BIG IMAGES OF TWO-DIMENSIONAL PSEUDOREPRESENTATIONS

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Abstract. Bellaïche has recently applied Pink-Lie theory to prove that, under mild conditions, the image of a continuous 2-dimensional pseudorepresentation ρ of a profinite group on a local pro-p domain A contains a nontrivial congruence subgroup of $\mathrm{SL}_2(B)$ for a certain subring B of A. We enlarge Bellaïche's ring and give this new B a conceptual interpretation both in terms of *conjugate self-twists* of ρ, symmetries that constrain its image, and in terms of the *adjoint trace ring* of ρ, which we show is both more natural and the optimal ring for these questions in general. Finally, we use our purely algebraic result to recover and extend a variety of arithmetic big-image results for GL_2 Galois representations arising from elliptic, Hilbert, and Bianchi modular forms and p-adic Hida or Coleman families of elliptic and Hilbert modular forms.

Let $\rho : \Pi \to \mathrm{GL}_2(A)$ be a continuous representation of a profinite group Π on a local pro-p domain A. How big or small can the image of ρ be? Arithmetic versions of this question, with ρ the p-adic Galois representation attached to a cuspidal modular eigenform — or more often a compatible family of such ρ considered adelically — have been considered since the 1960s, when Serre proved an adelic open-image result for the Galois representation on the Tate module of a non-CM elliptic curve [Ser68]. Serre's result was adapted by Ribet [Rib85][(i)] and then by Momose [Mom81] to the more delicate setting of modular forms, where certain symmetries naturally bound the size of the image. Recently Nekovář [Nek12] generalized their work to Hilbert modular forms.

In the 1990s Pink began a purely algebraic study of this kind of question by characterizing closed pro-p subgroups of $\mathrm{SL}_2(A)$ in terms of associated "Pink-Lie" algebras. Pink's investigations fueled further exploration of arithmetic big-image questions, this time for Galois representations attached to p-adic Hida families of modular forms, by Hida [Hid15] and then, accounting for Ribet–Momose-type symmetries, by Lang (second author here) [Lan16]. Simultaneously, Bellaïche began adapting Pink-Lie theory to the (pseudo)representation setting, obtaining abstract big-image results, but in a form that was difficult to compare to the symmetry formulations of Ribet, Momose, and Lang.

In the present work we finally unite the two approaches, refining Bellaïche-Pink-Lie theory to relate p-adic big-image results to natural symmetry bounds in an abstract algebraic setting. We thereby recover the p-adic big-image results of Ribet, Momose, Nekovář, and Lang, improving the latter, under mild conditions on $\bar{\rho}$. We also obtain the first big-image results for Galois representations attached to p-adic Coleman families of modular forms (rather than for associated rigid-analytic Lie algebras, as in Conti(first author here)–Iovita–Tilouine [CIT16]), p-adic families of Hilbert modular forms, and Bianchi modular forms. Along the way we propose shifting our perspective towards formulating big-image results in terms of rings of definition of adjoint representations rather than in terms of rings fixed by symmetries. We emphasize that our results require absolutely no arithmetic input and are provably optimal. We explain how our results apply to a wide variety of modular Galois representations, and we anticipate that this framework can yield even more arithmetic fruit, from understanding *even* Galois representations to relating reducibility/dihedrality ideals and automorphic congruence modules. Finally we hope that the algebraic nature of our results might portend similar phenomena in higher dimension.

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(i) Ribet's work appeared in a series of papers starting in 1975; [Rib85] is a convenient reference.

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1. Introduction

1.1. **The question.** Let p be an odd prime, A a local pro-p domain with maximal ideal $\mathfrak{m}$ and (finite) residue field $\mathbb{F} := A/\mathfrak{m}$, and Π a profinite group. Let $\rho\colon \Pi \to \mathrm{GL}_2(A)$ be a continuous representation[ii] with the property that the residual representation $\bar\rho := \rho \bmod \mathfrak{m}$ is semisimple and *multiplicity free*: either absolutely irreducible, or a sum of two distinct characters to $\mathbb{F}^\times$. Roughly, the objective is to show that the image of ρ is as big as possible.

Note that if ρ, or its restriction to an index-2 subgroup of Π, is reducible, then the image of ρ is both well understood and not big. Similarly, one cannot expect a big-image result when the image of ρ is up to twist isomorphic to that of $\bar\rho$, as happens when ρ arises from a modular form of weight one. Let us call these three kinds of representations *a priori small*. The a priori small representations are exactly those that are not strongly absolutely irreducible.

Suppose now that ρ is not a priori small. We cannot expect ρ to be surjective: even its determinant need not be surjective. Nor can we expect the image of ρ to contain all of $\mathrm{SL}_2(A)$, unless the image of $\bar\rho$ contains all of $\mathrm{SL}_2(\mathbb{F})$. Following ideas of Hida, we settle on the notion of *fullness*. If B is any ring and $\mathfrak{b} \subseteq B$ is any nonzero ideal, the subgroup of $\mathrm{SL}_2(B)$ given by the kernel of reduction modulo $\mathfrak{b}$ is a *congruence subgroup* of $\mathrm{SL}_2(B)$ (of *level* $\mathfrak{b}$):

$$\Gamma_B(\mathfrak{b}) := \ker\big(\mathrm{SL}_2(B) \to \mathrm{SL}_2(B/\mathfrak{b})\big).$$

[ii]In fact we consider 2-dimensional pseudorepresentations, but we stick with representations for the introduction.

If the image of ρ, up to conjugation, contains such a congruence subgroup, we say that ρ is *B-full*. A key part of the big-image game is the search for an optimal fullness ring — or rather for an optimal equivalence class of *fullness peers*, rings that each contain an ideal of the other.

Historically, the constraints on the fullness rings of a representation ρ have been described in terms of certain symmetries of ρ. If σ is an automorphism of A and η is a character of Π, the pair (σ, η) is a *conjugate self-twist* of ρ if applying the automorphism gives the same representation as twisting by the character: ${}^{\sigma}\rho \cong \eta \otimes \rho$. If ρ has a nontrivial conjugate self-twist (σ, η), then ρ cannot be A-full: indeed, the equation

$$\sigma\big(\operatorname{tr}\rho(g)\big) = \operatorname{tr}{}^{\sigma}\rho(g) = \eta(g)\operatorname{tr}\rho(g) \tag{1}$$

means that the trace of $\rho(g)$ is an eigenvector for σ viewed as a linear map over the σ-invariant scalars. But the trace of a congruence subgroup of A is not so constrained. Accordingly, the known arithmetic big-image results — of Ribet and Momose, of Nekovář, of Lang, described in Section 1.2 below — have all established fullness with respect to A^{Σ_ρ}, the subring of A fixed by the conjugate self-twists of ρ.

Stepping outside the constraints of the arithmetic setting, however, reveals shortcomings of the A^{Σ_ρ} perspective: A^{Σ_ρ} may simply be too big for ρ to be A^{Σ_ρ}-full. For one thing, A itself may not see all the conjugate self-twists of ρ, as in Example 4.3, a failure of normality. Enlarging A may still not suffice if A has inseparable elements, as in Example 4.10, or worse yet, transcendental ones. The limitations are exactly those of Galois theory: there may not be enough automorphisms to carve down deep enough to a fullness ring with conjugate self-twists alone.

Rather than carving down from above, we propose building a fullness ring from below. Let A_0 be the *adjoint trace ring* of ρ, the closed subring of A topologically generated by the elements $(\operatorname{tr}\rho(g))^2/\det\rho(g)$ for $g \in \Pi$, the traces of the adjoint representation.

It is easy to see that this generating set is both twist-invariant and fixed by all conjugate self-twists. Thus A_0 acts as a base ring for the conjugate self-twist automorphisms, and the question of whether A or its extensions have enough conjugate self-twists turns into the usual one of Galois theory: are there enough automorphisms to isolate the base? On the other hand, A_0 is a potential fullness ring free from the limitations of A^{Σ_ρ} outlined above. Moreover, in all arithmetic settings where we recover existing fullness results, we show that A_0 and A^{Σ_ρ} are always fullness peers, so that A_0- and A^{Σ_ρ}-fullness are equivalent. Our first theorem shows that A_0 is the optimal fullness ring in all cases.

Theorem A (Optimality theorem. See Theorem 5.3)**.**
If ρ is B-full for some ring B, then a fullness peer of B is contained in A_0.

Accompanying the A_0-optimality theorem, we present the main result of this paper: A_0-fullness. We assume that the pro-p part of Π is topologically finitely generated: for more on this *p-finiteness* condition of Mazur, always satisfied by both local and almost-everywhere-unramified global Galois groups, except in characteristic p, see Definition 2.6.

Theorem B′ (Main fullness theorem, preliminary version. See Theorem B and Theorem 10.1)**.**
Suppose that $\bar\rho$ satisfies a mild condition. If ρ is not a priori small, then ρ is A_0-full.

For a more precise formulation and a detailed discussion of the mild condition, see Theorem B in Section 1.3 below. For now we merely note that the most limiting condition for applications is a *regularity* assumption: we require that the image of $\bar\rho$ contain a matrix whose eigenvalue ratio differs from ± 1 but is contained in residue field of A_0. For the most general version of our theorem, which is formulated for images of *pseudorepresentations* in the sense of Chenevier [Che14] as studied by Bellaïche [Bel19], see Theorem 10.1.

The last stand-alone result of this paper is a refinement of Theorem B′ in the case where the residual image of ρ is large. Let $\mathbb{E}$ be the residue field of A_0; here we assume that $\#\mathbb{E} \geq 7$.

Theorem C (See Corollary 11.2)**.** *If* $\operatorname{Im}\bar\rho \supseteq \mathrm{SL}_2(\mathbb{E})$ *then* $\operatorname{Im}\rho$ *contains* $\mathrm{SL}_2(A_0)$ *up to conjugation.*

1.2. **History.** We now survey the history of big-image results, both arithmetic and algebraic, using the terminology introduced above, to situate Theorem B′ in context. In all of the theorems stated in Section 1.2, A_0-fullness is equivalent to A^{Σ_ρ}-fullness.

1.2.1. *Classical modular forms.* The big-image line of inquiry began in the late 1960s, when Serre showed that if ρ comes from the p-adic Tate module (including for $p = 2$) of a non-CM elliptic curve over a number field F, so that $\Pi = \mathrm{Gal}(\overline{F}/F)$ and $A = A_0 = \mathbb{Z}_p$, then ρ is $\mathbb{Z}_p$-full [Ser68, Theorem IV.2.2].[iii]

In the 80s, Ribet and Momose generalized Serre's theorem to elliptic modular forms. Let f be a cuspidal non-CM eigenform of weight at least 2. Given a prime p and an embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, one can associate to f a 2-dimensional Galois representation $\rho = \rho_{\iota_p}$ of $\Pi = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over the ring of integers A of a finite extension $\mathbb{Q}_p$.

Theorem 1.1 ([Rib85, Theorem 3.1], [Mom81, Theorem 4.1]; see also Theorem 12.2)**.** *For all but finitely many primes p, the representation ρ is A^{Σ_ρ}-full.*[iv]

More recently, Nekovář generalized Theorem 1.1 to representations coming from Hilbert modular forms, in which case Π is the absolute Galois group of a totally real number field and A is still a finite extension of $\mathbb{Z}_p$ [Nek12, Appendices B.3–B.6].

Our main theorem (Theorem B′) recovers the at-p statements of both the Ribet–Momose and the Nekovář results, under the assumption that residual representation satisfies our regularity condition. See Section 12.1 and Section 12.2 for details.

1.2.2. *Families of p-adic modular forms.* Although we have stated the work of Serre, Ribet, Momose, and Nekovář for a fixed prime p to better fit our p-adic framework, all of these theorems are actually adelic open-image results proved using geometric methods. Much work has been done to generalize such theorems to groups other than GL_2, but that is not the direction that interests us. Rather, we are interested in fixing p and deforming representations p-adically, which necessitates a completely different approach. There has been some progress in this direction in special cases. Recall that we are assuming throughout that $p \neq 2$.

First we suppose that ρ arises from a non-CM cuspidal Hida family. In this case $\Pi = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and A is a finite domain over $\Lambda := \mathbb{Z}_p[\![X]\!]$. When A is a constant extension of Λ and the image of $\bar{\rho}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$, Boston [MW86, Proposition 3] and Fischman [Fis02, Theorem 4.8] show that the image of ρ contains $\mathrm{SL}_2(A^{\Sigma_\rho})$; hence ρ is A^{Σ_ρ}-full.[v]

More recently, Hida proved that if $\bar{\rho}$ is locally-at-p multiplicity free then ρ is Λ-full [Hid15, Theorem I], but his work did not relate Λ to A_0 or conjugate self-twists of ρ. Lang then improved Hida's result from Λ-fullness to A^{Σ_ρ}-fullness under the assumption that $\bar{\rho}$ is absolutely irreducible, proving the following result.

Theorem 1.2 ([Lan16, Theorem 2.4]; see also Theorem 12.11)**.** *If ρ arises from a non-CM cuspidal Hida family, and $\bar{\rho}$ is absolutely irreducible and satisfies additional multiplicity-freeness conditions locally at p then ρ is A^{Σ_ρ}-full.*

The case when ρ arises from a Coleman family was studied by Conti–Iovita–Tilouine [CIT16]. In this case we again have $\Pi = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and A is a domain finite over Λ. In [CIT16, Theorem 6.2] it is proved that, under hypotheses similar to those in Theorem 1.2, a certain rigid analytic Lie algebra attached to $\mathrm{Im}\,\rho$ contains that of a congruence subgroup of A^{Σ_ρ}. This strongly suggests that ρ should be A^{Σ_ρ}-full, though this statement does not follow from [CIT16].

[iii] Serre's result is better known as an open-image theorem; and in fact he shows much more: the image of all the p-adic Tate modules for all p at once is open adelically.

[iv] Like Serre, Ribet and Momose prove stronger adelic big-image results. See Section 12.1 for more details.

[v] All the works in Section 1.2.2 consider only conjugate self-twists that fix Λ; see Section 12.4 for details.

1.2.3. *Abstract p-adic representations.* Both Hida [Hid15] and Lang [Lan16] rely in a key way on results of Pink [Pin93] classifying, for odd p, pro-p subgroups of $\mathrm{SL}_2(A)$ in terms of a correspondence with purely algebraically defined "Pink-Lie algebras". The analogous role in [CIT16] is played by rigid-analytic Lie theory, whence the different form of the conclusion in that case. Although the conclusions of the big-image theorems in all of [Hid15, Lan16, CIT16] are stated in terms of pure algebra — a feature that is most clear in the fullness results of [Hid15] and [Lan16] — nonetheless all of their proofs are arithmetic in nature: they rely on special information about the restriction of ρ to the local Galois group at p, and they use the results of Ribet and Momose as input.

In contrast, Bellaïche in [Bel19] studies the image of $\rho\colon \Pi \to \mathrm{GL}_2(A)$ in a purely algebraic way. More precisely, he applies Pink's theory from [Pin93] to images of 2-dimensional (pseudo)representations with *constant determinant*, that is, with $\det \rho$ equal to the Teichmüller lift of $\det \bar{\rho}$. Bellaïche's main application is to density results for mod-p modular forms, but along the way he also proves the following theorem, under the same p-finiteness assumption on Π (Definition 2.6).

Theorem 1.3 ([Bel19, Theorem 7.2.3]; see also Theorem 2.23)**.** *Suppose that the image of $\bar{\rho}$ contains an element with eigenvalues in $\mathbb{F}_p^\times$ whose ratio is not ± 1. If ρ has constant determinant and is not a priori small, then there is a subring $\mathcal{B}_\rho(\mathbb{F}_p)$ of A such that ρ is $\mathcal{B}_\rho(\mathbb{F}_p)$-full.*

Bellaïche's ring $\mathcal{B}_\rho(\mathbb{F}_p)$ is defined as the subring of A topologically generated by a $\mathbb{Z}_p$-module $I_1(\rho)$ created out of the Pink-Lie algebra of $\mathrm{Im}\,\rho$. See Theorem 2.23 and the discussion following it for the definition of the generating set $I_1(\rho)$ and the ring $\mathcal{B}_\rho(\mathbb{F}_p)$.

Unfortunately, as Bellaïche himself notes, it is not straightforward to relate the ring $\mathcal{B}_\rho(\mathbb{F}_p)$ to the rings A_0 or A^{Σ_ρ} from previous results. Indeed, he gives no conceptual interpretation of $\mathcal{B}_\rho(\mathbb{F}_p)$ at all. The goal of the present work is to refine the definition of $\mathcal{B}_\rho(\mathbb{F}_p)$ by enlarging scalars and then give it a conceptual interpretation. Under mild assumptions we thus recover, and in the case of p-adic families improve, the results mentioned above in a uniform and purely algebraic way. We point out that prior to Bellaïche's work, Hida's work was the only fullness result when $\bar{\rho}$ is reducible and ρ comes from a p-adic family of modular forms. In the case of Coleman families, a true fullness result was not previously known. Additionally, we obtain first results in other GL_2 contexts, including Galois representations attached to Bianchi modular modular forms and to p-adic families of Hilbert modular forms. See Section 12 for all the details.

1.3. **Main theorem.** We now state our main fullness theorem in more detail. Recall that A is a local pro-p domain with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$ and Π is a p-finite profinite group. Let $\rho : \Pi \to \mathrm{GL}_2(A)$ be a representation with mod-$\mathfrak{m}$ reduction $\bar{\rho}$. Let A_0 be the adjoint trace ring of ρ and $\mathbb{E}$ its residue field, so that $\mathbb{E} \subseteq \mathbb{F} \subseteq \overline{\mathbb{F}}_p$. We say that $\bar{\rho}$ is *regular* if there is some $g_0 \in \Pi$ such that $\bar{\rho}(g_0)$ has eigenvalues $\lambda_0, \mu_0 \in \overline{\mathbb{F}}^\times$ with $\lambda_0\mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$. See Remark 2.20 for an analysis of this condition. For the notion of *goodness*, see Definition 8.10.

Theorem B (Main fullness theorem; see also Theorem 10.1)**.**
Assume that $\bar{\rho}$ is regular. If the projective image of $\bar{\rho}$ is isomorphic to S_4, assume that $\bar{\rho}$ is good. If ρ is not a priori small, then ρ is A_0-full.

In fact we prove something slightly more general in that we can replace ρ by a pseudodeformation $(t, d)\colon \Pi \to A$ of $\bar{\rho}$: see Theorem 10.1 for a precise statement. Recall that Theorem B is provably optimal, in the sense that, if ρ is B-full for some subring B of A, then a fullness peer of B is contained in A_0 (see Theorem A or Theorem 5.3).

Let us point out some features of the statement of Theorem B. First, the group Π can be quite general. For that reason, representations coming from Hilbert modular forms and their p-adic families are no more difficult than representations coming from elliptic modular forms or their p-adic families. Similarly, since Π need not be the absolute Galois group of a number field, the notion of oddness does not play a role in the paper. In particular, Theorem B applies to deformations of $\bar{\rho}$ when $\bar{\rho}$ is an even Galois representation.

The proof of Theorem B proceeds via Bellaïche-Pink-Lie theory with various refinements and improvements. As in Bellaïche [Bel19], we linearize the pro-p normal core of the image of a constant-determinant ρ by considering its Pink-Lie algebra L. But rather than building up an A_0-structure on L directly and interpreting this for $\operatorname{Im}\rho$, we proceed by showing fullness in turn for a sequence of fullness peer rings, first $\mathcal{B}_\rho(\mathbb{E})$, and then A^{Σ_ρ}, and finally A_0, which then proves A_0-fullness for all the twists of ρ. More precisely, the argument proceeds in steps as follows.

(1) We show that fullness rings and adjoint trace rings are unchanged under twisting by a character (Corollary 3.12 and Proposition 4.8), so that we may assume that ρ has constant determinant.

(2) For constant-determinant ρ, we show that A_0 and A^{Σ_ρ} are fullness peers (Corollary 4.21). Therefore it suffices to prove A^{Σ_ρ}-fullness in the constant-determinant case.

(3) We refine Theorem 1.3 to show that a constant-determinant ρ is $\mathcal{B}_\rho(\mathbb{E})$-full (Corollary 6.6). Here $\mathcal{B}_\rho(\mathbb{E})$ is the $W(\mathbb{E})$-algebra generated by the same $\mathbb{Z}_p$-module $I_1(\rho)$ as Bellaïche's $\mathcal{B}_\rho(\mathbb{F}_p)$ — see discussion following Theorem 1.3 above. Although a small improvement, this is crucial for the next step, and it allows our regularity hypothesis to be weaker than that of Bellaïche.

(4) We show that for ρ with constant determinant, $\mathcal{B}_\rho(\mathbb{E})$ and A^{Σ_ρ} are fullness peers (Corollary 9.15), so that ρ is A^{Σ_ρ}-full.

More precisely, we show that $\mathcal{B}_\rho(\mathbb{E})$ is always contained in A^{Σ_ρ}, and the discrepancy between them is explained on the residue field by a failure of conjugate self-twists of $\bar{\rho}$ to lift to ρ. The residue field of $\mathcal{B}_\rho(\mathbb{E})$ is $\mathbb{E} = \mathbb{F}^{\Sigma_{\bar{\rho}}}$, the fixed field of residual twists (Corollary 4.13), whereas the residue field of A^{Σ_ρ} is $\mathbb{F}^{\Sigma_\rho}$, the field fixed by those residual twists that lift to twists of ρ. We show both that $A^{\Sigma_\rho} = \mathcal{B}_\rho(\mathbb{F}^{\Sigma_\rho})$ and that the elements of $\mathbb{F}^{\Sigma_\rho}$ "missing" in $\mathcal{B}_\rho(\mathbb{E})$ show up in its fraction field. This is the most delicate step of the argument.

We do not show that non-constant-determinant ρ are A^{Σ_ρ}-full: Examples 4.3 and 4.10 show that such a result is impossible in general, even under favorable regularity conditions. The key point is that A^{Σ_ρ} depends on ρ itself and may chance after twisting, whereas A_0 depends only on the projective representation of ρ. Nor do we show that $A_0 = A^{\Sigma_\rho}$ for constant-determinant ρ — they are merely fullness peers: see Example 4.22.

The circuitous nature of our argument may well be merely a matter of our historical bias: the known arithmetic big-image results are formulated in terms of conjugate self-twists, so our original motivation was to relate Bellaïche's $\mathcal{B}_\rho(\mathbb{F}_p)$ to A^{Σ_ρ}. The advantages of A_0 revealed themselves only later. A more direct argument for proving Theorem B is the subject of our current investigation.

1.4. **Structure of the paper.** This article is informally organized into parts as follows.

Background: Section 2. We review known material: pseudorepresentations, generalized matrix algebras, Pink-Lie theory, and Bellaïche's recent results from [Bel19]. We also introduce our notion of regularity.

Our philosophy: In Sections 3 to 5 we present and justify our approach: our goal is to show that ρ is full with respect to the adjoint trace ring A_0, which is both the optimal fullness result and equivalent to the historically familiar A^{Σ_ρ}-fullness of known applications.

– Section 3 discusses the basic properties of fullness and fullness peer rings. We prove that fullness is twisting-invariant (Corollary 3.12).

– Section 4 studies the relationship between A_0 and conjugate self-twists, paying particular attention to constant-determinant, and nearly so, settings where conjugate self-twists carve out the fraction field of A_0 (Corollary 4.20), so that A_0 and A^{Σ_ρ} are fullness peers.

– Section 5 proves two related optimality results: every fullness ring has a fullness peer contained in A_0 (Corollary 5.2) and is also fixed by all conjugate self-twists (Theorem 5.4).

Technical results: Sections 6 to 9 are the technical heart of the paper. We prove that constant-determinant ρ are $\mathcal{B}_\rho(\mathbb{E})$-full, which implies A^{Σ_ρ}-fullness, which in turn guarantees A_0-fullness.

- In Section 6 we show that the Bellaïche-Pink-Lie algebra L attached to ρ, a priori only a $\mathbb{Z}_p$-module, is in fact a $W(\mathbb{E})$-module. Under certain regularity assumptions, ρ is therefore $\mathcal{B}_\rho(\mathbb{E})$-full. This completes Step (3) of the proof.
- In Section 7 we study residual conjugate self-twists and their lifting properties via the universal deformation ring — how does $\Sigma_{\bar{\rho}}$ compare to Σ_ρ?
- Section 8 explores how the regularity of $\bar{\rho}$ imposes structure on residual conjugate self-twists in preparation for Section 9. We also introduce the *goodness* constraint.
- Section 9 is the technical heart of the technical heart of the paper. Its main goal is to prove that $\mathcal{B}_\rho(\mathbb{E})$ has the same field of fractions as A^{Σ_ρ} to show that they are fullness peers (Corollary 9.15), completing Step (4) of the proof. This turns out to be intimately related to the lifting properties of conjugate self-twists of $\bar{\rho}$ to ρ explored in Section 7.

Interpretation and applications: Sections 10 to 12 interpret our results for general ρ and explain how to apply them to various modular form contexts in detail.

- In Section 10 we derive our main fullness results: Theorem 10.1 or Theorem 10.3.
- Section 11 is independent of the main thrust of the paper; it gives an improvement on previous very-big-image results, showing that the image of ρ contains $\mathrm{SL}_2(A_0)$ if it does so residually.
- Section 12 explains in detail how and to what extent Theorem B recovers and refines known big-image results about Galois representations arising from modular forms and their p-adic families. We also apply Theorem B to obtain new results for Galois representations attached to Bianchi modular forms (Section 12.3), Coleman families of elliptic modular forms (Section 12.5), and p-adic families of Hilbert modular forms (Section 12.6).

Appendix: Appendix A houses a variety of lemmas on representation theory and commutative algebra for which we failed to find convenient references in the literature, in particular the statement that semisimple representations with isomorphic adjoints are isomorphic up to twist (Theorem A.10). No claims to originality here.

1.5. **Leitfaden suggestions.** We propose different levels of interaction with this paper for different readers. Those who are merely interested in results and applications should read the present Section 1 and skip to Section 12. Those also curious about our methods should additionally skim Section 2, and then read Sections 3 to 5 and Section 10. Readers who have the stomach for the technical weeds should begin the same way, but then also brave Sections 6 to 9. Finally, any of the previous types of readers who are interested in very-big-image results ($\mathrm{Im}\,\rho$ contains an SL_2 if it does so residually) should peek at Section 11.

1.6. **Notation.** We establish some notation and conventions. All rings are unital. Given any ring R (not necessarily commutative), we will let $R^\times$ denote the multiplicative group of invertible elements in R. For brevity, we call a commutative ring B containing a subring C an *extension* of C, *finite* if B is module-finite as a C-algebra. If B is a domain, then $Q(B)$ denotes its field of fractions. If F is a field, then $\overline{F}$ (respectively, F^{sep}) denotes a fixed algebraic (respectively, separable) closure.

If $C \subseteq B$ is an extension of rings, write $\mathrm{Aut}(B)$ for the group of ring automorphisms of B and $\mathrm{Aut}(B/C)$ for the subgroup fixing C pointwise. For $\Sigma \subseteq \mathrm{Aut}(B)$, write B^Σ for the subring pointwise fixed by every $\sigma \in \Sigma$. If $\sigma \in \mathrm{Aut}(B)$ and $f : X \to B$ is any set map, write ${}^\sigma f : X \to B$ for the map $x \mapsto \sigma(f(x))$.

For any integer n, let ζ_n denote a primitive n^{th} root of unity. Given a finite field $\mathbb{F}$ of size q a power of a prime p, its ring of Witt vectors, isomorphic to $\mathbb{Z}_p[\zeta_{q-1}]$, will be denoted by $W(\mathbb{F})$. Let $s\colon \mathbb{F}^\times \to W(\mathbb{F})^\times$ be the Teichmüller lift. We extend it to $s\colon \mathbb{F} \to W(\mathbb{F})$ by defining $s(0) := 0$. If A

is a $W(\mathbb{F})$-algebra, we use $W(\mathbb{F})$ to denote the image of $W(\mathbb{F})$ in A under the structure map. In particular, s may be viewed as being A-valued by composing with the structure map.

Throughout the paper we fix a prime $p \neq 2$. The ring A will always denote a local pro-p commutative ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$, which is then automatically a finite extension of $\mathbb{F}_p$. A closed subring of such an A is also automatically local and pro-p.[(vi)] Since 2 is invertible in A, we can always take square roots of elements $x \in 1+\mathfrak{m}$ via the formula

$$\sqrt{x} \coloneqq \sum_{n=0}^{\infty} \binom{1/2}{n}(x-1)^n.$$

In particular, when we write $\sqrt{x}$, we always choose the root congruent to 1 modulo $\mathfrak{m}$. In general, the profinite topology on A is coarser than the $\mathfrak{m}$-adic topology, but if such an A is noetherian, then the profinite topology coincides with the $\mathfrak{m}$-adic topology, so that A is a complete local noetherian ring [dSL97, Proposition 2.4]. In this case, every finite A-module is equipped with its natural $\mathfrak{m}$-adic A-module topology, which is compatible on submodules by the Artin-Rees lemma — see [AM69, Theorem 10.11]. In particular, every ideal in A is closed. By the Cohen structure theorem [Eis95, Theorem 7.7], if A is noetherian then it is a quotient of $W(\mathbb{F})[\![x_1,\dots,x_n]\!]$ for some n. If A is additionally a domain then A enjoys the so-called $N2$ (or sometimes "japanese") property: the integral closure of A in a finite extension of its field of fractions is finite over A [Mat70, Chapter 12, proof of Corollary 2], and hence a pro-p local noetherian domain in its own right. Note that in this case there need not be a topology on $Q(A)$ under which $Q(A)$ is a topological field containing A as a closed subring: since A is compact, the existence of any such topology would mean that $Q(A)$ is a locally compact field, so that $Q(A)$ is a finite extension of $\mathbb{F}_p$, $\mathbb{Q}_p$, or $\mathbb{F}_p(\!(X)\!)$ [RV91, Theorem 4.12]. But our A are more general.

If M is a subset of a $W(\mathbb{F})$-module N, then we will write $W(\mathbb{F})M$ for the $W(\mathbb{F})$-linear span of M in N. When R is a topological ring and S a subring of R, we say that S is *topologically generated* by a set X if S is the smallest closed subring of R containing X. Similarly, we can talk about an additive subgroup or a $W(\mathbb{F})$-algebra topologically generated by a set.

Finally, Π always denotes a p-finite profinite group (Definition 2.6). If $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ is a representation over a finite field $\mathbb{F}$, we can compose $\bar{\rho}$ with the natural projection $\mathbb{P}\colon\ \mathrm{GL}_2(\mathbb{F}) \to \mathrm{PGL}_2(\mathbb{F})$. We shall refer to the image of Π under the composition $\mathbb{P}\circ\bar{\rho}$ as *the projective image of* $\bar{\rho}$. It is well known that the projective image of $\bar{\rho}$ is cyclic, dihedral, or isomorphic to A_4, S_4, A_5 or one of $\mathrm{PSL}_2(\mathbb{F}')$ or $\mathrm{PGL}_2(\mathbb{F}')$ for some subfield $\mathbb{F}'$ of $\mathbb{F}$ ([Dic58, Chapter XII] or see [Bel19, Section 3.1]). If $\mathbb{P}\bar{\rho}(\Pi) \cong A_4$ (respectively, S_4, A_5), we say that $\bar{\rho}$ is *tetrahedral* (respectively, *octahedral*, *icosahedral*). If $\bar{\rho}$ is tetrahedral, octahedral, or icosahedral, then we say that $\bar{\rho}$ is *exceptional.* If $\mathbb{P}\bar{\rho}(\Pi)$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $\bar{\rho}$ is not exceptional, then we say that $\bar{\rho}$ is *large.* Be warned that there are exceptional isomorphisms $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4, \mathrm{PGL}_2(\mathbb{F}_3) \cong S_4, \mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$.

2. Bellaïche-Pink-Lie theory

In this section we introduce the basic objects of study — 2-dimensional pseudorepresentations and their associated realizations over generalized matrix algebras — along with the primary tools we use to study them: Pink-Lie algebras and Bellaïche's structure theorem (Theorem 2.23). The main reference for everything in this section is Bellaïche's paper [Bel19], which we refer to for most proofs. The only exception is that our definition of regularity (Definition 2.19) is weaker than that of Bellaïche.

(vi) Let B be closed subring of A. Its ideal $\mathfrak{m}_B := B\cap\mathfrak{m}$ is maximal since $\mathbb{F}_B := B/\mathfrak{m}_B$ is a subring of $\mathbb{F}$. Given any $\alpha \in A$ one can check that the sequence $\{\alpha^{p^n}\}_n$ converges and that its limit is $s(\bar{\alpha})$, the Teichmüller lift of the image of α in $\mathbb{F}$. Since B is closed, it thus contains both $s(\mathbb{F}_B^\times)$ and inverses of elements of $1+\mathfrak{m}_B$. Therefore every element of $B-\mathfrak{m}_B$ is invertible, and B is local. And as a closed subgroup of a pro-p group, B is automatically pro-p.

2.1. **Pseudorepresentations.** In this section we summarize the definitions and notation related to two-dimensional pseudorepresentations, algebraic gadgets introduced by Chenevier in [Che14] (where they are called "determinants") to mimic the behavior of trace and determinant functions of true 2-dimensional representation of groups. We follow [Che14, Example 1.8] and [Bel19, 2.1.1] for our definitions.

2.1.1. *Abstract pseudorepresentations.*

Definition 2.1. A (2-dimensional) *pseudorepresentation* of a group G over a commutative ring B is a pair of functions $t\colon G \to B$ and $d\colon \Pi \to B^\times$ such that

(1) $t(1) = 2$;
(2) $d(gh) = d(g)d(h)$ for all $g, h \in G$;
(3) $t(gh) = t(hg)$ for all $g, h \in G$;
(4) $t(gh) + d(h)t(gh^{-1}) = t(g)t(h)$ for all $g, h \in G$.

If G is a topological group and B is a topological ring, we say that a pseudorepresentation $(t, d)\colon G \to B$ is *continuous* if t and d are continuous maps.

One can verify that if $\rho\colon G \to \mathrm{GL}_2(B)$ is a (continuous) representation, then $(\operatorname{tr}\rho, \det\rho)$ is a (continuous) 2-dimensional pseudorepresentation. Conversely, if B is an algebraically closed field, then every pseudorepresentation $(t, d) : G \to B$ is carried by a unique semisimple representation $G \to \mathrm{GL}_2(B)$. If 2 is invertible in B, then a 2-dimensional pseudorepresentation (t, d) is determined by t alone: setting $h = g$ in (3) above yields $d(g) = \frac{t(g)^2 - t(g^2)}{2}$. In this way, pseudorepresentations generalize earlier work of Wiles [Wil90], Taylor [Tay91], and Rouquier [Rou96] on *pseudocharacters*, which mimic representations by keeping track only of a trace function.

If $(t, d)\colon \Pi \to B$ is a (continuous) pseudorepresentation and $\chi\colon \Pi \to B^\times$ is a (continuous) character, then $(\chi t, \chi^2 d)$ is also a (continuous) pseudorepresentation, called the *twist of* (t, d) *by* χ. We say (t, d) is *reducible* if $t = \chi_1 + \chi_2$ with $\chi_i\colon \Pi \to B^\times$ characters. Otherwise (t, d) is *irreducible*. We say (t, d) is *dihedral* if it is irreducible and there is a nontrivial character $\eta\colon \Pi \to B^\times$ such that $(\eta t, \eta^2 d) = (t, d)$.

The *kernel* of a pseudorepresentation $(t, d) : G \to B$ is

$$\ker(t, d) := \{g \in G : d(g) = 1 \text{ and } t(gx) = t(x) \text{ for all } x \in G\} \subseteq G.$$

This is a normal subgroup of G, closed if (t, d) is continuous. Moreover, (t, d) factors through the quotient $\Pi/\ker(t, d)$.

2.1.2. *Pseudorepresentations of profinite groups over pro-p rings.* We henceforth assume that all pseudorepresentations are continuous if both the group and the ring have topologies. Recall that A is a local pro-p commutative ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$ and Π is a profinite group.

If $(t, d) : \Pi \to A$ is a pseudorepresentation, its *residual* pseudorepresentation $(\bar{t}, \bar{d}) : \Pi \to \mathbb{F}$ is obtained by composing (t, d) with reduction modulo $\mathfrak{m}$. Because the Brauer group of $\mathbb{F}$ is trivial, the semisimple representation $\bar{\rho}$ carrying $(\bar{t}, \bar{d})$ is always realizable over $\mathbb{F}$ (see also Lemma A.12).

Definition 2.2. A pseudorepresentation $(t, d) : \Pi \to A$ has *constant determinant* if d is the Teichmüller lift of its reduction modulo $\mathfrak{m}$: that is, $d = s(\bar{d})$. Since $A^\times \cong s(\mathbb{F}^\times) \times 1 + \mathfrak{m}$, we see that d is always the product of $s(\bar{d})$ and a pro-p character $d_1 : \Pi \to 1 + \mathfrak{m}$. The twist of (t, d) by $d_1^{-1/2}$ is the unique constant-determinant pseudorepresentation with the same $\bar{\rho}$: this is the *constant determinant twist* of (t, d).

If $(t, d)\colon \Pi \to A$ is a pseudorepresentation, we call the subring of A topologically generated by $t(\Pi)$ the *trace ring* of (t, d). Note that the residue field of the trace ring is the trace ring of $(\bar{t}, \bar{d})$ or $\bar{\rho}$. Sometimes we need the residue field $\mathbb{F}$ of A to be a quadratic extension of the trace ring of

$(\bar{t}, \bar{d})$. Thus if $(t, d)\colon \Pi \to A$ is a pseudorepresentation and $\mathbb{F}$ is the residue field of A, we define the *trace algebra* of (t, d) to be the $W(\mathbb{F})$-subalgebra of A topologically generated by $t(\Pi)$. Thus the residue field does not change when restricting the codomain of a pseudorepresentation to its trace algebra. Both the trace ring and the trace algebra are pro-p local rings (Footnote (vi)).

Definition 2.3. We call a pseudorepresentation $(t, d) : \Pi \to A$ is *a priori small* if it is reducible or dihedral, or if the kernel of its constant-determinant twist is equal to the kernel of $\bar{\rho}$.

If A is a domain it turns out that the a priori small notion coincides with a certain weak kind of reducibility. Recall that a representation $\rho : G \to \mathrm{GL}_2(F)$ over a field F is *strongly (absolutely) irreducible* if $\rho|_H$ is (absolutely) irreducible for any finite-index subgroup H of G.

Proposition 2.4. *Let $(t, d) : \Pi \to A$ be a pseudorepresentation to a local pro-p domain A with field of fractions K, and let $\rho : \Pi \to \mathrm{GL}_2(\overline{K})$ be the semisimple representation carrying (t, d). The following are equivalent:*

(1) (t, d) is a priori small;

(2) ρ is reducible, dihedral, or its constant-determinant twist has finite image;

(3) ρ is not strongly irreducible.

Note that if any twist of $\rho : \Pi \to \mathrm{GL}_2(\overline{K})$ has finite image, then the constant-determinant twist does; equivalently, the image of the projective representation $\mathbb{P}\rho : \Pi \to \mathrm{PGL}_2(\overline{K})$ is finite.

Proof. Since all the notions in question are twist-invariant, we may replace (t, d) and ρ by constant-determinant twists. Clearly (1) $\implies$ (2) $\implies$ (3). To see that (3) $\implies$ (2), we follow [Rib75, Theorem 2.3]. Suppose $\rho|_H$ reducible for some finite-index subgroup H of Π. Up to replacing H with its normal core (i.e., the intersection of all conjugates, which is still of finite index in Π), we may assume that H is normal in Π, so that by Clifford's theorem [Cra19, Theorem 7.1.1], $\rho|_H$ is semisimple, and hence has abelian image. If $\rho(H)$ is not contained in the center of $\mathrm{GL}_2(\overline{K})$ then H contains a semisimple element h with distinct ρ-eigenvalues, and $\rho(H)$ is contained in the maximal torus centralizing $\rho(h)$. Moreover, since H is normal in Π, all of $\rho(\Pi)$ is contained in the normalizer of $\rho(h)$, so that $\rho(H)$ has index 1 or 2 in $\rho(\Pi)$, and ρ is either reducible or dihedral. On the other hand, if $\rho|_H$ is scalar, then since its trace is A-valued, $\rho|_H = \alpha \oplus \alpha$ for some character $\alpha : H \to A^\times$; since ρ has constant determinant, $\alpha^2 = s(\bar{\alpha}^2)$, so that $\rho|_H$ takes values in the finite set of prime-to-p roots of unity in A, whence ρ has finite image.

For (2) $\implies$ (1) it suffices to consider (t, d) residually exceptional or large, in which case by Rouquier-Nyssen ρ is the base change of a representation $\rho_A : \Pi \to \mathrm{GL}_2(A)$ with $\ker \rho_A = \ker(t, d)$ (see also Proposition 2.11 and Remark 2.12). We show that if ρ has finite image, then reduction modulo the pro-p subgroup $1 + M_2(\mathfrak{m})$ induces an isomorphism $\rho(\Pi) \cong \bar{\rho}(\Pi)$. If $A = \mathbb{F}$ there is nothing to show, so we can assume that A is infinite. (Indeed, if A has characteristic zero, then A contains $\mathbb{Z}_p$; otherwise, A is a local $\mathbb{F}$-algebra with residue field $\mathbb{F}$, and any such ring that is also finite over $\mathbb{F}$ is equal to $\mathbb{F}$.) We claim that the projective image of ρ is isomorphic to that of $\bar{\rho}$: If A has characteristic zero, then the finite subgroups of $\mathrm{GL}_2(K)$ are the same as those of $\mathrm{GL}_2(\mathbb{C})$ [Ser72, Proposition 16], hence the projective image of ρ is isomorphic to A_4, S_4 or A_5. And if A has characteristic p, then the finite subgroups of $\mathrm{GL}_2(A)$ are all defined over $\mathbb{F}$, because the eigenvalues of any finite-order element are roots of unity. In any case, the kernel of the reduction map $\rho_A(\Pi) \to \bar{\rho}(\Pi)$ is pro-p, so that it can be seen on the map $\mathbb{P}\rho_A(\Pi) \to \mathbb{P}\bar{\rho}(\Pi)$. But none of A_4, S_4, A_5, $\mathrm{PSL}_2(\mathbb{F}')$, or $\mathrm{PGL}_2(\mathbb{F}')$ have normal subgroups of p-power order for $p > 2$. ☐

2.1.3. *Pseudodeformations.* Fix a continuous semisimple representation $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$.

Definition 2.5. We say that a pseudorepresentation $(t, d)\colon \Pi \to A$ is a *pseudodeformation* of $\bar{\rho}$ if $(t, d) \equiv (\mathrm{tr}\,\bar{\rho}, \det \bar{\rho}) \bmod \mathfrak{m}$.

Let $\mathcal{C}$ be the category of local pro-p commutative rings with residue field $\mathbb{F}$, which have a natural $W(\mathbb{F})$-algebra structure, and with morphisms being local continuous $W(\mathbb{F})$-algebra homomorphisms. We are interested in the deformation functors

$$\begin{aligned} &F\colon \mathcal{C} \to \underline{\mathrm{SET}} \\ &A \mapsto \{(t,d)\colon \Pi \to A \text{ pseudodeformation of } \bar{\rho}\}. \end{aligned}$$

and

$$\begin{aligned} &G\colon \mathcal{C} \to \underline{\mathrm{SET}} \\ &A \mapsto \{(t,d) \in F(A)\colon d = s(\det \bar{\rho})\}. \end{aligned}$$

These functors are always representable. In order for the representing ring to be noetherian, we need to impose a finiteness condition on Π due to Mazur, which we now recall.

Definition 2.6. [Maz89, §1.1] A profinite group Π *satisfies the p-finiteness condition* or is *p-finite* if, for every open subgroup Π_0 of Π, the set $\mathrm{Hom}(\Pi_0, \mathbb{F}_p)$ is finite.

It is well known that F is represented by a pro-p local *noetherian* $W(\mathbb{F})$-algebra $\tilde{\mathcal{A}}$ whenever Π is a p-finite profinite group. See, for example, [Che14, Proposition 3.3] or [Böc13, Proposition 2.3.1]. In particular, the trace algebra of any pseudorepresentation of a p-finite profinite group on a local pro-p ring is a quotient of $\tilde{\mathcal{A}}$ and hence noetherian. Let $(t^{\mathrm{univ}}, d^{\mathrm{univ}})\colon \Pi \to \tilde{\mathcal{A}}$ be the universal pseudodeformation of $\bar{\rho}$. It is easy to see that the constant-determinant condition is a deformation condition. Indeed, let $\mathfrak{a}$ be the ideal of $\tilde{\mathcal{A}}$ topologically generated by $\{d^{\mathrm{univ}}(g) - s(\det \bar{\rho}(g))\colon g \in \Pi\}$. Then $\mathcal{A} \coloneqq \tilde{\mathcal{A}}/\mathfrak{a}$ represents G. In particular, $\mathcal{A}$ is also a pro-p local noetherian $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$. We use $(T,d)\colon \Pi \to \mathcal{A}$ to denote the universal constant-determinant pseudodeformation.

Definition 2.7. If $\mathbb{F}'$ is a subfield of $\mathbb{F}$, then we say that a 2-dimensional semisimple representation $\bar{\rho}$ is *multiplicity free over* $\mathbb{F}'$ if either $\bar{\rho}$ is absolutely irreducible or $\bar{\rho} \cong \chi_1 \oplus \chi_2$ such that $\chi_1, \chi_2\colon \Pi \to \mathbb{F}'^{\times}$ are distinct characters.

The following notion of admissibility, introduced by Bellaïche, plays a central role in [Bel19].

Definition 2.8. [Bel19, Section 5.2] A tuple $(\Pi, \bar{\rho}, t, d)$ is an *admissible pseudodeformation over A* if the following conditions are satisfied:

(1) Π is a p-finite profinite group;
(2) $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ is a continuous representation that is multiplicity free over $\mathbb{F}$;
(3) $(t,d)\colon \Pi \to A$ is a continuous pseudodeformation of $\bar{\rho}$;
(4) $d(g) \in s(\mathbb{F}^{\times})$ for all $g \in \Pi$, that is, (t,d) has constant determinant;
(5) A is the trace algebra of (t,d).

A local pro-p A accepting an admissible pseudodeformation is a complete noetherian local ring.

2.2. **GMAs and (t,d)-representations.** It is natural to ask when a given pseudodeformation $(t,d)\colon \Pi \to A$ arises as the trace and determinant of an actual representation $\rho\colon \Pi \to \mathrm{GL}_2(A)$. This has been studied in great generality; see the introduction of Chenevier's paper [Che14] for a thorough history. Bellaïche and Chenevier [BC09, Section 1.4] have shown that, under the residual multiplicity-free assumption, (t,d) always comes from a representation if one allows something more general than matrix algebras for the target. In Section 2.2 we summarize Bellaïche's [Bel19, Section 2], where he specializes his work with Chenevier to the 2-dimensional setting. All proofs that can be found in Bellaïche's work are omitted.

Definition 2.9. A *generalized matrix algebra (GMA)* over a commutative ring A is given by a tuple of data (A, B, C, m), where B and C are A-modules, $m\colon B \otimes_A C \to A$ is a morphism of A-modules satisfying

$$m(b \otimes c)b' = m(b' \otimes c)b \text{ and } m(b \otimes c')c = m(b \otimes c)c' \text{ for all } b, b' \in B, c, c' \in C.$$

Given such data, define the A-module $R := A \oplus B \oplus C \oplus A = \left(\begin{smallmatrix} A & B \\ C & A \end{smallmatrix}\right)$ and give R a multiplicative structure via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) := \left(\begin{smallmatrix} aa'+m(b\otimes c') & ab'+bd' \\ a'c+dc' & dd'+m(b'\otimes c) \end{smallmatrix}\right) \qquad \text{for } a, a', d, d' \in A, b, b' \in B, c, c' \in C,$$

so that R has the structure of an A-algebra via the ring homomorphism $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in R$. We refer to the GMA given by (A, B, C, m) simply by R.

A *morphism of GMAs* $(A, B, C, m) \to (A', B', C', m')$ (with associated A-algebras R and R') is a triple (f_A, f_B, f_C) consisting of a ring morphism $f_A\colon A \to A'$ and two A'-module morphisms $f_B\colon B \otimes_{A,f_A} A' \to B'$, $f_C\colon C \otimes_{A,f_A} A' \to C'$ such that $f_A \circ m = m' \circ (f_B \otimes f_C)$. The data (f_A, f_B, f_C) defines in a natural way an A-algebra morphism $\psi\colon R \to R'$; we say that ψ is associated with (f_A, f_B, f_C).

If A is a topological ring and B, C are topological A-modules, then R inherits a natural topology, and we call R a *topological GMA* if m is continuous. We say that R is *faithful* if m is nondegenerate as a pairing of A-modules. As with matrix algebras, we have the notion of a trace and determinant on a GMA R given by $\operatorname{tr}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a + d$ and $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - m(b \otimes c)$.

The following lemma shows that when A is a domain, faithful GMAs can be embedded into a matrix algebra over the field of fractions of A.

Lemma 2.10. [Bel19, Lemmas 2.2.2, 2.2.3] *Assume that A is a domain with field of fractions K and that $R = \left(\begin{smallmatrix} A & B \\ C & A \end{smallmatrix}\right)$ is a faithful GMA over A. Then there exist embeddings of A-modules $B, C \hookrightarrow K$ such that (identifying B, C with their images in K), $m\colon B \otimes_A C \to A$ is induced by multiplication in K. In particular, if $BC \neq 0$, then $R \otimes_A K$ is isomorphic over K as a GMA to $M_2(K)$.*

We recall the following result of Bellaïche, which explains that any residually multiplicity-free pseudorepresentation can be realized as the trace of a GMA-valued representation.

Proposition 2.11. [Bel19, Proposition 2.4.2] *Let $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be multiplicity free over $\mathbb{F}$. Let $(t, d)\colon \Pi \to A$ be a pseudodeformation of $\bar{\rho}$.*

(1) There exists a faithful GMA R over A and a morphism of groups $\rho\colon \Pi \to R^\times$ such that $\operatorname{tr}\rho = t, \det\rho = d$, and $A\rho(\Pi) = R$.
(2) If (ρ, R) and (ρ', R') are as in (1), then there is a unique isomorphism of A-algebras $\Psi\colon R \to R'$ such that $\Psi \circ \rho = \rho'$.
(3) If $g_0 \in \Pi$ such that $\bar{\rho}(g_0)$ has distinct eigenvalues $\lambda_0, \mu_0 \in \mathbb{F}^\times$, then there exists (ρ, R) as in (1) such that $\rho(g_0)$ is diagonal and $\rho(g_0) \equiv \begin{pmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{pmatrix} \bmod \mathfrak{m}$.
(4) If $g_0 \in \Pi$ and $(\rho, R), (\rho', R')$ are as in (3), then the unique isomorphism of A-algebras $\Psi\colon R \to R'$ such that $\Psi \circ \rho = \rho'$ is associated with an isomorphism of GMAs.
(5) If $\bar{\rho}$ is irreducible and (ρ, R) is as in (1), then $R = (A, B, C, m, R)$ is isomorphic to $M_2(A)$ as a GMA over A. If $\bar{\rho}$ is reducible, then $BC \subseteq \mathfrak{m}$.
(6) If (ρ, R) are as in (1), then $\ker\rho = \ker(t, d)$.
(7) Assume that A is noetherian and Π is p-finite. If (t, d) is continuous, then for (ρ, R) as in (1), R is of finite type as an A-module. If R is given its unique topology as an A-algebra, then ρ is continuous.

Remark 2.12. When $\bar{\rho}$ is absolutely irreducible, Proposition 2.11 allows us to identify the GMA R with the matrix algebra $M_2(A)$. We follow Bellaïche in always implicitly making such an identification. In particular, in the dihedral case, elements of B and C are viewed as elements of A. △

Following Bellaïche, we make the following definitions.

Definition 2.13. [Bel19, Definition 2.4.3] A representation $\rho\colon \Pi \to R^\times$ satisfying condition (1) in Proposition 2.11 is called a (t,d)*-representation*. If in addition ρ satisfies condition (3), then we say that ρ is *adapted* to (g_0, λ_0, μ_0).

In fact, it is often useful to have the following strengthening of Proposition 2.11(3).

Lemma 2.14. *Let $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be multiplicity free over $\mathbb{F}$ and $\lambda_0 \neq \mu_0 \in \mathbb{F}^\times$ be the eigenvalues of an element in $\operatorname{Im}\bar{\rho}$. Let $(t,d)\colon \Pi \to A$ be a pseudodeformation of $\bar{\rho}$. Then there exists $g_0 \in \Pi$ and a (t,d)-representation ρ adapted to (g_0, λ_0, μ_0) such that $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0)\end{smallmatrix}\right)$.*

Proof. Let $g_0' \in \Pi$ be any element such that $\bar{\rho}(g_0')$ has eigenvalues λ_0, μ_0. Then Proposition 2.11(3) guarantees the existence of a (t,d)-representation $\rho\colon \Pi \to R^\times$ adapted to (g_0', λ_0, μ_0). By [Bel19, Theorem 6.2.1], it follows that $\left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0)\end{smallmatrix}\right) \in \operatorname{Im}\rho$. Let g_0 be any element in $\rho^{-1}\left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0)\end{smallmatrix}\right)$. Then ρ is a (t,d)-representation adapted to (g_0, λ_0, μ_0) and $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0)\end{smallmatrix}\right)$. □

2.3. **Pink-Lie algebras.** In Section 2.3 we recall Pink's theory relating pro-p subgroups of $\mathrm{SL}_2(A)$ to closed Lie subalgebras of $\mathfrak{sl}_2(A)$ [Pin93]. In fact, we use Bellaïche's generalization to GMAs [Bel19, Section 4].

Recall that A is a local pro-p ring with $p \neq 2$. The assumption that $p \neq 2$ is critical for Pink's theory. We denote by $\mathfrak{m}$ the maximal ideal of A. Fix a compact topological GMA $R = \left(\begin{smallmatrix} A & B \\ C & A\end{smallmatrix}\right)$ over A. (The compactness condition is satisfied, for instance, when R is finite as an A-module.) Write

$$SR^\times \coloneqq \{r \in R^\times\colon \det r = 1\}.$$

Let $\operatorname{rad} R$ be the Jacobson radical of R, and $R^1 \coloneqq 1 + \operatorname{rad} R$. We let $SR^1 \coloneqq SR^\times \cap R^1$, which is a closed normal pro-p subgroup of $R^\times$. See [Bel19, Remark 4.2.1] for an explicit description of these objects. We mention here that in the case when $BC = A$ there is by [Bel19, Lemma 2.2.1] an isomorphism of GMAs $R \cong M_2(A)$ that we can use to identify $\operatorname{rad} R$ with $\mathfrak{m}M_2(A)$ and $R/\operatorname{rad} R$ with $M_2(\mathbb{F})$, while if $BC \subset \mathfrak{m}$ then $\operatorname{rad} R = \left(\begin{smallmatrix} \mathfrak{m} & B \\ C & \mathfrak{m}\end{smallmatrix}\right)$ and $R/\operatorname{rad} R = \left(\begin{smallmatrix} \mathbb{F} & 0 \\ 0 & \mathbb{F}\end{smallmatrix}\right)$.

Given any subset S of R, we write

$$S^0 \coloneqq \{s \in S\colon \operatorname{tr} s = 0\}.$$

Then $(\operatorname{rad} R)^0$ has a Lie algebra structure with bracket given by $[r_1, r_2] \coloneqq r_1 r_2 - r_2 r_1$.

For any topological group G and closed subgroup H of G, write (G, H) for the smallest closed subgroup of G containing $\{g^{-1}h^{-1}gh\colon g \in G, h \in H\}$. Fix a closed subgroup $\Gamma \subseteq SR^1$. Recall that the lower central series of Γ is defined by $\Gamma_1 \coloneqq \Gamma$ and define $\Gamma_{n+1} \coloneqq (\Gamma, \Gamma_n)$. We describe how Pink associates a filtration of Lie algebras to Γ [Pin93, Section 2].

Define a function

$$\begin{aligned} \Theta\colon R^\times &\to R^0 \\ r &\mapsto r - \frac{\operatorname{tr} r}{2}, \end{aligned}$$

where $(\operatorname{tr} r)/2$ is regarded as a scalar via the structure morphism $A \to R$. Let $L(\Gamma) = L_1(\Gamma)$ be the (additive) subgroup of $(\operatorname{rad} R)^0$ topologically generated by $\Theta(\Gamma)$. For $n \geq 2$, define $L_n(\Gamma)$ recursively as the subgroup of $(\operatorname{rad} R)^0$ topologically generated by the set

$$\{xy - yx\colon x \in L_1(\Gamma), y \in L_{n-1}(\Gamma)\}.$$

Although the $L_n(\Gamma)$ are a priori only subgroups of $(\operatorname{rad} R)^0$, Pink shows that they are closed under Lie brackets and form a descending filtration, as summarized in the following proposition, which is due to Pink when $R = M_2(A)$ [Pin93, Proposition 3.1, Proposition 2.3] and to Bellaïche in the GMA case [Bel19, Proposition 4.7.1].

Proposition 2.15. *For all $n \geq 1$, we have $L_{n+1}(\Gamma) \subseteq L_n(\Gamma)$. In particular, each $L_n(\Gamma)$ is a Lie subalgebra of* $(\operatorname{rad} R)^0$.

We emphasize that, a priori, each $L_n(\Gamma)$ is just a $\mathbb{Z}_p$-module, even if the ring A is very large. The point of Section 6 is to prove that, under mild conditions, $L_n(\Gamma)$ is in fact an algebra over an (in general) much larger ring.

Conversely, given a closed Lie subalgebra L of $(\operatorname{rad} R)^0$, define $H(L) \coloneqq \Theta^{-1}(L) \cap SR^1$. Let $H_n \coloneqq H(L_n(\Gamma))$. A priori, $H(L)$ is only a subset of SR^1. However, we have the following theorem of Pink [Pin93, Proposition 2.4, Theorem 2.7], which was generalized to GMAs by Bellaïche [Bel19, Theorem 4.7.3].

Theorem 2.16. *We have that H_n is a pro-p subgroup of SR^1. Furthermore, Γ is a normal subgroup of H_1, and H_1/Γ is abelian. For $n \geq 2$, we have $H_n = \Gamma_n$.*

Remark 2.17. Pink's construction satisfies the following two important properties.

(1) It is functorial with respect to surjective ring homomorphisms. Namely, let $\mathfrak{a}$ be a closed ideal of A and $\varphi\colon R \to R/\mathfrak{a}R$ the natural projection. Then for all $n \geq 1$ we have
$$\varphi(L_n(\Gamma)) = L_n(\varphi(\Gamma)).$$
(2) Pink's Lie algebra $L_n(\Gamma)$ is closed under conjugation by the normalizer of Γ in $R^\times$. This follows easily from the definitions since Θ is invariant under conjugation. △

See Lemma 3.9 for an example calculating $L_n(\Gamma)$ when Γ is a congruence subgroup.

2.4. **Decomposability and regularity.** In order to prove fullness theorems, it is useful to be able to decompose Pink's Lie algebra according to its entries. In Section 2.4 we define this precisely and then define regularity, which will turn out to ensure that the Lie algebras of the representations we work with are decomposable.

Definition 2.18. [Bel19, Section 4.9] Let R be a GMA over A and L a closed subspace of $(\operatorname{rad} R)^0$. We say that L is *decomposable* if
$$\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in L \text{ implies that } \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \in L \text{ and } \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in L.$$
We say that L is *strongly decomposable* if L is decomposable and
$$\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in L \text{ implies that } \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in L \text{ and } \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right) \in L.$$
If $L_n(\Gamma) \subseteq R = \left(\begin{smallmatrix} A & B \\ C & A \end{smallmatrix}\right)$ is decomposable, we write
$$\begin{aligned} I_n(\Gamma) &\coloneqq \{a \in A\colon \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \in L_n(\Gamma)\}, \\ \nabla_n(\Gamma) &\coloneqq \{\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in L_n(\Gamma)\}, \\ B_n(\Gamma) &\coloneqq \{b \in B\colon \exists c \in C \text{ such that } \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in L_n(\Gamma)\}, \\ C_n(\Gamma) &\coloneqq \{c \in C\colon \exists b \in B \text{ such that } \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in L_n(\Gamma)\}. \end{aligned}$$
Eventually, L will be a Pink-Lie algebra associated to some admissible pseudodeformation of $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Regularity is a condition on $\bar{\rho}$ that will allow us to decompose L, as we will see in Section 6.

Let $\mathbb{E}$ be the subfield of $\mathbb{F}$ generated by $\{(\operatorname{tr}\bar\rho(g))^2/\det\bar\rho(g)\colon g\in\Pi\}$; equivalently, $\mathbb{E}$ is generated by the traces of $\operatorname{ad}\bar\rho$. If λ_g,μ_g are the eigenvalues of $\bar\rho(g)$, then we see that $(\operatorname{tr}\bar\rho(g))^2/\det\bar\rho(g)=\lambda_g\mu_g^{-1}+\lambda_g^{-1}\mu_g+2$. Hence $\mathbb{E}$ is generated over $\mathbb{F}_p$ by the set

$$\{\lambda\mu^{-1}+\lambda^{-1}\mu\colon \lambda,\mu \text{ are the eigenvalues of } \bar\rho(g) \text{ for some } g\in\Pi\}. \tag{2}$$

In particular, g will not contribute to $\mathbb{E}$ if the multiplicative order of $\lambda_g\mu_g^{-1}$ is strictly less than 5. Using this reasoning, it is straightforward to calculate $\mathbb{E}$ when $\bar\rho$ exceptional. Namely, if $\bar\rho$ is tetrahedral or octahedral, then $\mathbb{E}=\mathbb{F}_p$. If $\bar\rho$ is icosahedral, then $\mathbb{E}=\mathbb{F}_p$ if $p=5$ and $\mathbb{E}=\mathbb{F}_p(\zeta_5+\zeta_5^{-1})=\mathbb{F}_p(\sqrt{5})$ otherwise.

Definition 2.19. Let $\bar\rho\colon\Pi\to\mathrm{GL}_2(\mathbb{F})$ be a semisimple representation. We say that $\bar\rho$ is *regular* if there exists $g_0\in\Pi$ such that $\bar\rho(g_0)$ has eigenvalues $\lambda_0,\mu_0\in\overline{\mathbb{F}}_p^\times$ satisfying $\lambda_0\mu_0^{-1}\in\mathbb{E}^\times\setminus\{\pm1\}$. We call g_0 a *regular element* for $\bar\rho$. If in addition $\lambda_0,\mu_0\in\mathbb{E}^\times$, then we say that $\bar\rho$ is *strongly regular*.

Definition 2.19 is weaker than Bellaïche's definition of regularity [Bel19, Definition 7.2.1], where the eigenvalues λ_0 and μ_0 are required in addition to belong to $\mathbb{F}_p$. Examining (2), we see that the only way $\bar\rho$ can fail to be regular is if, for every matrix in $\operatorname{Im}\bar\rho$ with eigenvalues λ,μ, either $\lambda\mu^{-1}=\pm1$ or the unique quadratic extension of $\mathbb{E}$ is $\mathbb{E}(\lambda\mu^{-1})$.

Remark 2.20. Let us analyze regularity depending on the projective image of $\bar\rho$. With notation as in Definition 2.19, note that the order of $\lambda_0\mu_0^{-1}$ in $\mathbb{E}^\times$ corresponds to the order of $\bar\rho(g_0)$ in the projective image of $\bar\rho$.

(1) If $\bar\rho$ is large, then $\bar\rho$ is regular. Indeed, $\mathbb{P}\bar\rho(\Pi)$ contains $\mathrm{PSL}_2(\mathbb{E})$ up to conjugation. Since $\bar\rho$ is not exceptional, $\mathbb{E}^\times$ contains an element x such that $x^2\not\equiv\pm1$. Then the image of $\bar\rho$ contains, up to conjugation, a scalar multiple of $\left(\begin{smallmatrix}x&0\\0&x^{-1}\end{smallmatrix}\right)$, which satisfies the regularity property.

(2) If $\bar\rho$ is tetrahedral and $p>3$, then a regular element must map to a 3-cycle in the projective image of $\bar\rho$, since the other elements of A_4 have order at most 2. Thus in this case regularity is equivalent to $\zeta_3\in\mathbb{E}=\mathbb{F}_p$, which is equivalent to $p\not\equiv 2 \bmod 3$. By a similar argument we see that if $\bar\rho$ is octahedral and $p>3$, then regularity is equivalent to one of ζ_3 or ζ_4 being in $\mathbb{E}=\mathbb{F}_p$, which is equivalent to $p\not\equiv 11 \bmod 12$. If $\bar\rho$ is icosahedral and $p\neq5$, then regularity is equivalent to one of ζ_3 or ζ_5 being in $\mathbb{E}=\mathbb{F}_p(\sqrt{5})$, which is equivalent to $p\not\equiv 14 \bmod 15$.

(3) If $\bar\rho\cong\varepsilon\oplus\delta$, then $\bar\rho$ is regular if and only if $\varepsilon\delta^{-1}$ takes values in $\mathbb{E}^\times$ (Lemma 8.1).

(4) If $\bar\rho=\operatorname{Ind}_{\Pi_0}^{\Pi}\chi$ is dihedral, then elements in $\Pi\setminus\Pi_0$ have projective order 2, and so any regular element must lie in Π_0. Furthermore, elements in $\Pi\setminus\Pi_0$ have trace 0, and so the field $\mathbb{E}$ associated to $\bar\rho$ is the same as the field $\mathbb{E}$ associated to $\bar\rho|_{\Pi_0}$. Hence we are reduced to the previous case when $\bar\rho$ is reducible.

(5) If the projective image of $\bar\rho$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$, or if $\mathbb{E}=\mathbb{F}_3$, then $\bar\rho$ is never regular. In particular, if $p=3$ and $\bar\rho$ is tetrahedral or octahedral, then $\bar\rho$ is not regular. If $p=5$ and $\bar\rho$ is icosahedral, then $\mathbb{P}\bar\rho(\Pi)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$. Thus $\mathbb{E}=\mathbb{F}_5$ and any potential regular element has eigenvalues in $(\mathbb{F}_5^\times)^2=\{\pm1\}$, so $\bar\rho$ is not regular in this case. △

2.5. **Bellaïche's results.** The purpose of this section is to state Bellaïche's main results that form the basis for our work in this paper. We state them in slightly less generality than [Bel19, Section 6]. As before, A denotes a local pro-p ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$. In particular, A is naturally a topological $W(\mathbb{F})$-algebra.

Let R be a faithful GMA over A. Recall the description of $R/\operatorname{rad}R$ that we gave in the beginning of Section 2.3. We define $s\colon R/\operatorname{rad}R\to R$ by

$$s\begin{pmatrix}a&b\\c&d\end{pmatrix}:=\begin{cases}\left(\begin{smallmatrix}s(a)&s(b)\\s(c)&s(d)\end{smallmatrix}\right) & \text{if } R=M_2(A)\\ \left(\begin{smallmatrix}s(a)&0\\0&s(d)\end{smallmatrix}\right) & \text{else.}\end{cases}$$

Note that in the latter case, we have a priori that $b = c = 0$.

Let us fix an admissible pseudodeformation $(\Pi, \bar\rho, t, d)$ over A. If $p = 3$, let us assume that $\bar\rho$ is not tetrahedral. By Proposition 2.11, there exists a (t, d)-representation $\rho\colon \Pi \to R^\times$. Given such a (t, d)-representation, write $G = G_\rho := \rho(\Pi)$ and $\Gamma = \Gamma_\rho := G \cap SR^1$. Furthermore, let $\overline{G}$ denote the image of G modulo $\operatorname{rad} R$. (Note that the image of $\overline{G}$ under an embedding $R/\operatorname{rad} R \to \mathrm{GL}_2(\mathbb{F})$ is a conjugate of $\bar\rho(\Pi)$.) We will write $L_n(\rho) := L_n(\Gamma_\rho)$ and analogously for $I_n(\rho), \nabla_n(\rho), B_n(\rho), C_n(\rho)$.

Bellaïche chooses his (t, d)-representations very carefully in order to give a nice description of their Pink-Lie algebras. How this is done depends upon the projective image of $\bar\rho$. Since $\bar\rho$ is multiplicity free over $\mathbb{F}$, we can let $\lambda_0 \neq \mu_0 \in \overline{\mathbb{F}}_p^\times$ be the eigenvalues of a matrix $x_0 \in \operatorname{Im}\bar\rho$ chosen such that the following conditions are satisfied:

- if $\bar\rho$ is large, then $(\lambda_0\mu_0^{-1})^2 \neq 1$ and $\lambda_0, \mu_0 \in \mathbb{F}_p^\times$;
- if $p = 3$ and $\bar\rho$ is octahedral, then $\lambda_0\mu_0^{-1}$ is a primitive fourth root of unity;
- if $p = 5$ and $\bar\rho$ is icosahedral, then $\lambda_0\mu_0^{-1}$ is a primitive third root of unity;
- if $\bar\rho$ is exceptional and does not belong to one of the previous to scenarios, then $\lambda_0\mu_0^{-1}$ is a primitive third, fourth, or fifth root of unity;
- otherwise, the multiplicative order of $\lambda_0\mu_0^{-1}$ is equal to the maximal order of an element in the projective image of $\bar\rho$.

Definition 2.21. Suppose $(\Pi, \bar\rho, t, d)$ is an admissible pseudodeformation. We say that a (t, d)-representation ρ is *well adapted* if

(1) ρ is adapted to an element g_0 such that $\rho(g_0) = \begin{pmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{pmatrix}$, where λ_0, μ_0 satisfy the relevant property listed above;

(2) if the projective image of $\bar\rho$ is dihedral and nonabelian, then $\overline{G}$ contains a matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc^{-1} \in \mathbb{F}_p^\times$ and $s(\overline{G}) \subseteq \operatorname{Im}\rho$.

Bellaïche shows that well-adapted (t, d)-representations always exist, provided that one is willing to replace $\mathbb{F}$ by a quadratic extension in the dihedral case [Bel19, Proposition 6.3.2, Lemma 6.8.2].

Define $\mathbb{F}_q$ as in the table below. We will see in Lemma 8.1 that if $\bar\rho$ is regular and reducible or dihedral, then $\mathbb{F}_q$ can be taken to be $\mathbb{E}$. If $\bar\rho$ is not projectively cyclic or dihedral, then $\mathbb{F}_q \subseteq \mathbb{E}$ by definition. (In the A_5 case, this follows from the calculation that $\mathbb{E} = \mathbb{F}_p(\sqrt{5})$ prior to Definition 2.19.)

Remark 2.22. Our definition of $\mathbb{F}_q$ differs from that of Bellaïche when $\bar\rho$ is exceptional. If $\bar\rho$ is tetrahedral, then Bellaïche defines $\mathbb{F}_q = \mathbb{F}_p(\zeta_3)$. If $\bar\rho$ is octahedral, he defines $\mathbb{F}_q$ to be $\mathbb{F}_p(\zeta_3)$ if the ratio $\lambda_0\mu_0^{-1}$ chosen prior to Definition 2.21 has order 3 and $\mathbb{F}_p(\zeta_4)$ if that ratio has order 4. If $\bar\rho$ is icosahedral, then he defines $\mathbb{F}_q = \mathbb{F}_p(\zeta_5)$. The key property that Bellaïche needs is that $\bar\rho$ can be conjugated so that its image lies in $Z\,\mathrm{GL}_2(\mathbb{F}_q)$ and $\begin{pmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{pmatrix} \in \operatorname{Im}\bar\rho$, where Z is the group of scalar matrices in $\mathbb{F}$ (cf. [Bel19, Lemma 6.8.5]). This change of definition will be justified in Lemma 8.12. △

the projective image of $\bar\rho$ is	$\mathbb{F}_q$
cyclic of order m or dihedral of order $2m$	any subfield of $\mathbb{F}$ such that $(m, q-1) > 2$
exceptional	$\mathbb{E}(\lambda_0\mu_0^{-1})$
otherwise	$\mathbb{F}_p$

The following theorem summarizes Bellaïche's results describing the structure of $W(\mathbb{F}_q)L_1(\rho)$ from [Bel19, Section 6]. We recall from Remark 2.12 that in the dihedral case, elements in B and C can be viewed as elements of A.

Theorem 2.23 (Bellaïche)**.** *Let $(\Pi, \bar\rho, t, d)$ be an admissible pseudodeformation such that the projective image of $\bar\rho$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$ nor $(\mathbb{Z}/2\mathbb{Z})^2$. Then every well-adapted (t,d)-representation $\rho\colon \Pi \to R^\times$ with $R = \left(\begin{smallmatrix} A & B \\ C & A \end{smallmatrix}\right)$ has the following properties:*

(1) $L_1(\rho)$ is decomposable;

(2) the ring A is equal to

$$\begin{cases} W(\mathbb{F}) + W(\mathbb{F})I_1(\rho) + W(\mathbb{F})I_1(\rho)^2 + W(\mathbb{F})B_1(\rho) & \textit{if } \bar\rho \textit{ is projectively dihedral} \\ W(\mathbb{F}) + W(\mathbb{F})I_1(\rho) + W(\mathbb{F})I_1(\rho)^2 & \textit{otherwise}; \end{cases}$$

(3) $W(\mathbb{F})C_1(\rho) = C$ and $W(\mathbb{F})B_1(\rho) = B$;

(4) up to possibly replacing ρ with its conjugate by a certain matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \in A^\times$ when $\bar\rho$ is exceptional or large, $W(\mathbb{F}_q)L_1(\rho)$ is equal to

$$\begin{pmatrix} W(\mathbb{F}_q)I_1(\rho) & W(\mathbb{F}_q)B_1(\rho) \\ W(\mathbb{F}_q)C_1(\rho) & W(\mathbb{F}_q)I_1(\rho) \end{pmatrix}^0.$$

Furthermore

(i) $(W(\mathbb{F}_q)I_1(\rho))^3 \subseteq W(\mathbb{F}_q)I_1(\rho)$;

(ii) if $\bar\rho$ is not reducible, then $W(\mathbb{F}_q)C_1(\rho) = W(\mathbb{F}_q)B_1(\rho)$;

(iii) if $\bar\rho$ is exceptional or large, then $W(\mathbb{F}_q)B_1(\rho) = W(\mathbb{F}_q)I_1(\rho)$ and $(W(\mathbb{F}_q)I_1(\rho))^2 \subset W(\mathbb{F}_q)I_1(\rho)$.

For a subfield $\mathbb{F}'$ of $\mathbb{F}$, we shall often refer to the $W(\mathbb{F}')$-subalgebra of A generated by $I_1(\rho)$. We denote it by $\mathcal{B}_\rho(\mathbb{F}')$, which is simply equal to $W(\mathbb{F}') + W(\mathbb{F}')I_1(\rho) + W(\mathbb{F}')I_1(\rho)^2$ whenever $\mathbb{F}_q \subseteq \mathbb{F}'$. When $\bar\rho$ is not reducible or dihedral, we have $\mathcal{B}_\rho(\mathbb{F}') = W(\mathbb{F}') + W(\mathbb{F}')I_1(\rho)$ by Theorem 2.23(ii, iii). In particular, Theorem 2.23 says that $A = \mathcal{B}_\rho(\mathbb{F})$ unless $\bar\rho$ is projectively dihedral, in which case $A = \mathcal{B}_\rho(\mathbb{F}) + W(\mathbb{F})B_1(\rho)$.

Bellaïche uses Theorem 2.23 to deduce that, under certain hypotheses, the representation ρ is $\mathcal{B}_\rho(\mathbb{F}_p)$-full. See Theorem 1.3 or [Bel19, Theorem 7.2.3] for a precise statement of his result. The first step in our main theorem is to improve this to $\mathcal{B}_\rho(\mathbb{E})$-fullness in Section 6. But first we discuss fullness, conjugate self-twists, and the connections between them in detail.

3. Fullness

In this section we explore the notion of *fullness*, our measure of the size of the image of a continuous (pseudo)representation on a noetherian local pro-p ring A. Here we additionally assume that A is a domain, with field of fractions K.

3.1. Fullness for (pseudo)representations.

Let B be any ring. For any nonzero B-ideal $\mathfrak{b}$, let

$$\Gamma_B(\mathfrak{b}) \coloneqq \ker(\mathrm{SL}_2(B) \to \mathrm{SL}_2(B/\mathfrak{b})) = \left\{ \left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right) \in \mathrm{SL}_2(B) \colon a, b, c, d \in \mathfrak{b} \right\}$$

be the *congruence subgroup* of $\mathrm{SL}_2(B)$ of *level* $\mathfrak{b}$.

Definition 3.1. Let G be a subgroup of $\mathrm{GL}_2(K)$. For a subring B of K we say that G is *B-full* if there exists a nonzero B-ideal $\mathfrak{b}$ and $x \in \mathrm{GL}_2(K)$ such that

$$x^{-1}Gx \supseteq \Gamma_B(\mathfrak{b}).$$

A $\mathrm{GL}_2(K)$-valued representation is *B-full* if its image is B-full. If $(t,d)\colon \Pi \to A$ is a pseudorepresentation, we say (t,d) is *B-full* if there exists a (t,d)-representation $\rho\colon \Pi \to R^\times$ such that $\iota \circ \rho$ is B-full, where ι is an embedding of R into $M_2(K)$. Such an ι exists by Lemma 2.10; note that by replacing ι by a conjugate embedding we may insist that $\Gamma_B(\mathfrak{b}) \subset \mathrm{GL}_2(K)$ is contained in $\iota(R^\times)$ on the nose. We will say B is a *(t,d)-fullness ring* if (t,d) is B-full.

The notion of fullness, which has appeared in earlier incarnations in [Hid15, last introductory paragraph] and [Lan16, Definition 2.2], is analogous to Bellaïche's notion of "congruence large-image" [Bel19, Definition 7.2.1]. We now show that fullness is well defined for pseudorepresentations and gives compatible notions for representations and pseudorepresentations.

Lemma 3.2. *Let $(t,d)\colon \Pi \to A$ be a pseudorepresentation and $\rho\colon \Pi \to \mathrm{GL}_2(K)$ a representation whose trace takes values in A. Let B be any subring of K.*

(1) If there exists a (t,d)-representation that is B-full, then every (t,d)-representation is B-full.

(2) The representation ρ is B-full if and only if its pseudorepresentation $(\operatorname{tr}\rho, \det\rho)$ is B-full.

Proof. To prove (1), let $\rho\colon \Pi \to R^\times$ and $\rho'\colon \Pi \to R'^\times$ be two (t,d)-representations. We just have to verify that the A-algebra isomorphism $\Psi\colon R \to R'$ such that $\rho' = \Psi \circ \rho$ from Proposition 2.11 is given by conjugation by an element of $\mathrm{GL}_2(K)$. Consider $\Psi \otimes 1\colon R \otimes_A K \to R' \otimes_A K$ and recall that $R \otimes_A K \cong M_2(K) \cong R' \otimes_A K$ by Lemma 2.10. By the Skolem-Noether theorem, it follows that $\Psi \otimes 1$ (and hence Ψ) is conjugation by an element of $\mathrm{GL}_2(K)$.

For (2), let $(t,d) = (\operatorname{tr}\rho, \det\rho)$, which is a pseudorepresentation over A by assumption. Let $r\colon \Pi \to R^\times$ be a (t,d)-representation, and embed R into $\mathrm{GL}_2(K)$ by Lemma 2.10, thus viewing r as valued in $\mathrm{GL}_2(K)$. Note that fullness of ρ (respectively, r) implies that ρ (respectively, r) is absolutely irreducible since this is true for the inclusion representation of a congruence subgroup in $\mathrm{GL}_2(K)$. As $(\operatorname{tr}\rho, \det\rho) = (t,d) = (\operatorname{tr} r, \det r)$ and either ρ or r is absolutely irreducible, by the Brauer-Nesbitt Theorem it follows that ρ and r are conjugate by a matrix in $\mathrm{GL}_2(K)$. Since fullness is defined up to conjugation in $\mathrm{GL}_2(K)$, it follows that ρ is B-full if and only if r, and hence (t,d), is B-full. $\square$

The next two propositions suggest that we can restrict our attention to fullness rings that are closed subrings of the trace algebra. These ideas are made precise Proposition 3.4 and Corollary 3.7 below. Fix a continuous pseudorepresentation $(t,d) : \Pi \to A$.

Proposition 3.3. *If (t,d) is B-full for some subring B of K, then the $\mathbb{Z}$-linear span of $t(\Pi)$ contains a nonzero B-ideal. In particular, the trace algebra A_t of t contains a nonzero B-ideal.*

Proof. Let $0 \neq \mathfrak{b}$ be an B-ideal such that $\Gamma_B(\mathfrak{b})$ is contained in the image of some (t,d)-representation. We claim that pairwise products of elements of $\mathfrak{b}$ are all in the set $\{t(g) - 2 : g \in \Pi\}$, so that $\mathfrak{b}^2$ is contained in the trace algebra A_t. Indeed, an element of $\Gamma_B(\mathfrak{b})$ is of the form $\left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right)$ with $a,b,c,d \in \mathfrak{b}$ such that $a + d + ad - bc = 0$. In particular, for any $b, c \in \mathfrak{b}$, taking $d = bc$ and $a = 0$ shows that

$$\operatorname{tr}\left(\begin{smallmatrix} 1 & b \\ c & 1+bc \end{smallmatrix}\right) = 2 + bc \in t(\Pi).$$

Since $2 = t(1) \in t(\Pi)$ and elements of the form bc generate $\mathfrak{b}^2$, we see that the $\mathbb{Z}$-span of $t(\Pi)$ contains $\mathfrak{b}^2$, which is nonzero since it is the square of a nonzero ideal of a domain. $\square$

Proposition 3.4. *Let $(t,d) : \Pi \to A$ be a continuous pseudorepresentation. If (t,d) is B-full for a subring B of A, then (t,d) is also full for the closure $\overline{B}$ of B in A. Conversely, suppose that (t,d) is $\overline{B}$-full. If the image of a (t,d)-representation contains a congruence subgroup of $\mathrm{SL}_2(\overline{B})$ whose level is the closure $\bar{\mathfrak{b}}$ of an ideal $\mathfrak{b}$ of B, then (t,d) is also B-full.*

Proof. We show that for a closed subgroup G of the unit group $R^\times$ of a faithful GMA R over A equipped with an embedding $\iota : R \hookrightarrow M_2(K)$, if $\iota(G)$ contains $\Gamma_B(\mathfrak{b})$ for some nonzero ideal $\mathfrak{b}$ of B, then $\iota(G)$ also contains $\Gamma_{\overline{B}}(\bar{\mathfrak{b}})$ for the closure $\bar{\mathfrak{b}}$ of $\mathfrak{b}$. Since A is noetherian, and both R and $M_2(A)$ are finite A-algebras, the preimage $S \coloneqq \iota^{-1}\big(M_2(A)\big)$ is a finite, hence closed, A-subalgebra of R. Moreover, the induced map $\iota|_S : S \to M_2(A)$ is a homeomorphism onto its image by the compatibility of topologies on finite A-modules (see p. 8). In particular, any closed subset of R containing $\iota^{-1}\big(\Gamma_B(\mathfrak{b})\big)$ will also contain its closure $\iota^{-1}\big(\Gamma_{\overline{B}}(\bar{\mathfrak{b}})\big)$. The converse claim is clear since $\bar{\mathfrak{b}}$ is nonzero only if $\mathfrak{b}$ is. $\square$

3.2. **Fullness peers.** A pseudorepresentation may be B-full for more than one choice of ring B, even if B is a closed subring of A. For example, any pseudorepresentation that happens to be full for $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$ is also full for the order $\mathbb{Z}_p + p\mathbb{Z}_{p^2} \subset \mathbb{Z}_{p^2}$. We say two subrings B_1, B_2 of K are *fullness peers* if every nonzero ideal of B_1 contains a nonzero ideal of B_2 and vice versa, in which case B_1-fullness is equivalent to B_2-fullness. One easily checks that fullness peerage is an equivalence relation on subrings of K. The next lemma gives a criterion for establishing when nested domains are fullness peers.

Lemma 3.5. *Let $B_1 \subseteq B_2$ be domains. The following conditions are equivalent:*

(1) B_1 contains a nonzero ideal of B_2;
(2) there exists $y \in B_1 \setminus \{0\}$ such that $yB_2 \subseteq B_1$;
(3) B_1 and B_2 are fullness peers.

These equivalent conditions imply that B_2 and B_1 have the same field of fractions. If moreover B_1 is noetherian, then conditions (1,2,3) are equivalent to:

(4) $Q(B_2) = Q(B_1)$ and B_2 is a finite B_1-algebra.

Proof. For (1) implies (2), take y to be any nonzero element of the nonzero ideal of B_2 contained in B_1. If (2) holds, then an arbitrary nonzero ideal $\mathfrak{b}$ of B_1 contains $(yB_2)\mathfrak{b}$, which is a nonzero ideal of B_2, implying (3). Clearly (3) implies (1).

To see that $Q(B_2) = Q(B_1)$ under any of (1,2,3), note that any $x \in B_2$ can be written as $(yx)/y \in Q(B_1)$ with y as in (2).

For the rest of the proof, assume that B_1 is noetherian. Suppose first that any of (1,2,3) holds. If J is a non-zero ideal of B_2 contained in B_1, then J is a finitely generated B_1-module. By replacing J with a smaller B_2-ideal, we can assume that J is principal in B_2, that is, $J = bB_2$ for some $b \in B_1$. Now choose a finite set of generators $\{bx_1, \dots, bx_n\}$ of bB_2 as an B_1-module, with $x_1, \dots, x_n$ in B_2. Then, for every y in B_1, by is a linear combination $\sum_i a_i b x_i$ for some $a_i \in B_1$, which means that $y = \sum_i a_i x_i$, so the set $\{x_1, \dots, x_n\}$ generates B_2 as an B_1-module.

Conversely, suppose that (4) is satisfied. Let $x_1, \dots, x_n$ be generators for B_2 as an B_1-module. Write $x_i = b_{i1}/b_{i2}$ with $b_{ij} \in B_1 \setminus \{0\}$. Set $b = \prod_{i=1}^n b_{i2} \in B_1 \setminus \{0\}$. Then $bx_i \in B_1$ for all i, and it follows that $bB_2 \subseteq B_1$, proving (2). ☐

Question 3.6. Note that $\mathbb{Z}_p[\![X]\!]$ and its non-noetherian subring $\mathbb{Z}_p + p\mathbb{Z}_p[\![X]\!]$ and are fullness peers even though the extension is not finite. Could profiniteness substitute for finiteness in (4) above? That is, if B_2 is a local noetherian pro-p domain, and $B_1 \subset B_2$ is a closed subring with the same field of fractions, are B_1 and B_2 necessarily fullness peers? △

Corollary 3.7. *Let $(t, d) : \Pi \to A$ be a pseudorepresentation. If (t, d) is B-full for some subring B of K, then $B \cap A$ is a fullness peer of B.*

Proof. By Proposition 3.3, the trace algebra of (t, d), and hence A, contains a nonzero ideal of B. Therefore so does $B \cap A$, and by Lemma 3.5, $B \cap A \subseteq B$ is an extension of fullness peers. ☐

Corollary 3.7 together with Proposition 3.4 allows us to restrict our attention to fullness for closed subrings of A, though we continue to point out features of the general case for completeness.

Corollary 3.8. *Let B be a complete local noetherian domain. Then all the extensions of B contained in the normalization B^{norm} are fullness peers of B.*

Proof. Since B^{norm} is the integral closure of B in its field of fractions, B^{norm} is finite over B by the $N2$ property (see p. 8), hence a noetherian B-module. Therefore any ring C with $B \subseteq C \subseteq B^{\mathrm{norm}}$ is finite over B. Fullness peerage then follows from Lemma 3.5. ☐

3.3. **Fullness and twisting.** A key property of fullness, shown in Corollary 3.12, is that it does not change when we twist a pseudorepresentation by a character. This will allow us to do all of our technical work in the setting of constant determinant pseudorepresentations where we have Bellaïche's Theorem 2.23 available. The proof of the twist invariance of fullness relies on a calculation of the Pink-Lie algebras of a congruence subgroup. Let B be a local pro-p domain. For a closed nonzero B-ideal $\mathfrak{b}$, define

$$\mathfrak{sl}_2(\mathfrak{b}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a, b, c \in \mathfrak{b} \right\} \subset M_2(\mathfrak{b}).$$

Lemma 3.9 (cf. [Bel19, Proposition 4.8.2])**.** *Let $\mathfrak{b}$ be a closed ideal of B. Then $\Gamma_B(\mathfrak{b})$ is a closed pro-p subgroup of $\mathrm{GL}_2(B)$ and $L_n(\Gamma_B(\mathfrak{b})) = \mathfrak{sl}_2(\mathfrak{b}^n)$.*

Proof. For $x = \left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right) \in \Gamma_B(\mathfrak{b})$ one has $\Theta(x) = \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} \in \mathfrak{sl}_2(\mathfrak{b})$, so $L_1(\Gamma_B(\mathfrak{b})) \subseteq \mathfrak{sl}_2(\mathfrak{b})$. In particular, for any $b, c \in \mathfrak{b}$, we have $\Theta\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ and $\Theta\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$. For $a \in \mathfrak{b}$ we have $\left(\begin{smallmatrix} 1+2a & -2a \\ 2a & 1-2a \end{smallmatrix}\right) \in \Gamma_B(\mathfrak{b})$, and so

$$\Theta\left(\begin{smallmatrix} 1+2a & -2a \\ 2a & 1-2a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -2a \\ 2a & -a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) + \Theta\left(\begin{smallmatrix} 1 & -2a \\ 0 & 1 \end{smallmatrix}\right) + \Theta\left(\begin{smallmatrix} 1 & 0 \\ 2a & 1 \end{smallmatrix}\right).$$

It follows that $\mathfrak{sl}_2(\mathfrak{b})$ is contained in the additive subgroup generated by $\Theta(\Gamma_B(\mathfrak{b}))$. Since $\mathfrak{sl}_2(\mathfrak{b})$ is closed in $\mathfrak{sl}_2(B)$, it follows that $\mathfrak{sl}_2(\mathfrak{b}) = L_1(\Gamma_B(\mathfrak{b}))$.

It is straightforward to calculate by induction on n that the subgroup topologically generated by

$$\{xy - yx \colon x \in \mathfrak{sl}_2(\mathfrak{b}), y \in \mathfrak{sl}_2(\mathfrak{b}^n)\}$$

is $\mathfrak{sl}_2(\mathfrak{b}^{n+1})$. That is, $L_n(\Gamma_B(\mathfrak{b})) = \mathfrak{sl}_2(\mathfrak{b}^n)$ for all $n \geq 1$. $\square$

Corollary 3.10. *Let $\mathfrak{b}$ be a closed B-ideal different from B. Then $\big(\Gamma_B(\mathfrak{b}), \Gamma_B(\mathfrak{b})\big) = \Gamma_B(\mathfrak{b}^2)$. If $\#\mathbb{B} > 3$, this also holds for $\mathfrak{b} = B$.*

Proof. First assume that $\mathfrak{b} \neq B$. By Theorem 2.16,

$$\big(\Gamma_B(\mathfrak{b}), \Gamma_B(\mathfrak{b})\big) = \Theta^{-1}(L_2(\Gamma_B(\mathfrak{b}))) \cap \Gamma_B(\mathfrak{m}).$$

By Lemma 3.9, $L_2(\Gamma_B(\mathfrak{b})) = \mathfrak{sl}_2(\mathfrak{b}^2)$.

Clearly $\Gamma_B(\mathfrak{b}^2) \subseteq \Theta^{-1}(\mathfrak{sl}_2(\mathfrak{b}^2)) \cap \Gamma_B A(\mathfrak{m})$. We compute $\Theta^{-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \cap \Gamma_B(\mathfrak{m})$ for $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathfrak{b}^2)$. If $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Theta^{-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \cap \Gamma_B(\mathfrak{m})$ then we must have $\beta = b, \gamma = c, \alpha - \delta = 2a$, and $1 = \alpha\delta - \beta\gamma$. From this one calculates that $\alpha = a \pm \sqrt{1 + a^2 + bc}$ and $\delta = -a \pm \sqrt{1 + a^2 + bc}$. But only one of these possibilities has $\alpha \equiv 1 \equiv \delta \bmod \mathfrak{m}$ and thus is in $\Gamma_B(\mathfrak{m})$. That is, there is a unique element in $\Theta^{-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \cap \Gamma_B(\mathfrak{m})$. It follows that $\Theta^{-1}(\mathfrak{sl}_2(\mathfrak{b}^2)) \cap \Gamma_B(\mathfrak{m}) = \Gamma_B(\mathfrak{b}^2)$, as desired.

We now prove that $\big(\mathrm{SL}_2(B), \mathrm{SL}_2(A)\big) = \mathrm{SL}_2(A)$ when $\#\mathbb{B} > 3$. By the first statement in the corollary, we know that $\Gamma_B(\mathfrak{m}^2) \subseteq \big(\mathrm{SL}_2(B), \mathrm{SL}_2(B)\big)$, so we may assume that $\mathfrak{m}^2 = 0$. Furthermore, the residual image of $\big(\mathrm{SL}_2(B), \mathrm{SL}_2(B)\big)$ is $\big(\mathrm{SL}_2(\mathbb{B}), \mathrm{SL}_2(\mathbb{B})\big)$, which is equal to $\mathrm{SL}_2(\mathbb{B})$. Therefore, it suffices to show that $\left(\begin{smallmatrix} 1+a & b \\ c & 1-a \end{smallmatrix}\right) \in \big(\mathrm{SL}_2(B), \mathrm{SL}_2(B)\big)$ for any with $a, b, c \in \mathfrak{m}$. Since $\mathfrak{m}^2 = 0$, we can decompose

$$\left(\begin{smallmatrix} 1+a & b \\ c & 1-a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+a & 0 \\ 0 & 1-a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ c & 1 \end{smallmatrix}\right).$$

Let $x \in B^\times$ such that $x^2 \not\equiv 1 \bmod \mathfrak{m}$, which exists since $\#\mathbb{B} > 3$. Note that for any $\beta, \gamma \in \mathfrak{m}$ we have

$$\begin{aligned} \left(\begin{smallmatrix} 1 & b \\ c & 1 \end{smallmatrix}\right) &= \begin{pmatrix} 1 & b(1-x^2)^{-1} \\ c(1-x^{-2})^{-1} & 1 \end{pmatrix} \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \begin{pmatrix} 1 & -b(1-x^2)^{-1} \\ -c(1-x^{-2})^{-1} & 1 \end{pmatrix} \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right) \\ &\in \big(\mathrm{SL}_2(B), \mathrm{SL}_2(B)\big) \end{aligned}$$

and

$$\left(\begin{smallmatrix} 1+a & 0 \\ 0 & 1-a \end{smallmatrix}\right) = \begin{pmatrix} 1+\frac{a}{2} & 0 \\ 0 & 1-\frac{a}{2} \end{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \begin{pmatrix} 1-\frac{a}{2} & 0 \\ 0 & 1+\frac{a}{2} \end{pmatrix} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \big(\mathrm{SL}_2(B), \mathrm{SL}_2(B)\big).$$

It follows that $\Gamma_B(\mathfrak{m}) \subseteq \big(\mathrm{SL}_2(B), \mathrm{SL}_2(B)\big)$ and hence that $\mathrm{SL}_2(B)$ is its own topological derived subgroup. $\square$

Having calculated the derived subgroup of a congruence subgroup, we can now prove that fullness for closed subrings of A is inherited by restrictions to finite-index and coabelian subgroups.

Proposition 3.11. *Suppose that the pseudorepresentation $(t,d) : \Pi \to A$ is B-full for a subring B of K. Let Π_0 be a closed normal subgroup of Π so that Π/Π_0 is abelian. Then $\big(t|_{\Pi_0}, d|_{\Pi_0}\big)$ is also B-full. The same is true if Π_0 is a closed finite-index subgroup of Π so long as B is not finite.*

Proof. First, assume that B is a closed subring of A.

Let $\rho\colon \Pi \to R^\times$ be a (t,d)-representation such that $\rho(\Pi)$ contains $\Gamma_B(\mathfrak{b})$ for some nonzero B-ideal $\mathfrak{b}$. Write $G := \rho(\Pi)$ and let $G_0 := \rho(\Pi_0)$. If Π_0 is coabelian, then G/G_0 is abelian, so that G_0 contains the derived subgroup (G,G). In particular, G_0 contains $(\Gamma_B(\mathfrak{b}), \Gamma_B(\mathfrak{b}))$, which is $\Gamma_B(\mathfrak{b}^2)$ by Corollary 3.10. Since A is a domain, $\mathfrak{b}^2$ is nonzero if $\mathfrak{b}$ is.

Suppose alternatively that Π_0 is finite index in Π and B is not finite. Replacing $\mathfrak{b}$ by $\mathfrak{m}_A \cap \mathfrak{b}$ (that is, if $\mathfrak{b} = B$, we replace $\mathfrak{b}$ by the maximal ideal $\mathfrak{m}_A \cap B$ of B, nonzero by the assumption on B), we note that $\Gamma_B(\mathfrak{b})$ is contained in $\rho(\Gamma)$ for $\Gamma := \ker \bar{\rho} \subseteq \Pi$. Let Γ_0 be the normal core of $\Gamma \cap \Pi_0$ inside Γ, so that Γ_0 is a finite-index normal subgroup of Γ contained in Π_0. Since Γ is pro-p, and hence pro-solvable, and Γ/Γ_0 is finite, Γ_0 must contain the n^{th} derived subgroup of Γ for some $n \geq 1$. Therefore $\rho(\Gamma_0)$, and hence $\rho(\Pi_0)$, contains the n^{th} closed derived subgroup of $\Gamma_B(\mathfrak{b})$, namely $\Gamma_B(\mathfrak{b}^{2^n})$ (Corollary 3.10). This last is again a nontrivial congruence subgroup of $\mathrm{SL}_2(B)$.

For arbitrary B, Corollary 3.7 allows us to assume that B is a subring of A. If (t,d) is B-full of level $\mathfrak{b}$ for some ideal $\mathfrak{b}$ of B, then the argument above and the first part of Proposition 3.4 tell us that $\big(t|_{\Pi_0}, d|_{\Pi_0}\big)$ is $\overline{B}$-full of level $\bar{\mathfrak{b}}^{2^n}$ for some $n \geq 1$. This ideal is the closure of the ideal $\mathfrak{b}^{2^n}$ of B, so we are done by the second part of Proposition 3.4. $\square$

We conclude that fullness is unchanged under twisting.

Corollary 3.12. *Let $(t,d) : \Pi \to A$ be a pseudorepresentation and $\chi : \Pi \to A^\times$ a continuous character. If (t,d) is B-full for some subring B of K, then $(\chi t, \chi^2 d)$ is also B-full.*

Proof. Follows from Proposition 3.11 by setting $\Pi_0 := \ker \chi$. $\square$

4. Adjoint trace rings and conjugate self-twists

Having established the terms of the investigation — finding congruence subgroups for *fullness rings* (Definition 3.1) contained in images of GMA-valued representations (Proposition 2.11) — in this section we search for optimal (fullness peerage equivalence classes of) fullness rings. Our first stop is the ring fixed by the conjugate self-twists of (t,d) (Definition 4.1 below), symmetries that naturally limit its image. Although this fixed-by-twist-automorphisms subring is a fullness ring for the historical big-image results that serve as our inspiration, one cannot expect fullness with respect to the ring fixed by conjugate self-twists in the general setting. Indeed, there may not be enough automorphisms to carve down to a fullness ring as illustrated in Examples 4.3 and 4.10, reflecting the limits of Galois theory.

Instead of trying to cut out a fullness ring from above, we build one from below by considering the trace ring A_0 of the adjoint pseudorepresentation. This *adjoint trace ring* (Definition 4.6 below) does not change when twisting (t,d) by a character, and moreover is morally expected to be pointwise fixed by all conjugate self-twists. Because of topological considerations, we do not actually show that A_0 is fixed by all conjugate self-twists (Corollary 4.24) until after we prove our main fullness result, so this idea is merely a guiding principle — except for (t,d) whose determinant is A_0-*constant* (Definition 4.14), a condition expected to be satisfied by all intended applications.

The main result of this section is Corollary 4.20: if (t,d) has A_0-constant–determinant, then A_0 and the ring fixed by conjugate self-twists are fullness peers. We crucially use this fullness peerage result when deriving our A_0-fullness results for certain constant-determinant pseudorepresentations satisfying our mild conditions (Corollary 9.16), which we then propagate to all such pseudorepresentations using the twist-invariance of A_0 (Theorem 10.1).

4.1. **Conjugate self-twists.** Recall that A is a local pro-p noetherian ring and Π is a p-finite profinite group. Fix a continuous pseudorepresentation $(t,d) : \Pi \to A$ with trace algebra A_t.

Definition 4.1. If A is a domain, let B be a domain extending it; otherwise let $B = A$. A *(B-valued) conjugate self-twist* of (t,d) is a pair (σ,η), where σ is an automorphism of B as a $\mathbb{Z}_p$-algebra and $\eta : \Pi \to B^\times$ is a character. We also consider $\widetilde{\Sigma}_t(B/C)$, the conjugate self-twists whose automorphisms $\Sigma_t(B/C)$ fix a subring C of B.

The set of all B-valued conjugate self-twists of a pseudorepresentation forms a group $\widetilde{\Sigma}_t(B)$, with composition law $(\sigma_1,\eta_1)\circ(\sigma_2,\eta_2) = (\sigma_1\sigma_2, \eta_1{}^{\sigma_1}\eta_2)$ and inverse $(\sigma,\eta)^{-1} = (\sigma^{-1}, {}^{\sigma^{-1}}\eta^{-1})$. Forgetting the character is therefore is a group homomorphism $\widetilde{\Sigma}_t(B) \to \mathrm{Aut}_{\mathbb{Z}_p}(B)$ whose image we denote $\Sigma_t(B)$. If A is a domain and $\rho : \Pi \to \mathrm{GL}_2(A)$ is a semisimple representation, we use the notation $\Sigma_\rho(B) := \Sigma_{\mathrm{tr}\,\rho}(B)$ and $\widetilde{\Sigma}_\rho(B) := \widetilde{\Sigma}_{\mathrm{tr}\,\rho}(B)$.

The kernel of the forget-the-character map are the *dihedral* conjugate self-twists Σ_t^{di}:

$$1 \to \Sigma_t^{\mathrm{di}} \to \widetilde{\Sigma}_t(B) \to \Sigma_t(B) \to 1.$$

If $(1,\eta)$ is a nontrivial dihedral conjugate self-twist, then one can check that $H := \ker\eta$ is an index-2 subgroup of G, that $(t|_H, d|_H)$ is a sum of two characters, and that (t,d) is carried by the induction of either of them. In particular η is quadratic so that Σ_t^{di} does not depend on B. Moreover, when B is a field, H, and hence η, is uniquely defined by (t,d) unless the projective image of the semisimple ρ carrying (t,d) is the Klein-4 group, in which case there are three possibilities for H. In other words, when B is a field, Σ_t^{di} is abelian dihedral if ρ is projectively dihedral, cyclic of order 2 if ρ is reducible of projective order 2, and trivial in all other cases. See Lemmas A.6 and A.7, and the proof of Lemma 7.1 for details.

Proposition 4.2. *If $\chi : \Pi \to A^\times$ is a continuous character and B is an extension of A, then*

$$\widetilde{\Sigma}_t(B) \cong \widetilde{\Sigma}_{\chi t}(B).$$

Proof. The map $(\sigma,\eta) \mapsto (\sigma, {}^\sigma\chi\chi^{-1}\eta)$ realizes the isomorphism. □

If a conjugate self-twist (σ,η) happens to be A-valued, then the automorphism σ is automatically continuous: since A is local, algebraic automorphisms automatically send the maximal ideal to itself, and since A is noetherian, the maximal ideal defines the topology. It turns out that an A-valued conjugate self-twist character η must also be continuous — see Proposition 4.4(3) below. But in general we cannot expect that all conjugate self-twists can be restricted to ones defined over A, as seen in Example 4.3 below illustrating a failure of normality in the sense of Galois theory.

Example 4.3. For any odd prime p, let Π be the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p[\sqrt[p]{p}])$ generated by $\mathrm{GL}_2(\mathbb{Z}_p)$ and the scalar matrix $1+\sqrt[p]{p}$. Let $\rho\colon \Pi \to \mathrm{GL}_2(\mathbb{Z}_p[\sqrt[p]{p}])$ be the inclusion, a representation with trace algebra $A = \mathbb{Z}_p[\sqrt[p]{p}]$. Note that $\Sigma_\rho(A)$ is trivial, since A has no automorphisms — but that's not the whole story. Let $B = \mathbb{Z}_p[\sqrt[p]{p}, \zeta_p]$ and consider the automorphism σ in $\mathrm{Aut}(B)$ sending $\sqrt[p]{p}$ to $\zeta_p\sqrt[p]{p}$, together with the character $\eta : \Pi \to B^\times$ with kernel $\mathrm{GL}_2(\mathbb{Z}_p)$ sending $1+\sqrt[p]{p}$ to $(1+\zeta_p\sqrt[p]{p})(1+\sqrt[p]{p})^{-1}$. Then (σ,η) is a B-valued conjugate self-twist of ρ. One can show that ρ is not A-full ($\mathrm{tr}\,\rho$ does not span an ideal of A; see Proposition 3.3). On the other hand, ρ is visibly $\mathbb{Z}_p$-full, corresponding to the fact that $\mathbb{Z}_p$ is the ring fixed by $\Sigma_\rho(B)$. △

Because it is not immediately clear that we can demand that relevant extensions of A be endowed with a sensible topology (if A is a domain we already cannot expect a topology on $Q(A)$: see p. 8), there is no way to require conjugate self-twists to be continuous. On the other hand, since trace algebras are *topologically* generated rings, they only behave well under conjugate self-twists satisfying continuity conditions. As a result, much of this section is devoted to finding settings where conjugate self-twists are A-valued and hence continuous. This will eventually allow us to show that all conjugate self-twists of constant-determinant pseudorepresentations satisfying the mild conditions of Theorem B are continuous: see Corollary 4.24. Write A_t^{alg} for the $W(\mathbb{F})$-subalgebra of A algebraically generated by $t(\Pi)$, so that the trace algebra A_t is the closure of A_t^{alg} in A.

Proposition 4.4. *Let (σ, η) be a B-valued conjugate self-twist.*

(1) If (t, d) has constant determinant, then η is $W(\mathbb{F})$-valued and finite order.

(2) If η is A_t^{alg}-valued (for example, if η is $W(\mathbb{F})$-valued), then σ restricts to an automorphism of A_t^{alg}, and there is an A_t-valued conjugate self-twist (σ', η) satisfying $\sigma'|_{A_t^{\mathrm{alg}}} = \sigma|_{A_t^{\mathrm{alg}}}$.

(3) If η is A-valued, then η is continuous.

Proof. (1) We follow [Mom81, 1.5]. Recall that $d = s(\bar{d})$ has finite order by the assumption. Since ${}^{\sigma}d = \eta^2 d$ we have $\eta^2 = {}^{\sigma}d d^{-1}$. Since d is finite order, ${}^{\sigma}d$ must be a power of d. We now claim that ${}^{\sigma}d d^{-1}$ has a power of d as a square root, so that η is differs from a power of d by at most a quadratic character and hence takes values in $W(\mathbb{F})$. Indeed, if d has odd order, then d itself has a power-of-d square root, so that any power of d has the same property. And if d has even order, then since σ preserves orders, ${}^{\sigma}d$ must be an odd power of d; which means that ${}^{\sigma}d d^{-1}$ is an even power of d and hence has a power-of-d square root.

(2) Given that η and t are both A_t^{alg}-valued, ${}^{\sigma}t = \eta t$ is A_t^{alg}-valued as well. Since B is either a domain or a local ring and σ is a $\mathbb{Z}_p$-algebra homomorphism, it follows that σ permutes $W(\mathbb{F})$, and hence it permutes A_t^{alg} as well. Moreover, the action of σ is continuous on A_t^{alg} in the topology from A_t: with $\mathfrak{m}^{\mathrm{alg}} := \mathfrak{m} \cap A_t^{\mathrm{alg}}$, we have $\sigma(\mathfrak{m}^{\mathrm{alg}}) \subseteq \eta(\Pi)\mathfrak{m}^{\mathrm{alg}} \subseteq \mathfrak{m}^{\mathrm{alg}}$. Therefore $\sigma|_{A_t^{\mathrm{alg}}}$ extends uniquely to a continuous automorphism σ' of the closure A_t of A_t^{alg}. Finally, since σ and σ' agree on $t(\Pi)$, the pair (σ', η) is still a conjugate self-twist, this time A_t-valued.

(3) First, we claim that $\ker(t, d) \subseteq \ker \eta$, so that η factors through $G := \Pi/\ker(t, d)$. Indeed, for $g \in \ker(t, d)$ we have in particular $t(g) = 2$ so that $\eta(g) = {}^{\sigma}t(g)/t(g) = 1$. Since $\ker(t, d)$ is closed, $G = \Pi/\ker(t, d)$ is still p-finite profinite, and we check continuity of η as a character on G. Moreover $\Gamma := \ker \bar{\rho}/\ker(t, d) \subseteq G$ is a finite-index pro-p subgroup of G [Che14, Lemma 3.8]. By p-finiteness, Γ is topologically finitely generated, so that any finite-index subgroup of Γ is open in Γ [Ser97, §4.2 exercise 6(d)], and hence in G. Now use the fact that $A^\times \cong \mathbb{F}^\times \times (1 + \mathfrak{m})$ to write $\eta = \eta^{(p)}\eta_p$, where $\eta_p : G \to 1 + \mathfrak{m}$ is a pro-p character and $\eta^{(p)} : G \to \mathbb{F}^\times$ has prime-to-p order. Then $\eta^{(p)}$ is continuous because its kernel contains the open pro-p subgroup Γ. And η_p is continuous because the preimage $U \subseteq \Gamma$ of any open subgroup of $1+\mathfrak{m}$ along $\eta_p|_\Gamma$ is finite index in Γ, hence open in G. Therefore η is continuous. $\square$

Corollary 4.5. *If (t, d) has constant determinant, then any conjugate self-twist (σ, η) of (t, d) has η continuous, finite-order, and $W(\mathbb{F})$-valued. Moreover, there exists an automorphism σ' of A_t agreeing with σ on A_t^{alg} so that (σ', η) is an A_t-valued conjugate self-twist of (t, d).*

A posteriori after our main fullness results, we will deduce that, at least for most constant-determinant (t, d) that are not a priori small, σ restricts to an automorphism of A_t, necessarily continuous, and consequently $\sigma' = \sigma|_{A_t}$: see Theorems 5.4 and 10.1.

4.2. **The adjoint trace ring.** Quite generally, if V is a 2-dimensional vector space over a field F, then the adjoint action of matrices $m \in \mathrm{GL}(V)$ on $\mathrm{End}_F(V)^0$ — that is, the conjugation action on the vector-space of trace-zero endomorphisms — has trace $(\operatorname{tr} m)^2/\det m$ and factors through $\mathrm{PGL}(V)$ since scalars act trivially. By analogy, we define the adjoint-trace elements and the adjoint-trace ring attached to (t,d).

Definition 4.6. The *adjoint trace ring* of (t,d), denoted $A_{0,t}$ or simply A_0, is the closed subring of A topologically generated by the *adjoint-trace elements* $\operatorname{tr}\operatorname{ad} t(g) := t(g)^2/d(g)$ for $g \in \Pi$.

Proposition 4.7. *The adjoint trace ring A_0 of any pseudorepresentation of a p-finite profinite group is a local noetherian pro-p ring. In particular, it is N2 (see p. 8).*

Proof. By definition, A_0 is a closed subring of the local pro-p ring A, so that A_0 is also local and pro-p. Moreover, by construction A_0 is the trace ring of a pseudodeformation of $\operatorname{ad}^0\bar\rho$, where $\bar\rho$ is the semisimple residual representation carrying (t,d). Since Π is p-finite, the universal deformation ring of the pseudorepresentation associated to $\operatorname{ad}^0\bar\rho$ is noetherian, hence so is its quotient A_0. ☐

Adjoint-trace elements don't change when $\big(t(g),d(g)\big)$ is replaced by $\big(\alpha t(g),\alpha^2 d(g)\big)$ for nonzero scalars α. This has two consequences. First, the adjoint-trace ring is unchanged under twisting.

Proposition 4.8. *If $\chi : \Pi \to A$ is a continuous character, then $A_{0,t} = A_{0,\chi t}$.*

Proof. The adjoint-trace elements of (t,d) and of $(\chi t, \chi^2 t)$ are the same: $\frac{t(g)^2}{d(g)} = \frac{(\chi(g)t(g))^2}{\chi^2(g)d(g)}$. ☐

To state the second, we define $A_{0,t}^{\mathrm{alg}} \subseteq A_{0,t}$ as the subring generated algebraically rather than topologically by the values of t^2/d, so that by definition $A_{0,t}$ is the closure of $A_{0,t}^{\mathrm{alg}}$ in A.

Proposition 4.9. *If (σ,η) is an arbitrary conjugate self-twist of (t,d), then σ fixes $A_{0,t}^{\mathrm{alg}}$ pointwise.*

Proof. The adjoint-trace elements are fixed by any σ. Indeed, for $g \in \Pi$,

$$^{\sigma}\big(t(g)^2/d(g)\big) = {}^{\sigma}t(g)^2/{}^{\sigma}d(g) = \big(\eta(g)t(g)\big)^2/\eta(g)^2 d(g) = t(g)^2/d(g).$$ ☐

In spite of Proposition 4.9, there may not be enough automorphisms to cut out A_0 or A_0^{alg}, as the following example illustrates.

Example 4.10. Let $G = \mathrm{GL}_2(\mathbb{F}_p[\![x]\!])$ and let $\rho : G \to \mathrm{GL}_2\left(\mathbb{F}_p[\![x]\!]\right)$ be the inclusion representation. Define $\chi : G \to \mathbb{F}_p[\![x^{1/p}]\!]^\times$ as follows: for $g \in G$ let $d_g := \det g \in \mathbb{F}_p[\![x]\!]$, and set $\chi(g) := d_g(x^{1/p})$. Then χ is a character to $A := \mathbb{F}_p[\![x^{1/p}]\!]$, and we consider the representation $\rho\otimes\chi : G \to \mathrm{GL}_2(A)$. Since ρ has no nontrivial conjugate self-twists — or, said more precisely, for any extension B of $A_0 := \mathbb{F}_p[\![x]\!]$, we have $\widetilde{\Sigma}_\rho(B/A_0) \cong \mathrm{Aut}(B/A_0)$, each appearing with trivial character — the conjugate self-twists of $\rho\otimes\chi$ defined over an extension B of A are all of the form $(\sigma, {}^\sigma\chi/\chi)$ for $\sigma \in \mathrm{Aut}(B/A_0)$ (Proposition 4.2). Since any automorphism that fixes A_0 also fixes A, the ring fixed by all conjugate self-twists of $\rho\otimes\chi$ is A itself. But in this case it's clear by inspection that A is not a fullness ring (see, for example Proposition 3.3). On the other hand A_0 is a fullness ring — the image of $\rho\otimes\chi$ contains $\mathrm{SL}_2(A_0)$. △

In the next section we show that, working over fields rather than rings, one can always cut out the analogue of A_0 with conjugate self-twists provided the usual conditions from Galois theory hold.

4.3. **Interlude: the theory over abstract fields.** In this subsection we switch gears to a topology-free setting, where we show that under Galois-theoretically favorable conditions, the *adjoint trace field* (defined below) is exactly the fixed field of all conjugate self-twists. This result allows us to give an interpretation of the residue field of A_0 as the fixed field of residual conjugate self-twists, and is both inspiration and key in deducing an analogous result for (a generalization of) constant-determinant pseudorepresentations in our pro-p setting in the next subsection — a lodestar and a tool.

Let F be an arbitrary field whose characteristic is not 2, G an abstract group, and $(t,d) : G \to F$ a pseudorepresentation. For an extension L of F, define an L-valued conjugate self-twist (σ, η) of (t,d) as in Definition 4.1, but where σ is simply a field automorphism rather than a morphism of $\mathbb{Z}_p$-algebras. Write $\widetilde{\Sigma}_t(L)$ for the set of L-valued conjugate self-twists of (t,d), and $\Sigma_t(L)$ for the group of automorphisms appearing in $\widetilde{\Sigma}_t(L)$. Define the *adjoint-trace field* of (t,d) to be the subfield $F_0 := F_{0,t}$ generated by the adjoint-trace elements $t(g)^2/d(g)$ over all $g \in G$. Then as in Proposition 4.9, the adjoint trace field $F_{0,t}$ is fixed by all conjugate self-twist automorphisms, so that $F_{0,t} \subseteq L^{\Sigma_t(L)}$. The following theorem gives conditions that guarantee reverse containment.

Theorem 4.11. *Let L be a separably closed extension of F, and $E \subseteq L$ an extension of $F_{0,t}$. Then*
$$\Sigma_t(L/E) = \mathrm{Aut}(L/E).$$
In particular, $\Sigma_t(F^{\mathrm{sep}}) = \mathrm{Aut}(F^{\mathrm{sep}}/F_{0,t})$.

Proof. To show that $\mathrm{Aut}(L/E) \subseteq \Sigma_t(L/E)$, we start with $\sigma \in \mathrm{Aut}(L/E)$ and produce a twist character. Let ρ be a semisimple representation over a finite extension F' of F carrying (t,d). Since σ fixes the adjoint trace algebra $F_0 \subseteq E$, we have $\mathrm{tr}\,\mathrm{ad}\,{}^\sigma\rho = \mathrm{tr}\,\mathrm{ad}\,\rho$. Moreover, $\mathrm{ad}\,\rho$ is semisimple if ρ is. By Brauer-Nesbitt, therefore, the multiplicities of irreducible representations of G inside $\mathrm{ad}\,\rho$ and inside $\mathrm{ad}({}^\sigma\rho)$ are equal in (the prime field of) F. If F has characteristic 0 or characteristic $p \geq 5$, then we can already conclude that $\mathrm{ad}\,\rho \cong \mathrm{ad}({}^\sigma\rho)$. If F has characteristic 3, then we split off a trivial character acting on the center using $\mathrm{ad}\,\rho = 1 \oplus \mathrm{ad}^0\,\rho$ and need only eliminate the case that $\mathrm{ad}^0\,\rho \cong \phi^{\oplus 3}$ and $\mathrm{ad}^0\,{}^\sigma\rho \cong {}^\sigma\phi^{\oplus 3}$ for some character ϕ with $\phi \neq {}^\sigma\phi$. But since at least one of the eigenvalues of $\mathrm{ad}^0\,\rho(g)$ is 1 for any $g \in G$ (see Lemma A.4), this is impossible. Finally, Theorem A.10 says that, since $\mathrm{ad}\,\rho \cong \mathrm{ad}\,{}^\sigma\rho$ as representations over L, there is a character $\eta \colon G \to L^\times$ such that ${}^\sigma\rho \cong \eta \otimes \rho$. Therefore (σ, η) is a conjugate self-twist!

The second statement is a special case of the first since every conjugate self-twist fixes $F_{0,t}$. $\square$

Corollary 4.12. *(1) If $F/F_{0,t}$ is a separable extension, then $(F^{\mathrm{sep}})^{\Sigma_t(F^{\mathrm{sep}})} = F_{0,t}$.*

(2) Suppose that L is a separably closed extension of F, and $E \subseteq L$ is an extension of $F_{0,t}$. If L/E is separable, then $L^{\Sigma_t(L/E)} = E$.

Both statements follow from Theorem 4.11. The conditions of Corollary 4.12(2) are nontrivially satisfied, for example, if $F_{0,t} = \mathbb{Q}$, $F = \mathbb{Q}(\sqrt[3]{2})$, $E = \mathbb{Q}_p$ and $L = \overline{\mathbb{Q}}_p$. Or see Theorem 10.3(4).

In particular Corollary 4.12(1) for $F = \mathbb{F}$ finite applies in our pro-p setting. In this case, every pseudorepresentation $(t,d) : \Pi \to F$ is carried by a unique semisimple $\bar\rho : \Pi \to \mathrm{GL}_2(\mathbb{F})$. Write $\mathbb{E}$ for the adjoint trace field $\mathbb{F}_{0,\bar\rho}$.

Corollary 4.13. $\mathbb{E} = \overline{\mathbb{F}}^{\Sigma_{\bar\rho}(\overline{\mathbb{F}})} = \mathbb{F}^{\Sigma_{\bar\rho}(\mathbb{F})}$

Proof. The first equality is Corollary 4.12(1). The second follows from Corollary 4.5, which implies that every conjugate self-twist of $\bar\rho$ restricts to one defined over $\mathbb{F}$. $\square$

Our next goal is to limn a setting where the pro-p analogue of the extension $F/F_{0,t}$ is Galois, so that we can obtain analogues of Theorem 4.11 and Corollary 4.12(1).

4.4. A_0-constant determinant and simple conjugate self-twists.

We now return to our pro-p setting and consider a restriction on the determinant of (t,d) that shares many properties with the constant-determinant case, but is mild enough to be expected to be satisfied by all our arithmetic applications. This constraint allows us to restrict our attention to conjugate self-twists valued in the trace algebra.

Definition 4.14. We say that (t,d) has A_0-*constant determinant* if d is the product of an A_0-valued character and a character of finite prime-to-p order. Said another way, d is A_0-constant if its pro-p part d_1 takes values in A_0. If (t,d) has A_0-constant determinant, we call the conjugate self-twists in $\widetilde{\Sigma}_t(A_t)$ *simple*. For brevity we write $\widetilde{\Sigma}_t$ and Σ_t for these simple conjugate self-twists in Sections 7 through 9.

By automatic continuity, all automorphisms in $\Sigma_t(A_t)$ fix A_0, so that $\Sigma_t(A_t) = \Sigma_t(A_t/A_0)$. The following lemma shows that twist characters are also continuous.

Lemma 4.15. *Let $(t,d) : \Pi \to A$ be an A_0-constant-determinant pseudorepresentation, and let (t',d') be its constant-determinant twist obtained by twisting off the pro-p part d_1 of d. Then:*

(1) $A_t = A_{t'}$

(2) $\widetilde{\Sigma}_t(A_t) = \widetilde{\Sigma}_{t'}(A_{t'})$

(3) If (σ,η) in $\widetilde{\Sigma}_t(A_t)$ is a simple conjugate self-twist, then η is $W(\mathbb{F})^\times$-valued and continuous.

Proof. First note that (t,d) and (t',d') have the same adjoint trace algebra A_0, since the latter is twist invariant (Proposition 4.8). Write χ for $d_1^{-1/2}$, a continuous A_0-valued character. Note that $A_0 \subseteq A_t \cap A_{t'}$. Since χ and χ^{-1} are both valued in $A_0 \subseteq A_t \cap A_{t'}$, we can move back and forth using $t' = \chi t$ and $t = \chi^{-1} t$. That is $A_{t'} = A_{\chi t} \subseteq A_t$ and $A_t = A_{\chi^{-1}t'} \subseteq A_{t'}$. Part (1) follows. For (2), because χ is fixed by σ, the map $\widetilde{\Sigma}_t(A_t) \to \widetilde{\Sigma}_{\chi t'}(A_t)$ from Proposition 4.2 sending (σ,η) to $(\sigma, {}^\sigma\chi\chi^{-1}\eta)$ is the identity. For (3), use (2) and Corollary 4.5, or redo Proposition 4.4(1) directly and use Proposition 4.4(3). $\square$

4.5. **K/K_0 as a Galois extension.** Finally we assume that A is a (local pro-p) <u>domain</u> and fix an <u>A_0-constant-determinant</u> pseudorepresentation $(t,d) : \Pi \to A$. By replacing A by A_t, we may assume that A is the trace algebra of (t,d); let K be the fraction field of A. Recall that $\bar\rho$ is the semisimple representation $\Pi \to \mathrm{GL}_2(\mathbb{F})$ carrying $(\bar t, \bar d)$. Let A_0 be the adjoint trace ring of (t,d) with fraction field K_0; write $\mathbb{E}$ for the adjoint trace ring of $\bar\rho$. Here $\mathbb{F}$ is the residue field of A and $\mathbb{E}$ is the residue field of A_0. Let $\Gamma := \ker\bar\rho \subset \Pi$. This is a finite-index normal subgroup of Π, and we can restrict (t,d) to Γ to obtain $(t|_\Gamma, d_1)$.

Lemma 4.16. *We have $t(\Gamma) \subset A_0$.*

Proof. For $\gamma \in \Gamma$, we have $t(\gamma) = 2 + m$ for some $m \in \mathfrak{m}$. The corresponding adjoint-trace element is $t(\gamma)^2/d_1(\gamma) = d_1(\gamma)^{-1}(4+4m+m^2) \in A_0$. Since $d_1(\gamma)^{-1} \in A_0$ by assumption, so is $4+4m+m^2$. But any closed subring containing $4+4m+m^2$ contains $2+m = t(\gamma)$ as well since 2 is invertible. $\square$

Proposition 4.17. *A is a finite A_0-algebra.*

Proof. The subgroup $\Gamma := \ker\bar\rho$ is a finite-index normal subgroup of Π. Moreover, $A_{t|_\Gamma} \subseteq A_0$ (Lemma 4.16). Thus K_0 contains the trace field of $t|_\Gamma$, so that by Lemma A.12 there is an at-most quadratic extension L_0 over K_0 over which we can define a representation carrying $(t|_\Gamma, d_1)$. By the proof of Proposition A.13, there is a finite extension L over L_0 containing $t(\Pi)$. In fact, letting A^{alg} be the $W(\mathbb{F})A_0$-submodule of A (algebraically, not topologically) generated by $t(\Pi)$, we have $A^{\mathrm{alg}} \subseteq L$. Note that A is the closure of A^{alg}.

We claim that A^{alg} is integral over A_0. For every $g \in \Pi$, the element $t(g)$ is a square root of $t(g)^2$, which is in $A_0[d] \subseteq W(\mathbb{F})A_0$. Since A^{alg} is generated by the $t(g)$ over $W(\mathbb{F})A_0$, the former is integral over the latter. The claim follows.

Finally, let B be the integral closure of A_0 in L. Since A_0 is noetherian and $N2$ (Proposition 4.7), B is a finite, hence noetherian, A_0-algebra. Because B contains A^{alg} by integrality, the latter is also finite over A_0. But this means that A^{alg} is a sum of finitely many compact submodules of A, so compact itself, hence closed. In other words, $A^{\mathrm{alg}} = A$. Therefore A is finite over A_0. $\square$

Corollary 4.18. *A is a multiquadratic extension of $W(\mathbb{F})A_0$. More precisely, there is an integer n prime to p, elements $a_1,\ldots,a_r$ of A_0, and integers $k_1,\ldots,k_r$ modulo n so that for $\zeta = \zeta_n$ a generator of $s(\mathbb{F}^\times)$ we have*

$$A = A_0[\zeta, \sqrt{\zeta^{k_1}a_1}, \ldots, \sqrt{\zeta^{k_r}a_r}]. \tag{3}$$

Proof. Let n be the order of $\mathbb{F}^\times$, so that ζ generates $W(\mathbb{F})$ and $A_0[\zeta] = W(\mathbb{F})A_0$. Since A_0 contains

$$a(g) := d_1(g)t^2(g)/d(g) = t^2(g)/s(\bar{d}(g)) = t^2(g)/\zeta^{k(g)}$$

for every $g \in \Pi$ and $k(g)$ depending on g, we see that A is topologically generated over $W(\mathbb{F})A_0$ by $t(g) = \sqrt{\zeta^{k(g)}a(g)}$. By Proposition 4.17 and compactness only finitely many of these are needed. ☐

Theorem 4.19. *The extension K over K_0 is finite abelian, with*

$$\mathrm{Gal}(K/K_0) = \mathrm{Aut}(A/A_0) = \Sigma_t(A).$$

We give two different arguments both fundamentally rooted in Theorem 4.11.

First proof of Theorem 4.19. Since $K = K_0(\zeta, \sqrt{\zeta^{k_1}a_1}, \ldots, \sqrt{\zeta^{k_r}a_r})$ by Corollary 4.18 (and using the same notation), it is clear that $\mathrm{Aut}(K/K_0) = \mathrm{Aut}(A/A_0)$. To see that K/K_0 is Galois, note that it is a subextension of $K_0(\zeta_{2n}, \sqrt{a_1}, \ldots, \sqrt{a_r})/K_0$, which is a compositum of abelian extensions and hence abelian as well. Here we use the fact that the characteristic of (the prime field of) K is either zero or an odd prime p, so that $K_0(\zeta_{2n})/K_0$ for n prime to p and the $K_0(\sqrt{a_i})/K_0$ are all separable and hence abelian. Therefore K/K_0 is abelian with $\mathrm{Gal}(K/K_0)$ a subgroup of $(\mathbb{Z}/2\mathbb{Z})^r \times (\mathbb{Z}/n\mathbb{Z})^\times$.

To see that $\mathrm{Aut}(A/A_0) = \Sigma_t(A)$, start with $\sigma \in \mathrm{Aut}(A/A_0)$ and proceed as in the proof of Theorem 4.11 to produce a character $\eta : \Pi \to K^\times$ with (σ, η) a conjugate self-twist of (t, d). By Lemma 4.15, η is A-valued, so that (σ, η) is in $\Sigma_t(A)$. ☐

Second proof of Theorem 4.19. As in the first proof, it is clear that K is separable over K_0. First suppose that (t, d) has constant determinant. To show that K/K_0 is normal, take an embedding σ of K into $\overline{K}$ over K_0. Proceed as in the proof of Theorem 4.11 to get a character $\eta : \Pi \to \overline{K}^\times$ so that (σ, η) is a $\overline{K}$-valued conjugate self-twist. Use Corollary 4.5 to find σ' so that (σ', η) is an A-valued conjugate self-twist. Note that $\sigma'|_{W(\mathbb{F})A_0} = \sigma|_{W(\mathbb{F})A_0}$ since both σ and σ' fix A_0 and act the same way on $W(\mathbb{F})$ by construction. Therefore $\sigma'\sigma^{-1}$ is an embedding of K that fixes $Q(W(\mathbb{F})A_0) = K_0(\zeta)$. Since $K/K_0(\zeta)$ is a multiquadratic extension, it is Galois, so $\sigma'\sigma^{-1}$ sends K to itself. But σ' also sends K to itself by construction, so σ must be an automorphism of K. In fact, using that $\sigma'\sigma^{-1}$ fixes K_0 and $t(\Pi)$, we see that $\sigma|_A = \sigma'$ and thus $\mathrm{Gal}(K/K_0) = \Sigma_t(A)$. The fact that $\Sigma_t(A)$ is abelian in the constant-determinant case follows from Corollary 7.2 and the diagram following Corollary 7.4. In general when (t, d) has A_0-constant determinant, the result follows from the constant determinant-case and Lemma 4.15. ☐

Corollary 4.20. *We have $K_0 = K^{\Sigma_t(A)}$. Moreover, $A^{\Sigma_t(A)}$ is a finite extension of A_0 with the same field of fractions and the same normalization.*

Proof. The first statement follows from Theorem 4.19. Since $A_0 \subseteq A^{\Sigma_t(A)} \subseteq K^{\Sigma_t(A)} = K_0$, the field of fractions of $A^{\Sigma_t(A)}$ is K_0. Finally, since A is integral over A_0, so is $A^{\Sigma_t(A)}$. ☐

Corollaries 3.8 and 4.20 imply the following fullness comparison result for two key subrings of A.

Corollary 4.21. *The rings A_0 and $A^{\Sigma_t(A)}$, as well as their normalizations, are all fullness peers.*

The example below shows that proper containment $A_0 \subsetneq A^{\Sigma_t(A)}$ is possible: A_0 and $A^{\Sigma_t(A)}$ need not have the same residue field.

Example 4.22. Define

$$\overline{G} := \left\{ \begin{pmatrix} a & b \\ b^p & a^p \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & b \\ -b^p & -a^p \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{F}_{p^2}),$$

so that in each set above a and b are in $\mathbb{F}_{p^2}$ satisfying $N(a) \neq N(b)$. Let $\widetilde{G} \subset \mathrm{GL}_2(\mathbb{F}_{p^2}[\![X]\!])$ be the set of matrices whose determinants are in $\mathbb{F}_{p^2}$, and let $G \subset \widetilde{G}$ be the set of invertible matrices that are residually in $\overline{G}$. Finally, let $\Pi := G$ and set $\rho : \Pi \to \mathrm{GL}_2(\mathbb{F}_{p^2}[\![X]\!])$ to be the inclusion map, a constant-determinant representation. Then the trace algebra A of ρ is $A = \mathbb{F}_{p^2}[\![X]\!]$ — indeed, fix a generator β of $\mathbb{F}_{p^2}^\times$, so that $N(\beta) = \beta^{p+1} \neq 1$. Then for every $a \in \mathbb{F}_{p^2}^\times$, G contains both

$$g_a := \begin{pmatrix} 1 & \beta \\ \beta^p + \beta^{-1}aX & 1 + aX \end{pmatrix} \qquad \text{and} \qquad h_a := \begin{pmatrix} a & 0 \\ 0 & -a^p \end{pmatrix},$$

so that A contains both $\operatorname{tr} g_a = 2 + aX$ and $\operatorname{tr} h_a = a - a^p$. A similar computation shows that the adjoint trace ring is $A_0 = \mathbb{F}_p + X\mathbb{F}_{p^2}[\![X]\!]$: on one hand, A_0 contains

$$(\operatorname{tr} g_a)^2 / \det g_a = (4 + 4aX + a^2X^2)(1 - N(\beta))^{-1}$$

for every $a \in \mathbb{F}_{p^2}^\times$, and on the other hand every element of A_0 is residually in $\mathbb{F}_p$, reflecting the fact that $\bar{\rho}$ admits a conjugate self-twist and illustrating Corollary 4.13. Since ρ itself has no conjugate self-twist — any automorphism of A appearing in a conjugate self-twist must fix A_0 and hence extends to the identity on K_0, which contains A — we have $A^{\Sigma_\rho(A)} = A$. △

We close with a technical observation necessary for our a posteriori justification for focusing on A-valued conjugate self-twists for A_0-constant–determinant pseudorepresentations.

Proposition 4.23. *Let (σ, η) be any conjugate self-twist of an A_0-constant–determinant pseudorepresenation (t, d). Its trace algebra A is σ-stable if and only if σ fixes A_0 pointwise. Both of these conditions are satisified if (t, d) is A_0-full.*

Proof. If σ restricts to an automorphism of A, then $\sigma|_A$ is automatically continuous, so that σ fixing A_0^{alg} pointwise per Proposition 4.9 is enough to conclude that σ fixes all of A_0. Conversely, if A_0 is fixed by σ, then since A is finite over A_0 in this setting (Proposition 4.17), A^{alg} is dense in A, and A is noetherian, we can express $A = a_1A_0 + \cdots + a_nA_0$ for $a_i \in A^{\mathrm{alg}}$, so that $\sigma(A) \subseteq A$.

If (t, d) is A_0-full, then A^{alg} contains some nonzero A_0-ideal $\mathfrak{b}_0$ (Proposition 3.3). The expression $A = a_1A_0 + \cdots + a_nA_0$ for $a_i \in A^{\mathrm{alg}}$ implies that $\mathfrak{b}_0A \subseteq A^{\mathrm{alg}}$, whence A^{alg} and A are fullness peers, and in particular have the same field of fractions (Lemma 3.5). Since $A \subseteq Q(A^{\mathrm{alg}})$, the values of σ on A are entirely determined by the restriction of σ to A^{alg}. Therefore the action of σ on A coincides with the action on A by the continuous extension of $\sigma|_{A^{\mathrm{alg}}}$ guaranteed by Corollary 4.5. Thus σ restricts to an automorphism of A, as claimed. □

Corollary 4.24. *Let $(t, d) : \Pi \to A$ be any pseudorepresentation. If (t, d) is full for its adjoint trace ring A_0, then every arbitrary conjugate self-twist of (t, d) fixes A_0 pointwise.*

Proof. Let (σ, η) be a conjugate self-twist of (t, d), and let (t', d') be its constant-determinant twist. Then (σ, η') is a conjugate self-twist of (t', d') for some character η' (Proposition 4.2). Since neither fullness nor A_0 is changed under twisting (Corollary 3.12 and Proposition 4.8), the constant-determinant twist (t', d') is still A_0-full, and hence by Proposition 4.23 σ fixes A_0 pointwise. □

Combined with our main A_0-fullness result (Theorem 10.1), Corollary 4.24 allows us to conclude a posteriori that under mild conditions on non–a-priori-small (t, d), all conjugate self-twists of (t, d) fix all of A_0, resolving the topological concerns we have danced around in Sections 4.1 and 4.2. Of course one continues to hope for a less circuitous argument that works for all (t, d).

5. Optimality

Armed with the notions of conjugate self-twists and the adjoint trace ring from Section 4, we prove two related optimality results. We show that the adjoint trace ring of a pseudorepresentation (t, d) contains a fullness peer of any fullness ring for (t, d) (Theorem 5.3). In this way we establish the adjoint trace ring as the optimal fullness ring up to fullness peerage. We also show that every fullness ring is fixed by all conjugate self-twists (Theorem 5.4), generalizing Corollary 4.24.

As usual, A is a local noetherian pro-p domain carrying a continuous pseudorepresentation $(t, d) : \Pi \to A$ of a p-finite profinite group Π. We assume that A is a domain with field of fractions K. Also let A_0 and K_0 be the adjoint trace ring of (t, d) (Definition 4.1) and its fraction field, respectively. Recall that (t, d) is B-full for any subring B of K if there exists a (t, d)-representation $\rho : \Pi \to R^\times$ and an embedding $R \hookrightarrow M_2(K)$ such that via this embedding $\operatorname{Im} \rho \supseteq \Gamma_B(\mathfrak{b})$ for some nonzero ideal $\mathfrak{b}$ of B (Definition 3.1).

We first show that fullness rings are fixed by conjugate self-twist automorphisms whose characters are continuous, or at least nearly so.

Proposition 5.1. *Suppose (t, d) is B-full for some subring B of A. Let (σ, η) be a conjugate self-twist of (t, d) valued in any domain extending A. If $\ker \eta$ contains a subgroup H, closed and normal in Π, such that Π/H is abelian, then σ fixes B pointwise.*

In particular, if (σ, η) is A-valued, then σ always fixes B pointwise.

Proof. By Proposition 3.4 we may replace B by its closure in A. On one hand, since $H \subseteq \ker \eta$, we have that $t(H)$ is fixed by σ. On the other hand, by Proposition 3.11, $(t|_H, d|_H)$ is still B-full, so that by Proposition 3.3 applied to $(t|_H, d|_H) \colon H \to A$, the $\mathbb{Z}$-span of $t(H)$ contains some nonzero B-ideal $\mathfrak{b}$. Therefore every element of $\mathfrak{b}$ is fixed by σ. Since any element of B is a ratio of elements of $\mathfrak{b}$ (if $b \neq 0$ is in $\mathfrak{b}$, then $x = \frac{xb}{b}$ for any $x \in B$), every element of B is fixed by σ.

If (σ, η) is A-valued, then η is continuous (Proposition 4.4(3)) so that we can take $H = \ker \eta$. ☐

Corollary 5.2. *If (t, d) is B-full for some subring B of K, then $B \subseteq K_0$.*

Proof. Replace (t, d) with its constant-determinant twist (Corollary 3.12 and Proposition 4.8) and let A_t be its trace algebra. Replace B by its fullness peer $B \cap A_t$ (Corollary 3.7 and Lemma 3.5). The result now follows from Proposition 5.1 using Theorem 4.19. ☐

Our main optimality theorem is an improvement on Corollary 5.2: any fullness ring is not only contained in K_0, it in fact has a fullness peer subring contained in A_0.

Theorem 5.3. *Suppose (t, d) is B-full for some subring B of K. Then $B \cap A_0$ is a fullness peer of B contained in A_0.*

Proof. First suppose that B is finite. Then B is a finite field; its only nonzero ideal is itself, and the only congruence subgroup of $\mathrm{SL}_2(B)$ is $\mathrm{SL}_2(B)$ itself. In this case, B has no proper subring fullness peers and we show directly that B is contained in A_0. Indeed, K, and hence A, is now an $\mathbb{F}_p$-algebra, and it follows that $\bar{\rho}$ is also B-full. Now Proposition 5.1 applied to $\bar{\rho}$ tells us that B is fixed by all conjugate self-twist automorphisms of $\bar{\rho}$: in other words, $B \subseteq \mathbb{E}$ (Corollary 4.13), which is the residue field of, and hence here contained in, A_0.

Now assume that B is not finite. Use Corollary 3.12 to replace (t, d) by its constant-determinant twist and let $A_t \subseteq A$ be its trace algebra. Replace B by its fullness peer $B \cap A_t$ (Corollary 3.12 and Proposition 4.8). Let $\Gamma = \ker \bar{\rho}$, a finite-index closed subgroup of Π. By Proposition 3.11, the restriction $(t|_\Gamma, d|_\Gamma)$ is B-full. By Proposition 3.3, A_0, which contains the trace algebra of Γ (Lemma 4.16), contains a nonzero ideal of B. Therefore so does $B \cap A_0$, making $B \cap A_0 \subseteq B$ an extension of fullness peers by Lemma 3.5. ☐

We do not know whether we can sharpen Theorem 5.3 to conclude that any closed fullness ring B contained in A_0 with $Q(B) = K_0$ must in fact be a fullness peer of A_0. This would follow from an affirmative answer to Question 3.6.

We close with an observation that fullness rings are always fixed by conjugate self-twist automorphisms. Corollary 4.24 has already established this for A_0; Theorem 5.4 below is a generalization.

Theorem 5.4. *Let $(t, d) \colon \Pi \to A$ be a pseudorepresentation. If (t, d) is B-full for some subring B of K, then B is fixed by any conjugate self-twists valued in any domain extending A containing B.*

Note that any automorphism valued in a domain E containing A extends uniquely to an automorphism of $Q(E)$, which contains K and hence B.

Proof. Let (t', d') be the constant determinant twist associated to (t, d), obtained by twisting off the pro-p part d_1 of d. As in Proposition 4.2, if (σ, η) is an E-valued conjugate self-twists (t, d) for some domain E extending A, then $\left(\sigma,\ \sigma(d_1^{-1/2})d_1^{1/2}\eta\right)$ is a E-valued conjugate self-twist of (t', d'). By Corollary 4.5, $\chi = \sigma(d_1^{-1/2})d_1^{1/2}\eta$ is continuous and $W(\mathbb{F})$-valued. Note that $\ker d_1 = \ker d_1^{1/2} = \ker \sigma(d_1^{-1/2})$ is closed since d_1 is also continuous. Thus $\ker d_1 \cap \ker \chi$ is a closed subgroup of $\ker \eta$. Moreover, since the order of χ is prime to p and d_1 is pro-p we get that

$$\Pi/(\ker d_1 \cap \ker \chi) \cong \Pi/\ker d_1 \times \Pi/\ker \chi$$

is abelian. Thus we can take $H = \ker d_1 \cap \ker \chi$ in Proposition 5.1 to deduce that $B \cap A$ is fixed by σ. As $B \cap A$ and B are fullness peers, σ thus fixes an ideal of B, hence all of B. $\square$

6. Fullness for $\mathcal{B}_\rho(\mathbb{E})$

Throughout this section we fix a local pro-p ring A, not necessarily a domain, with residue field $\mathbb{F}$. Fix an admissible pseudodeformation $(\Pi, \bar{\rho}, t, d)$ over A. Let A_0 be the adjoint trace ring of (t, d) and $\mathbb{E}$ the residue field of A_0.

6.1. $L_2(\rho)$ is a $W(\mathbb{E})$-module.

Recall that, a priori, Pink's construction only gives Lie algebras that are $\mathbb{Z}_p$-modules. The goal of Section 6.1 is to show that if ρ is a (t, d)-representation, then in fact its associated Lie algebras are modules over $W(\mathbb{E})$ (Proposition 6.2). Although this is a minor improvement on $\mathbb{Z}_p$ (indeed, it is no improvement at all if $W(\mathbb{E}) = \mathbb{Z}_p$), it is an essential input for proving the results of Section 9.

We assume throughout Section 6.1 that the eigenvalues of $\bar{\rho}(g)$ are in $\mathbb{F}^\times$ for all $g \in \Pi$. This requires at most replacing $\mathbb{F}$ by its unique quadratic extension.

Let $\lambda \neq \mu \in \mathbb{F}^\times$ be the eigenvalues of a matrix in $\operatorname{Im}\bar{\rho}$. By Lemma 2.14, there is a (t, d)-representation $\rho_{\lambda,\mu} \colon \Pi \to R_{\lambda,\mu}^\times$ and $g_{\lambda,\mu} \in \Pi$ such that

$$\rho_{\lambda,\mu}(g_{\lambda,\mu}) = \begin{pmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{pmatrix}.$$

Recall that $G_{\rho_{\lambda,\mu}} := \operatorname{Im}\rho_{\lambda,\mu}$, $\Gamma_{\rho_{\lambda,\mu}} := G_{\rho_{\lambda,\mu}} \cap SR^1_{\lambda,\mu}$, and $L_n(\rho_{\lambda,\mu}) := L_n(\Gamma_{\rho_{\lambda,\mu}})$. Since $\lambda \neq \mu$, the Lie algebra $L_1(\rho_{\lambda,\mu})$ is decomposable [Bel19, Corollary 6.2.2]. Note that although the Teichmüller map s is not additive, it is easy to check that $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu)) = \mathbb{Z}_p[s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu)]$, a fact that we make frequent use of in Lemma 6.1 below.

Lemma 6.1. *With the notation introduced at the beginning of Section 2.5, we have*

(1) $\nabla_1(\rho_{\lambda,\mu}), B_1(\rho_{\lambda,\mu}), C_1(\rho_{\lambda,\mu})$, and $L_2(\rho_{\lambda,\mu})$ are $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-modules;

(2) if the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $p \geq 7$, then $I_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module; after possibly replacing $\rho_{\lambda,\mu}$ with its conjugate by a certain $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with $a \in A^\times$, one has that $L_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module.

Proof. Note that

$$L_2(\rho_{\lambda,\mu}) = \left[I_1(\rho_{\lambda,\mu})\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \nabla_1(\rho_{\lambda,\mu})\right] + [\nabla_1(\rho_{\lambda,\mu}), \nabla_1(\rho_{\lambda,\mu})].$$

Furthermore, from their definitions on page 14, it's clear that $B_1(\rho_{\lambda,\mu})$ and $C_1(\rho_{\lambda,\mu})$ inherit whatever structure $\nabla_1(\rho_{\lambda,\mu})$ has. Therefore it suffices to show that $\nabla_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1}+\lambda^{-1}\mu))$-module.

To prove that

$$(s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu))\nabla_1(\rho_{\lambda,\mu}) \subseteq \nabla_1(\rho_{\lambda,\mu}),$$

recall that $L_1(\rho_{\lambda,\mu})$ is closed under conjugation by $G_{\rho_{\lambda,\mu}}$ (in fact, by any element in the normalizer of $\Gamma_{\rho_{\lambda,\mu}}$). In particular, it is closed under conjugation by $\begin{pmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{pmatrix}$ and $\begin{pmatrix} s(\lambda)^{-1} & 0 \\ 0 & s(\mu)^{-1} \end{pmatrix}$. Using this, a short matrix calculation shows that if $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho_{\lambda,\mu})$ then

$$\begin{pmatrix} 0 & s(\lambda)s(\mu)^{-1}b \\ s(\lambda)^{-1}s(\mu)c & 0 \end{pmatrix}, \begin{pmatrix} 0 & s(\lambda)^{-1}s(\mu)b \\ s(\lambda)s(\mu)^{-1}c & 0 \end{pmatrix} \in \nabla_1(\rho_{\lambda,\mu}).$$

Therefore $(s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu))\nabla_1(\rho_{\lambda,\mu}) \subseteq \nabla_1(\rho_{\lambda,\mu})$.

Finally, if the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $p \geq 7$, then by Theorem 2.23, up to replacing $\rho_{\lambda,\mu}$ with its conjugate by a certain $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \in A^\times$, we have

$$L_1(\rho_{\lambda,\mu}) = \begin{pmatrix} I_1(\rho_{\lambda,\mu}) & I_1(\rho_{\lambda,\mu}) \\ I_1(\rho_{\lambda,\mu}) & I_1(\rho_{\lambda,\mu}) \end{pmatrix}^0,$$

and thus $B_1(\rho_{\lambda,\mu}) = I_1(\rho_{\lambda,\mu}) = C_1(\rho_{\lambda,\mu})$. (Note that conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \in A^\times$ does not change $I_1(\rho_{\lambda,\mu})$.) In particular, $L_1(\rho_{\lambda,\mu})$ is strongly decomposable. By the second statement of the lemma, we see that $I_1(\rho_{\lambda,\mu})$ is a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module. The above description of $L_1(\rho_{\lambda,\mu})$ shows that it is also a $W(\mathbb{F}_p(\lambda\mu^{-1} + \lambda^{-1}\mu))$-module. ☐

Proposition 6.2. *Let $\rho\colon \Pi \to R^\times$ be a (t,d)-representation. Then $L_n(\rho)$ is a $W(\mathbb{E})$-module for all $n \geq 2$. If the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ and $p \geq 7$, then $L_1(\rho)$ is a $W(\mathbb{E})$-module.*

Proof. By the generating set for $\mathbb{E}$ given in (2) in Section 2.4, it suffices to show that $L_n(\rho)$ is closed under multiplication by $s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu)$ for all $\lambda, \mu \in \bar{\mathbb{F}}_p^\times$ that are distinct eigenvalues of an element in $\mathrm{Im}\,\bar{\rho}$. Fix such λ, μ. Let $\rho_{\lambda,\mu}\colon \Pi \to R_{\lambda,\mu}^\times$ be the (t,d)-representation over A described prior to Lemma 6.1. Let us assume furthermore that, in the case when $\bar{\rho}$ is not projectively cyclic or dihedral, that we have already replaced $\rho_{\lambda,\mu}$ by its relevant diagonal conjugate so that the description of $W(\mathbb{F}_q)L_1(\rho_{\lambda,\mu})$ from Theorem 2.23 applies to $\rho_{\lambda,\mu}$.

Since $\rho\colon \Pi \to R^\times$ and $\rho_{\lambda,\mu}\colon \Pi \to R_{\lambda,\mu}^\times$ are both (t,d)-representations over A, it follows from Proposition 2.11 that there is a unique A-algebra isomorphism $\Psi\colon R_{\lambda,\mu} \to R$ such that $\rho = \Psi \circ \rho_{\lambda,\mu}$. We claim that this implies that $L_n(\rho) = \Psi(L_n(\lambda,\mu))$ for all $n \geq 1$. If this is true, then $L_2(\rho)$ is closed under multiplication by $s(\lambda)s(\mu)^{-1} + s(\lambda)^{-1}s(\mu) \in A$ since $L_2(\rho_{\lambda,\mu})$ is by Lemma 6.1 and Ψ is an A-algebra homomorphism. Since $L_2(\rho)$ is a $W(\mathbb{E})$-module, it follows immediately from the definition that $L_n(\rho)$ is a $W(\mathbb{E})$-module for all $n \geq 2$. Furthermore, the argument in this paragraph applies to $L_1(\rho)$ under the assumption that the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ for $p \geq 7$.

To see that $L_n(\rho) = \Psi(L_n(\rho_{\lambda,\mu}))$, note that $G_\rho = G_{\Psi\circ\rho_{\lambda,\mu}} = \Psi(G_{\rho_{\lambda,\mu}})$. Since Ψ is an algebra morphism, it follows that $\Psi(\mathrm{rad}\, R_{\lambda,\mu}) = \mathrm{rad}\, R$. Furthermore, since ρ and $\rho_{\lambda,\mu}$ are both (t,d)-representations, it follows that Ψ preserves determinants. Therefore $\Psi(SR^1_{\lambda,\mu}) \supset SR^1$. Since Ψ is a continuous algebra homomorphism, it follows directly from the definition of Θ that $\Psi(L_1(\rho_{\lambda,\mu})) = L_1(\rho)$ and hence $\Psi(L_n(\rho_{\lambda,\mu})) = L_n(\rho)$ for all $n \geq 1$. ☐

6.2. $L_2(\rho)$ is a $\mathcal{B}_\rho(\mathbb{E})$-module.

In Section 6.2 we use Bellaïche's work to show that, for any well-adapted (t,d)-representation ρ, $L_n(\rho)$ is a module over a ring comparable to A. This is the key input into Corollary 6.6, which is our improvement on Bellaïche's fullness theorem.

Proposition 6.3. *Let ρ be a (t,d)-representation adapted to (g_0, λ_0, μ_0). Then*
(1) $L_2(\rho)$ is a module over $W(\mathbb{E})[I_1(\rho)^2] \coloneqq W(\mathbb{E}) + W(\mathbb{E})I_1(\rho)^2$;
(2) if $n \geq 1$ and $L_n(\rho)$ is strongly decomposable, then $L_{n+1}(\rho)$ is a module over $\mathcal{B}_\rho(\mathbb{E})$;
(3) if the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ for $p \geq 7$, then up to replacing ρ with its conjugate by some $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \in A^\times$, $L_1(\rho)$ is a module over $\mathcal{B}_\rho(\mathbb{E})$.

Proof. Since ρ is adapted to (g_0, λ_0, μ_0), it follows that $L_1(\rho)$ is decomposable [Bel19, Corollary 6.2.2]. Note that $[I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \nabla_1(\rho)] \subset \nabla_1(\rho)$ since $L_1(\rho)$ is a Lie algebra. That is, for all $a \in I_1(\rho)$ and $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho)$, we have $2a\left(\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho)$. To prove the first statement, we can apply this fact a second time to $\alpha \in I_1(\rho)$ and $2a\left(\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right)$ to see that $4a\alpha\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \nabla_1(\rho)$. Therefore $\nabla_1(\rho)$ is closed under multiplication by $I_1(\rho)^2$. Since

$$L_2(\rho) = \left[I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \nabla_1(\rho)\right] + [\nabla_1(\rho), \nabla_1(\rho)],$$

we see that $L_2(\rho)$ is closed under multiplication by $I_1(\rho)^2$.

For the second statement, if $L_n(\rho)$ is strongly decomposable, then we can write

$$L_n(\rho) = I_n(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \oplus B_n(\rho)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \oplus C_n(\rho)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

By calculating $\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right]$ and $\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right]$, we find that $I_1(\rho)B_n(\rho) = B_{n+1}(\rho) \subset B_n(\rho)$ and $I_1(\rho)C_n(\rho) = C_{n+1}(\rho) \subset C_n(\rho)$. Therefore $B_n(\rho), C_n(\rho)$ are closed under multiplication by $I_1(\rho)$. Since

$$\begin{aligned} L_{n+1}(\rho) = [B_n(\rho)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), C_n(\rho)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)] + [I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), B_n(\rho)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)] + \\ + [I_1(\rho)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), C_n(\rho)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)], \end{aligned}$$

it follows that $L_{n+1}(\rho)$ is closed under multiplication by $I_1(\rho)$.

The first two results now follow from Proposition 6.2. The last statement follows from Theorem 2.23 and Proposition 6.2. ☐

Remark 6.4. It would be nice to remove the assumption that $L_1(\rho)$ is strongly decomposable and still conclude that $L_2(\rho)$ is a $\mathcal{B}_\rho(\mathbb{E})$-module, but we do not see a way to do this. △

6.3. **Regularity implies $\mathcal{B}_\rho(\mathbb{E})$-fullness.** The goal of Section 6.3 is to establish a slightly stronger version of [Bel19, Theorem 7.2.3], which is Bellaïche's Theorem 1.3 of the introduction. We do so in Corollary 6.6 below. Our result is different from that of Bellaïche mainly in that we can weaken his definition of regularity and enlarge his ring $\mathcal{B}_\rho(\mathbb{F}_p)$ to $\mathcal{B}_\rho(\mathbb{E})$.

Throughout Section 6.3 the ring A will be a local pro-p domain with residue field $\mathbb{F}$ and field of fractions K. We fix an admissible pseudodeformation $(\Pi, \bar{\rho}, t, d)$ over A throughout this section. If ρ is a (t, d)-representation that is adapted to some (g_0, λ_0, μ_0), then $L_1(\rho)$ is decomposable by [Bel19, Corollary 6.2.2]. Thus $I_1(\rho)$ is defined. We write K_1 for the field of fractions of $\mathcal{B}_\rho(\mathbb{E})$.

Proposition 6.5. *Assume that $\bar{\rho}$ is regular. Let $\rho\colon \Pi \to R^\times$ be a (t, d)-representation adapted to (g_0, λ_0, μ_0) for a regular element g_0 such that $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$. If $B_1(\rho), C_1(\rho) \neq 0$, then ρ is $\mathcal{B}_\rho(\mathbb{E})$-full.*

Proof. It is easy to see that the eigenvalues of $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$ acting on $L_n(\rho)$ by conjugation are $1, s(\lambda_0)s(\mu_0^{-1}), s(\lambda_0^{-1})s(\mu_0)$, which are distinct elements of $W(\mathbb{E})^\times$ since g_0 is a regular element. Since $L_n(\rho)$ is a $W(\mathbb{E})$-module for $n \geq 2$ by Proposition 6.2, it follows that $L_n(\rho)$ is the direct sum of the eigenspaces for the conjugation action of $\rho(g_0)$. Thus, $L_n(\rho)$ is strongly decomposable for $n \geq 2$. By Proposition 6.3, it follows that $L_n(\rho)$ is an $\mathcal{B}_\rho(\mathbb{E})$-module for $n \geq 3$.

Since A is a domain, we may view R inside of $M_2(K)$ by Lemma 2.10. Note that if $B_1(\rho), C_1(\rho) \neq 0$, then since $I_n(\rho), B_n(\rho), C_n(\rho) \subset K$, it follows that $I_n(\rho), B_n(\rho)$, and $C_n(\rho)$ are nonzero for all $n \geq 1$. In particular, $I_3(\rho), B_3(\rho), C_3(\rho)$ are nonzero $\mathcal{B}_\rho(\mathbb{E})$-modules.

Define

$$R_1 := \begin{pmatrix} \mathcal{B}_\rho(\mathbb{E}) & B_3(\rho) \\ C_3(\rho) & \mathcal{B}_\rho(\mathbb{E}) \end{pmatrix}.$$

Then R_1 is a faithful GMA over $\mathcal{B}_\rho(\mathbb{E})$. By the proof of [Bel19, Lemma 2.2.2], if $0 \neq b_0 \in B_3(\rho)$ and $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b_0 \end{smallmatrix}\right)$, it follows that $xR_1x^{-1} \subseteq \mathrm{GL}_2(K_1)$. Thus, by replacing ρ with $x\rho x^{-1}$, which is still a (t, d)-representation adapted to (g_0, λ_0, μ_0) that sends g_0 to $\left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$, we may assume that $B_3(\rho), C_3(\rho) \subseteq K_1$. (Note that $I_1(\rho) = I_1(x\rho x^{-1})$.)

Note that any nonzero $\mathcal{B}_\rho(\mathbb{E})$-submodule of K_1 contains a nonzero element of $\mathcal{B}_\rho(\mathbb{E})$ and thus contains a non-zero $\mathcal{B}_\rho(\mathbb{E})$-ideal. Therefore there exists a nonzero $\mathcal{B}_\rho(\mathbb{E})$-ideal $\mathfrak{b}$ contained in $I_3(\rho) \cap$

$B_3(\rho) \cap C_3(\rho)$. Hence $\mathfrak{sl}_2(\mathfrak{b}) \subseteq L_3(\rho)$. Using Theorem 2.16 we deduce that $\Gamma_{\mathcal{B}_\rho(\mathbb{E})}(\mathfrak{b}) \subset \operatorname{Im}\rho$ and ρ is $\mathcal{B}_\rho(\mathbb{E})$-full. ☐

Corollary 6.6. *Assume that $\bar{\rho}$ is regular. Let ρ be a well-adapted (t,d)-representation adapted to (g_0, λ_0, μ_0) for a regular element g_0 such that $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$. If (t,d) is not a priori small, then (t,d) is $\mathcal{B}_\rho(\mathbb{E})$-full.*

Proof. By Proposition 6.5 it suffices to show that $B_1(\rho), C_1(\rho) \neq 0$. We do this by analyzing the different possibilities for $\bar{\rho}$. By Lemma A.5, we see that either $\bar{\rho}$ is reducible, dihedral, or $\operatorname{ad}^0 \bar{\rho}$ is irreducible. Assume first that we are in the last case. The group Γ is equipped with a decreasing normal filtration

$$\Gamma_n := \Gamma \cap \Gamma_A(\mathfrak{m}^n)$$

whose quotients Γ_n/Γ_{n+1} have the structure of $\mathbb{F}_p$-vector spaces with an action of $\overline{G}$ by conjugation. Moreover, the map $\gamma \mapsto \gamma - 1$ gives a $\overline{G}$-equivariant embedding $\Gamma_n/\Gamma_{n+1} \hookrightarrow \mathfrak{sl}_2(\mathfrak{m}^n/\mathfrak{m}^{n+1})$; write V_n for its image. The assumption that (t,d) is not a priori small implies that Γ is nontrivial, so that Γ_n/Γ_{n+1}, and hence V_n, is nontrivial for some $n \geq 1$. To see that $B_1(\rho)$ (respectively, $C_1(\rho)$) is nonzero, it suffices to show that this V_n contains an element whose upper right (respectively, lower left) entry is nonzero. This can be checked on the $\mathbb{F}$-span of V_n. Choosing an $\mathbb{F}$-basis $x_1, \dots, x_d$ of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ gives a $\overline{G}$-equivariant splitting $\mathfrak{sl}_2(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \oplus_{i=1}^d \mathfrak{sl}_2(\mathbb{F}) x_i$. Since V_n is nontrivial, there is some i such that the projection W of $\mathbb{F}V_n$ to $\mathfrak{sl}_2(\mathbb{F})x_i$ is nonzero. Then W is a stable subspace of $\mathfrak{sl}_2(\mathbb{F})x_i$, which is simple since $\operatorname{ad}^0 \bar{\rho}$ is irreducible. Thus $W = \mathfrak{sl}_2(\mathbb{F})x_i$ and W, and hence $\mathbb{F}V_n$, contains an element whose upper right (respectively, lower left) entry is nonzero.(vii)

Now suppose that $\bar{\rho}$ is reducible. Since ρ is well adapted by assumption, it follows that ρ is adapted to (g_0, λ_0, μ_0), where $\bar{\rho}(g_0)$ generates the projective image of $\bar{\rho}$. In particular, ρ is automatically adapted to a regular element. Suppose for contradiction that $C_1(\rho) = 0$ (respectively, $B_1(\rho) = 0$). Then Γ_ρ is contained in the upper (respectively, lower) triangular matrices. By [Bel19, Theorem 6.2.1], we know that $s(\overline{G}) \subset G_\rho$ since ρ is well adapted. Thus $G_\rho = s(\overline{G})\Gamma_\rho$. But then G_ρ is contained in the upper (respectively, lower) triangular matrices, and hence ρ is reducible. Therefore t is the sum of two continuous characters $\Pi \to A^\times$, which contradicts our assumption that it is not a priori small. Thus $B_1(\rho), C_1(\rho) \neq 0$ if $\bar{\rho}$ is reducible.

Finally suppose that $\bar{\rho}$ is dihedral. By Lemma A.7 there is a unique subgroup Π_0 of index 2 in Π such that $\bar{\rho} \cong \operatorname{Ind}_{\Pi_0}^{\Pi} \chi$ for some character $\chi \colon \Pi_0 \to \mathbb{F}^\times$. Applying the reducible case to $\bar{\rho}|_{\Pi_0}$, we see that either $\rho|_{\Pi_0}$ is reducible or $B_1(\rho|_{\Pi_0}), C_1(\rho|_{\Pi_0}) \neq 0$. The first possibility is not allowed by hypothesis, so we must have $B_1(\rho|_{\Pi_0}), C_1(\rho|_{\Pi_0}) \neq 0$. But $B_1(\rho|_{\Pi_0}) \subseteq B_1(\rho)$ and $C_1(\rho|_{\Pi_0}) \subseteq C_1(\rho)$, which proves the desired result when $\bar{\rho}$ is dihedral. ☐

7. Lifting residual conjugate self-twists

In Section 7 we study the (A_t-valued) conjugate self-twists of constant-determinant pseudorepresentations. In particular, we show in Section 7.2 that they are all controlled by those of $\bar{\rho}$, and all residual conjugate self-twists lift to the universal constant-determinant pseudodeformation ring. Having shown in Section 7.1 that the group of residual conjugate self-twists is finite and abelian, we deduce the same for any constant determinant pseudorepresentation. Finally, in Section 7.3 we study special phenomena that arise when $\bar{\rho}$ is dihedral and hence there may be conjugate self-twists that are residually dihedral.

Throughout Section 7 we only consider simple conjugate self-twists. Thus we write $\widetilde{\Sigma}_t = \widetilde{\Sigma}_t(A_t)$ and $\widetilde{\Sigma}_{\bar{\rho}}$ for $\widetilde{\Sigma}_{\bar{\rho}}(\mathbb{F})$; similarly for Σ_t and $\Sigma_{\bar{\rho}}$.

(vii)Together with the observation that $B_n(\rho), C_n(\rho) \neq 0$ for all n if it is true for $n = 1$ from Proposition 6.5, the argument here gives an independent proof of the (2) $\implies$ (1) implication in Proposition 2.4.

7.1. **Residual conjugate self-twists.** Fix a semisimple representation $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$, and recall that $\mathbb{P}\colon \mathrm{GL}_2(\mathbb{F}) \to \mathrm{PGL}_2(\mathbb{F})$ denotes the natural projection. For Section 7.1 only, we do not require $\bar{\rho}$ to be residually multiplicity-free. We begin by studying $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ and use that to show that $\widetilde{\Sigma}_{\bar{\rho}}$ is finite and abelian.

Lemma 7.1.

(1) If $\operatorname{Im}\mathbb{P}\bar{\rho}$ is not dihedral or cyclic of order 2, then $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ is trivial.

(2) If $\operatorname{Im}\mathbb{P}\bar{\rho}$ is either a nonabelian dihedral group or has order 2, then $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ has order 2.

(3) If $\operatorname{Im}\mathbb{P}\bar{\rho}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, then $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Proof. We claim that if $\bar{\rho}$ is irreducible, then the following sets are in bijection:

(a) $\Sigma_{\bar{\rho}}^{\mathrm{di}} \setminus \{(1,1)\}$;

(b) subgroups Π_0 of Π such that $[\Pi\colon \Pi_0] = 2$ and $\bar{\rho}(\Pi_0)$ is abelian;

(c) subgroups H of $\mathbb{P}\bar{\rho}(\Pi)$ such that $[\mathbb{P}\bar{\rho}(\Pi)\colon H] = 2$ and H is abelian.

Indeed, the maps between them can be described as follows. Given $(1,\eta) \in \Sigma_{\bar{\rho}}^{\mathrm{di}} \setminus \{(1,1)\}$, let $\Pi_0 \coloneqq \ker\eta$. The fact that $[\Pi\colon \Pi_0] = 2$ follows from Lemma A.7, and Lemma A.6 shows that $\bar{\rho}(\Pi_0)$ is abelian. Conversely, given Π_0 as in (b), let $\eta_{\Pi_0}\colon \Pi \to \Pi/\Pi_0 \cong \{\pm 1\}$ be the natural projection. Note that $\bar{\rho}|_{\Pi_0}$ is reducible since $\bar{\rho}(\Pi_0)$ is abelian. Let $\chi\colon \Pi_0 \to \mathbb{F}^\times$ be a constituent of $\bar{\rho}|_{\Pi_0}$. Then $\bar{\rho} \cong \operatorname{Ind}_{\Pi_0}^{\Pi}\chi$ by Frobenius reciprocity since $\bar{\rho}$ is irreducible. Thus $(1,\eta_{\Pi_0}) \in \Sigma_{\bar{\rho}}^{\mathrm{di}} \setminus \{(1,1)\}$ by Lemma A.7.

Given Π_0 as in (b), let $H \coloneqq \mathbb{P}\bar{\rho}(\Pi_0)$. Given H as in (c), let $\Pi_0 \coloneqq \mathbb{P}\bar{\rho}^{-1}(H)$. It is clear that $[\Pi\colon \Pi_0] = 2$. That $\bar{\rho}(\Pi_0)$ is abelian follows from the fact that H is abelian and scalar matrices commute with everything.

When $\bar{\rho}$ is irreducible, the lemma now follows from counting subgroups as in (c) in each of the possible projective images of $\bar{\rho}$. (The fact that elements in $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ have order at most 2 follows from the fact that $\det\bar{\rho} = \eta^2\det\bar{\rho}$ and so $\eta^2 = 1$.)

Finally, suppose that $\bar{\rho} = \varepsilon \oplus \delta$. If $(1,\eta) \in \Sigma_{\bar{\rho}}^{\mathrm{di}}$ and η is nontrivial, then we must have $\eta\varepsilon = \delta$ and $\eta\delta = \varepsilon$. Thus

$$\varepsilon\delta^{-1} = \eta = \delta\varepsilon^{-1},$$

which implies that $\varepsilon\delta^{-1}$ has order 2. But the projective image of $\bar{\rho}$ is isomorphic to the image of $\varepsilon\delta^{-1}$. Thus $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ is trivial unless the projective image of $\bar{\rho}$ has order 2, in which case there is one nontrivial element. □

Corollary 7.2. *The group $\widetilde{\Sigma}_{\bar{\rho}}$ is finite and abelian.*

Proof. Since $\mathbb{F}$ is a finite field, there are only finitely many automorphisms of $\mathbb{F}$. Any two elements in $\widetilde{\Sigma}_{\bar{\rho}}$ with the same automorphism differ by an element of $\Sigma_{\bar{\rho}}^{\mathrm{di}}$, which is finite by Lemma 7.1.

To see that $\widetilde{\Sigma}_{\bar{\rho}}$ is abelian, fix a generator σ of the cyclic group $\Sigma_{\bar{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$. Let η be a character such that $(\sigma,\eta) \in \widetilde{\Sigma}_{\bar{\rho}}$. Then $\widetilde{\Sigma}_{\bar{\rho}}$ is generated by $\{(\sigma,\eta\eta')\colon (1,\eta') \in \Sigma_{\bar{\rho}}^{\mathrm{di}}\}$. Since η' is at most quadratic by Lemma 7.1, the action of σ on η' is trivial and hence one easily checks that any two of the generators commute. □

7.2. **Lifting conjugate self-twists.** Let Π be a profinite group satisfying the p-finiteness condition. Fix a multiplicity-free representation $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Recall from Section 2.1 that there is a local pro-p noetherian $W(\mathbb{F})$-algebra $\mathcal{A}$ with maximal ideal $\mathfrak{m}_{\mathcal{A}}$ and residue field $\mathbb{F}$ and a pseudodeformation $(T,d)\colon \Pi \to \mathcal{A}$ that is universal among all constant-determinant pseudodeformations of $\bar{\rho}$. The purpose of Section 7.2 is to show that every conjugate self-twist of $\bar{\rho}$, and in fact of every constant-determinant pseudodeformation of $\bar{\rho}$, can be lifted to a conjugate self-twist of (T,d) (see Proposition 7.3 and Corollary 7.4 below).

Since we are working only with constant-determinant pseudodeformations, we shall identify any $\mathbb{F}$-valued character η with the $W(\mathbb{F})$-valued character $s(\eta)$. Furthermore, we will consider η as being valued in any $W(\mathbb{F})$-algebra via the structure map. If σ is an automorphism of $\mathbb{F}$, we write $W(\sigma)$ the automorphism of $W(\mathbb{F})$ induced by σ.

We introduce some notation that will be used in the proof of Proposition 7.3. For any $W(\mathbb{F})$-algebra A, let $A^\sigma := A \otimes_{W(\mathbb{F}),W(\sigma)} W(\mathbb{F})$, where $W(\mathbb{F})$ is considered as a $W(\mathbb{F})$-algebra via $W(\sigma)$. We can equip A^σ with two different $W(\mathbb{F})$-algebra structures by letting $W(\mathbb{F})$ act either on the first or second factor of the tensor product. In what follows, we refer to these actions respectively as the *first* or *second* $W(\mathbb{F})$*-algebra structure* on A^σ. Let $\iota(\sigma, A)\colon A \to A^\sigma$ be the natural map given by $\iota(\sigma, A)(a) = a \otimes 1$. It is an isomorphism of rings with inverse given by $\iota(\sigma^{-1}, A^\sigma)$ since $(A^\sigma)^{\sigma^{-1}}$ can be naturally identified with A as a $W(\mathbb{F})$-algebra. Furthermore, $\iota(\sigma, A)$ is a morphism of $W(\mathbb{F})$-algebras with respect to the first structure on A^σ. Note that if we view $\mathcal{A}^\sigma$ with respect to its second $W(\mathbb{F})$-algebra structure, its residue field is $\mathbb{F} \otimes_{\mathbb{F},\sigma} \mathbb{F}$, which is identified with $\mathbb{F}$ via $x \otimes y \mapsto \sigma(x)y$. The proof of the following proposition is a more streamlined treatment of the arguments in [Lan16, Section 2].

Proposition 7.3. *Let $(\sigma, \eta) \in \widetilde{\Sigma}_{\bar{\rho}}$. Then there is an automorphism $\tilde{\sigma}$ of $\mathcal{A}$ such that $(\tilde{\sigma}, \eta) \in \widetilde{\Sigma}_T$ and $\tilde{\sigma}$ induces σ modulo $\mathfrak{m}_{\mathcal{A}}$. Furthermore, for any w in the image of $W(\mathbb{F})$ in $\mathcal{A}$, we have $\tilde{\sigma}(w) = W(\sigma)(w)$.*

Note that any such $\tilde{\sigma}$ is necessarily unique, because it is determined by the character η.

Proof. Note that $\eta T\colon \Pi \to \mathcal{A}$ is the universal constant-determinant pseudodeformation of $\eta \otimes \bar{\rho} \cong {}^\sigma\bar{\rho}$. We claim that, considering $\mathcal{A}^{\sigma^{-1}}$ with its second $W(\mathbb{F})$-algebra structure, $\iota(\sigma^{-1}, \mathcal{A}) \circ (\eta T)$ is a constant-determinant pseudodeformation of $\bar{\rho}$. Indeed, reducing $\iota(\sigma^{-1}, \mathcal{A}) \circ (\eta T)$ modulo the maximal of ideal of $\mathcal{A}^{\sigma^{-1}}$ gives

$$\iota(\sigma^{-1}, \mathbb{F}) \circ (\eta \operatorname{tr} \bar{\rho}) = {}^\sigma\!\operatorname{tr} \bar{\rho} \otimes_{\mathbb{F},\sigma^{-1}} 1 = 1 \otimes \operatorname{tr} \bar{\rho},$$

which is identified with $\operatorname{tr}\bar{\rho}$ under the identification of $\mathbb{F} \otimes_{\mathbb{F},\sigma^{-1}} \mathbb{F}$ with $\mathbb{F}$ discussed prior to the proposition.

By universality, there is a unique $W(\mathbb{F})$-algebra homomorphism $\alpha\colon \mathcal{A} \to \mathcal{A}^{\sigma^{-1}}$, where $\mathcal{A}^{\sigma^{-1}}$ is given its second $W(\mathbb{F})$-algebra structure, such that

$$\alpha \circ T = \iota(\sigma^{-1}, \mathcal{A}) \circ (\eta T).$$

Since $\iota(\sigma, \mathcal{A}^{\sigma^{-1}})$ is the inverse of $\iota(\sigma^{-1}, \mathcal{A})$, we have that

$$\iota(\sigma, \mathcal{A}^{\sigma^{-1}}) \circ \alpha \circ T = \eta T. \tag{4}$$

Define $\tilde{\sigma} := \iota(\sigma, \mathcal{A}^{\sigma^{-1}}) \circ \alpha$, which is a ring endomorphism of $\mathcal{A}$. The relation (4) implies that $\tilde{\sigma}$ is an automorphism of $\mathcal{A}$ since the image of T topologically generates $\mathcal{A}$ as a $W(\mathbb{F})$-module [Bel19, Proposition 5.3.3] and η takes values in $W(\mathbb{F})$. The relation (4) also shows that $(\tilde{\sigma}, \eta) \in \widetilde{\Sigma}_T$.

Finally, let $w \in W(\mathbb{F})$. Since α is a $W(\mathbb{F})$-algebra homomorphism with respect to the second $W(\mathbb{F})$-algebra structure on $\mathcal{A}^{\sigma^{-1}}$, we have that

$$\begin{aligned}\tilde{\sigma}(w) &= \iota(\sigma, \mathcal{A}^{\sigma^{-1}}) \circ \alpha(w) = \iota(\sigma, \mathcal{A}^{\sigma^{-1}})(1 \otimes w)\\ &= \iota(\sigma^{-1}, \mathcal{A})^{-1}(W(\sigma)(w) \otimes 1) = W(\sigma)(w).\end{aligned}$$

□

For the rest of Section 7.2, fix a local pro-p $W(\mathbb{F})$-algebra A with residue field $\mathbb{F}$ and a constant-determinant pseudodeformation $(t, d)\colon \Pi \to A$ of $\bar{\rho}$. Assume that A is the $W(\mathbb{F})$-algebra generated by $t(\Pi)$. Let $\alpha_t\colon \mathcal{A} \to A$ be the unique $W(\mathbb{F})$-algebra homomorphism such that $\alpha \circ T = t$ given by universality. The following corollary shows that conjugate self-twists of (t, d) also lift to conjugate self twists of (T, d).

Corollary 7.4. *Given $(\sigma, \eta) \in \widetilde{\Sigma}_t$, there is a unique $(\tilde{\sigma}, \eta) \in \widetilde{\Sigma}_T$ such that $\alpha_t \circ \tilde{\sigma} = \sigma \circ \alpha_t$.*

Proof. Let $\bar{\sigma}$ denote the automorphism of $\mathbb{F}$ induced by σ. Let $\tilde{\sigma}$ be the automorphism of $\mathcal{A}$ given by Proposition 7.3 lifting $\bar{\sigma}$ to $\mathcal{A}$. Then we just have to show that $\alpha_t \circ \tilde{\sigma} = \sigma \circ \alpha_t$. Note that σ acts by $W(\bar{\sigma})$ on the image of $W(\mathbb{F})$ in A, so $\sigma^{-1} \circ \alpha \circ \tilde{\sigma}$ is a $W(\mathbb{F})$-algebra homomorphism. Thus by universality, it suffices to show that $t = \sigma^{-1} \circ \alpha_t \circ \tilde{\sigma} \circ T$. Since η takes values in $W(\mathbb{F})$ and α_t is a $W(\mathbb{F})$-algebra homomorphism, we have that

$$\sigma^{-1} \circ \alpha_t \circ \tilde{\sigma} \circ T = \sigma^{-1} \circ \alpha_t(\eta T) = \sigma^{-1}(\eta t) = \sigma^{-1}(\sigma(t)) = t. \qquad \square$$

We end Section 7.2 with some observations about the consequences of Proposition 7.3 and Corollary 7.4. They give the following commutative diagram with exact rows.

$$\begin{tikzcd}
1 \arrow[r] & \Sigma_t^{\mathrm{di}} \arrow[r] \arrow[d, hook] & \widetilde{\Sigma}_t \arrow[r] \arrow[d, hook] & \Sigma_t \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \Sigma_T^{\mathrm{di}} \arrow[r] \arrow[d, hook] & \widetilde{\Sigma}_T \arrow[r] \arrow[d, "\cong"] & \Sigma_T \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Sigma_{\bar{\rho}}^{\mathrm{di}} \arrow[r] & \widetilde{\Sigma}_{\bar{\rho}} \arrow[r] & \Sigma_{\bar{\rho}} \arrow[r] & 1
\end{tikzcd}$$

We write

$$\beta_t \colon \Sigma_t \to \Sigma_{\bar{\rho}}$$

for the composition of the vertical maps on the right in the above diagram. It is induced by the composition $\tilde{\beta}_t \colon \widetilde{\Sigma}_t \to \widetilde{\Sigma}_{\bar{\rho}}$ of the middle maps, which reflects the fact that every conjugate self-twist of (t, d) induces a conjugate self-twist of $\bar{\rho}$. Combining Corollary 7.4 with Corollary 7.2, we see that $\widetilde{\Sigma}_t$ is a finite abelian group for any constant-determinant pseudodeformation (t, d) of $\bar{\rho}$.

In this paper, we will only be concerned with pseudodeformations (t, d) of $\bar{\rho}$ that are not a priori small. Under this assumption, if $t \neq \operatorname{tr} \bar{\rho}$ then $\Sigma_t^{\mathrm{di}} = 1$ and $\Sigma_T^{\mathrm{di}} = 1$ by Lemma 7.1(1). In particular, $\widetilde{\Sigma}_t = \Sigma_t$ and $\widetilde{\Sigma}_T = \Sigma_T$, so (except for $\bar{\rho}$) a conjugate self-twist (σ, η) is determined uniquely by the automorphism σ.

7.3. **The dihedral case.** As usual, let Π be a p-finite profinite group, A a local pro-p ring and $(t, d) \colon \Pi \to A$ a constant determinant pseudorepresentation with trace algebra A. Throughout this section we assume that its associated residual representation $\bar{\rho}$ is dihedral with nonabelian projective image. We also fix any well-adapted (t, d)-representation ρ, which can be taken to be valued in $\mathrm{GL}_2(A)$ since $\bar{\rho}$ is absolutely irreducible Proposition 2.11(5). This case requires special care for two related reasons. First, it is the only case when A is not generated simply by $I_1(\rho)$ as a $W(\mathbb{F})$-algebra; one also needs to include $B_1(\rho)$ in the generating set by Theorem 2.23. (As explained in Remark 2.12, it makes sense to view $B_1(\rho)$ as a subset of A in this case.) Second, this is the only case when $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ is nontrivial and hence $\ker \beta_t$ can be nontrivial. As we will see, a nontrivial element in $\ker \beta_t$ necessarily behaves quite differently from all other conjugate self-twists, because its action cannot be seen on the residue field. In Proposition 7.5 we explain how a nontrivial element in $\ker \beta_t$ interacts with $I_1(\rho)$ and $B_1(\rho)$. In contrast, when $\ker \beta_t = 1$ we show that $\mathcal{B}_\rho(\mathbb{F})$ has the same fraction field as $A = \mathcal{B}_\rho(\mathbb{F}) + W(\mathbb{F}) B_1(\rho)$.

We use τ to refer to the nontrivial element of $\ker \beta_t$, should it exist. For $\varepsilon \in \{+, -\}$, let

$$A^\varepsilon := \{a \in A \colon {}^\tau a = \varepsilon a\}.$$

Proposition 7.5. *Assume that $\bar{\rho}$ is projectively dihedral and nonabelian. Suppose there exists $1 \neq \tau \in \ker \beta_t$. If $\rho \colon \Pi \to \mathrm{GL}_2(A)$ is a well-adapted (t, d)-representation, then $A^+ = \mathcal{B}_\rho(\mathbb{F})$ and $A^- = W(\mathbb{F}) B_1(\rho)$.*

Proof. Since $A = A^+ \oplus A^-$ and $A = \mathcal{B}_\rho(\mathbb{F}) + W(\mathbb{F})B_1(\rho)$ by Bellaïche's Theorem 2.23, it suffices to show that τ acts trivially on $I_1(\rho)$ and by -1 on $B_1(\rho)$. Let $\eta\colon \Pi \to \{\pm 1\}$ be the unique quadratic character such that $\bar\rho \cong \bar\rho \otimes \eta$. (It is unique by Lemma 7.1 since the projective image of $\bar\rho$ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$). Since $\tau \in \ker \beta_t$, it follows that η must be the character such that $(\tau, \eta) \in \widetilde{\Sigma}_t$.

We first prove that $I_1(\rho)$ is fixed by τ. As usual, let $\Gamma := \operatorname{Im}\rho \cap \Gamma_A(\mathfrak{m})$. Recall that by definition $I_1(\rho)$ is the $\mathbb{Z}_p$-module topologically generated by $\{\alpha - \delta \colon \left(\begin{smallmatrix} 1+\alpha & b \\ c & 1+\delta \end{smallmatrix}\right) \in \Gamma\}$. Let $g \in \ker \eta$. Since ρ is well adapted, we can write

$$\rho(g) = \gamma \cdot \begin{pmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{pmatrix}$$

for some $\gamma \in \Gamma$ and $\lambda, \mu \in \mathbb{F}^\times$. Write $\gamma = \left(\begin{smallmatrix} 1+\alpha & b \\ c & 1+\delta \end{smallmatrix}\right)$ with $\alpha - \delta = 2a$, $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in L_1(\rho)$ and $0 = \alpha + \delta + \alpha\delta - bc$. Then we have

$$\rho(g) = \begin{pmatrix} s(\lambda)(1+\alpha) & s(\mu)b \\ s(\lambda)c & s(\mu)(1+\delta) \end{pmatrix}.$$

Since $g \in \ker \eta$, it follows that

$$s(\lambda)(1+\alpha) + s(\mu)(1+\delta) = \operatorname{tr}\rho(g) = {}^{\tau}(\operatorname{tr}\rho(g)) = s(\lambda)(1+{}^{\tau}\alpha) + s(\mu)(1+{}^{\tau}\delta),$$

since τ acts trivially on $W(\mathbb{F})$. Thus, we obtain

$$s(\lambda_0\mu_0^{-1})(\alpha - {}^{\tau}\alpha) = {}^{\tau}\delta - \delta$$

for all λ, μ such that $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$. As the projective image of $\bar\rho$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$, it follows that when g varies $\lambda\mu^{-1}$ takes at least two distinct values in $\mathbb{F}^\times$. Thus it follows that $\alpha - {}^{\tau}\alpha = 0 = {}^{\tau}\delta - \delta$. Since $I_1(\rho)$ is generated by $\alpha - \delta$ with α, δ as above, it follows that $I_1(\rho)$ is fixed by τ.

The proof that τ acts by -1 on $B_1(\rho)$ is similar. Namely, recall that $B_1(\rho)$ is topologically generated by $\{b, c \in A \colon \left(\begin{smallmatrix} 1+\alpha & b \\ c & 1+\delta \end{smallmatrix}\right) \in \Gamma\}$. Let $g \in \Pi \setminus \ker \eta$. Again since ρ is well adapted, we can write

$$\rho(g) = \gamma \cdot \begin{pmatrix} 0 & s(\lambda) \\ s(\mu) & 0 \end{pmatrix}$$

for some $\gamma \in \Gamma, \lambda, \mu \in \mathbb{F}^\times$. As above, write $\gamma = \left(\begin{smallmatrix} 1+\alpha & b \\ c & 1+\delta \end{smallmatrix}\right)$. Then we have

$$\rho(g) = \begin{pmatrix} s(\mu)b & s(\lambda)(1+\alpha) \\ s(\mu)(1+\delta) & s(\lambda)c \end{pmatrix}.$$

Since $g \notin \ker \eta$, it follows that

$$s(\mu)b + s(\lambda)c = \operatorname{tr}\rho(g) = -{}^{\tau}(\operatorname{tr}\rho(g)) = -s(\mu){}^{\tau}b - s(\lambda){}^{\tau}c.$$

Thus

$$s(\mu\lambda^{-1})(b + {}^{\tau}b) = -(c + {}^{\tau}c)$$

for all $\left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$. Once again, since the projective image of $\bar\rho$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$, it follows that $\lambda\mu^{-1}$ takes at least two distinct values in $\mathbb{F}^\times$. Therefore $b + {}^{\tau}b = 0 = c + {}^{\tau}c$. Since $B_1(\rho)$ is generated by such b and c, it follows that τ acts on $B_1(\rho)$ by -1. ∎

In particular, we can always apply Proposition 7.5 to the universal pseudorepresentation $(T, d)\colon \Pi \to \mathcal{A}$ of $\bar\rho$ since $\Sigma_T \cong \widetilde{\Sigma}_{\bar\rho}$ and thus $\ker \beta_T$ is always nontrivial in the dihedral case whenever a nondihedral deformation exists. Fix a well-adapted (T, d)-representation $\rho^{\mathrm{univ}}\colon \Pi \to \mathrm{GL}_2(\mathcal{A})$. By universality, we have a $W(\mathbb{F})$-algebra homomorphism $\alpha_t\colon \mathcal{A} \to A$ such that $t = \alpha_t \circ T$. Let $\rho = \rho_t$ be the well-adapted (t, d)-representation obtained by composing ρ^{univ} with the map $\mathrm{GL}_2(\mathcal{A}) \to \mathrm{GL}_2(A)$ induced by α_t. Since the Pink-Lie algebra is functorial with respect to surjective ring homomorphisms, we see that $I_1(\rho^{\mathrm{univ}})$ (respectively, $B_1(\rho^{\mathrm{univ}})$) surjects onto $I_1(\rho)$ (respectively, $B_1(\rho)$).

For any subfield $\mathbb{F}' \subseteq \mathbb{F}$, let $\mathcal{A}' = \mathcal{B}_{\rho^{\text{univ}}}(\mathbb{F}') + W(\mathbb{F}')B_1(\rho^{\text{univ}})$. We have $\mathcal{A}' = (\mathcal{A}')^+ \oplus (\mathcal{A}')^-$ with $(\mathcal{A}')^+ = \mathcal{B}_{\rho^{\text{univ}}}(\mathbb{F}')$ and $(\mathcal{A}')^- = W(\mathbb{F}')B_1(\rho^{\text{univ}})$ by Proposition 7.5.

Proposition 7.6. *Suppose that A is a local pro-p domain and $(t, d)\colon \Pi \to A$ is a constant-determinant pseudodeformation of a dihedral $\bar{\rho}$ such that $\ker \beta_t = 1$. Let ρ be a well-adapted (t, d)-representation obtained from a universal one as described above. Then for any subfield $\mathbb{F}' \subseteq \mathbb{F}$:*

(1) $\mathcal{A}'$ and $\mathcal{B}_\rho(\mathbb{F}')$ are noetherian rings;

(2) $W(\mathbb{F}')B_1(\rho)$ and hence $\mathcal{B}_\rho(\mathbb{F}') + W(\mathbb{F}')B_1(\rho)$ are noetherian $\mathcal{B}_\rho(\mathbb{F}')$-modules;

(3) $\mathcal{B}_\rho(\mathbb{F}') + W(\mathbb{F}')B_1(\rho)$ and $\mathcal{B}_\rho(\mathbb{F}')$ have the same field of fractions.

Proof. For (1) note that $\mathcal{B}_\rho(\mathbb{F}')$ is the image of $(\mathcal{A}')^+$ under α_t, which is noetherian if $\mathcal{A}'$ is by Lemma A.16. To see that $\mathcal{A}'$ is noetherian when $\mathbb{F}' = \mathbb{F}$, we have $\mathcal{A}' = \mathcal{A} = \mathcal{B}_{\rho^{\text{univ}}}(\mathbb{F}) + W(\mathbb{F})B_1(\rho^{\text{univ}})$ by Theorem 2.23, and hence $\mathcal{A}$ is noetherian by the p-finiteness of Π. Note that $\mathcal{A}$ is finite and integral over $\mathcal{B}_{\rho^{\text{univ}}}(\mathbb{F}') + W(\mathbb{F}')B_1(\rho^{\text{univ}})$ since this is true of $W(\mathbb{F})$ over $W(\mathbb{F}')$. Then $\mathcal{A}' = \mathcal{B}_{\rho^{\text{univ}}}(\mathbb{F}') + W(\mathbb{F}')B_1(\rho^{\text{univ}})$ is noetherian by [Eak86, Theorem 2].

The statements in (2) follow from the corresponding statements for ρ^{univ}, which in turn follow from Proposition A.18 since $\mathcal{A}'$ is noetherian.

As $\mathcal{B}_\rho(\mathbb{F}')$ is the image of $(\mathcal{A}')^+$ under α_t while $W(\mathbb{F}')B_1(\rho)$ is he image of $(\mathcal{A}')^-$ and $\mathcal{A}'$ is noetherian, (3) follows from Proposition A.19. $\square$

8. Regularity and residual conjugate self-twists

In this section we prepare the groundwork for Section 9 by studying conjugate self-twists of $\bar{\rho}$, particularly how they interact with regularity (Definition 2.19). In particular, $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ is a semisimple regular representation throughout this section, and after Section 8.1 it is always absolutely irreducible. We only consider simple conjugate self-twists in this section and thus write $\widetilde{\Sigma}_{\bar{\rho}}$ for $\widetilde{\Sigma}_{\bar{\rho}}(\mathbb{F})$ and similarly for $\Sigma_{\bar{\rho}}$. In Section 8.1 we see that when $\bar{\rho}$ is regular we may assume that it has no conjugate self-twists. We consider the restriction of $\bar{\rho}$ to the kernels of twist characters in Section 8.2, followed by some technical lemmas about the extension $\mathbb{F}/\mathbb{E}$ in Section 8.3. In Section 8.4 we define the condition of *goodness* when $\bar{\rho}$ is octahedral, which weighs on our main theorem. Section 8.4 culminates in Proposition 8.13 where we choose the basis of $\bar{\rho}$ that will be used throughout Section 9.

Recall that $\bar{\rho}$ is *regular* if $\operatorname{Im} \bar{\rho}$ contains an element with eigenvalues $\lambda_0, \mu_0 \in \bar{\mathbb{F}}^\times$ such that $\lambda_0 \mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$.

8.1. **Reducible regular representations.** We show that if $\bar{\rho}$ is reducible and regular, then one can eliminate the conjugate self-twists of $\bar{\rho}$ by twisting $\bar{\rho}$ by a character, making the proof of our main theorem especially easy in that case (cf. Section 9.1).

Lemma 8.1. *Suppose that $\bar{\rho} = \varepsilon \oplus \delta$ and $\bar{\rho}$ is regular. If $(\sigma, \eta) \in \widetilde{\Sigma}_{\bar{\rho}}$, then ${}^\sigma\varepsilon = \eta\varepsilon$ and ${}^\sigma\delta = \eta\delta$. In particular, $\varepsilon\delta^{-1}$ takes values in $\mathbb{E}$.*

Proof. It suffices to show that if $\bar{\rho}$ is regular then we cannot have ${}^\sigma\varepsilon = \eta\delta$ and ${}^\sigma\delta = \eta\varepsilon$. If this were true, then we would have ${}^\sigma\varepsilon\delta^{-1} = \eta = {}^\sigma\delta\varepsilon^{-1}$, which implies that

$$ {}^\sigma(\varepsilon\delta^{-1}) = \delta\varepsilon^{-1}. \tag{5} $$

Since $\bar{\rho}$ is regular, there is some $g \in \Pi$ such that $\varepsilon(g)\delta(g)^{-1} \in \mathbb{E} \setminus \{\pm 1\}$. As $\mathbb{E}$ is fixed by σ by Proposition 4.9, it follows from (5) that $\varepsilon(g)\delta(g)^{-1} = \pm 1$, a contradiction.

The last sentence in the statement of the lemma follows from the fact that, for any $\sigma \in \Sigma_{\bar{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$, we have ${}^\sigma\varepsilon\varepsilon^{-1} = \eta = {}^\sigma\delta\delta^{-1}$ and hence $\varepsilon\delta^{-1}$ is fixed by $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$. $\square$

Corollary 8.2. *Suppose $\bar{\rho} = \varepsilon \oplus \delta$ and $\bar{\rho}$ is regular. Then $\bar{\rho}' := \bar{\rho} \otimes \delta^{-1}$ has no conjugate self-twists.*

Proof. Since $\bar{\rho}$ is regular, its projective image cannot have order 2. Therefore it suffices to show that $\Sigma_{\bar{\rho}'}$ is trivial by Lemma 7.1. Let $\mathbb{F}'$ be the extension of $\mathbb{F}_p$ generated by the trace of $\bar{\rho}'$. Then $\Sigma_{\bar{\rho}'} = \mathrm{Gal}(\mathbb{F}'/\mathbb{E})$, so it suffices to show that $\mathbb{F}' \subseteq \mathbb{E}$. But $\mathbb{F}'$ is generated by the values of $\varepsilon\delta^{-1}$, which takes values in $\mathbb{E}$ by Lemma 8.1. □

8.2. **Kernels of twist characters and regularity.** In this section we introduce the subgroup of Π given by intersecting the kernels of all twist characters. It is often useful to restrict to this subgroup because doing so kills the conjugate self-twists and but retains fullness. In this section we study how this restriction affect the residual representation and regularity, first in the exceptional/large image case, then when $\bar{\rho}$ is dihedral. Define

$$\Pi_0(\bar{\rho}) \coloneqq \bigcap_{(\sigma,\eta)\in\widetilde{\Sigma}_{\bar{\rho}}} \ker\eta. \tag{6}$$

Remark 8.3. The quotient $\Pi/\Pi_0(\bar{\rho})$ is abelian. Indeed, $\Pi_0(\bar{\rho})$ is the kernel of the natural map diagonal map of Π to $\prod_{(\sigma,\eta)\in\widetilde{\Sigma}_{\bar{\rho}}} \Pi/\ker\eta$, so $\Pi/\Pi_0(\bar{\rho})$ can be embedded into an abelian group. △

As we will now see, it is easiest to control $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ when the order of $\det\bar{\rho}$ is a power of 2, which is an assumption we are forced to make in Section 9. We may always twist $\bar{\rho}$ by a character to assume that the order of $\det\bar{\rho}$ is a power of 2:

Lemma 8.4. *Let $d\colon \Pi \to \mathbb{F}^\times$ be a character. Then there is a character $\chi\colon \Pi \to \mathbb{F}^\times$ such that the order of $d\chi^2$ is a power of 2.*

Proof. The odd-order part of d has a square root ψ. Take $\chi = \psi^{-1}$. □

Lemma 8.5. *Assume that $\bar{\rho}$ is exceptional or large. If the order of $\det\bar{\rho}$ is a power of 2, then $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ is absolutely irreducible.*

Proof. If $(\sigma,\eta) \in \widetilde{\Sigma}_{\bar{\rho}}$, then η^2 is equal to a power of $\det\bar{\rho}$ and hence the order of η is a power of 2. Thus $[\Pi\colon \Pi_0(\bar{\rho})]$ is a power of 2, and $\Pi/\Pi_0(\bar{\rho})$ is abelian by Remark 8.3.

By hypothesis, the projective image of $\bar{\rho}$ is isomorphic to one of $A_4, S_4, A_5, \mathrm{PSL}_2(\mathbb{E}), \mathrm{PGL}_2(\mathbb{E})$. None of $A_4, A_5, \mathrm{PSL}_2(\mathbb{E})$ has a subgroup of 2-power index with abelian quotient. Both S_4 and $\mathrm{PGL}_2(\mathbb{E})$ have a unique proper 2-power index subgroup with abelian quotient, namely A_4 and $\mathrm{PSL}_2(\mathbb{E})$, respectively. Therefore the possible projective images of $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ are the same as for $\bar{\rho}$, so $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ is absolutely irreducible. □

Proposition 8.6. *Assume that $\bar{\rho}$ is regular dihedral, say $\bar{\rho} = \mathrm{Ind}_{\Pi_0}^{\Pi}\chi$. Then $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ is multiplicity free over $\mathbb{E}$. Furthermore, given $g \in \Pi_0$, we have $g \in \Pi_0(\bar{\rho})$ if and only if $\chi(g) \in \mathbb{E}^\times$.*

Proof. Since $\bar{\rho}$ is regular, it follows from Lemma A.7 that there is a unique subgroup Π_0 of Π of index 2 such that $\bar{\rho} \cong \mathrm{Ind}_{\Pi_0}^{\Pi}\chi$ for some character $\chi\colon \Pi_0 \to \mathbb{F}^\times$. For any $h \in \Pi$, define $\chi^h\colon \Pi_0 \to \mathbb{F}^\times$ by $\chi^h(g) \coloneqq \chi(h^{-1}gh)$. The character χ^h only depends on the class of h in Π/Π_0. Fix an element $c \in \Pi \setminus \Pi_0$. Fix a generator $\sigma \in \Sigma_{\bar{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$, and choose η such that $(\sigma,\eta) \in \widetilde{\Sigma}_{\bar{\rho}}$. (Note that there are two choices for η, and they differ by the character $\eta_0\colon \Pi \twoheadrightarrow \Pi/\Pi_0 \cong \{\pm 1\}$.) Then $\Pi_0(\bar{\rho}) = \ker\eta_0 \cap \ker\eta$ since σ generates $\Sigma_{\bar{\rho}}$. Therefore $\Pi_0(\bar{\rho}) = \ker\eta|_{\Pi_0}$.

Note that any regular element for $\bar{\rho}$ must be in Π_0 since elements in $\Pi\setminus\Pi_0$ have projective order 2. By applying Lemma 8.1 to $\bar{\rho}|_{\Pi_0}$, we find that ${}^\sigma\chi = \eta\chi$ and ${}^\sigma\chi^c = \eta\chi^c$. Hence $\eta|_{\Pi_0} = {}^\sigma\chi\chi^{-1}$, so $g \in \Pi_0(\bar{\rho})$ if and only if $\chi(g) \in \mathbb{E}^\times$. In particular, $\ker\chi \subseteq \Pi_0(\bar{\rho})$. Furthermore, using the fact that ${}^\sigma\chi\chi^{-1} = \eta|_{\Pi_0} = {}^\sigma\chi^c(\chi^c)^{-1}$, we find that the character χ/χ^c takes values in $\mathbb{E}^\times$.

We know that $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ is multiplicity free over $\mathbb{E}$ if and only if there is some $g \in \Pi_0(\bar{\rho})$ such that $\chi(g) \neq \chi^c(g)$. If $\ker\chi \neq \ker\chi^c$, then we can choose $g \in \ker\chi \setminus \ker\chi^c$. Then $\chi(g) = 1 \neq \chi^c(g)$ and $g \in \Pi_0(\bar{\rho})$ by the previous paragraph. Therefore we may assume that $\ker\chi = \ker\chi^c$.

Let n denote the order of χ. Since $\ker\chi = \ker\chi^c$, we have that $\chi^c = \chi^a$ for some $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. Note that $\chi^{c^2} = \chi$ since $c^2 \in \Pi_0$. Therefore

$$\chi = \chi^{c^2} = (\chi^c)^c = (\chi^a)^c = (\chi^c)^a = (\chi^a)^a = \chi^{a^2}.$$

Fix $g_0 \in \Pi_0$ such that $\bar\rho(g_0)$ generates the projective image of $\bar\rho(\Pi_0)$. We will show that $h := g_0^{a-1}$ is in $\Pi_0(\bar\rho)$ and $\chi(h) \in \mathbb{E}^\times$ with $\chi(h) \neq \chi^c(h)$. First we calculate, using the fact that $\chi^{a^2} = \chi$,

$$\chi^c(h) = \chi^a(g_0^{a-1}) = \chi^{a^2}(g_0)\chi^{-1}(g_0) = 1.$$

Hence $h \in \ker\chi^c = \ker\chi \subseteq \Pi_0(\bar\rho)$. On the other hand,

$$\chi(h) = \chi(g_0^{a-1}) = \chi^c(g_0)\chi^{-1}(g_0).$$

We saw in the second paragraph that χ/χ^c is an $\mathbb{E}$-valued character. Furthermore, $\chi^c(g_0)/\chi(g_0) \neq 1$ since g_0 was chosen has a generator of the projective image of $\bar\rho(\Pi_0)$, which is isomorphic to the image of χ/χ^c. □

Corollary 8.7. *Assume that $\bar\rho$ is regular and dihedral and that the order of $\det\bar\rho$ is a power of 2. Let σ be a generator of $\Sigma_{\bar\rho}$ and $\eta\colon \Pi \to \mathbb{F}^\times$ a character such that $(\sigma,\eta) \in \widetilde{\Sigma}_{\bar\rho}$. Then either $\Sigma_{\bar\rho}$ is trivial or $\bar\rho|_{\ker\eta}$ is absolutely irreducible.*

Proof. Write $\bar\rho = \mathrm{Ind}_{\Pi_0}^{\Pi}\chi$ and fix $c \in \Pi\backslash\Pi_0$. We shall make frequent use of Lemma A.7 in this proof without referencing it every time. We saw in the proof of Proposition 8.6 that $\Pi_0(\bar\rho) = \Pi_0 \cap \ker\eta$. Thus Proposition 8.6 implies that $\chi|_{\Pi_0\cap\ker\eta} \neq \chi^c|_{\Pi_0\cap\ker\eta}$.

If $\ker\eta \neq \Pi_0\cap\ker\eta$, then $[\ker\eta\colon \Pi_0\cap\ker\eta] = 2$ since $[\Pi\colon \Pi_0] = 2$. Thus $\bar\rho|_{\ker\eta} \cong \mathrm{Ind}_{\Pi_0\cap\ker\eta}^{\ker\eta}\chi|_{\Pi_0\cap\ker\eta}$. Since $\chi|_{\Pi_0\cap\ker\eta} \neq \chi^c|_{\Pi_0\cap\ker\eta}$ it follows that $\bar\rho|_{\ker\eta}$ is irreducible.

If $\ker\eta = \Pi_0\cap\ker\eta$ then $\Pi_0 \supseteq \ker\eta$ and $\Pi/\ker\eta$ is a cyclic group whose order is a power of 2 since η^2 is a power of $\det\bar\rho$. If $\Pi_0 \neq \ker\eta$, then there is a subgroup $\ker\eta \subseteq \Pi' \subset \Pi_0$ such that $[\Pi_0\colon \Pi'] = 2$. Note that $\chi|_{\Pi'} \neq \chi^c|_{\Pi'}$ since $\chi|_{\ker\eta} \neq \chi^c|_{\ker\eta}$. Then $\bar\rho|_{\Pi_0} \cong \mathrm{Ind}_{\Pi'}^{\Pi_0}\chi|_{\Pi'}$ is irreducible, a contradiction since $\bar\rho \cong \mathrm{Ind}_{\Pi_0}^{\Pi}\chi$. Thus we must have $\Pi_0 = \ker\eta$. Therefore $\bar\rho \cong \bar\rho\otimes\eta$ and so σ, and hence $\Sigma_{\bar\rho}$, is trivial. □

8.3. $\mathbb{F}/\mathbb{E}$ **when** $\det\bar\rho$ **is a power of** 2. In Section 9 we will assume that the order of $\det\bar\rho$ is a power of 2. A large part of the reason for that assumption is that it guarantees that $[\mathbb{F}\colon \mathbb{E}]$ can be taken to be a power of 2 as well, as the next lemma shows. We need this in an induction argument in Section 9. Given any $\mathbb{F}$-valued function f and any subfield $\mathbb{F}'$ of $\mathbb{F}$, let us write $\mathbb{F}'(f)$ for the subfield of $\mathbb{F}$ generated over $\mathbb{F}'$ by the values of f.

Lemma 8.8. *Assume that the order of $\det\bar\rho$ is a power of 2. Then the degree of $\mathbb{F}_p(\mathrm{tr}\,\bar\rho)$ over $\mathbb{E}$ is a power of 2.*

Proof. Let $d := \det\bar\rho$. Since the order of d is a power of 2, the degree of $\mathbb{E}(d)$ over $\mathbb{E}$ is a power of 2. But, for an arbitrary $g' \in \Pi$, the extension $\mathbb{E}(\mathrm{tr}\,\bar\rho(g'))$ is at most quadratic over $\mathbb{E}(d)$ because $\mathrm{tr}\,\bar\rho(g')$ satisfies

$$d(g')x^2 - (\mathrm{tr}\,\bar\rho(g'))^2/d(g') \in \mathbb{E}(d)[x].$$

The field $\mathbb{F}_p(\mathrm{tr}\,\bar\rho)$ is obtained from $\mathbb{E}(d)$ by adding finitely many values of $\mathrm{tr}\,\bar\rho$. □

In Section 9 we will be interested in gradings coming from conjugate self-twists. To be able to apply Lemma A.22 in those situations, we now verify one of the hypotheses.

Lemma 8.9. *Assume that both the order of $\det\bar\rho$ and $[\mathbb{F}\colon \mathbb{E}]$ are powers of 2. If $n = \#\Sigma_{\bar\rho}$, then $\mathbb{F}$ contains a primitive n^{th} root of unity. In particular, condition $(*)$ from Appendix A.4 is satisfied.*

Proof. Let $d := \det\bar\rho$, and write 2^s for the order of d. We have that $\mathbb{E}(d)$ contains a primitive $(2^s)^{\text{th}}$ root of unity. If $[\mathbb{F}\colon \mathbb{E}(d)] = 2^r$, then $\mathbb{F}$ contains a primitive $(2^{r+s})^{\text{th}}$ root of unity. On the other hand,

$$n = \#\Sigma_{\bar\rho} = [\mathbb{F}\colon \mathbb{E}] = 2^r[\mathbb{E}(d)\colon \mathbb{E}].$$

Since d has order 2^s, it follows that $[\mathbb{E}(d)\colon \mathbb{E}]$ divides 2^{s-1}. Thus n divides 2^{r+s-1}, and so $\mathbb{F}$ contains a primitive n^{th} root of unity. □

8.4. **A good basis for $\bar\rho$.** We need to carefully choose a basis for $\bar\rho$ that has many good properties and will allow us to choose a good (t,d)-representation in Section 9.2. In this section we explain how to find this basis when $\bar\rho$ is exceptional or large. Let us first define an extra condition on octahedral representations.

Definition 8.10. We say a regular octahedral representation $\bar\rho$ is *good* if at least one of the following properties is satisfied:

(1) $p \equiv 1 \bmod 3$;
(2) $\bar\rho$ is strongly regular;
(3) there is a regular element $g_0 \in \Pi$ such that $g_0^2 \in \Pi_0(\bar\rho)$.

We shall need to know that if $\bar\rho$ is good, then twisting away the odd part of the determinant of $\bar\rho$ gives a representation that is also good.

Lemma 8.11. *Let $\bar\rho\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a good representation. Let $\chi\colon \Pi \to \mathbb{F}^\times$ be the unique odd-order character such that the order of $\chi^2\det\bar\rho$ is a power of 2. Then $\bar\rho\otimes\chi$ is good.*

Proof. First note that twisting by any character does not change the projective image, so $\bar\rho\otimes\chi$ is octahedral. Regularity is also invariant under twisting. The claim is clear if $p \equiv 1 \bmod 3$, so we assume that $p \equiv 2 \bmod 3$. The regularity assumption then implies that $\zeta_4 \in \mathbb{F}_p$ by Remark 2.20. As in the proof of Lemma 8.4, decompose $\det\bar\rho = d_1 d_2$, where $d_i\colon \Pi \to \mathbb{F}^\times$ are characters such that the order of d_1 is odd and the order of d_2 is a power of 2.

First suppose that $\bar\rho$ is strongly regular. Then there is a matrix $g_0 \in \Pi$ such that $\bar\rho(g_0)$ has eigenvalues $\lambda_0, \mu_0 \in \mathbb{E}^\times$ such that $\lambda_0\mu_0^{-1} = \zeta_4$. We have $\lambda_0\mu_0 = \det\bar\rho(g_0) = d_1(g_0)d_2(g_0)$. Note that any $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{E})$ fixes $\lambda_0\mu_0$ since $\lambda_0,\mu_0 \in \mathbb{E}$. Therefore $\sigma(d_1(g_0)d_2(g_0)) = d_1(g_0)d_2(g_0)$. But since $d_1(g_0)$ is an odd order root of unity and $d_2(g_0)$ is a 2-power order root of unity, it follows that σ must fix both $d_1(g_0)$ and $d_2(g_0)$. Write a for the order of d_1. Then $\chi = d_1^{-(a+1)/2}$ by the proof of Lemma 8.4. In particular, $\chi(g_0) \in \mathbb{E}^\times$. Thus the eigenvalues $\chi(g_0)\lambda_0$ and $\chi(g_0)\mu_0$ of $(\bar\rho\otimes\chi)(g_0)$ are in $\mathbb{E}$. Thus g_0 is a strongly regular element for $\bar\rho\otimes\chi$, as desired.

Finally, suppose that there is a regular element $g_0 \in \Pi$ such that $g_0^2 \in \Pi_0(\bar\rho)$. Let σ be a generator for $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$ and let $\eta\colon \Pi \to \mathbb{F}^\times$ such that $(\sigma,\eta) \in \widetilde\Sigma_{\bar\rho}$. Then $\Pi_0(\bar\rho) = \ker\eta$ and $\Pi_0(\bar\rho\otimes\chi) = \ker{}^\sigma\chi\chi^{-1}\eta$. Since $g_0^2 \in \Pi_0(\bar\rho)$ and ${}^\sigma\!\det\bar\rho = \eta^2\det\bar\rho$, it follows that $\det\bar\rho(g_0) \in \mathbb{E}$. But $\det\bar\rho(g_0) = d_1(g_0)d_2(g_0)$, and since $d_1(g_0)$ is an odd order root of unity and $d_2(g_0)$ has 2-power order, it follows that both $d_1(g_0)$ and $d_2(g_0)$ are in $\mathbb{E}$. Therefore $\chi(g_0) = d_1^{-(a+1)/2}(g_0) \in \mathbb{E}$. Thus $g_0^2 \in \ker{}^\sigma\chi\chi^{-1}\eta = \Pi_0(\bar\rho\otimes\chi)$. □

Finally we describe the basis of $\bar\rho$ that we shall work with in Section 9. Let Z denote the group of scalar matrices in $\mathrm{GL}_2(\mathbb{F})$. The following lemma justifies our definition of $\mathbb{F}_q$ in Section 2.5 for exceptional representations.

Lemma 8.12. *Up to conjugation, the image of $\bar\rho$ is contained in $Z\,\mathrm{GL}_2(\mathbb{E})$. If $\mathbb{F}_q$ is an extension of $\mathbb{E}$ and $\lambda_0,\mu_0 \in \overline{\mathbb{F}}_p^\times$ are eigenvalues of a matrix in the image of $\bar\rho$ such that $\lambda_0\mu_0^{-1} \in \mathbb{F}_q$, then we may further conjugate $\bar\rho$ to assume that $\left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$ and $\operatorname{Im}\bar\rho \subseteq Z\,\mathrm{GL}_2(\mathbb{F}_q)$.*

Proof. By Corollary 4.13, $\mathbb{E} = \mathbb{F}^{\Sigma_{\bar\rho}}$. First we show that $\bar\rho$ can be conjugated to land in $Z\,\mathrm{GL}_2(\mathbb{E})$. Let $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{E})$ be a generator and η a character such that $(\sigma, \eta) \in \Sigma_{\bar\rho}$. Then there is some $x \in \mathrm{GL}_2(\mathbb{F})$ such that for all $g \in \Pi$, we have ${}^{\sigma}\bar\rho(g) = x^{-1}\eta(g)\bar\rho(g)x$. By a theorem of Serge Lang [Lan56, Corollary to Theorem 1], it follows that there is some $y \in \mathrm{GL}_2(\mathbb{F})$ such that $x = {}^{\sigma}yy^{-1}$. Thus ${}^{\sigma}(y^{-1}\bar\rho(g)y) = \eta(g)(y^{-1}\bar\rho(g)y)$. Replacing $\bar\rho$ by its conjugate by y, we have that the projective image of $\bar\rho$ is fixed by $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$, and hence the image of $\bar\rho$ lands in $Z\,\mathrm{GL}_2(\mathbb{E})$, as desired.

If $\mathbb{F}_q, \lambda_0, \mu_0$ are as in the statement of the lemma, then $\bar\rho$ can be further conjugated such that $\left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right) \in \mathrm{Im}\,\bar\rho$ while preserving the property that the image of $\bar\rho$ is in $Z\,\mathrm{GL}_2(\mathbb{F}_q)$. ☐

Proposition 8.13. *Let $\bar\rho\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be regular and either exceptional or large. If $\bar\rho$ is octahedral, assume further that $\bar\rho$ is good. Assume that the order of $\det\bar\rho$ is a power of 2. Then there is a regular element $g_0 \in \Pi$ and a basis for $\bar\rho$ such that the following are simultaneously true:*

(1) $\mathrm{Im}\,\bar\rho \subseteq Z\,\mathrm{GL}_2(\mathbb{E})$;
(2) $\bar\rho(g_0) = \left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right)$ for some $\lambda_0, \mu_0 \in \mathbb{F}$;
(3) if $p \geq 7$ and $\bar\rho$ is large, then $\lambda_0, \mu_0 \in \mathbb{F}_p^\times$;
(4) there is a positive integer n such that $g_0^n \in \Pi_0(\bar\rho)$ and $\bar\rho(g_0^n)$ is not scalar.

Proof. By Lemma 8.12 we can always conjugate $\bar\rho$ so that $\mathrm{Im}\,\bar\rho \subseteq Z\,\mathrm{GL}_2(\mathbb{E})$. If $g_0 \in \Pi$ is a regular element and λ_0 and μ_0 are the eigenvalues of $\bar\rho(g_0)$, then we may assume further that $\bar\rho(g_0) = \left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right)$.

If $\bar\rho$ is large, then up to conjugation, $\mathrm{Im}\,\bar\rho \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. Indeed, up to conjugation we may assume that the projective image of $\bar\rho$ contains $\mathrm{PSL}_2(\mathbb{E})$. Therefore there is some $\lambda \in \mathbb{F}^\times$ such that $\lambda\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Im}\,\bar\rho$. Note that the n^{th} power of this matrix is $\lambda^n\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. Since $\lambda \in \mathbb{F}^\times$, its order m is prime to p. Therefore we can write $1 = am + bp \equiv am \bmod p$ for some $a, b \in \mathbb{Z}$. Thus

$$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \lambda^{am}\left(\begin{smallmatrix} 1 & am \\ 0 & 1 \end{smallmatrix}\right) = \left(\lambda\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)^{am} \in \mathrm{Im}\,\bar\rho.$$

Similarly, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{Im}\,\bar\rho$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{F}_p)$, it follows that $\mathrm{SL}_2(\mathbb{F}_p) \subseteq \mathrm{Im}\,\bar\rho$.

If $p \geq 7$, then we can choose $\alpha \in \mathbb{F}_p^\times$ such that $\alpha^2 \neq \pm 1$. Then any $g_0 \in \Pi$ such that $\bar\rho(g_0)$ has eigenvalues α, α^{-1} satisfies the first three conditions. Note that $\mathbb{P}\bar\rho(g_0) \in \mathrm{PSL}_2(\mathbb{E})$. Recall that $\Pi_0(\bar\rho)$ is a normal subgroup of 2-power index in Π since the order of $\det\bar\rho$ is a power of 2. Furthermore, $\Pi/\Pi_0(\bar\rho)$ is abelian. Therefore $\mathbb{P}\bar\rho(\Pi_0(\bar\rho))$ is either $\mathrm{PGL}_2(\mathbb{E})$ or $\mathrm{PSL}_2(\mathbb{E})$. In either case, we can find $g_0 \in \Pi_0(\bar\rho)$ such that $\bar\rho(g_0)$ has eigenvalues α, α^{-1}. Thus all of the properties of the proposition are satisfied for this choice of g_0.

Next suppose that $\bar\rho$ is either tetrahedral or icosahedral. Once again, $\mathbb{P}\bar\rho(\Pi_0(\bar\rho))$ is a normal subgroup of $\mathbb{P}\bar\rho(\Pi)$ with 2-power index and abelian quotient. Since $\mathbb{P}\bar\rho(\Pi)$ is isomorphic to one of A_4 or A_5, it follows that $\mathbb{P}\bar\rho(\Pi_0(\bar\rho)) = \mathbb{P}\bar\rho(\Pi)$. In particular, one can choose the regular element g_0 to be in $\Pi_0(\bar\rho)$, and the resulting representation satisfies all of the desired conditions.

Finally, suppose that $\bar\rho$ is octahedral and good. If $p \equiv 1 \bmod 3$ then any $g_0 \in \Pi$ such that $\mathbb{P}\bar\rho(g_0)$ has order 3 is a regular element. Since $\mathbb{P}\bar\rho(\Pi_0(\bar\rho))$ is a normal subgroup of $\mathbb{P}\bar\rho(\Pi)$ with 2-power index and abelian quotient, it follows that $\Pi_0(\bar\rho)$ contains an element g_0 such that $\mathbb{P}\bar\rho(g_0)$ has order 3. Such a g_0 satisfies all of the necessary conditions.

Next suppose that $p \equiv 2 \bmod 3$ and that $\bar\rho$ is strongly regular. Let $g_0 \in \Pi$ be a strongly regular element. Then $\bar\rho(g_0) = \left(\begin{smallmatrix} \lambda\zeta_4 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ for some $\lambda \in \mathbb{E}^\times$. (Note that $\zeta_4 \in \mathbb{F}_p$ since $\bar\rho$ is regular and $p \equiv 2 \bmod 3$.) We claim that $g_0 \in \Pi_0(\bar\rho)$. Indeed, let σ be a generator of $\mathrm{Gal}(\mathbb{F}/\mathbb{E})$ and η a character such that $(\sigma, \eta) \in \widetilde{\Sigma}_{\bar\rho}$. Then $\Pi_0(\bar\rho) = \ker\eta$. Since $\lambda, \zeta_4 \in \mathbb{E}^\times$ we have

$$\lambda(\zeta_4 + 1) = {}^{\sigma}(\lambda(\zeta_4 + 1)) = {}^{\sigma}\mathrm{tr}\,\bar\rho(g_0) = \eta(g_0)\,\mathrm{tr}\,\bar\rho(g_0) = \eta(g_0)\lambda(\zeta_4 + 1).$$

As $\zeta_4 + 1 \neq 0$ it follows that $\eta(g_0) = 1$, and so $g_0 \in \Pi_0(\bar\rho)$, as claimed. Therefore g_0 satisfies all of the necessary conditions.

Finally suppose that $p \equiv 2 \bmod 3$ and there is a regular element $g_0 \in \Pi$ such that $g_0^2 \in \Pi_0(\bar{\rho})$. Note that $\mathbb{P}\bar{\rho}(g_0)$ has order 4 since $p \not\equiv 1 \bmod 3$. Therefore $\mathbb{P}\bar{\rho}(g_0^2)$ is nontrivial, so $\bar{\rho}(g_0^2)$ is not scalar. Therefore g_0 satisfies all of the conditions of the proposition. $\square$

Note that the g_0 chosen in Proposition 8.13 satisfies all of the conditions prior to Definition 2.21. In particular, if $(t,d)\colon \Pi \to A$ is any admissible pseudodeformation of $\bar{\rho}$, then any (t,d)-representation that is adapted to the element g_0 from Proposition 8.13 is well adapted.

9. Fullness peers: $\mathcal{B}_\rho(\mathbb{E})$ and A^{Σ_ρ}

Throughout Section 9 we fix a local pro-p domain A and an admissible pseudodeformation $(\Pi, \bar{\rho}, t, d)$ over A. We only consider A-valued conjugate self-twists throughout this section and thus write Σ_t for $\Sigma_t(A)$ and similarly for $\widetilde{\Sigma}_t$. The goal of Section 9 is to prove that (t,d) is A_0-full whenever (t,d) is not a priori small and regularity is satisfied. In view of Corollary 4.21 and Corollary 6.6, it suffices to show that A^{Σ_t} and $\mathcal{B}_\rho(\mathbb{E})$ are fullness peers for some well chosen (t,d)-representation ρ. Let us point out an easy case when this is possible. If $\bar{\rho}$ has no conjugate self-twists, then $\mathbb{E} = \mathbb{F}$ by Corollary 4.13 and $\Sigma_t = 1$ by the diagram following Corollary 7.4. Furthermore, the assumption that $\widetilde{\Sigma}_{\bar{\rho}} = 1$ implies that $\bar{\rho}$ is not dihedral and so $A = \mathcal{B}_\rho(\mathbb{F})$ by Theorem 2.23. Therefore we have

$$\mathcal{B}_\rho(\mathbb{E}) = \mathcal{B}_\rho(\mathbb{F}) = A = A^{\Sigma_t}.$$

In general, the proof that (t,d) is A^{Σ_t}-full, hence A_0-full, is structured as follows. The case when $\bar{\rho}$ is reducible is easily done in Proposition 9.1, so from Section 9.2 onwards we always assume that $\bar{\rho}$ is irreducible. In light of Corollary 6.6, the strategy is to prove that, under certain conditions on $\bar{\rho}$ and a good choice of a (t,d)-representation ρ, the two rings $\mathcal{B}_\rho(\mathbb{E})$ and A^{Σ_t} have the same fields of fractions and A^{Σ_t} is finitely generated as a $\mathcal{B}_\rho(\mathbb{E})$-module. This is done in Corollary 9.15, although key parts of it are proved in Corollary 9.9 and Proposition 9.14. Lemma 3.5 then implies that A^{Σ_t} and $\mathcal{B}_\rho(\mathbb{E})$ are fullness peers. In Corollary 9.16, we combine Corollary 6.6, which established $\mathcal{B}_\rho(\mathbb{E})$-fullness, with Corollary 9.15 to show that (t,d) is A^{Σ_t}-full under mild assumptions on $\bar{\rho}$. Since A^{Σ_t} and A_0 are fullness peers in the constant-determinant setting, we conclude that our admissible pseudodeformation (t,d) is A_0-full.

Let us now establish some assumptions on our fixed residual representation $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Assume that $\bar{\rho}$ is regular and, after Section 9.1, absolutely irreducible. Whenever $\bar{\rho}$ is absolutely irreducible, assume further that $\det \bar{\rho}$ is a power of 2, which can always be achieved by twisting $\bar{\rho}$ by a character by Lemma 8.4. Furthermore, the twisting operation does not change the field $\mathbb{E}$ by Proposition 4.8. Assume that $[\mathbb{F} \colon \mathbb{E}]$ is a power of 2, which is possible by Lemma 8.8. Note that we do not require $\mathbb{F}$ to be the trace ring of $\bar{\rho}$ since one may need to make a quadratic extension of the trace ring in order to make representations well-adapted in the dihedral case.

9.1. The reducible case.

When $\bar{\rho}$ is reducible, we can use Corollary 6.6 to show that (t,d) is A^{Σ_t}- and A_0-full.

Proposition 9.1. *Suppose that $\bar{\rho} = \varepsilon \oplus \delta$ and that $\bar{\rho}$ is regular. If (t,d) is not a priori small then (t,d) is A^{Σ_t}- and A_0-full.*

Proof. Let $(t', d') = (s(\delta^{-1})t, s(\delta^{-1})^2 d)$, which is a pseudodeformation of $\bar{r} := \bar{\rho} \otimes \delta^{-1}$. Let $A_{t'}$ be the trace ring of (t', d'); its residue field $\mathbb{E}$ since $\bar{r}$ has no conjugate self-twists by Corollary 8.2. Then $(\Pi, \bar{r}, t', d')$ is an admissible pseudorepresentation over $A_{t'}$. Note that (t', d') is not a priori small since (t,d) is not. By Corollary 6.6, there is a well-adapted (t', d')-representation r such that (t', d') is $\mathcal{B}_r(\mathbb{E})$-full. But $A_{t'} = \mathcal{B}_r(\mathbb{E})$ by Theorem 2.23. Since $\bar{r}$ has no conjugate self-twists by Corollary 8.2, it follows that $\Sigma_{t'}$ is trivial. Thus $A_{t'}^{\Sigma_{t'}} = A_{t'} = \mathcal{B}_r(\mathbb{E})$.

By Corollary 3.12 it follows that (t,d) is $A_{t'}^{\Sigma_{t'}}$-full. We know that A^{Σ_t} and $A_{t'}^{\Sigma_{t'}}$ have the same fields of fractions, namely K_0, by Corollary 4.20. Furthermore A is obtained by adjoining the values of $s(\delta)$ and $W(\mathbb{F})$ to $A_{t'}$. Therefore A, and hence A^{Σ_t}, is finitely generated over A', so (t,d) is A^{Σ_t}-full by Lemma 3.5. Finally, A_0 and A^{Σ_t} are fullness peers in this setting by Corollary 4.21. $\square$

9.2. **Choosing a good (t,d)-representation.** Throughout Section 9.2 through Section 9.4 we fix an absolutely irreducible regular representation $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ such that the order of $\det \bar{\rho}$ is a power of 2. We assume that $[\mathbb{F}\colon \mathbb{E}]$ is a power of 2 by Lemma 8.8. If $\bar{\rho}$ is octahedral, we assume further that $\bar{\rho}$ is good. Furthermore, we fix a good basis for $\bar{\rho}$ as follows. If $\bar{\rho}$ is exceptional or large, choose a basis and a regular element $g_0 \in \Pi$ such that Proposition 8.13 holds. If $\bar{\rho} = \mathrm{Ind}_{\Pi_0}^{\Pi} \chi$ is dihedral, assume that $\bar{\rho}(\Pi_0)$ is diagonal and $\mathrm{Im}\,\bar{\rho}$ contains a matrix $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ such that $bc^{-1} \in \mathbb{F}_p$, which is possible by [Bel19, Proposition 6.3.2].

Recall that $(T,d)\colon \Pi \to \mathcal{A}$ is the universal constant-determinant pseudorepresentation. Part of our arguments will require appealing to a universal (T,d)-representation. This requires choosing a good (T,d)-representation ρ^{univ} and also choosing our (t,d)-representation to be compatible with ρ^{univ}. In particular, we want $I_1(\rho^{\mathrm{univ}})$ to be fixed by all conjugate self-twists of (T,d). In Section 9.2 we make these choices and compatibilities precise. Since we need Proposition 9.2 and Corollary 9.3 for the universal ring $\mathcal{A}$, in Section 9.2 we do not require A to be a domain, only a local pro-p ring.

Fix a generator σ_1 of $\Sigma_{\bar{\rho}} = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$. We want to choose a character $\eta_1\colon \Pi \to \mathbb{F}^\times$ such that $(\sigma_1, \eta_1) \in \widetilde{\Sigma}_{\bar{\rho}}$. There is a unique choice for η_1 when $\bar{\rho}$ is not dihedral. If $\bar{\rho}$ is dihedral and $\Sigma_t = 1$, choose η_1 to be the trivial character. Recall from the end of Section 7.2 that $\beta_t\colon \Sigma_t \to \Sigma_{\bar{\rho}}$ is given by reducing automorphisms of A modulo $\mathfrak{m}$. If $\bar{\rho}$ is dihedral and $\ker \beta_t = 1$ but $\Sigma_t \neq 1$, then there is a unique complement to $\Sigma_{\bar{\rho}}^{\mathrm{di}}$ in $\widetilde{\Sigma}_{\bar{\rho}}$ that contains $\tilde{\beta}_t(\widetilde{\Sigma}_t)$. Choose η_1 such that (σ_1, η_1) generates that complement. Otherwise, when $\bar{\rho}$ is dihedral, we may take η_1 to be either of the two characters such that $(\sigma_1, \eta_1) \in \widetilde{\Sigma}_{\bar{\rho}}$. Recall from (6) in Section 7.1 that $\Pi_0(\bar{\rho})$ is the intersection of the kernels of all twist characters of $\bar{\rho}$. Define

$$\Pi_1 \coloneqq \begin{cases} \ker \eta_1 & \text{if } \bar{\rho} \text{ is dihedral} \\ \Pi_0(\bar{\rho}) & \text{else.} \end{cases}$$

Let A_1 be the subring of A topologically generated by $t(\Pi_1)$. Note that $\bar{\rho}|_{\Pi_1}$ is absolutely irreducible by Lemma 8.5 and Corollary 8.7.

Proposition 9.2. *There exists a well-adapted (t,d)-representation $\rho\colon \Pi \to \mathrm{GL}_2(A)$ such that $\rho|_{\Pi_1}$ takes values in $\mathrm{GL}_2(A_1)$ and such that ρ is adapted to a regular element.*

Proof. With the exception of the well-adaptedness statement, the proof is well known since $\bar{\rho}|_{\Pi_1}$ is absolutely irreducible. Indeed, a theorem of Rouquier [Rou96, Theorem 5.1] and Nyssen [Nys96] tells us that there are representations $\rho\colon \Pi \to \mathrm{GL}_2(A)$ and $\rho_1\colon \Pi_1 \to \mathrm{GL}_2(A_1)$ such that $\operatorname{tr} \rho = t$ and $\operatorname{tr} \rho_1 = t|_{\Pi_1}$. By a theorem of Carayol and Serre, $\rho|_{\Pi_1}$ and ρ_1 are conjugate by a matrix in $\mathrm{GL}_2(A)$ [Car94, Théorème 1].

For the well-adaptedness statement, let us first assume that $\bar{\rho}$ is not dihedral. Choose ρ adapted to g_0 and ρ_1 adapted to g_0^n with g_0 and n as in Proposition 8.13. Then the matrix $M \in \mathrm{GL}_2(A)$ such that $M^{-1}\rho|_{\Pi_1} M = \rho_1$ commutes with $\rho(g_0^n) = \left(\begin{smallmatrix} s(\lambda_0^n) & 0 \\ 0 & s(\mu_0^n) \end{smallmatrix}\right) = \rho_1(g_0^n)$. Since $\lambda_0^n \neq \mu_0^n$ by Proposition 8.13, it follows that M must be diagonal. In particular, M commutes with $\rho(g_0)$. Hence $M^{-1}\rho M$ is still adapted to g_0 and satisfies the properties in the statement of the proposition.

The idea is similar when $\bar{\rho}$ is dihedral, except we can no longer assume that ρ_1 is adapted to the $g_0 \in \Pi_0$ such that $\bar{\rho}(g_0)$ generates the unique index-2 subgroup of the projective image of $\bar{\rho}$, because g_0 may not be in Π_1. Let ρ be a well-adapted (t,d)-representation, say adapted to g_0 with $\rho(g_0) = \left(\begin{smallmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{smallmatrix}\right)$.

Since $\bar{\rho}$ is regular, it follows that $\bar{\rho}|_{\Pi_0(\bar{\rho})}$ is multiplicity free over $\mathbb{E}$ by Proposition 8.6. Therefore, since ρ is well adapted, the image of ρ contains a matrix of the form $\left(\begin{smallmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{smallmatrix}\right)$ with $\lambda \neq \mu$ and $\lambda, \mu \in \mathbb{E}^\times$. Let $h \in \Pi$ such that $\rho(h) = \left(\begin{smallmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{smallmatrix}\right)$.

We claim that $h \in \Pi_1$. It suffices to prove that $h \in \Pi_0(\bar{\rho})$ since $\Pi_0(\bar{\rho}) = \Pi_0 \cap \ker \eta_1 \subset \ker \eta_1 = \Pi_1$. Note that $h \in \Pi_0$ as $\bar{\rho}(h)$ is diagonal. By Proposition 8.6, $h \in \Pi_0(\bar{\rho})$ if and only if the eigenvalues of $\bar{\rho}(h)$ are in $\mathbb{E}^\times$. But the eigenvalues λ, μ of $\bar{\rho}(h)$ were chosen to be in $\mathbb{E}^\times$. Therefore $h \in \Pi_1$.

By [Bel19, Proposition 2.4.2] we may assume that ρ_1 in the first paragraph of this proof is adapted to h. Therefore the matrix $M \in \mathrm{GL}_2(A)$ such that $M^{-1}\rho|_{\Pi_1} M = \rho_1$ commutes with $\rho(h) = \left(\begin{smallmatrix} s(\lambda) & 0 \\ 0 & s(\mu) \end{smallmatrix}\right) = \rho_1(h)$. Since $\lambda \neq \mu$, it follows that M is a diagonal matrix. Note that the second property in Definition 2.21 is unchanged by conjugation by a diagonal matrix. Therefore $M^{-1}\rho M$ is still well adapted and satisfies the statement of the proposition. □

We recall that when $\bar{\rho}$ is dihedral, we may view elements in B as elements of A (Remark 2.12).

Corollary 9.3. *There exists a well-adapted (t,d)-representation $\rho\colon \Pi \to R^\times$ such that $I_1(\rho) \subseteq A^{\Sigma_t}$. If $\bar{\rho}$ is dihedral and $\sigma \in \Sigma_t$ such that σ and $\ker \beta_t$ generate Σ_t, then we may assume furthermore that $B_1(\rho)$ is pointwise fixed by σ.*

Proof. Let ρ be the (t,d)-representation from Proposition 9.2. Since the order of $\det \bar{\rho}$ is a power of 2, it follows that $[\Pi\colon \Pi_0(\bar{\rho})]$ is a power of 2. Since Γ is pro-p and $p \neq 2$, it follows that $\Gamma \subseteq \rho(\Pi_0(\bar{\rho})) \subseteq \mathrm{GL}_2(A_1)$. Therefore $L_1(\rho) \subseteq \mathfrak{sl}_2(A_1)$, and so $I_1(\rho), B_1(\rho) \subseteq A_1$.

Let $(\sigma, \eta) \in \widetilde{\Sigma}_t$ such that $\Pi_1 \subseteq \ker \eta$. Then for all $g \in \Pi_1$ we have

$$^{\sigma}t(g) = \eta(g)t(g) = t(g),$$

and thus A_1 is contained in the subring of A fixed by σ.

If $\ker \beta_t = 1$, then every $(\sigma, \eta) \in \widetilde{\Sigma}_t$ satisfies $\Pi_1 \subseteq \ker \eta$ by definition of Π_1. Thus if $\ker \beta_t = 1$, then $A_1 \subseteq A^{\Sigma_t}$, and hence $I_1(\rho), B_1(\rho) \subseteq A^{\Sigma_t}$.

Now suppose that $\bar{\rho}$ is dihedral and $\ker \beta_t \neq 1$. Then half of the elements $(\sigma, \eta) \in \widetilde{\Sigma}_t$ satisfy $\ker \eta \subseteq \ker \eta_1 = \Pi_1$, namely all those in the preimage under β_t of the subgroup generated by (σ_1, η_1) in $\widetilde{\Sigma}_{\bar{\rho}}$. This proves the statement about $B_1(\rho)$ in the dihedral case. To see that $I_1(\rho)$ is fixed by all conjugate self-twists, it remains to show that $I_1(\rho)$ is fixed by the nontrivial element in $\ker \beta_t$. This follows from Proposition 7.5. □

In light of Corollary 9.3, let us fix a well-adapted (T,d)-representation $\rho^{\mathrm{univ}}\colon \Pi \to \mathrm{GL}_2(\mathcal{A})$ such that $I_1(\rho^{\mathrm{univ}}) \subseteq \mathcal{A}^{\Sigma_T}$. Assume furthermore in the case when the projective image of $\bar{\rho}$ is not dihedral that we have conjugated ρ^{univ} by the relevant diagonal element so that Theorem 2.23 applies to ρ^{univ}, and thus to any quotient of ρ^{univ}. Recall from the commutative diagram following Corollary 7.4 that we have a natural reduction map $\beta_T\colon \Sigma_T \to \Sigma_{\bar{\rho}}$ that sends an automorphism σ of $\mathcal{A}$ to the automorphism it induces on $\mathbb{F} = \mathcal{A}/\mathfrak{m}_{\mathcal{A}}$. In the case when $\bar{\rho}$ is dihedral, we need to choose a complement to $\ker \beta_T$ in Σ_T, whose generator we will denote by ν. We choose ν such that $(\nu, \eta_1) \in \widetilde{\Sigma}_T$, where η_1 is the character fixed prior to Proposition 9.2. By Corollary 9.3, we may and do assume that $B_1(\rho^{\mathrm{univ}})$ is fixed by ν.

The universal property of $(\mathcal{A}, (T,d))$ gives a surjective $W(\mathbb{F})$-algebra homomorphism $\alpha_t\colon \mathcal{A} \to A$. Let $\rho_t := \alpha_t \circ \rho^{\mathrm{univ}}\colon \Pi \to \mathrm{GL}_2(A)$. It is a (t,d)-representation such that $I_1(\rho_t) \subseteq A^{\Sigma_t}$ by the diagram following Corollary 7.4. Furthermore, if $\bar{\rho}$ is dihedral and $\ker \beta_t = 1$, then $B_1(\rho_t) \subseteq A^{\Sigma_t}$ as well. By the functoriality of Pink-Lie algebras with respect to quotient maps, we have that $\alpha_t(I_1(\rho^{\mathrm{univ}})) = I_1(\rho_t)$ and $\alpha_t(B_1(\rho^{\mathrm{univ}})) = B_1(\rho_t)$. All of our theorems below will be specifically for this well-chosen representation ρ_t. To ease notation, write $\rho = \rho_t$.

Recall that by Theorem 2.23

$$A = \begin{cases} \mathcal{B}_\rho(\mathbb{F}) + W(\mathbb{F})B_1(\rho) & \text{if } \bar\rho \text{ is dihedral} \\ \mathcal{B}_\rho(\mathbb{F}) & \text{else.} \end{cases}$$

By Proposition 7.5 and the fact that $B_1(\rho) \subseteq A^{\Sigma_t}$ if $\bar\rho$ is dihedral and $\ker\beta_t = 1$, it follows that

$$A^{\Sigma_t} = \begin{cases} \mathcal{B}_\rho(\mathbb{F}^{\beta_t(\Sigma_t)}) + W(\mathbb{F}^{\beta_t(\Sigma_t)})B_1(\rho) & \text{if } \bar\rho \text{ dihedral and } \ker\beta_t = 1 \\ \mathcal{B}_\rho(\mathbb{F}^{\beta_t(\Sigma_t)}) & \text{else.} \end{cases}$$

We therefore define

$$J = J(\rho) := \begin{cases} W(\mathbb{E})I_1(\rho) + W(\mathbb{E})I_1(\rho)^2 + W(\mathbb{E})B_1(\rho) & \text{if } \bar\rho \text{ is dihedral and } \ker\beta_t = 1 \\ W(\mathbb{E})I_1(\rho) + W(\mathbb{E})I_1(\rho)^2 & \text{else.} \end{cases}$$

We claim that $J \subset \mathfrak{m}$ is a multiplicatively closed $W(\mathbb{E})$-module by Theorem 2.23. The key is to note that, since $\bar\rho$ is regular and ρ is well adapted, it follows that Bellaïche's field $\mathbb{F}_q$ from p. 16 is contained in $\mathbb{E}$. Therefore it follows from Theorem 2.23 that $(W(\mathbb{E})I_1(\rho))^3 \subseteq W(\mathbb{E})I_1(\rho)$ and $W(\mathbb{E})I_1(\rho)B_1(\rho) \subseteq W(\mathbb{E})B_1(\rho)$ and $(W(\mathbb{E})B_1(\rho))^2 \subseteq W(\mathbb{E})I_1(\rho)$, which proves that J is multiplicatively closed. Define

$$\mathfrak{A} := W(\mathbb{F}) + W(\mathbb{F})J.$$

We have $\mathfrak{A} = A$ unless $1 \neq \ker\beta_t$, in which case $\mathfrak{A} = A^+$ by Proposition 7.5.

Remark 9.4. The rings $W(\mathbb{E}) + J$ and A^{Σ_t} differ only in their constants, $W(\mathbb{E})$ versus $W(\mathbb{F}^{\beta_t(\Sigma_t)})$. Furthermore, $W(\mathbb{E}) + J$ is often equal to $\mathcal{B}_\rho(\mathbb{E})$, and the goal of this section is to relate $\mathcal{B}_\rho(\mathbb{E})$ with A^{Σ_t}. Assume for a moment that $W(\mathbb{E}) + J = \mathcal{B}_\rho(\mathbb{E})$. Then the difference between $\mathcal{B}_\rho(\mathbb{E})$ and A^{Σ_t} is entirely governed by understanding which elements of $\Sigma_{\bar\rho}$ lift to elements in Σ_t under β_t. In particular, when there are elements in $\Sigma_{\bar\rho}$ that do not lift to Σ_t, we will be interested in writing the extra elements in $W(\mathbb{F}^{\beta_t(\Sigma_t)})$ as quotients of elements in J to show that $Q(\mathcal{B}_\rho(\mathbb{E})) = Q(A^{\Sigma_t})$. △

9.3. **Lifting conjugate self-twists to $\mathfrak{A}$.** In Section 9.3 we study a condition on J, called *smallness* (Definition 9.6), that dictates which conjugate self-twists of $\bar\rho$ lift to conjugate self-twists of (t, d). This study culminates in Theorem 9.8. As a consequence, we prove in Corollary 9.9 that under such a smallness assumption, $A^{\Sigma_t} = \mathcal{B}_\rho(\mathbb{E})$. The reader is advised that, with the exception of the motivational remark following Definition 9.6, the assumption that A is a domain is never used in Section 9.3.

Throughout Section 9.3, fix a subgroup $\Sigma \subseteq \Sigma_{\bar\rho}$, and let $\mathbb{F}' := \mathbb{F}^\Sigma$. Write $W := W(\mathbb{F})$ and $W' := W(\mathbb{F}')$. For an arbitrary ring $\mathcal{R}$ and a finite group X of ring automorphisms of $\mathcal{R}$, for any $\varphi \in \mathrm{Hom}(X, \mathcal{R}^\times)$, we write

$$\mathcal{R}^\varphi := \{s \in \mathcal{R} \colon {}^\sigma s = \varphi(\sigma)s, \forall \sigma \in X\}.$$

As explained at the beginning of Section 9, we assume that $[\mathbb{F}\colon \mathbb{E}]$ is a power of 2. By Lemma 8.9 we may apply Lemma A.22 to conclude that $\mathbb{F} = \oplus_{\varphi\in\Sigma^*}\mathbb{F}^\varphi$, where $\Sigma^* := \mathrm{Hom}(\Sigma, \mathbb{F}^\times)$. Note that since $\Sigma = \mathrm{Gal}(W/W')$, it follows that this decomposition lifts to W. More precisely, viewing elements of Σ as automorphisms of W and elements of Σ^* as valued in $W^\times$ by composing with the Teichmüller map, we can define W^φ for each $\varphi \in \Sigma^*$. Then Lemma A.22 gives $W = \oplus_{\varphi\in\Sigma^*}W^\varphi$.

For all $\varphi \in \Sigma$, define

$$\mathfrak{A}(\varphi) := W^\varphi + W^\varphi J,$$

where $W^\varphi J := \{\sum_i \alpha_i j_i | \alpha_i \in W^\varphi, j_i \in J\}$. Since $\mathfrak{A} = W + WJ$ it follows immediately that $\mathfrak{A} = \sum_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$. We will be interested in understanding when this sum is direct, because in that case we will show that it is possible to find lifts of elements of Σ in Σ_t. If $\mathfrak{a}$ is an ideal of $\mathfrak{A}$ and $\varphi \in \Sigma^*$, let $\mathfrak{a}(\varphi) := \mathfrak{A}(\varphi) \cap \mathfrak{a}$ and let $(\mathfrak{A}/\mathfrak{a})(\varphi) \subset \mathfrak{A}/\mathfrak{a}$ be the image of $\mathfrak{A}(\varphi)$ under the natural projection $\mathfrak{A} \to \mathfrak{A}/\mathfrak{a}$.

Lemma 9.5. *The following are equivalent:*

(1) $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$.

(2) For every $\mathfrak{A}$*-ideal* $\mathfrak{a}$ *such that* $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$*, we have* $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$*. Furthermore, there exists at least one such ideal* $\mathfrak{a}$*.*

(3) There exists an $\mathfrak{A}$*-ideal* $\mathfrak{a}$ *such that* $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$ *and* $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$*.*

Proof. First we show that (1) implies (2). We can take $\mathfrak{a} = 0$ for the existence statement in (2). Now suppose that $\mathfrak{a}$ is an $\mathfrak{A}$-ideal such that $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$. If $\sum_{\varphi\in\Sigma^*}\bar{a}_\varphi = 0 \in \mathfrak{A}/\mathfrak{a}$ with each $\bar{a}_\varphi \in (\mathfrak{A}/\mathfrak{a})(\varphi)$, then letting $a_\varphi \in \mathfrak{A}(\varphi)$ be a lift of $\bar{a}_\varphi$, we see that $\sum_{\varphi\in\Sigma^*} a_\varphi \in \mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$. Thus, there are $\alpha_\varphi \in \mathfrak{a}(\varphi)$ such that $\sum_{\varphi\in\Sigma^*} a_\varphi = \sum_{\varphi\in\Sigma^*}\alpha_\varphi$. Since $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$, it follows that $a_\varphi = \alpha_\varphi$ for all $\varphi \in \Sigma^*$. Thus $\bar{a}_\varphi = 0 \in \mathfrak{A}/\mathfrak{a}$ for all $\varphi \in \Sigma^*$ and hence $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$.

The fact that (2) implies (3) is trivial.

To see that (3) implies (1), suppose that $\mathfrak{a}$ is an $\mathfrak{A}$-ideal such that $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$ and $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$. For each $\varphi \in \Sigma^*$ fix a set $S_\varphi \subset \mathfrak{A}(\varphi)$ of representatives of $(\mathfrak{A}/\mathfrak{a})(\varphi)$ such that $0 \in S_\varphi$. Suppose that $\sum_{\varphi\in\Sigma^*} a_\varphi = 0$ with $a_\varphi \in \mathfrak{A}(\varphi)$. Then there is a unique way to write each a_φ as

$$a_\varphi = s_\varphi + \alpha_\varphi$$

with $s_\varphi \in S_\varphi$ and $\alpha_\varphi \in \mathfrak{a}(\varphi)$. Modulo $\mathfrak{a}$, we see that

$$\sum_{\varphi\in\Sigma^*} \bar{s}_\varphi = 0.$$

Since $\mathfrak{A}/\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/\mathfrak{a})(\varphi)$, it follows that $\bar{s}_\varphi = 0$ for all φ. As $0 \in S_\varphi$, it follows that $s_\varphi = 0$ for all $\varphi \in \Sigma^*$. Therefore $a_\varphi = \alpha_\varphi \in \mathfrak{a}(\varphi)$. As $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$, it follows that each $a_\varphi = 0$. Thus $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$. ☐

Definition 9.6. Let $\mathbb{L}_2 \subset \mathbb{L}_1$ be subfields of $\mathbb{F}$. We say that J is *small with respect to* $\mathbb{L}_1/\mathbb{L}_2$ if

$$\ker(W(\mathbb{L}_1) \otimes_{W(\mathbb{L}_2)} W(\mathbb{L}_2)J \to W(\mathbb{L}_1)J) = 0,$$

where the map is multiplication inside $\mathfrak{A}$. Otherwise, we say that J is *big with respect to* $\mathbb{L}_1/\mathbb{L}_2$.

To motivate Definition 9.6, recall from Remark 9.4 that we need to be able to write elements of $W(\mathbb{F}^{\beta_t(\Sigma_t)})$ as quotients of elements in J whenever $\mathbb{F}^{\beta_t(\Sigma_t)} \neq \mathbb{E}$. Suppose that $\mathbb{L}_2 = \mathbb{E}, \mathbb{L}_1 = \mathbb{F}^{\beta_t(\Sigma_t)}$, and $[\mathbb{L}_1 \colon \mathbb{L}_2] = 2$. Write $\mathbb{L}_1 = \mathbb{L}_2(\alpha)$. Then $W(\mathbb{L}_1) = W(\mathbb{L}_2) \oplus s(\alpha)W(\mathbb{L}_2)$ and so

$$W(\mathbb{L}_1) \otimes_{W(\mathbb{L}_2)} W(\mathbb{L}_2)J = W(\mathbb{L}_2)J \oplus (s(\alpha)W(\mathbb{L}_2) \otimes_{W(\mathbb{L}_2)} W(\mathbb{L}_2)J).$$

If J is big with respect to $\mathbb{L}_1/\mathbb{L}_2$, then we can find $x, y \in W(\mathbb{L}_2)J \setminus \{0\}$ such that $x + s(\alpha)y = 0$. Thus $s(\alpha) = x/y$, and hence $W(\mathbb{L}_1)$ is in the field of fractions of any domain containing $W(\mathbb{L}_2)J$. In contrast, the following proposition shows that when J is small with respect to $\mathbb{F}/\mathbb{F}'$, elements of Σ can be lifted to automorphisms of $\mathfrak{A}$.

Proposition 9.7. *If* J *is small with respect to* $\mathbb{F}/\mathbb{F}'$*, then* $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$*. In this case, every* $\bar{\sigma} \in \Sigma$ *can be lifted to an automorphism* σ *of* $\mathfrak{A}$ *such that* σ *acts trivially on* J*, and a lift with this property is unique.*

Proof. Note that $\mathfrak{a} := WJ$ is an $\mathfrak{A}$-ideal since $\mathfrak{A} = W + WJ$ and J is multiplicatively closed as discussed prior to Remark 9.4. The assumption that J is small with respect to $\mathbb{F}/\mathbb{F}'$ implies that $WJ = \oplus_{\varphi\in\Sigma^*} W^\varphi J$. Indeed,

$$\bigoplus_{\varphi\in\Sigma^*}(W^\varphi \otimes_{W'} W'J) = \Big(\bigoplus_{\varphi\in\Sigma^*} W^\varphi\Big) \otimes_{W'} W'J = W \otimes_{W'} W'J \hookrightarrow WJ,$$

and the image of $W^\varphi \otimes_{W'} W'J$ is exactly $W^\varphi J$. Since $WJ = \sum_{\varphi\in\Sigma^*} W^\varphi J$, it follows that the multiplication map is an isomorphism and thus WJ is graded by Σ^*. Note that $\mathfrak{a}(\varphi) = W^\varphi J$, so $\mathfrak{a} = \oplus_{\varphi\in\Sigma^*}\mathfrak{a}(\varphi)$.

By Lemma 9.5, for the first statement of the proposition it suffices to show that $\mathfrak{A}/WJ = \oplus_{\varphi\in\Sigma^*}(\mathfrak{A}/WJ)(\varphi)$. Note that

$$\mathfrak{A}/WJ = (W + WJ)/WJ \cong W/(W \cap WJ)$$

and $W \cap WJ$ is a closed W-submodule of pW since $J \subseteq \mathfrak{m}_{\mathfrak{A}}$. Thus we have $W \cap WJ = p^n W$ and $\mathfrak{A}/WJ \cong W/p^n W$ for some $1 \le n \le \infty$, where $p^\infty W := \{\infty\}$. Since W is graded by Σ^*, it follows from Lemma 9.5 that $W/p^n W$ is graded by Σ^* as well. Therefore $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$.

For the second statement, we let σ act by $W(\bar\sigma)$ on W and trivially on J. The only question is to verify that this is well defined. Since $\mathfrak{A} = \oplus_{\varphi\in\Sigma^*}\mathfrak{A}(\varphi)$, it suffices to show that σ is well defined on each $\mathfrak{A}(\varphi)$. That is, we must show

$$\sum_{i=1}^n \alpha_i j_i = 0 \implies \sum_{i=1}^n W(\bar\sigma)(\alpha_i) j_i = 0,$$

where $\alpha_i \in W^\varphi, j_i \in J$. Since J is small with respect to $\mathbb{F}/\mathbb{F}'$, $\sum_{i=1}^n \alpha_i j_i = 0$ implies that $\sum_{i=1}^n \alpha_i \otimes j_i = 0 \in W \otimes_{W'} W'J$. Since $\alpha_i \in W^\varphi$, we know that $W(\bar\sigma)(\alpha_i) = s(\varphi(\bar\sigma))\alpha_i$ for all i. Hence

$$0 = \sum_{i=1}^n \alpha_i \otimes j_i \implies 0 = s(\varphi(\bar\sigma)) \sum_{i=1}^n \alpha_i \otimes j_i.$$

Therefore $\sum_{i=1}^n W(\bar\sigma)(\alpha_i) j_i = 0$, since it is the image of $s(\varphi(\bar\sigma)) \sum_{i=1}^n \alpha_i \otimes j_i$ under $W \otimes_{W'} W'J \to WJ$. $\square$

Now that we have lifted elements of Σ to automorphisms of $\mathfrak{A}$ under the smallness assumption, we would like to verify that the lifts are conjugate self-twists of (t, d) when $\mathfrak{A} = A$ and that they come from conjugate self-twists when $\mathfrak{A} = A^+$. (Recall that A^+ is only defined when $\ker \beta_t$ is nontrivial; see Proposition 7.5.)

Theorem 9.8. *If J is small with respect to $\mathbb{F}/\mathbb{F}'$ then Σ is contained in the image of $\beta_t \colon \Sigma_t \to \Sigma_{\bar\rho}$. Furthermore, every lift of $\bar\sigma \in \Sigma$ to Σ_t acts trivially on J.*

Proof. Fix $\bar\sigma \in \Sigma$. By Proposition 9.7, there is a unique $\sigma \in \operatorname{Aut}\mathfrak{A}$ that acts as $W(\bar\sigma)$ on W and fixes J. If $\ker \beta_t = 1$, then $\mathfrak{A} = A$. If $\ker \beta_t \neq 1$, then $\mathfrak{A} = A^+$ and we need to extend σ to $A = A^+ \oplus A^-$. We do this by declaring that σ fixes A^-; we will still denote the automorphism of A by σ. We already know that σ acts trivially on J, so it is enough to prove that $\sigma \in \Sigma_t$.

Our strategy is to show that σ comes from an appropriate element of Σ_T. More precisely, we claim that there is some $(\tilde\sigma, \eta) \in \widetilde{\Sigma}_T$ such that $\sigma \circ \alpha_t = \alpha_t \circ \tilde\sigma$, where $\alpha_t \colon \mathcal{A} \to A$ is the W-algebra homomorphism given by universality. If this is true, then for all $g \in \Pi$ we have

$${}^\sigma t(g) = \sigma \circ \alpha_t(T(g)) = \alpha_t \circ \tilde\sigma(T(g)) = \alpha_t(\eta(g)T(g)) = \eta(g)\alpha_t(T(g)) = \eta(g)t(g)$$

since α_t is a W-algebra homomorphism and $\eta(g) \in W$. Thus $\sigma \in \Sigma_t$.

First suppose that $\bar\rho$ is not dihedral. Then there is a unique lift $\tilde\sigma$ of $\bar\sigma$ in Σ_T by Lemma 7.1 and Proposition 7.3. By Proposition 7.3, we know that $\tilde\sigma$ acts as $W(\bar\sigma)$ on the image of W in $\mathcal{A}$. Furthermore, $\tilde\sigma$ acts trivially on $I_1(\rho^{\mathrm{univ}})$ by our fixed choice of ρ^{univ} after Corollary 9.3. Since $\bar\rho$ is not dihedral, it follows that $\mathcal{A} = W + WJ(\rho^{\mathrm{univ}})$. By the construction of ρ^{univ} and ρ_t, we have $\alpha_t(I_1(\rho^{\mathrm{univ}})) = I_1(\rho)$ and thus $\alpha_t(J(\rho^{\mathrm{univ}})) = J(\rho) = J$. Recall that σ acts trivially on J. Both $\tilde\sigma$ and σ act on W by $W(\bar\sigma)$. Thus for any $\sum_{i=1}^n a_i x_i$ with $a_i \in W, x_i \in J(\rho^{\mathrm{univ}}) \cup \{1\}$, we have

$$\sigma \circ \alpha_t\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n W(\bar\sigma)(a_i)\alpha_t(x_i) = \alpha_t \circ \tilde\sigma\left(\sum_{i=1}^n a_i x_i\right).$$

If $\bar\rho$ is dihedral, then there are two lifts of $\bar\sigma$ in Σ_T by Lemma 7.1 and Proposition 7.3. One acts on $\mathcal{A}^-$ by $+1$ and the other acts by -1 by Proposition 7.5 and since we chose ρ^{univ} such that $B_1(\rho^{\mathrm{univ}})$ is fixed by ν, which generates a complement of $\ker \beta_T$. Let $\tilde\sigma \in \Sigma_T$ be the lift of $\bar\sigma$ that is in $\langle \nu \rangle$. Thus $\tilde{\sigma}$ acts trivially on $J(\rho^{\mathrm{univ}})$ and $B_1(\rho^{\mathrm{univ}})$. Then an argument similar to that in the previous paragraph shows that $\sigma \circ \alpha_t = \alpha_t \circ \tilde{\sigma}$. $\square$

Corollary 9.9. *If J is small with respect to $\mathbb{F}/\mathbb{E}$ then $A^{\Sigma_t} = W(\mathbb{E}) + J$. Suppose furthermore that either $\bar{\rho}$ is not dihedral or $\ker \beta_t \neq 1$. Then $A^{\Sigma_t} = \mathcal{B}_\rho(\mathbb{E})$.*

Proof. By Theorem 9.8 applied to $\Sigma = \Sigma_t$, the map β_t is a surjection and Σ_t acts trivially on J. If $\ker \beta_t = 1$, then $A = \mathfrak{A} = W + WJ$, so $A^{\Sigma_t} = W(\mathbb{E}) + J$.

If $\ker \beta_t \neq 1$, then $A = W + WI_1(\rho) + WI_1(\rho)^2 + WB_1(\rho)$ and $J = W(\mathbb{E})I_1(\rho) + W(\mathbb{E})I_1(\rho)^2$. Note that $A^{\Sigma_t} \subseteq A^+ = W + WJ$ since the nontrivial element in $\ker \beta_t$ acts by -1 on $B_1(\rho)$ by Proposition 7.5. As above, we have that

$$A^{\Sigma_t} = (W + WJ)^{\Sigma_t} = W(\mathbb{E}) + J.$$

The last sentence in the statement of the corollary follows from the definition of J. $\square$

Remark 9.10. Note that none of the arguments in Section 9.3 require that A is a domain. In particular, when J is small with respect to $\mathbb{F}/\mathbb{E}$ and either $\bar{\rho}$ is not dihedral or $\ker \beta_t \neq 1$, Corollary 9.9 gives a conceptual interpretation of the ring $\mathcal{B}_\rho(\mathbb{E})$. $\triangle$

9.4. **When J is big with respect to $\mathbb{F}/\mathbb{E}$.** Corollary 9.9 requires the assumption that J is small with respect to $\mathbb{F}/\mathbb{E}$. We do not always expect this to be true. The purpose of Section 9.4 is to show that A^{Σ_t} and $\mathcal{B}_\rho(\mathbb{E})$ have the same fraction field and A^{Σ_t} is a finite type $\mathcal{B}_\rho(\mathbb{E})$-module even without the smallness assumption. This is done in Corollary 9.15, although the two key inputs to that theorem are Proposition 9.13 and Proposition 9.14. Then we can apply Lemma 3.5 and Corollary 6.6 to conclude that ρ_t, and thus (t, d), is A^{Σ_t}-full in Corollary 9.16.

The discussion following Definition 9.6 shows why one may expect to get $Q(A^{\Sigma_t}) = Q(\mathcal{B}_\rho(\mathbb{E}))$ when smallness fails and $[\mathbb{F}^{\beta_t(\Sigma_t)} \colon \mathbb{E}] = 2$. Unfortunately, the assumption that $[\mathbb{F}^{\beta_t(\Sigma_t)} \colon \mathbb{E}] = 2$ is rather critical to that argument. This is the primary reason we insist that $[\mathbb{F} \colon \mathbb{E}]$ be a power of 2 throughout this section. It allows us to split up the extension $\mathbb{F}^{\beta_t(\Sigma_t)}/\mathbb{E}$ into a series of quadratic extensions, and thus we can apply the argument following Definition 9.6 inductively. This is the essential idea of the argument; we now prepare some notation to formalize it.

Write $[\mathbb{F}^{\beta_t(\Sigma_t)} \colon \mathbb{E}] = 2^n$ for some $n \geq 0$. For integers $0 \leq i \leq n$, let $\mathbb{E}_i$ be the unique extension of $\mathbb{E}$ of degree 2^i. In particular, $\mathbb{E}_0 = \mathbb{E}$ and $\mathbb{E}_n = \mathbb{F}^{\beta_t(\Sigma_t)}$, and $[\mathbb{E}_i \colon \mathbb{E}_{i-1}] = 2$ for all $1 \leq i \leq n$. For $0 \leq i \leq n$, let W_i denote the image of $W(\mathbb{E}_i)$ in $\mathfrak{A}$. Define

$$\mathfrak{A}_i := W_i + W_i J \subseteq \mathfrak{A}.$$

In particular, $\mathfrak{A}_0 = W(\mathbb{E}) + J$ and $\mathfrak{A}_n = W(\mathbb{F}^{\beta_t(\Sigma_t)}) + W(\mathbb{F}^{\beta_t(\Sigma_t)})J$. Since A is a domain, so are all of the $\mathfrak{A}_i$, and we write $Q(\mathfrak{A}_i)$ for the field of fractions of $\mathfrak{A}_i$.

In the case when $\mathfrak{A} = A^+$, there is a 2-to-1 group homomorphism $\Sigma_t \to \operatorname{Aut} A^+$ given by restricting elements of Σ_t to A^+. Let $\Sigma_t(\mathfrak{A})$ denote the image of this map when $\mathfrak{A} = A^+$, and otherwise (that is, whenever $\ker \beta_t = 1$) let $\Sigma_t(\mathfrak{A}) = \Sigma_t$. In either case we can identify $\Sigma_t(\mathfrak{A})$ with a subgroup of $\Sigma_{\bar{\rho}}$ via β_t, and we have $\mathbb{E}_n = \mathbb{F}^{\beta_t(\Sigma_t(\mathfrak{A}))}$. We write $\Sigma_t(\mathfrak{A})^* := \operatorname{Hom}(\Sigma_t(\mathfrak{A}), \mathfrak{A}^\times)$.

We begin with two preliminary lemmas about the relationship between smallness and the $\mathbb{E}_i$.

Lemma 9.11. *We have that J is small with respect to $\mathbb{F}/\mathbb{E}_n$; that is,* $\ker(W \otimes_{W_n} W_n J \to WJ) = 0$.

Proof. Recall that WJ is an $\mathfrak{A}$-ideal that is stable under the action of $\Sigma_t(\mathfrak{A})$ since Σ_t fixes J by the construction of ρ_t. By Lemma 8.9, we can apply Lemma A.22 with $X = \Sigma_t(\mathfrak{A})$. Therefore

$$WJ = \bigoplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} (WJ)^\varphi,$$

where $(WJ)^\varphi := \{x \in WJ \colon {}^\sigma x = \varphi(\sigma)x, \forall \sigma \in \Sigma_t(\mathfrak{A})\}$. Recall that $W^\varphi J$ was defined prior to Lemma 9.5. We claim that

$$(WJ)^\varphi = W^\varphi J.$$

Clearly $(WJ)^\varphi \supseteq W^\varphi J$ since $\Sigma_t(\mathfrak{A})$ acts trivially on J. On the other hand,

$$WJ = \bigoplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} (WJ)^\varphi = \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} (WJ)^\varphi \supseteq \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} W^\varphi J = WJ,$$

so we must have equality.

For each $\varphi \in \Sigma_t(\mathfrak{A})^*$, choose $x_\varphi \in \mathbb{F}^\varphi \setminus \{0\}$. Then $\{s(x_\varphi) \colon \varphi \in \Sigma_t(\mathfrak{A})^*\}$ is a W_n-basis for W. Thus we have

$$W \otimes_{W_n} W_n J = \bigoplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} W_n s(x_\varphi) \otimes_{W_n} W_n J.$$

If $x \in \ker(W \otimes_{W_n} W_n J \to WJ)$, then we can write

$$x = \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} s(x_\varphi) \otimes y_\varphi$$

for some $y_\varphi \in W_n J$. Then we have

$$0 = \sum_{\varphi \in \Sigma_t(\mathfrak{A})^*} s(x_\varphi) y_\varphi$$

and $s(x_\varphi) y_\varphi \in W^\varphi J$. Since $WJ = \oplus_{\varphi \in \Sigma_t(\mathfrak{A})^*} W^\varphi J$, it follows that each $s(x_\varphi) y_\varphi = 0$. As $\mathfrak{A}$ is a domain and $s(x_\varphi) \neq 0$, it follows that $y_\varphi = 0$ for all $\varphi \in \Sigma_t(\mathfrak{A})^*$. ☐

Lemma 9.12. *We have*

$$\ker(W \otimes_{W_{n-1}} W_{n-1} J \to WJ) = W \otimes_{W_n} \ker(W_n \otimes_{W_{n-1}} W_{n-1} J \to W_n J).$$

Proof. Let $K \coloneqq \ker(W_n \otimes_{W_{n-1}} W_{n-1} J \to W_n J)$. We have an exact sequence of W_n-modules

$$0 \to K \to W_n \otimes_{W_{n-1}} W_{n-1} J \to W_n J \to 0.$$

Since W is free over W_n, tensoring with W over W_n gives an exact sequence

$$0 \to W \otimes_{W_n} K \to W \otimes_{W_{n-1}} W_{n-1} J \to W \otimes_{W_n} W_n J \to 0.$$

We can identify the last nonzero term in this sequence with WJ by Lemma 9.11.
Thus $W \otimes_{W_n} K = \ker(W \otimes_{W_{n-1}} W_{n-1} J \to WJ)$. ☐

Proposition 9.13. *If J is big with respect to $\mathbb{F}/\mathbb{E}$ then $Q(\mathfrak{A}_n) = Q(\mathfrak{A}_{n-1})$.*

Proof. We claim that J is big with respect to $\mathbb{E}_n/\mathbb{E}_{n-1}$. Indeed, if J were small with respect to $\mathbb{E}_n/\mathbb{E}_{n-1}$, then J would be small with respect to $\mathbb{F}/\mathbb{E}_{n-1}$ by Lemma 9.12. Therefore we could apply Theorem 9.8 with $\Sigma = \mathrm{Gal}(\mathbb{F}/\mathbb{E}_{n-1})$, which implies that $\mathbb{E}_n \subseteq \mathbb{E}_{n-1}$, a contradiction.

Let $\{1, \alpha\}$ be an $\mathbb{E}_{n-1}$-basis for $\mathbb{E}_n$. Then $\{1, s(\alpha)\}$ is a W_{n-1}-basis for W_n and so

$$W_n \otimes_{W_{n-1}} W_{n-1} J = (W_{n-1} \otimes_{W_{n-1}} W_{n-1} J) \oplus (W_{n-1} s(\alpha) \otimes_{W_{n-1}} W_{n-1} J).$$

Since J is big with respect to $\mathbb{E}_n/\mathbb{E}_{n-1}$, there exist $x, y \in W_{n-1} J \setminus \{0\}$ such that

$$x + s(\alpha) y = 0.$$

Thus, $s(\alpha) = -x/y \in Q(\mathfrak{A}_{n-1})$. It follows that $W_n \subset Q(\mathfrak{A}_{n-1})$ and hence $Q(\mathfrak{A}_n) = Q(\mathfrak{A}_{n-1})$. ☐

Finally, we descend from $Q(\mathfrak{A}_n)$ to $Q(\mathfrak{A}_0)$ by induction on n.

Proposition 9.14. *For all $2 \leq k \leq n$, if $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$ then $Q(\mathfrak{A}_{k-1}) = Q(\mathfrak{A}_{k-2})$. In particular, if J is big with respect to $\mathbb{F}/\mathbb{E}$, then $Q(A^{\Sigma_t}) = Q(W(\mathbb{E}) + J)$.*

Proof. Note that for any $k \geq 1$ we have $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$ if and only if $W_k \subseteq Q(\mathfrak{A}_{k-1})$. Assume that $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$ for some k, $2 \leq k \leq n$. Choose $\bar{\alpha} \in \mathbb{E}_{k-2}, \bar{\beta} \in \mathbb{E}_{k-1}$ such that $\mathbb{E}_{k-1} = \mathbb{E}_{k-2}(\sqrt{\bar{\alpha}})$ and $\mathbb{E}_k = \mathbb{E}_{k-1}(\sqrt{\bar{\beta}})$. Define $\alpha := s(\bar{\alpha})$ and $\beta := s(\bar{\beta})$, so $W_{k-1} = W_{k-2}(\sqrt{\alpha})$ and $W_k = W_{k-1}(\sqrt{\beta})$. It suffices to show that $\sqrt{\alpha} \in Q(\mathfrak{A}_{k-2})$.

Since $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-1})$, we can write $\sqrt{\beta} = x/y$ with $x, y \in \mathfrak{A}_{k-1} \setminus \{0\}$. By multiplying x and y by any nonzero element of $W_{k-1}J$, we may assume that $x, y \in W_{k-1}J \setminus \{0\}$.

Note that we can write $y = i_1 + \sqrt{\alpha} i_2$ with $i_1, i_2 \in W_{k-2}J$. If $i_1 - \sqrt{\alpha} i_2 \neq 0$, then by multiplying x and y by $i_1 - \sqrt{\alpha} i_2$, we may assume that $y \in W_{k-2}J \setminus \{0\}$. If $i_1 - \sqrt{\alpha} i_2 = 0$ and $y \notin W_{k-2}J$, then we must have $i_2 \neq 0$ since $y \neq 0$ and $\sqrt{\alpha} = i_1/i_2 \in Q(\mathfrak{A}_{k-2})$, as desired. We assume henceforth that $y \in W_{k-2}J$.

Write $x = a + b\sqrt{\alpha}$ for some $a, b \in W_{k-2}J$. Then we have $y\sqrt{\beta} = a + b\sqrt{\alpha}$ and thus

$$y^2\beta = a^2 + \alpha b^2 + 2ab\sqrt{\alpha}. \tag{7}$$

Since $\beta \in W_{k-1}$, we may write $\beta = e + f\sqrt{\alpha}$ for some $e, f \in W_{k-2}$. Note that $f \not\equiv 0 \bmod p$ since $[\mathbb{E}_k \colon \mathbb{E}_{k-1}] = 2$ and $\mathbb{E}_k = \mathbb{E}_{k-1}(\sqrt{\bar{\beta}})$. Substituting this into equation (7), we see that

$$(y^2 f - 2ab)\sqrt{\alpha} = a^2 + \alpha b^2 - y^2 e \in W_{k-2}J.$$

Note that $y^2 f - 2ab \in W_{k-2}J$ since all of $y, f, a, b \in W_{k-2}J$. If $y^2 f - 2ab \neq 0$, then we can conclude that $\sqrt{\alpha} \in Q(\mathfrak{A}_{k-2})$ as desired.

Henceforth, assume that $y^2 f = 2ab$. Then we also have $y^2 e = a^2 + \alpha b^2$. Thus $2f^{-1}eab = a^2 + \alpha b^2$. Note that $a, b \neq 0$ since $2ab = y^2 f$ and we know $y, f \neq 0$. Then we have

$$2f^{-1}e = \frac{a}{b} + \alpha\frac{b}{a}.$$

Therefore $\frac{a}{b}$ is a root of $t^2 - 2f^{-1}et + \alpha \in W_{k-2}[t]$. The discriminant of this polynomial is $4(f^{-2}e^2 - \alpha)$. We claim that $W_{k-1} \supseteq W_{k-2}(\sqrt{f^{-2}e^2 - \alpha})$. Note that since $\beta = s(\bar{\beta})$ and s is multiplicative we have that $e^2 - f^2\alpha = s(N_{\mathbb{E}_{k-1}/\mathbb{E}_{k-2}}(\bar{\beta}))$, and therefore $\sqrt{e^2 - f^2\alpha} = s(\sqrt{N_{\mathbb{E}_{k-1}/\mathbb{E}_{k-2}}(\bar{\beta})}) \in s(\mathbb{E}_{k-1}^\times)$. Therefore $\frac{a}{b} \in W_{k-1}$. Thus we can write $\sqrt{\beta} = \frac{x/b}{y/b} = \frac{\frac{a}{b} + \sqrt{\alpha}}{y/b}$, and so

$$\frac{y}{b} = \left(\frac{a}{b} + \sqrt{\alpha}\right)\sqrt{\beta}^{-1}.$$

Note that $\frac{a}{b} + \sqrt{\alpha} \neq 0$ since $y \neq 0$. It follows that $(\frac{a}{b} + \sqrt{\alpha})\sqrt{\beta}^{-1}$ generates W_k over W_{k-1} since $\frac{a}{b} + \sqrt{\alpha} \in W_{k-1}$ and $\sqrt{\beta}$ generates W_k over W_{k-1}. Thus $(\frac{a}{b} + \sqrt{\alpha})\sqrt{\beta}^{-1} = \frac{y}{b} \in Q(\mathfrak{A}_{k-2})$ and so $W_k \subset Q(\mathfrak{A}_{k-2})$. Therefore $Q(\mathfrak{A}_k) = Q(\mathfrak{A}_{k-2})$.

For the second statement of the proposition, note that by Proposition 9.13 we have $Q(\mathfrak{A}_n) = Q(\mathfrak{A}_0)$ for all $0 \leq k \leq n$. We have $\mathfrak{A}_0 = W(\mathbb{E}) + J$ by definition. Since Σ_t acts trivially on J, it follows that

$$A^{\Sigma_t} = \mathfrak{A}^{\Sigma_t(\mathfrak{A})} = W_n + W_n J = \mathfrak{A}_n. \qquad \square$$

Corollary 9.15. *We have*

(1) $A^{\Sigma_t} \supseteq \mathcal{B}_\rho(\mathbb{E})$;

(2) A^{Σ_t} is a finitely generated $\mathcal{B}_\rho(\mathbb{E})$-module;

(3) A^{Σ_t} has the same field of fractions as $\mathcal{B}_\rho(\mathbb{E})$.

In particular, A^{Σ_t} and $\mathcal{B}_\rho(\mathbb{E})$ are fullness peers.

Proof. The definition of $\mathbb{E}$ and Corollary 9.3 imply (1).

For (2), if either $\bar{\rho}$ is not dihedral or $\ker \beta_t \neq 1$, then $A^{\Sigma_t} = \mathcal{B}_\rho(\mathbb{F}^{\beta_t(\Sigma_t)})$, which is finitely generated over $\mathcal{B}_\rho(\mathbb{E})$ since $W(\mathbb{F})^{\Sigma_t}$ is finitely generated over $W(\mathbb{E})$. In the case when $\bar{\rho}$ is dihedral and $\ker \beta_t = 1$, recall that $\mathcal{B}_\rho(\mathbb{E})$ is a noetherian ring (Proposition 7.6(1)) and $A = \mathcal{B}_\rho(\mathbb{F}) + W(\mathbb{F})B_1(\rho)$

is a noetherian $\mathcal{B}_\rho(\mathbb{E})$-module by Proposition 7.6(2) and the fact that $W(\mathbb{F})$ is noetherian over $W(\mathbb{E})$. Thus the $\mathcal{B}_\rho(\mathbb{E})$-submodule A^{Σ_t} of A is necessarily noetherian and hence finitely generated.

The third point has largely been established already. When J is small with respect to $\mathbb{F}/\mathbb{E}$, it follows from Corollary 9.9. When J is big with respect to $\mathbb{F}/\mathbb{E}$ and either $\bar{\rho}$ is not dihedral or $\ker \beta_t \neq 1$, this follows from Proposition 9.14 since in those cases $W(\mathbb{E}) + J = \mathcal{B}_\rho(\mathbb{E})$. Finally, when $\bar{\rho}$ is dihedral, $\ker \beta_t = 1$, and J is big with respect to $\mathbb{F}/\mathbb{E}$ we have

$$Q(A^{\Sigma_t}) = Q(W(\mathbb{E}) + J) = Q(\mathcal{B}_\rho(\mathbb{E})),$$

where the first equality follows from Proposition 9.14 and they second from Proposition 7.6(3).

The final statement now follows from Lemma 3.5. $\square$

We now have the following corollary, which summarizes the most general theorem we have for images of admissible pseudodeformations with 2-power determinant.

Corollary 9.16. *Let $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a regular representation such that the order of $\det \bar{\rho}$ is a power of 2. If $\bar{\rho}$ is octahedral, assume furthermore that $\bar{\rho}$ is good. Let A be a domain and $(t,d)\colon \Pi \to A$ an admissible pseudodeformation of $\bar{\rho}$. If (t,d) is not a priori small, then (t,d) is A^{Σ_t}-full, hence A_0-full.*

Proof. By Corollary 6.6, ρ_t is $B_\rho(\mathbb{E})$-full. Corollary 9.15 implies that ρ_t is A^{Σ_t}-full, hence A_0-full by Corollary 4.21. $\square$

10. Main fullness results

In Section 10 we draw conclusions from Corollary 9.16 that are useful in applications and when comparing our work with previous results in the literature. Although the constant determinant assumption is important to be able to use Bellaïche's work in Section 9, in practice one rarely works in a constant-determinant setting. Here we give the most general fullness result we can prove; in particular, we remove the constant-determinant assumption present in Corollary 9.16. We then recast our main theorem and other highlights of the theory of fullness in the language of representation theory rather than pseudorepresentations.

To ensure that our main result can be read independent of much of the rest of the paper, we briefly recall our notation and terminology. Let p be an odd prime, A a local pro-p domain with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$, and Π a p-finite profinite group (Definition 2.6). We are interested in a continuous pseudodeformation $(t,d)\colon \Pi \to A$ of a semisimple representation $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$. Unlike much of the paper, in Section 10 we never require A to be the trace algebra of (t,d). We say (t,d) is not a priori small if it is not reducible, dihedral, or equal to a twist of its Teichmüller lift (Definition 2.3), and it is A_0-full if the image of some representation carrying (t,d) contains, up to conjugation, an A_0-congruence subgroup (Definition 3.1).

Recall that A_0 is the adjoint trace ring of (t,d) (Definition 4.6); its residue field is the trace field of $\bar{\rho}$ and is denoted $\mathbb{E}$. We say that $\bar{\rho}$ is *regular* if its image contains a matrix whose eigenvalue ratio is in $\mathbb{E}^\times \setminus \{\pm 1\}$: see Definition 2.19 and Remark 2.20 immediately following. When $\bar{\rho}$ is octahedral (that is, projective image S_4), see Definition 8.10 for the definition of *goodness*.

Theorem 10.1. *Let $p > 2$ be prime, A a local pro-p domain with residue field $\mathbb{F}$, and Π be a p-finite profinite group. Let $\bar{\rho}\colon \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a regular semisimple representation that is good if $\bar{\rho}$ is octahedral. If $(t,d)\colon \Pi \to A$ is a pseudodeformation of $\bar{\rho}$ that is not a priori small, then (t,d) is A_0-full.*

Proof. Let $\chi\colon \Pi \to A^\times$ be a character such that $(t', d') := (\chi t, \chi^2 d)$ is a constant-determinant pseudorepresentation, and write $\bar{\rho}' := \bar{\chi} \otimes \bar{\rho}$, where $\bar{\chi}\colon \Pi \to \mathbb{F}^\times$ is the reduction of χ modulo $\mathfrak{m}$. Assume that χ is chosen such that $\bar{\rho}'$ has no conjugate self-twists if $\bar{\rho}$ is reducible and the order of $\det \bar{\rho}'$ is a power of 2 if $\bar{\rho}$ is absolutely irreducible. This is possible by Corollary 8.2 in the reducible case and Lemma 8.4 in the absolutely irreducible case. Furthermore, note that if $\bar\rho$ is octahedral and good, then so is $\bar\rho'$ by Lemma 8.11. Let A' be the subring of A topologically generated by $t'(\Pi)$. We have seen in Proposition 9.1 and Corollary 9.16 that if $\bar\rho'$ is regular (and under the further assumption that $\bar\rho$ is good when $\bar\rho$ is octahedral) and (t', d') is not a priori small, then (t', d') is A_0-full. This is sufficient by Corollary 3.12. ☐

Remark 10.2. Let (t, d) be as in Theorem 10.1. Then its constant-determinant twist $(t', d') : \Pi \to A$ satisfies the conditions of Corollary 9.16, so that it is $A_{t'}^{\Sigma_{t'}(A_{t'})}$-full. By Corollary 3.12, (t, d) itself is also $A_{t'}^{\Sigma_{t'}(A_{t'})}$-full. However it does not follow that that (t, d) is $A_t^{\Sigma_t(A)}$-full. Indeed, (t, d) may be affected by the pathologies pointed out in Examples 4.3 and 4.10, or even worse, the pro-p-part of d may be transcendental over A_0. Such obstacles are not faced by well-behaved A_0. △

We end this section by recasting all our headline results, including our main result (Theorem 10.1) in the language of representation theory, convenient for comparing applications to results in the literature in Section 12.

Building on the notation recalled above, let $\rho : \Pi \to \mathrm{GL}_2\big(Q(A)\big)$ be a representation whose trace lands in A and is continuous, and let $(t, d) = (\operatorname{tr}\rho, \det\rho)$. Recall that ρ is not a priori small if ρ is strongly absolutely irreducible (Proposition 2.4). Write A_ρ for the trace algebra of ρ — the $W(\mathbb{F})$-subalgebra of A topologically generated by $t(\Pi)$ — and K for its field of fractions. Let K_0 denote the field of fractions of A_0, the adjoint trace ring of ρ. There is a unique semisimple representation $\bar\rho$ whose trace is equal to the reduction of t modulo $\mathfrak{m}$. Finally, recall that we say that the determinant of ρ is A_0-constant if the pro-p part of $\det\rho$ is A_0-valued. In this case K/K_0 is Galois (Theorem 4.19).

Theorem 10.3. *Let ρ be not a priori small. Suppose $\bar\rho$ is regular, and good if octahedral.*

(1) **A_0-fullness:** *ρ is A_0-full.*
(2) **Optimality:** *If ρ is A'-full for a subring $A' \subseteq A$, then A_0 contains a fullness peer of A'.*
(3) **All CSTs fix A_0:** *For any extension B of A we have $\Sigma_\rho(B) = \Sigma_\rho(B/A_0)$.*

If further K is a separable extension of K_0, then:

(4) **CSTs carve out K_0:** $(K^{\mathrm{sep}})^{\Sigma_\rho(K^{\mathrm{sep}})} = K_0$.

If further still ρ has A_0-constant determinant, then:

(5) **All CSTs are simple:** *Every K^{sep}-valued conjugate self-twist restricts to a simple A_ρ-valued one: $\Sigma_\rho(K^{\mathrm{sep}}) \twoheadrightarrow \Sigma_\rho(A_\rho)$. Moreover, $K^{\Sigma_\rho(A_\rho)} = K_0$.*
(6) **CST-invariants fullness:** *ρ is $A_\rho^{\Sigma_\rho(A_\rho)}$-full.*

Proof. Recall that ρ is A'-full if and only if $(\operatorname{tr}\rho, \det\rho)$ is A'-full by Lemma 3.2(2). Thus the first statement follows from Theorem 10.1 and the second from Theorem 5.3. The third statement follows from (1) by Corollary 4.24. The fourth statement follows from Corollary 4.12(2) with $L = K^{\mathrm{sep}}$, $E = K_0$, $F = K$ and F_0 the (nontopological) adjoint trace field of (t, d) viewed as valued in K. For the fifth statement, use Corollary 4.24 to obtain the restriction map $\Sigma_t(K^{\mathrm{sep}}/A_0) \to \Sigma_t(A_\rho)$, surjective since K^{sep} is normal over K. Alternatively, K is Galois over K_0 with $\Sigma_t(A_\rho) = \mathrm{Gal}(K/K_0)$ by Theorem 4.19 and then $\Sigma_t(K^{\mathrm{sep}}) = \mathrm{Gal}(K^{\mathrm{sep}}/K_0)$ by Theorem 4.11. The last statement follows from (1) and Corollary 4.21. ☐

11. Residually large representations

Here we show that by imposing stronger conditions on the residual image when $\bar\rho$ is large, we obtain a more precise description of the image of ρ. That is, we assume that $\rho\colon \Pi \to \mathrm{GL}_2(A)$ is a continuous representation such that $\operatorname{Im}\bar\rho \supseteq \mathrm{SL}_2(\mathbb{E})$ and $\bar\rho$ is large. Under this assumption we have a more precise understanding of the image of ρ than simply fullness. Unlike our main fullness result, in this section A is any local pro-p ring; it need not be a domain.

Historically, this is the case that has been studied the most, starting with the work of Boston in [MW86, Appendix]. Boston shows that if A is a complete local noetherian ring and H is a closed subgroup of $\mathrm{SL}_2(A)$ that projects onto $\mathrm{SL}_2(A/\mathfrak{m}^2)$, then $H = \mathrm{SL}_2(A)$ [MW86, Appendix Proposition 2]. Bellaïche has pointed out that Boston's result follows from his work [Bel19, Remark 6.8.4]. As an application of the description of the image found in Proposition 11.1, we show in Theorem 11.3 that one can replace the hypothesis that H projects onto $\mathrm{SL}_2(A/\mathfrak{m}^2)$ with the hypothesis that H projects onto $\mathrm{SL}_2(A/\mathfrak{m})$ and A is the trace ring of H to obtain the same conclusion. Theorems of this form have been obtained in special cases, for instance for the Galois representation attached to the mod-p Hecke algebra [Amo21, Theorem, Introduction]. Note that assuming H projects onto $\mathrm{SL}_2(A/\mathfrak{m})$, the hypothesis about A being the trace algebra can always be arranged while Boston's hypothesis about projecting onto $\mathrm{SL}_2(A/\mathfrak{m}^2)$ may fail. Moreover, our description of the image in Proposition 11.1 does not require that the residual image contain all of $\mathrm{SL}_2(\mathbb{F})$, rather only $\mathrm{SL}_2(\mathbb{E})$, and is thus more general that the setting of Boston's work.

Let $\rho\colon \Pi \to \mathrm{GL}_2(A)$ be a continuous representation such that $\operatorname{Im}\bar\rho \supseteq \mathrm{SL}_2(\mathbb{E})$. Note that such a $\bar\rho$ is large if and only if $\mathbb{E} \neq \mathbb{F}_3, \mathbb{F}_5$. Let $\chi\colon \Pi \to A^\times$ be the character described in the proof of Theorem 10.1, and let $r \coloneqq \chi \otimes \rho$. Assume that ρ is conjugated in such a way that Theorem 2.23 applies to r. Recall that $\mathcal{B}_r(\mathbb{E}) = W(\mathbb{E})[I_1(r)]$.

Proposition 11.1. *Assume $\#\mathbb{E} \geq 7$. Then*

(1) $\operatorname{Im} r \supseteq \mathrm{SL}_2(\mathcal{B}_r(\mathbb{E}))$ *as a finite index subgroup;*

(2) $\operatorname{Im}\rho \supseteq \mathrm{SL}_2(\mathcal{B}_r(\mathbb{E}))$*;*

(3) $\mathcal{B}_r(\mathbb{E})$ *is the largest subring B of A for which* $\operatorname{Im}\rho \supseteq \mathrm{SL}_2(B)$.

Proof. For ease of notation, write $\mathfrak{m}_r$ for the maximal ideal of $\mathcal{B}_r(\mathbb{E})$.

Since $\operatorname{Im}\bar r \supseteq \mathrm{SL}_2(\mathbb{E})$, it follows that $\operatorname{Im} r \supseteq \mathrm{SL}_2(W(\mathbb{E}))$ by [Man15, Main Theorem]. In particular, $p \in I_1(r)$ and so $\mathfrak{m}_r = I_1(r)$ by Theorem 2.23. Let H be the subgroup of $G \coloneqq \operatorname{Im} r$ generated by $\Gamma = \Gamma(r)$ and $\mathrm{SL}_2(W(\mathbb{E}))$. Then H is a finite index subgroup of G since Γ is.

We claim that $H = \mathrm{SL}_2(\mathcal{B}_r(\mathbb{E}))$. Indeed, note that $\Gamma = \Gamma_{\mathcal{B}_r(\mathbb{E})}(\mathfrak{m}_r)$ by [Bel19, Corollary 6.8.3] and the fact that $\mathfrak{m}_r = I_1(r)$. In particular, this shows that $H \subseteq \mathrm{SL}_2(\mathcal{B}_r(\mathbb{E}))$. In fact, H is a subgroup of $\mathrm{SL}_2(\mathcal{B}_r(\mathbb{E}))$ such that $H/\Gamma = \mathrm{SL}_2(\mathbb{E}) = \mathrm{SL}_2(\mathcal{B}_r(\mathbb{E}))/\Gamma$. Thus we must have equality.

Now suppose that $\operatorname{Im} r \supseteq \mathrm{SL}_2(B)$ for some subring B of A. Without loss of generality, we may assume B is closed, hence local. Then $\Gamma \supseteq \Gamma_B(\mathfrak{m}_B)$, which implies that $I_1(r) \supseteq \mathfrak{m}_B$. On the other hand, if $\operatorname{Im} r \supseteq \mathrm{SL}_2(B)$ then $\operatorname{Im}\bar r \supseteq \mathrm{SL}_2(B/\mathfrak{m}_B)$. By definition of $\mathbb{E}$, we know that $\mathbb{E}$ is the largest subfield of $\mathbb{F}$ such that $\operatorname{Im}\bar r \supseteq \mathrm{SL}_2(\mathbb{E})$. Thus we must have $B/\mathfrak{m}_B \subseteq \mathbb{E}$. It follows that $B \subseteq \mathcal{B}_r(\mathbb{E})$.

As for ρ, note that there is a character $\tilde\chi\colon \operatorname{Im}\rho \to A^\times$ such that $\operatorname{Im} r = \{x\tilde\chi(x) \colon x \in \operatorname{Im}\rho\}$. Now $\tilde\chi$ must be trivial on $\mathrm{SL}_2(B)$ for any ring B whose residue field has more than three elements by Corollary 3.10. Therefore $\operatorname{Im}\rho$ and $\operatorname{Im} r$ contain the same copies of SL_2. ☐

Corollary 11.2. *If $\#\mathbb{E} \geq 7$ and $\operatorname{Im}\bar\rho \supseteq \mathrm{SL}_2(\mathbb{E})$, then $\operatorname{Im}\rho$ contains $\mathrm{SL}_2(A_0)$ up to conjugation.*

Proof. By Proposition 11.1 it suffices to show that $\mathcal{B}_r(\mathbb{E}) \supseteq A_0$, and this only needs to be shown when $r = \rho$ is the constant-determinant universal representation. Without loss of generality we may twist to assume that the order of $\det\rho$ is a power of 2. By Theorem 2.23 we have $\mathcal{A} = \mathcal{B}_r(\mathbb{F})$ since $\bar\rho$ is large. By Corollary 9.3 we may conjugate ρ to assume that $I_1(\rho)$ is fixed by all ($\mathcal{A}$-valued) conjugate-self twists of ρ, and by Corollary 7.4 all conjugate-self twists of $\bar\rho$ lift to conjugate self-twists of ρ since ρ is universal. Therefore

$$\mathcal{B}_r(\mathbb{E}) = W(\mathbb{F})^{\Sigma_\rho}[I_1(\rho)] = \mathcal{A}^{\Sigma_\rho} \supseteq \mathcal{A}_0.$$ ☐

Theorem 11.3. *Let A be a pro-p local noetherian ring with $p \neq 2$ and residue field $\mathbb{F}$. Let H be a closed subgroup of $\mathrm{SL}_2(A)$ that projects onto $\mathrm{SL}_2(\mathbb{F})$. If A is the trace ring of H, that is, A is topologically generated by $\operatorname{tr} H$, then $H = \mathrm{SL}_2(A)$.*

Proof. Let $\rho\colon H \to \mathrm{SL}_2(A)$ be the natural inclusion. Note that ρ has constant determinant. Since A is the trace ring of ρ, Theorem 2.23 implies that $A = \mathcal{B}_\rho(\mathbb{F})$. Thus by Proposition 11.1, it suffices to show that $\mathbb{E} = \mathbb{F}$. Since $\mathrm{Im}\,\overline{\rho} = \mathrm{SL}_2(\mathbb{F})$, it follows that $\mathbb{E}$ is the subfield of $\mathbb{F}$ generated by the squares of traces of $\mathrm{SL}_2(\mathbb{F})$. A straightforward matrix calculation shows that $\mathbb{E} = \mathbb{F}$. $\square$

Remark 11.4. In the preprint [AB], Aryas-de-Reina and Böckle prove a large image result for a residually full representation $\Pi \to G(A)$, where G is an adjoint group and A is the ring of definition of the representation. It seems possible to recover Theorem 11.3 by applying their result to the projective representation $\mathbb{P}\rho\colon \Pi \to \mathrm{PGL}_2(A)$ attached to ρ and using the fact that the ring of definition of $\mathbb{P}\rho$ is the ring fixed by the conjugate self-twists of ρ. $\triangle$

12. Applications to Galois representations

In this section we specialize Theorem 10.1 to some arithmetic settings, more specifically to representations coming from elliptic, Hilbert, and Bianchi cuspidal eigenforms (Section 12.1 through Section 12.3) and cuspidal p-adic families of elliptic and Hilbert eigenforms (Section 12.4 through Section 12.6). We explain how to recover, and in some cases improve, the results already present in the literature. In particular, since our methods are entirely agnostic about the group Π, they reveal that many of the classical big-image results are fundamentally *algebraic* in nature: they do not rely on the arithmetic input, such as local information at the places where a Galois representation is ramified, that went into the original proof.

12.1. **Classical modular eigenforms.** Let f be a non-CM cuspidal modular eigenform of some level and some weight $k \geq 2$ defined over a number field K. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and let $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For any prime $\mathfrak{p}$ of K lying over a rational prime p, let $K_{\mathfrak{p}}$ be the completion of K at $\mathfrak{p}$ and $\mathcal{O}_{\mathfrak{p}}$ its ring of integers. A construction of Deligne attaches to this data an irreducible continuous representation $\rho_{f,\mathfrak{p}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{\mathfrak{p}})$, unramified almost everywhere and hence factoring through a p-finite extension, whose traces of Frobenius elements at unramified primes correspond to Hecke eigenvalues of f. Because $G_{\mathbb{Q}}$ is compact we may view $\rho_{f,\mathfrak{p}}$ as taking values in $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$.

The following result about the image of $\rho_{f,\mathfrak{p}}$ was proved by Ribet and Momose in the 1980s, generalizing an earlier theorem of Serre about Tate modules of elliptic curves. Let $K_{\mathfrak{p},0}$ be the subfield of $K_{\mathfrak{p}}$ fixed by all the generalized conjugate self-twists of $\rho_{f,\mathfrak{p}}$, and $\mathcal{O}_{\mathfrak{p},0}$ its ring of integers.

Theorem 12.1 (Ribet, Momose at $\mathfrak{p}$: first version [Rib85, Mom81])**.**
For all but finitely many primes $\mathfrak{p}$ of K, the representation $\rho_{f,\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p},0}$-full.

To show the extent to which our work recovers the result of this theorem at $\mathfrak{p}$, we first make more explicit Ribet and Momose's condition on $\mathfrak{p}$. Let K_0 be the subfield of K fixed by the conjugate self-twists of f; the p-adic field $K_{\mathfrak{p},0}$ defined above is the completion of K_0 at the prime under $\mathfrak{p}$. Let $H \subseteq G_{\mathbb{Q}}$ be the intersection of the kernels of all the conjugate self-twist characters of f:

$$H := \bigcap_{(\sigma,\eta)\in\widetilde{\Sigma}_f} \ker\eta. \tag{8}$$

Then H is a finite-index normal subgroup of $G_{\mathbb{Q}}$. Because all the conjugate self-twist characters of $\rho_{f,\mathfrak{p}}$ are trivial on H, the trace of $\rho_{f,\mathfrak{p}}|_H$ lands in $K_{\mathfrak{p},0}$. As described just after Definition A.11, there is therefore a $K_{\mathfrak{p},0}$-quaternion algebra $D_{\mathfrak{p}}$ splitting over $K_{\mathfrak{p}}$ with $\rho_{f,\mathfrak{p}}(H) \subset D_{\mathfrak{p}}^\times \subset \mathrm{GL}_2(K_{\mathfrak{p}})$.

Ribet and Momose describe a global analogue to this picture. They define a global K_0-quaternion algebra D split over K with $D(K_{\mathfrak{p},0}) \cong D_{\mathfrak{p}}$: that is, for each prime $\mathfrak{p}$, the restriction to H of $\rho_{f,\mathfrak{p}}$ can be viewed as taking values in $D(K_{\mathfrak{p},0})^\times$. By compactness again we may view $\rho_{f,\mathfrak{p}}(H)$ as a subgroup of the units of a maximal order $\mathcal{O}_{D,\mathfrak{p}}$ of $D(K_{\mathfrak{p},0})$. Ribet and Momose's adelic open-image theorems say that the image of $\rho_{f,\mathfrak{p}}$ always contains an open subgroup of the norm-1 units of $\mathcal{O}_{D,\mathfrak{p}}^\times$, and for all but finitely many $\mathfrak{p}$ it contains all of those norm-1 units. In particular, if $D(K_{\mathfrak{p},0})$ is split, then up to conjugation $\mathcal{O}_{D,\mathfrak{p}}^\times = \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p},0})$; and we can therefore make Theorem 12.1 more precise.

Theorem 12.2 (Ribet, Momose at $\mathfrak{p}$: second version [Rib85, Mom81])**.** *If $D(K_{\mathfrak{p},0})$ is split, then the representation $\rho_{f,\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p},0}$-full.*

Remark 12.3. In fact, the statement in Theorem 12.2 is an if-and-only-if. Indeed, no element of $D(K_{\mathfrak{p},0})$ can have distinct GL_2-eigenvalues in $K_{\mathfrak{p},0}$. A matrix g with eigenvalues α, β satisfies $(g-\alpha)(g-\beta) = 0$. If α, β are in the center of a division algebra containing g, and at the same time are eigenvalues of g in any matrix setting, then the Cayley-Hamilton equation $(g-\alpha)(g-\beta) = 0$ means that either $g = \alpha$ or $g = \beta$. Thus no embedding of $D(K_{\mathfrak{p},0})$ into $\mathrm{GL}_2(K_{\mathfrak{p}})$ can contain any congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_{\mathfrak{p},0})$, or even of $\mathrm{SL}_2(\mathbb{Z}_p)$. The same is true for all of $\rho(\Pi)$: see Theorem 4.19 and Proposition 3.11 or Proposition 12.5 below. In other words, no nonsplit $\rho_{f,\mathfrak{p}}$ can ever satisfy our present definition of fullness. As we've defined it, fullness is fundamentally a GL_2 property; a fitting notion for more general algebraic groups generalizing Ribet and Momose's openness beyond Krull dimension 1 is outside the scope of this investigation. △

We now show that our results recover Ribet and Momose's theorem at $\mathfrak{p}$ in most cases. Let $\mathbb{F}$ be the residue field of $\mathcal{O}_{\mathfrak{p}}$, a finite extension of $\mathbb{F}_p$, and let $\bar{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be the semisimplification $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{\mathfrak{p}}$. Recall that $\bar{\rho}$ is *regular* if its image contains a matrix whose eigenvalue ratio is not ± 1 but is contained in the trace algebra $\mathbb{E}$ of $\mathrm{ad}\,\bar{\rho}$ (which is a subfield of the residue field of $\mathcal{O}_{\mathfrak{p},0}$): see Definition 2.19 and Remark 2.20 immediately following. If $\bar{\rho}$ is octahedral (that is, projective image S_4), see Definition 8.10 for the notion of *goodness*.

Theorem 12.4 (Our results recovering Ribet and Momose at $\mathfrak{p}$)**.** *Assume that p is odd and that $\bar{\rho}$ is regular; if $\bar{\rho}$ is octahedral, assume further that $\bar{\rho}$ is good. Then $\rho_{f,\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p},0}$-full.*

Proof. Since f is cuspidal, non-CM, and has weight $k \geq 2$, its associated representation $\rho_{f,\mathfrak{p}}$ is strongly absolutely irreducible ([Rib77, Proposition 4.4]) and hence not a priori small. By Theorem 10.3, $\rho_{f,\mathfrak{p}}$ is A_0-full, where as usual $A_0 \subseteq \mathcal{O}_{\mathfrak{p}}$ is the adjoint trace ring. By Corollary 4.21, A_0 and $\mathcal{O}_{\mathfrak{p},0}$ are fullness peers, and $\rho_{f,\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p},0}$-full. □

The regularity assumption in Theorem 12.4 a posteriori forces $D(K_{\mathfrak{p},0})$ to split (Remark 12.3). We can also see that a nonsplit $D(K_{\mathfrak{p},0})$ means an irregular $\bar{\rho}$ directly:

Proposition 12.5. *If $D(K_{\mathfrak{p},0})$ is a division algebra, then $\bar{\rho}$ is reducible and not regular.*

Proof. We first show that $\bar{\rho}|_H$ is reducible and not regular. Let $L/K_{\mathfrak{p},0}$ be the unique quadratic unramified extension, π is a uniformizer of either, and σ the nontrivial element of $\mathrm{Gal}(L/K_{\mathfrak{p},0})$. Note that we do not assume that L is a subfield of $K_{\mathfrak{p}}$. Write ℓ, k for the residue fields of L, $K_{\mathfrak{p},0}$, respectively; then $[\ell : k] = 2$ and $\mathbb{E} \subseteq k$. By compactness $\rho(H)$ can be viewed as a subgroup of

$$\left\{ \left(\begin{smallmatrix} \alpha & \pi\beta \\ \sigma(\beta) & \sigma(\alpha) \end{smallmatrix} \right) : \alpha \in \mathcal{O}_L^\times, \beta \in \mathcal{O}_L \right\} \subset \mathrm{GL}_2(\mathcal{O}_L),$$

the maximal order of $D(K_{\mathfrak{p},0})$ as viewed inside $\mathrm{GL}_2(L)$, which gives a representation ρ' of H over L. By inspection, it is clear that its residual representation $\bar{\rho}'$ is, up to semisimplification, a sum of two characters to ℓ conjugate over k. This means that the eigenvalue ratio r of any element in $\bar{\rho}'(H)$ is in the form $r = a^{\#k-1}$ for some $a \in \ell^\times$. Such an element is in k if and only if $r^{\#k-1} = 1$; in other words, if and only if

$$1 = a^{(\#k)^2 - 2(\#k) + 1} = a^{-2(\#k)-2} = r^{-2}.$$

But this last is only possible if $r = \pm 1$. In other words, ρ' is residually neither absolutely irreducible nor regular. Since ρ' is isomorphic to $\rho|_H$ over $\overline{\mathbb{Q}}_p$, the same is true for $\rho|_H$ as well.

We now follow [Nek12, B.4.8(1)] to claim that the same is true for ρ on all of Π. Change notation to let L be any subextension of $K_{\mathfrak{p}}$ that is quadratic over $K_{\mathfrak{p},0}$ and hence splits $D(K_{\mathfrak{p},0})$, with σ a generator of $\mathrm{Gal}(L/K_{\mathfrak{p},0})$ and π a uniformizer of $K_{\mathfrak{p},0}$ that is not a norm from L. Since H is normal in Π and $\rho(H)$ spans $D(K_{\mathfrak{p},0})$ over $K_{\mathfrak{p},0}$, the image of ρ, up to conjugation, will be contained in the normalizer of the subgroup $Q = \left\{ \left(\begin{smallmatrix} \alpha & \pi\beta \\ \sigma(\beta) & \sigma(\alpha) \end{smallmatrix} \right) : (\alpha, \beta) \in L^2 - \{(0,0)\} \right\}$, isomorphic to $D(K_{\mathfrak{p},0})^\times$, in $\mathrm{GL}_2(K_\mathfrak{p})$. One can show that this normalizer is just $K_\mathfrak{p}^\times Q$[(viii)], so that passing from H to Π does not affect the projective image of $\bar\rho$. Therefore $\bar\rho$ on all of Π remains reducible and not regular. $\square$

In other words, the regularity assumption in Theorem 12.4 eliminates the division algebra case. One might hope for a converse, so that our methods could recover all of Theorem 12.2. But alas this is not so: there are certainly cases where $D(K_{\mathfrak{p},0})$ is a matrix algebra but regularity is not satisfied, so that our methods do not apply. In addition to $p = 2$, we do not conclude fullness if $\mathbb{F} = \mathbb{F}_3$, even if f has no conjugate self-twists and hence D is globally split, as is the case for a non-CM elliptic curve over $\mathbb{Q}$. If the image of $\bar\rho$ is too small to accommodate the regularity assumption, our methods cannot handle it.

12.2. **Hilbert modular eigenforms.** Everything in Section 12.1 has been generalized to Hilbert modular forms. In particular, our results recover, in much the same manner and to much the same extent, the big-image results of Nekovář generalizing Ribet and Momose's work over $\mathbb{Q}$. We summarize the situation very briefly.

Let F be a totally real field and f a non-CM cuspidal Hilbert modular eigenform over F all of whose weights are at least 2. Fix an algebraic closure $\overline{F}$ of F, and let $G_F \coloneqq \mathrm{Gal}(\overline{F}/F)$. Fix a prime p and an embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\rho_{f,\iota_p} \colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the Galois representation attached to f in the usual way, which we may view as having coefficients in the ring of integers $\mathcal{O}$ of some finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_0 \subseteq \mathcal{O}$ be the ring of integers of the fixed field of all the conjugate self-twists of ρ_{f,ι_p}. Like Ribet and Momose, Nekovář constructs a division algebra D over the fixed field K_0 of Σ_f and proves an adelic open-image result, which implies $\mathcal{O}_0$-fullness when D splits.

Theorem 12.6 ([Nek12])**.** *For all but finitely many ι_p, the representation ρ_{f,ι_p} is $\mathcal{O}_0$-full.*

Our results depend on hypotheses on the reduction $\bar\rho_{f,\iota_p}$ of ρ_{f,ι_p} modulo the maximal ideal of $\mathcal{O}$.

Theorem 12.7 (Our results recovering Nekovář at ι_p)**.** *Suppose that p is odd and $\bar\rho_{f,\iota_p}$ is regular; if $\bar\rho_{f,\iota_p}$ is octahedral suppose further that it is good. Then ρ_{f,ι_p} is $\mathcal{O}_0$-full.*

The proof is analogous to that of Theorem 12.4. In particular, the fact that the weight of f is at least 2 at each infinite place means that ρ_{f,ι_p} has distinct Hodge-Tate weights, which implies that no twist of it has finite image (see Proposition 2.4 for context). Or apply [CEG, Lemma 3.2.12].

12.3. **Bianchi modular forms and generalizations.** Unlike Hilbert modular forms, which are automorphic forms on GL_2 over totally real fields, Galois representations associated to automorphic forms of GL_2 over CM fields have only been constructed relatively recently, and even then only under some technical assumptions. We briefly summarize how our results can be applied to that context.

Let E be a CM field with maximal totally real subfield F. Fix an algebraic closure $\overline{E}$, and let $G_E \coloneqq \mathrm{Gal}(\overline{E}/E)$. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$, where $\mathbb{A}_E$ denotes

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[(viii)] Certainly, $K_\mathfrak{p}^\times Q$ normalizes. Conversely, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{p})$ normalizes $\left(\begin{smallmatrix} 0 & \pi \\ 1 & 0 \end{smallmatrix}\right)$, then the off-diagonal relation gives us $\sigma((\pi a^2 - \pi^2 c^2)\delta^{-1}) = (\pi d^2 - b^2)\delta^{-1}$, where $\delta = ad - bc$; if it normalizes $\left(\begin{smallmatrix} 0 & \pi t \\ -t & 0 \end{smallmatrix}\right)$, where $t \in L$ with $\sigma(t) = -t$, then the off-diagonal gives $\sigma((\pi a^2 + \pi^2 c^2)\delta^{-1}) = (\pi d^2 + b^2)\delta^{-1}$; if it normalizes $\left(\begin{smallmatrix} t & 0 \\ & -t \end{smallmatrix}\right)$ then the off-diagonal is $\sigma(a(\pi c)\delta^{-1}) = db\delta^{-1}$. Combining the first two, we obtain $\sigma(a^2\delta^{-1}) = d^2\delta^{-1}$ and $\sigma((\pi c)^2\delta^{-1}) = b^2\delta^{-1}$; adding the third gives us, for example if a is invertible, $\sigma(\pi c/a) = b/d$. By considering the diagonal relations, we also get $\sigma(b/a) = \pi c/d$ and $\sigma(a/d) = d/a$. In other words any normalizing element with a nonzero entry in the upper left looks like $\left(\begin{smallmatrix} a & ua\,\sigma(\pi c/a) \\ c & ua \end{smallmatrix}\right)$, with $a \in K_\mathfrak{p}^\times$ arbitrary, $c \in aL$, and $u := d/a \in L$ of norm 1. From Hilbert 90, any norm-1 u is $\sigma(\alpha)/\alpha$ for some $\alpha \in L$; letting $x := a/\alpha$ and $\beta := c\alpha/a$ puts our matrix in the desired form $x\left(\begin{smallmatrix} \alpha & \pi\sigma(\beta) \\ \beta & \sigma(\alpha) \end{smallmatrix}\right) \in K_\mathfrak{p}^\times Q$. Or see [Nek12, B.1.6] for a more conceptual argument.

the adeles of E; when E is imaginary quadratic, π is called a Bianchi modular form. Assume that π is of cohomological type with central character ω. Following Mok [Mok14], assume moreover that ω arises from an algebraic idele class character $\tilde{\omega}$ on $\mathbb{A}_F^\times$ via the norm map and that $\tilde{\omega} = \bigotimes_v \tilde{\omega}_v$ such that $\tilde{\omega}_v(-1)$ takes the same value for all archimedean places of F. (When $F = \mathbb{Q}$, this is simply the condition that ω is invariant under complex conjugation.) Suppose there is no nontrivial quadratic character δ of E such that $\pi \cong \pi \otimes \delta$; this is analogous to the non-CM assumption present in Section 12.1 and Section 12.2.

For each rational prime p and fixed embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, associated to π there is a continuous irreducible representation $\rho_{\pi,\iota_p} \colon G_E \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, which we may view as having coefficients in the ring of integers $\mathcal{O}$ of some finite extension of $\mathbb{Q}_p$. In this generality, the existence of ρ_{π,ι_p} is due to Mok [Mok14], who generalized the construction of Taylor in the imaginary quadratic case [Tay94]. Mok also shows, building on work of Berger and Harcos in the imaginary quadratic case [BH07], that ρ_{π,ι_p} is unramified outside a finite set of places and hence factors through a p-finite group Π.

Let $\mathcal{O}_0$ be the ring of integers of the fixed field of all the conjugate self-twists of ρ_{π,ι_p}.

Theorem 12.8. *Suppose p is odd and $\bar{\rho}_{\pi,\iota_p}$ is regular and good if octahedral. Then ρ_{π,ι_p} is $\mathcal{O}_0$-full.*

The proof is analogous to that of Theorem 12.4 and Theorem 12.7. The Hodge-Tate weights of ρ_{π,ι_p} are distinct by [Mok14, Theorem 5.17].

To our knowledge, Theorem 12.8 is the most general fullness result in the literature in this context, though Taylor proves the weaker theorem that the image of ρ_{π,ι_p} is Zariski dense in the imaginary quadratic case [Tay94, Corollary 2]. Our Theorem 12.8 may be well known to experts.

Remark 12.9. In contrast to the case of elliptic or Hilbert modular forms that can be p-adically interpolated in families with dense classical points and thus have associated “big” Galois pseudorepresentations (see Sections 12.4 to 12.6), a p-adic family of Bianchi modular forms often has only finitely many classical points [CM09, Theorem 8.9], [Ser19, Theorem 1.1]. Therefore no Galois pseudorepresentations have been attached to Bianchi families by conventional methods. △

12.4. **Hida p-adic families of modular forms.** In this section, we explore the extent to which our methods recover known big-image results for Galois representations attached to ordinary p-adic families of modular forms, often called Hida families.

Fix $p > 2$, and let A be the ring corresponding to a primitive non-CM irreducible component of Hida's cuspidal shallow Hecke algebra parametrizing p-ordinary cuspforms of some fixed tame level and weight k such that $k-1 \equiv i \bmod p-1$ for some fixed $0 \le i \le p-1$ (for $i = 0$ this is the ring $\mathbb{I}'$ on [Lan16, p. 158]). Then A is a finite extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\![T]\!]$, which parametrizes the corresponding component of weight space. Let $\mathbb{F}$ be the residue field of A and K the fraction field of A.

Let $(t,d) \colon G_{\mathbb{Q}} \to A$ be the pseudorepresentation obtained from gluing together those attached to the classical cuspforms in the family, all of which have the same semisimplifed residual representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$. Let $\varepsilon_p \colon G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character and $\langle\cdot\rangle \colon \mathbb{Z}_p^\times \to 1+p\mathbb{Z}_p$ the projection onto the pro-p part of $\mathbb{Z}_p^\times$. The weight character $\kappa \colon G_{\mathbb{Q}} \to \Lambda^\times$ is given by $\kappa(g) = (1+T)^{\langle \varepsilon_p(g)\rangle}$. Let χ be the tame Dirichlet character associated to the family. The determinant of ρ is given by

$$d = \kappa\chi s(\bar{\varepsilon}_p)^i, \tag{9}$$

Both $\bar{\rho}$ and (t,d) factor through Π, the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside p and the level, a p-finite profinite group. Let $\Pi_p \subset \Pi$ be a decomposition group at p and $I_p \subset \Pi_p$ its inertia subgroup. The ordinary condition guarantees that there exists a (t,d)-representation ρ, which we view as $\mathrm{GL}_2(K)$-valued by Lemma 2.10, with $\rho|_{\Pi_p} = \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$, where δ an unramified character and ϵ coincides with κ on wild inertia and therefore surjects onto $(1+T)^{1+p\mathbb{Z}_p}$ [Hid12, Theorem 4.3.2].

The image of ρ has been studied by Boston [MW86, Appendix], Fischman [Fis02], Hida [Hid15], and Lang [Lan16]. The latter two are the more recent and most general results, so we focus there.

Theorem 12.10 (Λ-fullness for Hida families, Hida [Hid15])**.** *If $\bar{\rho}$ restricted to Π_p is multiplicity free and ρ is realizable by a representation over A then ρ is Λ-full.*

Hida's Λ-fullness strongly suggested that every conjugate self-twist of (t,d) should fix Λ (see Theorem 5.4 here for a proof of this fact). Following Hida, Lang analyzes how the image of ρ is constrained by $\Sigma_t(A/\Lambda)$, the conjugate self-twists that fix Λ pointwise. To state her result, we let $H_\Lambda \subseteq \Pi$ be the intersection of all the $\ker \eta$ for (σ, η) in $\Sigma_t(A/\Lambda)$.

Theorem 12.11 (Big image for Hida families, [Lan16, Theorem 2.4])**.** *Suppose that $\mathbb{F} \neq \mathbb{F}_3$, that $\bar{\rho}$ is absolutely irreducible, and that there is an element in $\bar{\rho}(H_\Lambda \cap \Pi_p)$ whose eigenvalue ratio is in $\mathbb{E}^\times \setminus \{1\}$.*[(ix)] *Then ρ is $A^{\Sigma_\rho(A/\Lambda)}$-full.*

A posteriori Lang's fullness result by itself justifies considering only those conjugate self-twists that fix Λ, even without Hida's Λ-fullness: see Theorem 5.4. Our work both recovers virtually all of Lang's result (exception: [Lan16] is able to handle some cases where $\mathbb{P}\bar{\rho}$ is the Klein-4 group) and extends it to include residually reducible $\bar{\rho}$. Let A_0 be the adjoint trace ring of ρ; see Definition 4.6. For the notion of A_0-constant determinant, see Definition 4.14; for good octahedral $\bar{\rho}$ see Definition 8.10.

Theorem 12.12 (Our work recovering [Lan16])**.**

(1) If $\bar{\rho}$ is regular and good if octahedral, then ρ is A_0-full.

(2) If Π_p contains a regular element for $\bar{\rho}$, then A_0 contains Λ.

Consequently, if Π_p contains a regular element for $\bar{\rho}$ and $\bar{\rho}$ is further good if octahedral, then

(3) ρ is Λ-full;

(4) ρ is $A^{\Sigma_\rho(A)}$-full;

(5) every conjugate self-twist of ρ fixes Λ, so that $\Sigma_\rho(A) = \Sigma_\rho(A/\Lambda)$ and ρ is $A^{\Sigma_\rho(A/\Lambda)}$-full.

Proof. For (1), the representation ρ is not reducible since the Hida family is cuspidal, and it is not dihedral since the Hida family is not CM. The fact that $\rho(\Pi) \not\cong \bar{\rho}(\Pi)$ follows from the fact that a Hida family has classical specializations of weight at least 2. Therefore we know that ρ is A_0-full by Theorem 10.3.

For (2), let $d_1 : \Pi \to A^\times$ be the pro-p part of $d = \det \rho$, and let $\rho' = d_1^{-1/2} \otimes \rho$ with $(t', d') = (\operatorname{tr} \rho', \det \rho')$ the constant-determinant (pseudo)representation of ρ. Note that $\rho'|_{\Pi_p}$ is still upper triangular since $\rho|_{\Pi_p}$ is. Let $g_0 \in \Pi_p$ be a regular element with residual eigenvalues λ_0, μ_0, and let r be a (t', d')-representation adapted to (g_0, λ_0, μ_0). By the proof of [Bel19, Theorem 6.2.1], we see that, up to replacing g_0 with the limit of a sequence of its powers, we may assume that $r(g_0) = \begin{pmatrix} s(\lambda_0) & 0 \\ 0 & s(\mu_0) \end{pmatrix}$. Viewing both ρ' and r as $\mathrm{GL}_2(K)$-valued by Lemma 2.10, we see that they are isomorphic since they have the same trace and are irreducible. In particular, $\rho'(\Pi_p)$ contains an element with eigenvalues $s(\lambda_0), s(\mu_0)$, which (up to swapping λ_0 and μ_0) is necessarily of the form $M := \begin{pmatrix} s(\lambda_0) & * \\ 0 & s(\mu_0) \end{pmatrix}$. On the other hand, using the description of ϵ and δ above, we see that $\rho'(\Pi_p)$ contains $J := \begin{pmatrix} (1+T)^{1/2} & * \\ 0 & (1+T)^{-1/2} \end{pmatrix}$.

(ix) There is a small error in [Lan16], which we correct here. Theorem 2.4 as stated *loc. cit.* requires merely that $\bar{\rho}$ restricted to $H_\Lambda \cap \Pi_p$ be multiplicity free, but in fact the result relies on the stronger regularity condition given here. Indeed, on [Lan16, p. 174] the definition of $\overline{L}[\lambda]$ only makes sense if one knows $\overline{L}$ is closed under multiplication by λ, where λ is an adjoint eigenvalue of the regular element.

We compute adjoint-trace elements. Both $a := \frac{(\operatorname{tr} M)^2}{\det M} = \frac{(s(\lambda_0)+s(\mu_0))^2}{s(\lambda_0)s(\mu_0)} = 2 + \frac{s(\lambda_0)}{s(\mu_0)} + \frac{s(\mu_0)}{s(\lambda_0)}$ and

$$b := \frac{(\operatorname{tr} MJ)^2}{\det MJ} = \frac{\big(s(\lambda_0)(1+T) + s(\mu_0)\big)^2}{s(\lambda_0)s(\mu_0)(1+T)} = \left(a + \Big(2 + 2\frac{s(\lambda_0)}{s(\mu_0)}\Big)T + \frac{s(\lambda_0)}{s(\mu_0)}T^2\right)(1+T)^{-1}.$$

are in A_0 by construction. The last expression shows that b is in $W(\mathbb{E})[\![T]\!]$, since $\lambda_0\mu_0^{-1} \in \mathbb{E}$ by the regularity assumption. Moreover, the T-coefficient of b is $2 + 2\frac{s(\lambda_0)}{s(\mu_0)} - a = \frac{s(\lambda_0)}{s(\mu_0)} - \frac{s(\mu_0)}{s(\lambda_0)}$, which is in $W(\mathbb{E})^\times$ since $\lambda_0\mu_0^{-1} \neq \pm 1$. It follows that the closed $W(\mathbb{E})$-algebra generated by b in A_0 is all of $W(\mathbb{E})[\![T]\!]$. In other words $\Lambda \subseteq W(\mathbb{E})[\![T]\!] \subseteq A_0$, as claimed.

For (3), combine (1) and (2). For (4), from (2) and the expression for d in (9) (t,d) has A_0-constant determinant. Now use Corollary 4.21. For (5), combine (3), Theorem 5.4, and (4). $\square$

Remark 12.13. The regularity-on-Π_p hypothesis in Theorem 12.12(2) can easily be check in terms of the data of the tame Nebentypus character χ and the mod-p eigenvalue a_p of U_p. Indeed, since δ sends Frobenius to U_p, to verify regularity we need to check whether $\bar{\epsilon}\bar{\delta}^{-1}$ takes on a value in $\mathbb{E}^\times \setminus \{\pm 1\}$. Writing $\chi|_{\Pi_p} = \chi^{\mathrm{unr}}\chi^{\mathrm{tame}}$ with χ^{unr} unramified and χ^{tame} a character on μ_{p-1} by local class field theory, we see that the tame part of $\bar{\epsilon}\bar{\delta}^{-1}$ is $\chi^{\mathrm{tame}}\bar{\varepsilon}_p^i$ and the unramified part is $\chi^{\mathrm{unr}}\bar{\delta}^{-2}$. Note that the tame part necessarily takes values in $\mathbb{F}_p^\times$, so if $\chi^{\mathrm{tame}}\bar{\varepsilon}_p^i$ has order greater than 2, then the regularity-on-Π_p hypothesis is automatically satisfied. Otherwise, one must look to the unramified part and check whether some power of $\chi^{\mathrm{unr}}(p)a_p^{-2}$ lies in $\mathbb{E}^\times \setminus \{\pm 1\}$. $\triangle$

Remark 12.14. **Do all conjugate self-twists of p-adic families fix weight space?** In an abstract algebraic setting, given a representation of a profinite group Π over a ring A that is finite over $\Lambda = \mathbb{Z}_p[\![T]\!]$, we cannot expect to prove that ρ is Λ-full — equivalently, that every conjugate self-twist of ρ fixes Λ — because it is simply not true.

On the other hand, one intuitively expects Λ, which parametrizes weight space, to be preserved by any conjugate self-twist of a p-adic family. Indeed, if (σ, χ) is a conjugate self-twist that doesn't fix Λ, then modulo any prime ideal of Λ, the character χ will relate pairs of eigenforms of different p-adic weights — an implausible scenario. Therefore, although our A_0-fullness result in Theorem 12.12(1) is purely algebraic, in order to recover the full strength of [Lan16], and to match our intuition of how conjugate self-twists in p-adic families behave, we necessarily need to use some modular-form-theoretic input. For Hida families the ordinary condition suffices: see the proof of Theorem 12.12 above, especially part (2). We have not extended this result to Coleman families, which is a drawback on our big-image result in Theorem 12.15. Ultimately one hopes to formalize the geometric intuition alluded to above. $\triangle$

12.5. **Coleman p-adic families of classical modular forms.** We now relax the ordinary assumption present in Section 12.4 and derive the consequences of our main theorem in the context of pseudorepresentations arising from the Coleman-Mazur eigencurve, comparing with known results in the literature.

Let X be a cuspidal irreducible component of the p-adic Coleman-Mazur eigencurve of some fixed tame level ([Buz07], [CM98]); having dealt with the ordinary case in Section 12.4, we assume that X is nonordinary. Let A be the ring of analytic functions on X bounded by 1. It is a compact $\mathbb{Z}_p$-algebra (hence pro-p) since X is nested [BC09, Lemma 7.2.11(ii), Corollary 7.2.12]. In fact, the map from X to weight space endows A with a Λ-algebra structure. As usual, A is a local domain since X is irreducible. As in the Hida family setting, one obtains a 2-dimensional pseudorepresentation $(t,d)\colon G_{\mathbb{Q}} \to A$ by gluing together those attached to classical cuspforms in the family, all of which have the same semisimplified residual representation $\bar{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$. Then (t,d) is unramified outside of a finite set of primes, namely p and the tame level, and thus factors through a p-finite quotient Π of $G_{\mathbb{Q}}$. Unlike the theorems presented in Section 12.1 through Section 12.4, there are no previous fullness results known for (t,d) (though see Remark 12.16 below), so we proceed directly to a statement of our result in this setting.

Theorem 12.15. *If $\bar{\rho}$ is regular and good if octahedral, then (t,d) is A_0-full.*

Proof. By Theorem 10.1 it suffices to show that (t,d) is not a priori small. It is not reducible since it is cuspidal. Any (t,d)-representation does not have finite image since X admits classical specializations of weight at least 2. Finally, since X is nonordinary its CM points are isolated and hence X necessarily admits a classical non-CM positive slope specialization: see [CIT16, Corollary 3.6]. Thus (t,d) is not dihedral. □

Remark 12.16. Besides Bellaïche's work [Bel19], the only previous work on images of Galois representations of finite slope p-adic families of modular forms was done in [CIT16]. We briefly compare Theorem 12.15 to their main result [CIT16, Theorem 1.3].

- **Setup:** Rather than working with A as above, they restrict to an irreducible component $\mathbb{I}^\circ$ of what they call the "adapted slope $\leq h$ Hecke algebra" — essentially a bounded-slope piece of X as above: see [CIT16, §3.1]. Note that one can replace A above by $\mathbb{I}^\circ$ and retain the veracity of Theorem 12.15.
- **Assumptions on $\bar{\rho}$:** In [CIT16, Theorem 1.3] the authors assume that $\bar{\rho}$ is absolutely irreducible, even when restricted to the intersection of the kernels of twist characters. Moreover, their regularity assumption is stronger than ours in that it requires the mod-p eigenvalues of the regular element to be in $\mathbb{F}_p^\times$ rather than requiring their ratio to lie in $\mathbb{E}^\times$.
- **Conclusion:** As mentioned above, [CIT16, Theorem 1.3] is not a true fullness result. Rather, it shows "rigid-Lie fullness" — a certain rigid analytic Lie algebra attached to the image of a (t,d)-representation contains the rigid analytic Lie algebra of a congruence subgroup. While highly suggestive, one does not know how to recover an actual congruence subgroup in the image from this result. Following [Lan16], Conti and his coauthors show rigid-Lie fullness with respect to the ring fixed by $\Sigma_t(\mathbb{I}^\circ/\Lambda)$. △

Remark 12.17. Although the determinant d has a form similar to (9) — the universal character κ times a finite-order character — in this setting we have not proved that d is A_0-constant: we do not know whether the image of Λ is contained in A_0. See Remark 12.14 for why one expects this to be true nonetheless. △

12.6. **p-Adic families of Hilbert modular forms.** Since our methods are agnostic about the group Π, one can proceed with a similar analysis in the context of p-adic families of Hilbert modular forms, which we briefly outline here. We believe these are the first big image results in this context.

As in Section 12.2, fix a totally real field F. Let X be a cuspidal irreducible component of a p-adic eigenvariety interpolating classical Hilbert modular forms over F of a fixed tame level; there are several possible constructions, for instance [Urb11] or [AIP16]. Let A be the ring of analytic functions on X bounded by 1, which is again a pro-p local domain. Gluing together the pseudorepresentations attached to the classical cusp forms parametrized by X, all of which have the same semisimplified residual representation $\bar{\rho}\colon G_F \to \mathrm{GL}_2(\mathbb{F})$, yields a 2-dimensional pseudorepresentation $(t,d)\colon G_F \to A$. It is unramified outside the tame level and p and hence factors through a p-finite quotient Π of G_F.

Theorem 12.18. *Suppose that $\bar{\rho}$ is regular and good if octahedral. If X admits a non-CM classical specialization, then (t,d) is A_0-full.*

Proof. By Theorem 10.1 it suffices to check that (t,d) is not a priori small, which follows from the fact that X admits classical specializations that are cuspidal, not CM, and whose weights are at least 2. □

Remark 12.19. As in Section 12.5, we do not know whether (t,d) has A_0-constant determinant in this case and hence lack an $A_t^{\Sigma_t}$-fullness result. As in Remark 12.14, we expect the image in A of the ring Λ_F of analytic functions on weight space to be contained in A_0. Note that in this case, Λ_F is a power series ring over $\mathbb{Z}_p$; the number of variables depends on the totally real field F. △

Appendix A. Algebraic sundries

A.1. **Representations with isomorphic adjoint differ by a character.** Throughout Appendix A.1, let G be a group and F a separably closed field of odd characteristic. All representations are assumed to be finite-dimensional. Let $\mathfrak{sl}_n(F)$ denote the F-vector space of $n \times n$-matrices of trace 0 and $\mathrm{ad}^0\colon \mathrm{GL}_n(F) \to \mathrm{GL}_{n^2-1}(F)$ the representation obtained by letting $\mathrm{GL}_n(F)$ act on $\mathfrak{sl}_n(F)$ by conjugation. The primary goal of this section is to prove that if $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(F)$ are semisimple representations such that $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, then $\rho_1 \cong \rho_2 \otimes \eta$ for some character $\eta\colon G \to F^\times$. This is done in Theorem A.10. The easier case when the ρ_i are not dihedral is treated first in Appendix A.1.1. Appendix A.1.2 is an analysis of dihedral representations that allows us to conclude Theorem A.10 in full generality. The results of this section are probably well known to experts, but we give proofs for lack of a reference in the generality we need. We were guided by Venkatarama's answer to MathOverflow question 297746. In the nondihedral case, this result can be found in [KMP00, Lemma 2.9]. When the representations ρ_1 and ρ_2 arise from classical modular forms, the result can be found in [DK00, Appendix].

A.1.1. *The nondihedral case.* Given a representation $\rho\colon G \to \mathrm{GL}_n(F)$, we write ρ^* for its dual representation. That is, if V is the representation space of ρ, then $V^* := \mathrm{Hom}(V, F)$ is the representation space of ρ^* with G-action given by $(g\varphi)(v) := g\varphi(g^{-1}v)$. In terms of matrices, if we fix a basis for V and take the dual basis for V^*, then $\rho^*(g)$ is the inverse transpose of $\rho(g)$.

If ρ is 2-dimensional, then an explicit calculation shows that $\rho^* \cong \rho \otimes \Lambda^2\rho^*$, where Λ^2 denotes the second exterior power of ρ. (The conjugating matrix can be taken to be $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.) We have that

$$1 \oplus \mathrm{ad}^0 \rho \cong \rho \otimes \rho^*.$$

In particular, $\mathrm{ad}^0 \rho$ is self dual. Furthermore,

$$1 \oplus \mathrm{ad}^0 \rho \cong \rho \otimes \rho^* \cong \rho \otimes \rho \otimes \Lambda^2\rho^* \cong 1 \oplus (\mathrm{Sym}^2 \rho \otimes \Lambda^2\rho^*),$$

and so $\mathrm{ad}^0 \rho \cong \mathrm{Sym}^2 \rho \otimes \Lambda^2\rho^* = \mathrm{Sym}^2 \rho \otimes \det \rho^{-1}$.

The following lemma is essentially a version of Schur's lemma that will be useful in what follows.

Lemma A.1. *If $\rho\colon G \to \mathrm{GL}_n(F)$ is a semisimple representation such that $\mathrm{ad}^0 \rho$ contains a copy of the trivial representation, then ρ is reducible.*

Proof. Let V be the F-vector space on which G acts via ρ. Then $\mathrm{End}\, V$ is the representation space for $1 \oplus \mathrm{ad}^0 \rho$, where 1 is the trivial representation, which corresponds to scalar endomorphisms of V. If $\mathrm{ad}^0 \rho$ contains a copy of the trivial representation, then there is a nonscalar $\varphi \in \mathrm{End}\, V$ that commutes with the action of G. By Schur's lemma, ρ must be reducible. □

Lemma A.2. *Let $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(F)$ be semisimple reducible representations. If $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, then there exists a character $\eta\colon G \to F^\times$ such that $\rho_1 \cong \eta \otimes \rho_2$.*

Proof. By assumption there exist for $i = 1, 2$ characters $\lambda_i, \mu_i\colon G \to F^\times$ such that $\rho_i \cong \lambda_i \oplus \mu_i$. It is straightforward to calculate

$$\lambda_1\mu_1^{-1} \oplus 1 \oplus \lambda_1^{-1}\mu_1 \cong \mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2 \cong \lambda_2\mu_2^{-1} \oplus 1 \oplus \lambda_2^{-1}\mu_2.$$

Thus, up to switching λ_2 and μ_2, we must have $\lambda_1\mu_1^{-1} = \lambda_2\mu_2^{-1}$. Let $\eta = \mu_1\mu_2^{-1}$. Then

$$\rho_1 \cong \lambda_1 \oplus \mu_1 = \lambda_2\mu_1\mu_2^{-1} \oplus \mu_1 = (\mu_1\mu_2^{-1}) \otimes (\lambda_2 \oplus \mu_2) \cong \eta \otimes \rho_2.$$ □

Lemma A.3. *Let $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(F)$ be semisimple representations such that both $\mathrm{ad}^0 \rho_i$ are irreducible. If $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, then there exists a character $\eta\colon G \to F^\times$ such that $\rho_1 \cong \eta \otimes \rho_2$.*

Proof. We begin by showing that $\rho_1 \otimes \rho_2$ must be reducible (which does not make use of the assumption that $\mathrm{ad}^0 \rho_i$ is irreducible). Indeed, by Lemma A.1 if $\rho_1 \otimes \rho_2$ were irreducible then its endomorphism ring would contain a single copy of the trivial representation. But

$$\begin{aligned}\mathrm{End}(\rho_1 \otimes \rho_2) = (\rho_1 \otimes \rho_2) \otimes (\rho_1 \otimes \rho_2)^* &\cong (\rho_1 \otimes \rho_1^*) \otimes (\rho_2 \otimes \rho_2^*)\\ &\cong (1 \oplus \mathrm{ad}^0 \rho_1) \otimes (1 \oplus \mathrm{ad}^0 \rho_2)\\ &\cong 1 \oplus \mathrm{ad}^0 \rho_1 \oplus \mathrm{ad}^0 \rho_1 \oplus (\mathrm{ad}^0 \rho_1 \otimes \mathrm{ad}^0 \rho_1)\\ &\cong 1 \oplus \mathrm{ad}^0 \rho_1 \oplus \mathrm{ad}^0 \rho_1 \oplus (\mathrm{ad}^0 \rho_1 \otimes (\mathrm{ad}^0 \rho_1)^*),\end{aligned}$$

and $\mathrm{ad}^0 \rho_1 \otimes (\mathrm{ad}^0 \rho_1)^* \cong \mathrm{End}(\mathrm{ad}^0 \rho_1)$ contains a copy of the trivial representation, a contradiction.

Next we show that $\rho_1 \otimes \rho_2$ cannot be the sum of two 2-dimensional representations. Indeed, suppose that $\rho_1 \otimes \rho_2 \cong r_1 \oplus r_2$, where $r_1, r_2 \colon G \to \mathrm{GL}_2(F)$ are representations. Take the second exterior product on both sides. We have

$$\Lambda^2(\rho_1 \otimes \rho_2) \cong (\Lambda^2 \rho_1 \otimes \mathrm{Sym}^2 \rho_2) \oplus (\mathrm{Sym}^2 \rho_1 \otimes \Lambda^2 \rho_2)$$

and

$$\Lambda^2(r_1 \oplus r_2) \cong \Lambda^2 r_1 \oplus \Lambda^2 r_2 \oplus (r_1 \otimes r_2).$$

Since $\mathrm{ad}^0 \rho_i \cong \mathrm{Sym}^2 \rho_i \otimes \Lambda^2 \rho_i^*$, we have $\mathrm{Sym}^2 \rho_1 \otimes \Lambda^2 \rho_2 \cong \mathrm{Sym}^2 \rho_2 \otimes \Lambda^2 \rho_1$. But if

$$(\Lambda^2 \rho_1 \otimes \mathrm{Sym}^2 \rho_2)^{\oplus 2} \cong \Lambda^2 r_1 \oplus \Lambda^2 r_2 \oplus (r_1 \otimes r_2),$$

then this contradicts irreducibility of $\mathrm{ad}^0 \rho_i$. Thus $\rho_1 \otimes \rho_2$ must contain a 1-dimensional representation; call it χ. Then we claim that $\rho_2 \cong \rho_1^* \otimes \chi \cong \rho_1 \otimes \det \rho_1^{-1} \otimes \chi$, and so ρ_1 and ρ_2 differ by a twist.

To see that $\rho_2 \cong \rho_1^* \otimes \chi$, recall that $\rho_1 \otimes \rho_2 \cong \mathrm{Hom}(\rho_1^*, \rho_2)$. Thus having a 1-dimensional G-stable subspace corresponds to a nonzero linear map $\varphi \colon \rho_1^* \to \rho_2$ such that $g\varphi = \lambda(g)\varphi$ for some $\lambda(g) \in F^\times$ for all $g \in G$. Define $f \colon \rho_1^* \to \rho_2 \otimes \chi^{-1}$ by $v \mapsto \varphi(v) \otimes e$, where e is a basis for the 1-dimensional vector space on which G acts by χ. Note that $f \neq 0$ since $\varphi \neq 0$. It is straightforward to check that $f(gv) = gf(v)$ for all $g \in G$. Therefore $\mathrm{Hom}(\rho_1^*, \rho_2 \otimes \chi^{-1}) \neq 0$. Since ρ_1^* and $\rho_2 \otimes \chi^{-1}$ are irreducible, it follows that they must be isomorphic. $\square$

The following observation can be checked easily via a direct calculation on 2×2 matrices.

Lemma A.4. *For any $g \in \mathrm{GL}_2(F)$ with (not necessarily distinct) eigenvalues λ, μ, the eigenvalues of $\mathrm{ad}^0 g$ are $1, \lambda\mu^{-1}, \lambda^{-1}\mu$. In particular, we have*

$$\mathrm{tr}\,\mathrm{ad}^0 g = \frac{\mathrm{tr}(g)^2}{\det(g)} - 1.$$

A.1.2. *The dihedral case.* In Appendix A.1.2 we assume for simplicity that the characteristic of F is not equal to 2. The goal of Appendix A.1.2 is to remove the assumption that both ρ_i are reducible or both $\mathrm{ad}^0 \rho_i$ are irreducible from Lemmas A.2 and A.3. We begin with a lemma that shows that, in light of Lemmas A.2 and A.3, we only need to consider the case when both ρ_1 and ρ_2 are dihedral representations.

Lemma A.5. *If $\rho \colon G \to \mathrm{GL}_2(F)$ is irreducible but $\mathrm{ad}^0 \rho$ is reducible, then ρ is dihedral.*

Proof. If $\mathrm{ad}^0 \rho$ is reducible, then so is $\mathrm{Sym}^2 \rho$ and $\mathrm{Sym}^2 \rho^*$ since $\mathrm{ad}^0 \rho \cong \mathrm{Sym}^2 \rho \otimes \det \rho^{-1}$. But $\mathrm{Sym}^2 \rho^*$ can be identified with the action of G on the F-vector space of quadratic forms on F^2. Thus, there is a quadratic form Q on which G acts by a scalar. Since F is separably closed and $\mathrm{char} F \neq 2$, all quadratic forms are equivalent. In particular, we may assume that $Q(x, y) = xy$. But one checks immediately that the only matrices that preserve Q up to scalars are diagonal and antidiagonal. Thus ρ must be dihedral. $\square$

The rest of this section is devoted to an analysis of dihedral representations.

Lemma A.6. *Assume that $\rho\colon G \to \mathrm{GL}_2(F)$ is a semisimple representation. If $\rho \cong \eta \otimes \rho$ for some nontrivial character $\eta\colon G \to F^\times$, then the image of $\rho|_{\ker \eta}$ is abelian.*

Proof. This argument essentially comes from [Rib77, Proposition 4.4]. Note that $\det \rho = \eta^2 \det \rho$ and so $\eta^2 = 1$. Set $H \coloneqq \ker \eta$. Thus $[G\colon H] = 2$ since η is nontrivial. By assumption, there is a matrix $M \in \mathrm{GL}_2(F)$ such that $M\rho(g)M^{-1} = \eta(g)\rho(g)$ for all $g \in G$. In particular, $\rho(H)$ is contained in the commutant of M.

We claim that M is semisimple. It suffices to show that M has distinct eigenvalues. Up to a change of basis for ρ, we may assume that M is upper triangular, say $M = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. The eigenvalues of M acting on $M_2(F)$ by conjugation are $1, 1, ac^{-1}, a^{-1}c$ by Lemma A.4. Note that for any $g \in G \setminus H$, we have

$$M\rho(g)M^{-1} = -\rho(g).$$

Thus $-1 = ac^{-1}$, which implies that $a \neq c$ and thus M has distinct eigenvalues, as claimed. Therefore M is semisimple and so its commutant, and hence $\rho(H)$, is abelian. ☐

If H is a subgroup of G of index 2, then we use c to denote a fixed element in $G \setminus H$. For a character $\chi\colon H \to F^\times$ and $g \in G$, we write $\chi^g\colon H \to F^\times$ for the character defined by $\chi^g(h) \coloneqq \chi(g^{-1}hg)$. It is not difficult to check that χ^g depends only on the coset of g in G/H. Set $\chi^- \coloneqq \chi/\chi^c$. We will write $\eta_H\colon G \to G/H \cong \{\pm 1\}$ for the canonical projection map. With this notation, we recall an explicit description of $\mathrm{Ind}_H^G \chi$. Namely, $\mathrm{Ind}_H^G \chi$ is isomorphic to the representation

$$g \mapsto \begin{cases} \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi^c(g) \end{pmatrix} & \text{if } g \in H \\ \begin{pmatrix} 0 & \chi(gc) \\ \chi^c(gc^{-1}) & 0 \end{pmatrix} & \text{otherwise.} \end{cases} \tag{10}$$

Using Frobenius reciprocity it is easy to see that $\mathrm{Ind}_H^G \chi$ is irreducible if and only if $\chi \neq \chi^c$.

Lemma A.7.

(1) If $\rho = \mathrm{Ind}_H^G \chi$ for a character $\chi\colon H \to F^\times$ and $[G\colon H] = 2$, then $\rho \cong \rho \otimes \eta_H$.

(2) Conversely, if $\rho\colon G \to \mathrm{GL}_2(F)$ is a dihedral representation, then there is a subgroup H of G of index 2 and a character $\chi\colon H \to F^\times$ such that $\rho \cong \mathrm{Ind}_H^G \chi$ and $\chi \neq \chi^c$.

(3) Furthermore, H as in (2) is unique unless $\chi^2 = (\chi^c)^2$.

(4) If $\chi^2 = (\chi^c)^2$ then there are exactly three index 2 subgroups H_i of G for $i = 1, 2, 3$ for which there exist characters $\chi_i\colon H_i \to F^\times$ such that $\rho \cong \mathrm{Ind}_{H_i}^G \chi_i$.

Proof. For the first point, note that χ is a constituent of $(\rho \otimes \eta_H)|_H = \rho|_H$. By Frobenius reciprocity and dimension counting, it follows that $\mathrm{Ind}_H^G \chi \cong \rho \otimes \eta_H$.

If ρ is dihedral, then there is a nontrivial character $\eta\colon G \to F^\times$ such that $\operatorname{tr} \rho = \eta \operatorname{tr} \rho$ and $\det \rho = \eta^2 \det \rho$. In particular, $\eta^2 = 1$ and so η is a quadratic character. Let $H \coloneqq \ker \eta$. Then H is a subgroup of G of index 2 and $\rho|_H$ is reducible by Lemma A.6. Let $\chi\colon H \to F^\times$ be one of the constituents of $\rho|_H$. By Frobenius reciprocity, $\mathrm{Ind}_H^G \chi$ is a constituent of ρ and we deduce equality for dimension reasons. Thus we have $\rho|_H = \chi \oplus \chi^c$. Since ρ is irreducible by the definition of being dihedral, it follows by Frobenius reciprocity that $\chi \neq \chi^c$. This finishes the proof of the second point.

For the third point, suppose that $\rho = \mathrm{Ind}_{H'}^G \chi'$ for some character $\chi'\colon H' \to F^\times$ and $[G\colon H'] = 2$. Let $c' \in G \setminus H'$. Then by restricting to H we have $\chi \oplus \chi^c = (\eta_{H'})|_H \cdot \chi \oplus (\eta_{H'})|_H \cdot \chi^c$. Thus we either have $\chi = (\eta_{H'})|_H \cdot \chi$ or $\chi = (\eta_{H'})|_H \cdot \chi^c$. In the first case, we see that $H = \ker \eta_{H'} = H'$. In the second case we conclude that $\chi^2 = (\chi^c)^2$ since $\eta_{H'}$ is quadratic.

Finally, suppose that $\chi^2 = (\chi^c)^2$. Then $H_0 \coloneqq \ker(\chi/\chi^c)$ is a subgroup of index 2 in H. We claim that H_0 is normal in G. Recall that χ^c is independent of the choice of $c \in G \setminus H$. If $h \in H_0$ and $g \in G \setminus H$ then

$$\begin{aligned}\chi(g^{-1}hg)/\chi^c(g^{-1}hg) &= \chi(g^{-1}hg)/\chi^g(g^{-1}hg) = \chi^g(h)/\chi(h)\\ &= (\chi/\chi^g)(h)^{-1} = (\chi/\chi^c)(h)^{-1} = 1.\end{aligned}$$

Furthermore, the above calculation shows that the class of c generates a subgroup of G/H_0 of order 2 distinct from H. Thus G/H_0 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. We claim that if H' is any of the three subgroups of G of index 2 containing H_0, then there is a character $\chi' \colon H' \to F^\times$ such that $\rho \cong \operatorname{Ind}_{H'}^G \chi'$. By Frobenius reciprocity, it suffices to show that $\rho|_{H'}$ is reducible. Since $\rho|_{H_0} = \chi|_{H_0} \oplus \chi^c|_{H_0}$, it follows from Frobenius reciprocity that $\rho|_{H'} = \operatorname{Ind}_{H_0}^{H'} \chi|_{H_0}$. But $\chi|_{H_0} = \chi^c|_{H_0}$ and so it follows (again by Frobenius reciprocity) that $\rho|_{H'}$ is reducible. $\square$

Combining the following lemma with Frobenius reciprocity, we see that the irreducibility of $\operatorname{Ind}_H^G \chi$ is related to the question of whether the character $\chi \colon H \to F^\times$ extends to a character of G.

Lemma A.8. *Let H be a subgroup of G of index 2 and $\chi \colon H \to F^\times$ a character. Then there exists an extension of χ to a character $G \to F^\times$ if and only if $\chi = \chi^c$. If χ extends to a character of G, then there are exactly two different extensions, and they differ by η_H.*

Proof. If such an L and extension of χ exist, then certainly $\chi = \chi^c$. On the other hand, since $c^2 \in H$, we know that $\chi(c^2)$ is well defined. Since F is algebraically closed, we may choose a square root r of $\chi(c^2)$ in F. Define a new character $\tilde{\chi} \colon G \to L^\times$ by

$$\tilde{\chi}(g) \coloneqq \begin{cases} \chi(g) & \text{if } g \in H \\ r\chi(c^{-1}g) & \text{if } g \notin H. \end{cases}$$

To see that $\tilde{\chi}$ is a character, it suffices to verify that it is multiplicative. That is, one must check that $\tilde{\chi}(h)\tilde{\chi}(ch') = \tilde{\chi}(hch')$ and $\tilde{\chi}(ch)\tilde{\chi}(ch') = \tilde{\chi}(chch')$ for $h, h' \in H$. It is easy to see by direct computation that these are satisfied if $\chi = \chi^c$. $\square$

Lemma A.9. *Let $\rho = \operatorname{Ind}_H^G \chi$ be a dihedral representation. Then $\operatorname{ad}^0 \rho \cong \eta_H \oplus \operatorname{Ind}_H^G \chi^-$. If $\operatorname{ad}^0 \rho$ is the sum of three characters, then $\chi^2 = (\chi^c)^2$ and $\operatorname{ad}^0 \rho \cong \eta_{H_1} \oplus \eta_{H_2} \oplus \eta_{H_3}$, where the H_i are the index 2 subgroups of G given in Lemma A.7.*

Proof. The first claim is an explicit calculation. Let $e_1 \coloneqq \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e_2 \coloneqq \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), e_3 \coloneqq \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Assume that ρ is given by (10). Then with respect to the basis e_1, e_2, e_3 we see that

$$\operatorname{ad}^0(g) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \chi^-(g) & 0 \\ 0 & 0 & \chi^-(g)^{-1} \end{pmatrix} & \text{if } g \in H \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \chi^-(gc) \\ 0 & \chi^-(gc^{-1})^{-1} & 0 \end{pmatrix} & \text{otherwise.} \end{cases}$$

We observe that η_H appears in the upper left corner. Furthermore, $(\chi^-)^c = (\chi^-)^{-1}$. Therefore the lower right 2×2-matrix in $\operatorname{ad}^0 \rho$ is isomorphic to $\operatorname{Ind}_H^G \chi^-$ by (10). Thus $\operatorname{ad}^0 \rho \cong \eta_H \oplus \operatorname{Ind}_H^G \chi^-$.

If $\operatorname{ad}^0 \rho$ is the sum of three characters, then $\operatorname{Ind}_H^G \chi^-$ is reducible and thus $\chi^- = (\chi^-)^c$. That is, $\chi^2 = (\chi^c)^2$. By Lemma A.7, it follows that there are exactly three subgroups H_i of G of index 2 for which $\rho \cong \operatorname{Ind}_{H_i}^G \chi_i$. By the above calculation, each η_{H_i} must be a constituent of $\operatorname{ad}^0 \rho$. By counting dimensions, we find that $\operatorname{ad}^0 \rho \cong \eta_{H_1} \oplus \eta_{H_2} \oplus \eta_{H_3}$. $\square$

Theorem A.10. *Let F be a field whose characteristic is not 2. Let $\rho_1, \rho_2 \colon G \to \mathrm{GL}_2(F)$ be semisimple representations. If $\operatorname{ad}^0 \rho_1 \cong \operatorname{ad}^0 \rho_2$ then there is a character $\eta \colon G \to L^\times$ such that $\rho_1 \cong \eta \otimes \rho_2$.*

Proof. The case when either of ρ_1 or ρ_2 is not dihedral is settled by Lemmas A.2, A.3 and A.5. Therefore we may assume that both ρ_1 and ρ_2 are dihedral. By Lemma A.9 there are index-2 subgroups H_i of G and characters $\chi_i\colon H_i \to F^\times$ such that $\rho_i \cong \mathrm{Ind}_{H_i}^G \chi_i$. Note that the set of possible such H_i can be read off from $\mathrm{ad}^0 \rho_i$ since η_{H_i} is a constituent of $\mathrm{ad}^0 \rho_i$ by Lemma A.9 and $H_i = \ker \eta_{H_i}$. In particular, since $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 \rho_2$, we may assume that $H := H_1 = H_2$. By Lemma A.9 we have $\mathrm{Ind}_H^G \chi_1^- \cong \mathrm{Ind}_H^G \chi_2^-$. By restricting to H it follows that $\chi_1^- \oplus (\chi_1^-)^c \cong \chi_2^- \oplus (\chi_2^-)^c$, and so up to replacing χ_2 with χ_2^c (which is okay since $\mathrm{Ind}_H^G \chi_2 \cong \mathrm{Ind}_H^G \chi_2^c$), it follows that $\chi_1^- = \chi_2^-$. That is, $\chi_1\chi_2^{-1} = (\chi_1\chi_2^{-1})^c$. By Lemma A.8 there is a character $\eta\colon G \to L^\times$ such that $\eta|_H = \chi_1\chi_2^{-1}$. We claim that $\rho_1 \cong \eta \otimes \rho_2$. Indeed, this is true upon restriction to H since

$$\rho_1|_H = \chi_1 \oplus \chi_1^c = \eta|_H \otimes (\chi_2 \oplus \chi_2^c) = (\eta \otimes \rho_2)|_H.$$

Therefore $\rho_1 \cong \eta \otimes \rho_2$ by Frobenius reciprocity since ρ_1 is irreducible and thus $\chi_1 \neq \chi_1^c$. □

A.2. **Trace field extensions.** In this brief subsection, let G be an abstract group, F an abstract field, and $(t,d) : G \to F$ a (2-dimensional) pseudorepresentation.

Definition A.11. The *trace field* of (t,d) is the subfield of F generated by the image $t(G)$ over the prime subfield of F. A pseudorepresentation $(t,d) : G \to F$ is *realizable* over an extension L of F if there exists a semisimple representation $\rho : G \to \mathrm{GL}_2(L)$ that *carries* (t,d) — that is, with $t = \mathrm{tr}\,\rho$ and $d = \det \rho$.

Lemma A.12. *Let $(t,d) : G \to F$ is a pseudorepresentation with trace field F. If the characteristic of F is not 2, then (t,d) realizable over an at-most quadratic extension of F.*

Proof. To start with, (t,d) is always realizable by a semisimple representation V over $\overline{F}$ [Che14, Theorem 2.12]. If we suppose that V is irreducible, then the image of the associated F-algebra map $\rho : F[G] \to \mathrm{End}_{\overline{F}}(V)$ surjects onto the full matrix algebra $\mathrm{End}_{\overline{F}}(V)$ after extending scalars to $\overline{F}$, and is therefore a quaternion algebra D over F. There are now two possibilities. Either D is split, in which case $D^\times \cong \mathrm{GL}_2(F)$ is a realization of (t,d) over F. Or D is an F-division algebra, in which case $\rho|_G : G \to D^\times$ carries (t,d), in the sense that $\rho(g)$ has reduced trace $t(g)$ and reduced norm $d(g)$, and any quadratic extension L of F that splits D carries a realization of (t,d) as an irreducible representation $G \to \mathrm{GL}_2(L)$.

On the other hand, suppose (t,d) splits into a sum of two characters $\chi, \chi' : G \to \overline{F}^\times$. The image of χ is a subgroup of $\overline{F}^\times$ whose every element is contained in an at-most-quadratic extension of F. Suppose that $\alpha = \chi(a)$ and $\beta = \chi(b)$ for $a, b \in G$ generate *different* quadratic extensions of F. Then on one hand, $\alpha\beta = \chi(ab)$ must generate the third quadratic subextension of $F(\alpha,\beta)$. But on the other hand, we claim that $\chi'(ab) = \chi'(a)\chi'(b)$ is equal to $\alpha\beta$: indeed, let c_α be the generator of $\mathrm{Gal}\big(F(\alpha,\beta)/F(\beta)\big)$ viewed as an element of $\mathrm{Gal}\big(F(\alpha,\beta)/F\big)$, so that $\chi'(a) = c_\alpha(\alpha)$; define c_β similarly. Then $c_\alpha c_\beta$ generates $\mathrm{Gal}\big(F(\alpha,\beta)/F(\alpha\beta)\big)$ and hence fixes $\alpha\beta$. Therefore $\chi(ab) + \chi'(ab) = 2\alpha\beta$, which is not in F[(x)] — a contradiction. □

The following proposition is true in any dimension, but we state it here for dimension 2.

Proposition A.13. *Let $(t,d) : G \to F$ be a pseudorepresentation and $H \subseteq G$ a finite-index normal subgroup. Then the trace field of (t,d) is a finite extension of the trace field of $(t|_H, d|_H)$.*

Proof. Replace F by the trace field of (t,d), and let $E \subseteq F$ be the trace field of $(t|_H, d|_H)$. Let ρ be a semisimple representation carrying (t,d) over an extension of F. Note that F is algebraic over E: every $g \in G$ satisfies $g^{[G:H]} \in H$, so that every eigenvalue of $\rho(g)$, and hence $t(g)$, is algebraic over the finite extension of E containing the eigenvalues of $g^{[G:H]}$. Since F is contained in the field

[(x)] The constraint on the characteristic is necessary: consider $G = \mathbb{Z}^2$ and $F = \mathbb{F}_2(x,y)$, with (t,d) the pseudocharacter corresponding to the scalar 2-dimensional representation $(1,0)$ to $\sqrt{x}$ and $(0,1)$ to $\sqrt{y}$.

generated by all these eigenvalues, F is algebraic over E, and in particular ρ is realizable over an algebraic extension of E.

By Clifford's theorem [Cra19, Theorem 7.1.1], $\rho|_H$ is still semisimple, so it is carried by a representation V over a finite extension E' of E. Since F/E is algebraic, $V_{\overline{E}} = V \otimes_{E'} \overline{E}$ carries all of ρ. Let $g_1, \dots, g_n$ be coset representatives for H in G; write each $\rho(g_i)$ as a matrix in a fixed E'-basis of V extended to $V_{\overline{E}}$. Let M be the subfield of $\overline{E}$ generated over E' by the matrix coefficients of all the $\rho(g_i)$. Then M is a finite extension of E', and hence of E, containing the values of $\operatorname{tr}\rho$. Therefore F/E is finite. □

A.3. **Rings with involution.** Throughout Appendix A.3, let A be a commutative noetherian ring equipped with an involution $*$. Note that we will need to apply the results in this section to the universal constant-determinant pseudodeformation ring $\mathcal{A}$, so we cannot assume that A is a domain. Let $A^{\varepsilon} = \{a \in A \colon a^* = \varepsilon a\}$ for $\varepsilon \in \{+,-\}$. We will assume throughout that $*$ is not the identity on A so that $A^- \neq 0$. It is easy to see that A^+ is a subring of A and A^- is an A^+-module. The following results have been adapted from [Lan75] and [CL77], where they are presented in the context when A may be noncommutative.

Definition A.14. We say that an A-ideal $\mathfrak{a}$ is a *$*$-ideal* if $\mathfrak{a}^* = \mathfrak{a}$. We say that A is *$*$-prime* if whenever $\mathfrak{a}$ and $\mathfrak{b}$ are $*$-ideals such that $\mathfrak{a}\mathfrak{b} = 0$ then either $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$.

Lemma A.15. *If A is $*$-prime then A is reduced.*

Proof. Let $0 \neq a \in A$ be nilpotent. Then there is a smallest integer $n > 1$ such that $a^n = 0$. Let $\mathfrak{a} = aA$ and $\mathfrak{b} = a^{n-1}A$. Note that $\mathfrak{a} \neq 0$ and $\mathfrak{b} \neq 0$ by the minimality of n. If $\mathfrak{a}$ and $\mathfrak{b}$ are $*$-ideals then we have reached a contradiction since $\mathfrak{a}\mathfrak{b} = a^n A = 0$. In particular, if $a + a^* = 0$ then $a^* = -a \in aA$ and so $\mathfrak{a}, \mathfrak{b}$ are $*$-ideals.

If $a + a^* \neq 0$, then $a + a^*$ is still nilpotent since A is commutative. By replacing a with $a + a^*$ in the above argument, we find that $\mathfrak{a}$ and $\mathfrak{b}$ are $*$-ideals and thus we reach a contradiction. □

Lemma A.16. *If A is a noetherian commutative ring with $2 \in A^{\times}$, then A^+ is a noetherian ring.*

Proof. The following argument comes from [CL77, Lemma]. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in A^+. Then $I_1 A \subseteq I_2 A \subseteq \cdots$ is an ascending chain of ideals in A. Since A is noetherian, there is some n such that $I_n A = I_m A$ for all $m \geq n$.

Fix $m \geq n$ and $a \in I_m \subseteq A^+$. Since $a \in I_m A = I_n A$ we may write

$$a = \sum_i b_i x_i$$

with $b_i \in I_n$ and $x_i \in A$. Applying the involution $*$ yields

$$a = a^* = \sum_i b_i x_i^*.$$

Thus

$$2a = \sum_i b_i (x_i + x_i^*).$$

Since $x_i + x_i^* \in A^+$ and $2 \in A^{\times}$ it follows that $a = \frac{1}{2}\sum_i b_i(x_i + x_i^*) \in I_n$. In particular, $I_m = I_n$. □

We would like to show that A is finitely generated as an A^+-module, which is equivalent to A being a noetherian A^+-module since A^+ is a noetherian ring by Lemma A.16. The following lemma follows the proof of [Lan75, Lemma 6].

Lemma A.17. *If there is an element $d \in A^-$ that is not a zero divisor in A, then A is noetherian as an A^+-module.*

Proof. Since d is not a zero divisor, it follows that A is isomorphic to dA as an A^+-module. On the other hand, for any $a \in A$ we can write

$$da = \frac{1}{2}(d(a - a^*)) + \frac{1}{2}(d(a + a^*)) \in A^+ + dA^+.$$

Thus dA is a submodule of the finitely generated A^+-module $A^+ + dA^+$. Since A^+ is noetherian by Lemma A.16, it follows that dA, and hence A, is a finitely generated (and hence noetherian) A^+-module. ☐

Proposition A.18. *If A is a commutative noetherian ring with $2 \in A^\times$, then A is a noetherian A^+-module.*

Proof. This proof combines elements of the proofs of [CL77, Theorem] and [Lan75, Theorem 7].

Suppose not. Let $\mathfrak{a}_0$ be the largest $*$-ideal of A such that $A/\mathfrak{a}_0$ is not a noetherian A^+-module, which exists since A is a noetherian ring and is not noetherian as an A^+-module. Thus, by replacing A with $A/\mathfrak{a}_0$, we may assume that $A/\mathfrak{a}$ is a noetherian A^+-module for any $*$-ideal $\mathfrak{a} \neq 0$.

We claim that, under this assumption, A is reduced. It suffices to show that A is $*$-prime by Lemma A.15. Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero $*$-ideals of A such that $\mathfrak{ab} = 0$. Note that we can view $\mathfrak{a}$ as an $A/\mathfrak{b}$-module since $\mathfrak{ab} = 0$. We know that $\mathfrak{a}$ is noetherian as an $A/\mathfrak{b}$-module since $\mathfrak{a}$ is noetherian as an A-module. Furthermore, $A/\mathfrak{b}$ is a noetherian A^+-module since $\mathfrak{b} \neq 0$. Thus $\mathfrak{a}$ is noetherian as an A^+-module. We also know that $A/\mathfrak{a}$ is a noetherian A^+-module since $\mathfrak{a} \neq 0$. Therefore A is a noetherian A^+-module, a contradiction. Thus A is $*$-prime and hence reduced.

Since A is a noetherian ring, it has only finitely many minimal prime ideals; call them $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Since A is reduced, we have that

$$\bigcap_{i=1}^{n} \mathfrak{p}_i = 0.$$

Note that $n = 1$ corresponds to the case when A is a domain, and in that case we have already seen that A is a noetherian A^+-module by Lemma A.17. Thus we assume henceforth that $n > 1$ and thus each $\mathfrak{p}_i \neq 0$.

If $\mathfrak{p}_i^* \cap \mathfrak{p}_i \neq 0$, then $\mathfrak{p}_i \cap \mathfrak{p}_i^*$ is a $*$-ideal and so $A/(\mathfrak{p}_i \cap \mathfrak{p}_i^*)$ is a noetherian A^+-module. If every $\mathfrak{p}_i$ satisfies $\mathfrak{p}_i \cap \mathfrak{p}_i^* \neq 0$ then we can view A as a subring of

$$\bigoplus_{i=1}^{n} A/(\mathfrak{p}_i \cap \mathfrak{p}_i^*),$$

which is noetherian as an A^+-module. In particular, A is a noetherian A^+-module, a contradiction, which proves the proposition.

Suppose there is some k such that $\mathfrak{p}_k \cap \mathfrak{p}_k^* = 0$. It is easy to check that $\mathfrak{p}_k^*$ is another minimal prime ideal of A. We claim that $n = 2$ in this case. Indeed, if $\mathfrak{p}$ is any minimal prime ideal of A, then we have $\mathfrak{p}_k\mathfrak{p}_k^* \subseteq \mathfrak{p}_k \cap \mathfrak{p}_k^* = 0$ and thus $\mathfrak{p}_k\mathfrak{p}_k^* = 0 \in \mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{p}_k$ or $\mathfrak{p} = \mathfrak{p}_k^*$.

Let us write $\mathfrak{p} = \mathfrak{p}_k$ henceforth. We can embed A into $A/\mathfrak{p} \times A/\mathfrak{p}^*$ by identifying $a \in A$ with $(a + \mathfrak{p}, a + \mathfrak{p}^*)$. Note that $A^+ \cap \mathfrak{p} = 0$ since if $a \in A^+ \cap \mathfrak{p}$ then $a = a^* \in \mathfrak{p}^* \cap \mathfrak{p} = 0$. Similarly, $A^+ \cap \mathfrak{p}^* = 0$. In particular, A^+ injects into $A/\mathfrak{p}$ and is therefore a domain.

Note that by Lemma A.17, we may assume that every element of A^- is a zero divisor in A. However, both $A/\mathfrak{p}$ and $A/\mathfrak{p}^*$ are domains, so the only zero divisors in $A/\mathfrak{p} \times A/\mathfrak{p}^*$ are elements of the form $(a + \mathfrak{p}, \mathfrak{p}^*)$ or $(\mathfrak{p}, a + \mathfrak{p}^*)$. Recall that $(A^-)^2 \subseteq A^+$. In particular, if $(a + \mathfrak{p}, \mathfrak{p}^*) \in A^-$, then $(a^2 + \mathfrak{p}, \mathfrak{p}^*) \in A^+$. That is, there is some $a_+ \in A^+$ such that $a_+ - a^2 \in \mathfrak{p}$ and $a_+ \in \mathfrak{p}^*$. But we have already seen that $A^+ \cap \mathfrak{p}^* = 0$. Similarly, any $(\mathfrak{p}, a + \mathfrak{p}^*) \in A^-$ must be trivial. In other words, $A^- = 0$, a contradiction. Therefore A must be noetherian as an A^+-module. ☐

Given any ideal $\mathfrak{a}$ of A, we define $\mathfrak{a}^\varepsilon := \mathfrak{a} \cap A^\varepsilon$. We call $\mathfrak{a}$ a *graded* ideal if $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$.

Proposition A.19. *Let A be a commutative local noetherian ring such that A and A^+ have the same residue field. Assume that $2 \in A^\times$. If A' is the quotient of A by a nongraded prime ideal, then A' has the same field of fractions as the image of A^+ in A'.*

Proof. Write $f\colon A \to A'$ for the quotient map. It suffices to show that every element of $f(A^-)$ can be written as a quotient of elements in $f(A^+)$. Since the prime ideal $\mathfrak{p} = \ker f$ is assumed to be nongraded, it follows that there is some $a \in \mathfrak{p}$ such that, if we decompose $a = a^+ + a^-$ with $a^+ \in A^+$ and $a^- \in A^-$, then neither a^+ nor a^- is in $\mathfrak{p}$. It follows that $f(a^-) = -f(a^+)$, and so $f(a^-) \in f(A^+)$. Note that $f(a^-) \neq 0$ since $a^- \notin \mathfrak{p}$. For any $x \in A^-$ we have that $xa^- \in A^+$ since $(A^-)^2 \subseteq A^+$. Thus $f(x) = f(xa^-)/f(a^-) \in Q(f(A^+))$, as desired. $\square$

A.4. **Automorphisms and gradings.** We recall how ring automorphisms give rise to gradings.

Let A be a complete local ring and X a finite abelian subgroup of the group of ring automorphisms of A. We write $\mu_n(A) := \{a \in A^\times : a^n = 1\}$. Given a character $\varphi\colon X \to A^\times$, we define

$$A^\varphi := \{a \in A : {}^\sigma a = \varphi(\sigma)a, \forall \sigma \in X\}.$$

The following lemma is standard, so we leave the proof to the reader.

Lemma A.20. *Let $\mathbb{F}$ be a finite field of characteristic p and A a pro-p local ring with residue field $\mathbb{F}$. If $p \nmid n$, then $\mu_n(A) = s(\mu_n(\mathbb{F}))$.*

Assume the following:

($*$) for every positive integer n, if X contains an element of order n, then $\#\mu_n(A) = n$.

Then one has $\#X = \#\operatorname{Hom}(X, A^\times)$. (It is easily checked when X is cyclic, and then for general X one applies the structure theorem of finite abelian groups.)

Corollary A.21. *Assume ($*$). If $p \nmid \#X$, then for any $1 \neq \sigma \in X$ we have*

$$\sum_{\varphi \in \operatorname{Hom}(X, A^\times)} \varphi(\sigma) = 0.$$

Proof. First suppose that X is cyclic of order n and σ is a generator for X. Then $\operatorname{Hom}(X, A^\times)$ is cyclic, generated by any φ_0 such that $\varphi_0(\sigma)$ is a primitive n^{th} root of unity. Let $H := \langle \varphi_0^k \rangle$ be a nontrivial subgroup of $\operatorname{Hom}(X, A^\times)$. Then by Lemma A.20 we have

$$\sum_{\varphi \in H} \varphi(\sigma) = \sum_{i=0}^{n/k} \varphi_0^{ki}(\sigma) = \sum_{\omega \in \mu_{n/k}(A)} \omega = \sum_{\omega \in \mu_{n/k}(\mathbb{F})} s(\omega) = 0.$$

Now we allow X to be any finite abelian group such that $p \nmid \#X$ and σ any nontrivial element of X. Then we have an exact sequence

$$0 \to \operatorname{Hom}(X/\langle\sigma\rangle, A^\times) \to \operatorname{Hom}(X, A^\times) \to \operatorname{Hom}(\langle\sigma\rangle, A^\times).$$

Thus $\sum_{\varphi \in \operatorname{Hom}(X, A^\times)} \varphi(\sigma)$ is an integral multiple of $\sum_{\varphi \in H} \varphi(\sigma)$, where H is the image of $\operatorname{Hom}(X, A^\times)$ in $\operatorname{Hom}(\langle\sigma\rangle, A^\times)$. This sum is 0 by the first paragraph. $\square$

Lemma A.22. *Let A and X be as above. Assume that $\#X \in A^\times$ and that condition ($*$) holds. Then A admits a grading given by $A = \bigoplus_{\varphi \in \operatorname{Hom}(X, A^\times)} A^\varphi$. Furthermore, for any $\mathbb{Z}[1/\#X][X]$-submodule $M \subseteq A$, letting $M^\varphi := M \cap A^\varphi$, there is a decomposition*

$$M = \bigoplus_{\varphi \in \operatorname{Hom}(X, A^\times)} M^\varphi.$$

Proof. For $\varphi \in \operatorname{Hom}(X, A^\times)$, define $e_\varphi := \frac{1}{\#X} \sum_{\sigma \in X} \varphi(\sigma)\sigma^{-1} \in \mathbb{Z}[1/\#X][X]$. A straightforward computation shows that $\{e_\varphi : \varphi \in \operatorname{Hom}(X, A^\times)\}$ is an orthogonal system of idempotents in $\mathbb{Z}[1/\#X][X]$. (Note that Corollary A.21 is needed to show that $\sum_\varphi e_\varphi = 1$.) There is a natural ring homomorphism $\mathbb{Z}[1/\#X][X] \to \operatorname{End} A$; pushing forward the e_φ to $\operatorname{End} A$ gives the result. $\square$

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