematic failure in obtaining accurate results. The validity, accuracy and runtime of HSNC are compared to NI to illustrate the capabilities of the method.

The NI method used in this work is the Adams-Bashforth four-step method with a fixed time increment. For the first scenario, the response of the system is obtained using the HSNC method and then compared with the NI method with two different step sizes ($\varepsilon = 1e^{-3}$ and $\varepsilon = 1e^{-6}$). The response computed by NI for both time steps is shown in Fig. (5) and highlights the change in solution from the two different time steps. A more complicated adaptive time step approach would be required to accurately solve this problem using NI, which would be computationally expensive and require a very small time step to obtain an accurate solution. It should be noted that the initial conditions of the simulation for both scenarios are $x_1(0) = 0.2, x_2(0) = -0.1, \dot{x}_1(0) = -5$ and $\dot{x}_2(0) = -2$.

As mentioned in the introduction, PWLDFE systems are not only important from a modeling perspective but are also theoretically significant since they are capable of producing a wide range of dynamical motions. Simple PWL systems are known to illustrate chaotic motion as study [20] suggests. The prior example of this section also demonstrates a case where this system produces a steady-state periodic motion. In addition to the aforementioned motions, this work will show that PWLDFE systems are also capable of illustrating weakly chaotic motion where the divergence of two infinitesimally close trajectories does not occur exponentially. It is important to note that this weakly chaotic motion has not previously been reported for mechanical systems. The response of the system exhibiting weakly chaotic motion is presented in Fig. 6

Figure 6 shows the trajectories of the HSNC response start to diverge from the NI response; however, unlike chaos that diverges exponentially, this weakly chaotic system diverges slowly and is only clearly seen after nearly 100 seconds. It is also important to note that the runtime of HSNC is significantly smaller; for the simulation in Fig. 6, NI takes 900 seconds while HSNC take 6.31 seconds, which is about **142 times faster** than NI. Also as the dimension of the system grows, HSNC's computational

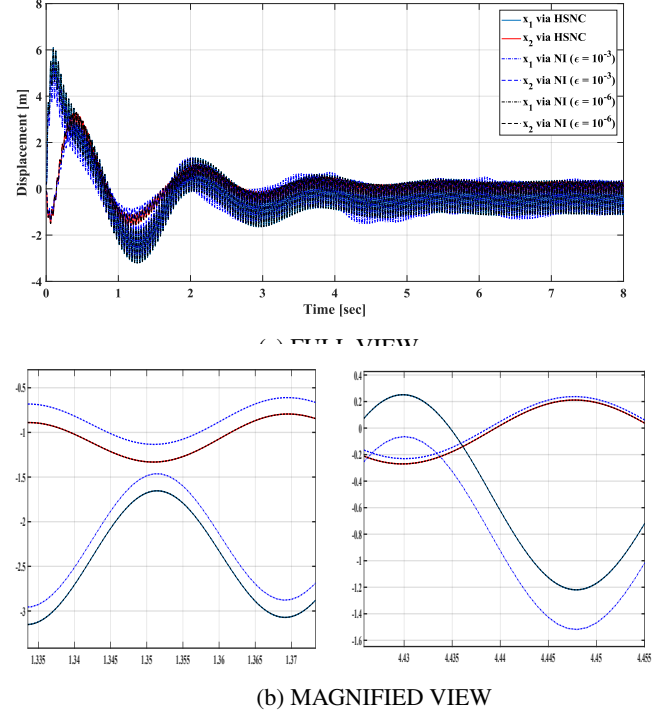
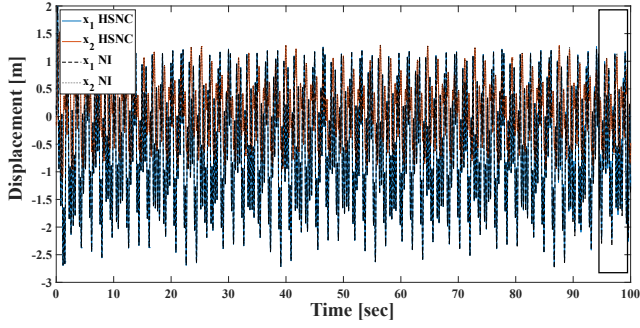


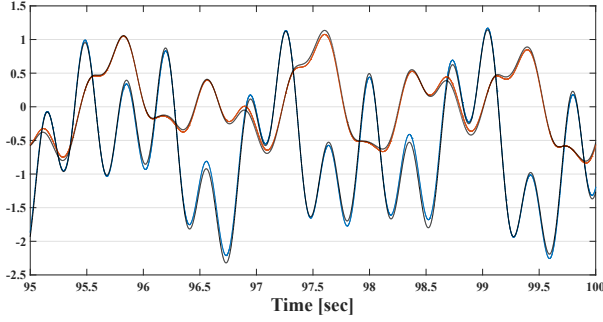
FIGURE 5: TIME EVALUATION OF THE SYSTEM VIA HSNC AND NI WITH 2 DIFFERENT STEP SIZES.

performance with respect to NI continues to increase significantly [20]. Although one might think that the divergence of NI from the response of HSNC is a result of numerical error (and not weak chaos), but this is not the case, which will be demonstrated in the next example.

To better demonstrate the weakly chaotic behavior, the phase portrait of the system is presented in Fig. 7. In fact, Fig. 7 illustrates the existence of the elements of chaos in the system. Since the majority of the response computed by HSNC is done analytically, with only computational tools to determine switches in the system, its response is closer to the exact response and weak chaos will not affect the accuracy of its outcome as severely as NI, which introduces computational errors at every time step. To highlight the weakly chaotic behavior of the PWLDFE system



(a) FULL VIEW



(b) MAGNIFIED VIEW

FIGURE 6: TIME EVALUATION OF THE SYSTEM VIA HSNC AND NI TO ILLUSTRATE WEAK CHAOS.

under study, the initial conditions are varied slightly ($\delta = 1e^{-6}$) and the response is obtained using HSNC. As it can be observed from Fig. (8), the divergence occurs very slowly.

All the simulations and scenarios of this section are computed using a msi GL63 8RC laptop and it should be noted that the runtime of HSNC varies based on the choice of numerical methods in this algorithm. Nonetheless, regardless of the processing unit, HSNC will be hundreds of times faster than conventional NI methods.

CONCLUSION AND DISCUSSION

The importance of piecewise linear systems with discontinuous force elements (PWLDFFE), due to their modeling capabilities and their significance from theoretical perspectives, are highlighted in this work. They are known to be very accurate models of numerous physical phenomena or to be highly compatible simplifications of more sophisticated mechanical assemblies. Their importance necessitates methods that can obtain their responses efficiently and accurately. Consequently a novel method was proposed for this class of nonlinear systems that relied on a fusion of numerical and analytical methods. The proposed method is shown to be significantly faster than conventional NI methods. Moreover, the HSNC method presses its advantage in

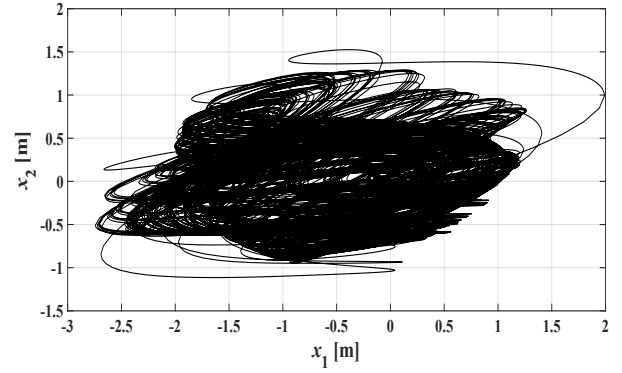
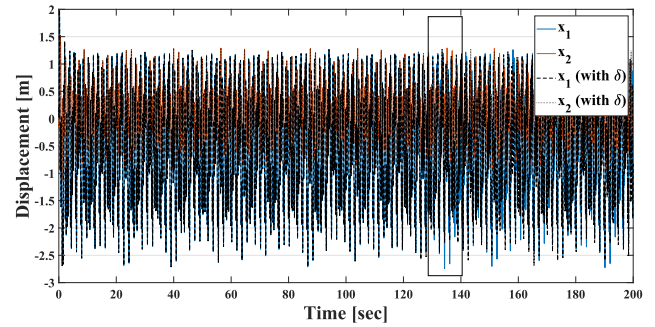
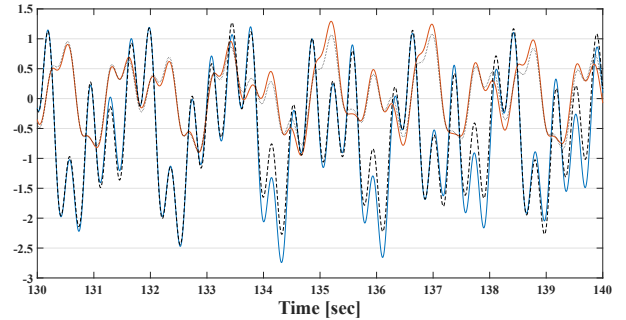


FIGURE 7: PHASE PORTRAIT OF THE SYSTEM UNDER STUDY INDICATING A STRANGE ATTRACTOR OBTAINED USING HSNC.



(a) FULL VIEW



(b) MAGNIFIED VIEW

FIGURE 8: DIVERGENCE OF TWO INFINITESIMALLY CLOSE TRAJECTORIES DUE TO SLOW CHAOS ($\delta = 1e^{-6}$).

cases where the PWLDFFE systems demonstrate signs of chaos. The existence of chaos makes the use of NI methods ineffective, since the errors in these methods will cause the system to eventually diverge from its true response. It is also noteworthy to state that the PWLDFFE systems have been observed to demonstrate weakly chaotic motion in this work, which has not been

previously observed in mechanical systems.

The occurrence of weak chaos is a phenomenon that requires more attention since its characteristics vary from conventional chaos. Weak chaos is a relatively untouched subject since it has been only observed to happen in limited cases [24]. The observation of this phenomenon in this work, in a mechanical assembly, can lead to a more detailed exploration and their possible application in numerous fields ranging from nonlinear dynamics to cryptography.

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