

Finding Long Cycles in Balanced Tripartite Graphs: A First Step

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1 Introduction

In this chapter, all graphs are simple. For undefined terms, see [6]. The lack of loops or parallel edges means that if a graph has enough edges, then it must contain long cycles. We use $e(G)$ to denote the number of edges in the graph G . In this chapter, we specifically consider the problem of finding long cycles in balanced tripartite graphs, and prove the following result.

Theorem 1.1 *Let G be a balanced tripartite graph on $3n$ vertices such that $n = 1, 2$ or $n \geq 14$. If $e(G) \geq 3n^2 - 4n + 5$, then G contains a cycle of length at least $3n - 1$.*

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The edge bound in Theorem 1.1 is tight; see Figure 1. While the cases $3 \leq n \leq 13$ are not addressed in Theorem 1.1, we suspect that the theorem also holds for these values of n ; the current proof just does not work for them.

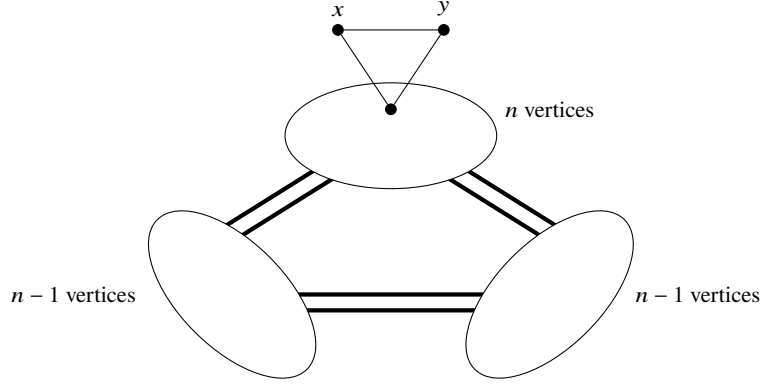


Fig. 1 This graph is a $K_{n,n-1,n-1}$ with two additional degree-2 vertices, x and y , with a common neighbor in the largest partite set. It has $3n^2 - 4n + 4$ edges, and its longest cycle has $3n - 2$ vertices, thus demonstrating that Theorem 1.1 is sharp.

In the next section we survey degree and edge conditions for Hamiltonicity and long cycles in graphs, including bipartite and k -partite results where they exist. This survey provides context for Theorem 1.1 as the first step in the larger problem of determining edge conditions for long cycles in tripartite and k -partite graphs. The proof of Theorem 1.1 appears in Section 3 of this chapter.

As a matter of notation, the variable p will denote the number of vertices of a graph, and n will denote the number of vertices in one part of a (typically balanced) multipartite graph.

2 Background

Given an arbitrary graph, the problem of finding a *Hamiltonian cycle*, i.e. a cycle containing every vertex of the graph, is known to be NP-complete ([19]). However, much study has been devoted to finding sufficient conditions for *Hamiltonicity* (i.e. for a graph to contain a Hamiltonian cycle).

2.1 Degree conditions

The most famous of these results involve degree conditions.

2.1.1 Degree conditions and Hamiltonicity

In 1952, Dirac gave the following simple minimum degree condition for Hamiltonicity.

Theorem 2.1 [10] *For $p \geq 3$, if a p -vertex graph G has $\delta(G) \geq p/2$, then G is Hamiltonian.*

In 1960, Ore proved a variation on Dirac's famous theorem.

Theorem 2.2 [14] *Let G be a p -vertex graph with $p \geq 3$ such that for any pair of nonadjacent vertices u and v , $\deg(u) + \deg(v) \geq p$. Then G is Hamiltonian.*

Ore's result led to the definition of a new graph parameter:

$$\sigma_2(G) = \min\{\deg(u) + \deg(v) \mid u, v \in V(G), u \neq v, uv \notin E(G)\},$$

for $G \neq K_n$, and $\sigma_2(K_n) = \infty$. Results which involve a bound on this parameter are called σ_2 results or Ore-type results.

In 1963, Moon and Moser considered balanced bipartite graphs and were able to lower Dirac's minimum degree bound by a factor of two.

Theorem 2.3 [13] *If G is a balanced bipartite graph on p vertices with $p \geq 4$ and $\delta(G) > p/4$, then G is Hamiltonian.*

They also proved the following Ore-type result with an improved lower σ_2 bound.

Theorem 2.4 [13] *If G is a balanced bipartite graph on p vertices with $p \geq 4$ and $\sigma_2(G) > p/2$, then G is Hamiltonian.*

Chen, Faudree, Gould, Jacobson, and Lesniak extended the Dirac-type Hamiltonicity results to balanced k -partite graphs.

Theorem 2.5 [7] *If G is a balanced k -partite graph on kn vertices and*

$$\delta(G) > \begin{cases} (\frac{k}{2} - \frac{1}{k+1})n & \text{if } k \text{ is odd} \\ (\frac{k}{2} - \frac{2}{k+2})n & \text{if } k \text{ is even,} \end{cases}$$

then G is Hamiltonian.

Chen and Jacobson proved an Ore-type result for Hamiltonicity in balanced k -partite graphs in 1997. Note this result does not use the σ_2 notation because the Chen-Jacobson degree-sum condition is restricted to nonadjacent vertices in different partite sets; thus it implies a σ_2 condition.

Theorem 2.6 [8] *Let G be a balanced k -partite graph of order kn with $k \geq 2$. If*

$$\deg(u) + \deg(v) > \begin{cases} \left(k - \frac{2}{k+1}\right)n & \text{if } k \text{ is odd} \\ \left(k - \frac{4}{k+2}\right)n & \text{if } k \text{ is even} \end{cases}$$

for every pair of nonadjacent vertices u and v which are in different partite sets, then G is Hamiltonian.

In 2019, DeBiasio et al. [9] addressed k -partite graphs that are not necessarily balanced. Assuming a k -partite graph has the necessary condition that no part contains more than half the vertices, they gave a minimum degree condition that implies Hamiltonicity. This minimum degree condition, although somewhat cumbersome to describe, asymptotically implies the result of Chen et al. (above) for balanced k -partite graphs.

A different type of degree condition for Hamiltonicity was given by Pósa in 1962.

Theorem 2.7 [16] *Let G be a graph of order p . If, for every integer r with $1 \leq r < \frac{p}{2}$, the number of vertices of degree at most r is less than r , then G is Hamiltonian.*

Moon and Moser gave a similar result for balanced bipartite graphs in 1963.

Theorem 2.8 [13] *Let G be a balanced bipartite graph of order $2n \geq 4$. If, for each r with $1 \leq r \leq n/2$, the number of vertices of degree at most r is less than r , then G is Hamiltonian.*

2.1.2 Degree conditions and long cycles

The circumference of a graph, denoted $c(G)$, is the length of the longest cycle in G . All of the results we have discussed so far concern Hamiltonian graphs, which always have $c(G) = |V(G)|$. However, minimum degree, σ_2 , and Pósa-type conditions can also imply the existence of more general long cycles in graphs. Dirac's 1952 paper states a result of this type.

Theorem 2.9 (Dirac [10]) *Suppose G is a 2-connected graph on p vertices and $3 \leq c \leq p$. If $\delta(G) \geq c/2$, then G has a cycle of length at least c .*

In 1975, Bermond published a generalization of Ore's theorem.

Theorem 2.10 [5] *Suppose G is a 2-connected graph on p vertices and $3 \leq c \leq p$. If $\sigma_2(G) \geq c$, then G has a cycle of length at least c .*

Long cycle variations of bipartite Hamiltonicity results have also been proved. For example, in 2009 J. Adamus and L. Adamus proved an Ore-type result for long cycles in balanced bipartite graphs.

Theorem 2.11 [2] *Let G be a 2-connected balanced bipartite graph on $2n \geq 4$ vertices such that $\sigma_2(G) \geq n$. Then G contains an even cycle of length at least $2n - 2$.*

In 1963, Pósa generalized his own Hamiltonicity result with the following theorem that gives a degree condition for the existence of long cycles. We will use this theorem to prove our main result in Section 3.

Theorem 2.12 [17] *Suppose G is a 2-connected graph on p vertices, and let $3 \leq c \leq p$. If $|\{v : \deg(v) \leq j\}| < j$ for all $1 \leq j \leq \frac{c-1}{2}$, then G has a cycle of length at least c .*

2.2 Edge conditions

In addition to the degree conditions, it is also possible to guarantee Hamiltonicity or long cycles in a graph using a global edge count. The seminal result on edge conditions for Hamiltonicity was given by Ore in 1961.

Theorem 2.13 (Ore [15]) *If G is a simple graph of order p and $e(G) \geq \frac{1}{2}(p-1)(p-2) + 2$, then G is Hamiltonian.*

A natural problem inspired by Ore's result is the following: For any pair of nonnegative integers p and $k < \frac{p}{2} - 1$, find the minimum integer $g(p, k)$ such that every graph of order p and size at least $g(p, k)$ contains a cycle of length $p - k$. In 1972, Woodall proved the following theorem.

Theorem 2.14 [20] *Let G be a graph of order $p \geq 2k + 3$, $k \in \mathbb{N}$, and*

$$e(G) \geq g(p, k) = \binom{p-k-1}{2} + \binom{k+2}{2} + 1.$$

Then G contains a cycle of length ℓ for each ℓ such that $3 \leq \ell \leq p - k$.

This result is best possible. An extremal graph for this problem is the graph obtained by identifying one vertex of a K_{p-k-1} with one vertex of a K_{k+2} . Note that this graph has exactly $g(p, k) - 1$ edges and cannot contain any cycle of length $p - k$.

Analogous edge conditions have also been found for multipartite graphs. In 1985, Mitchem and Schmeichel proved the following result about balanced bipartite graphs. A bipartite graph is *bipancyclic* if it contains even cycles of all possible lengths. Thus, a bipancyclic balanced bipartite graph is also Hamiltonian.

Theorem 2.15 [12] *Let G be a balanced bipartite graph of order $2n$. If $E(G) \geq n^2 - n + 2$, then G is bipancyclic.*

In 1999, Bagga and Varma proved a variation of this result for bipartite graphs that may not be balanced.

Theorem 2.16 [4] *Let G be a bipartite graph with partite sets X and Y , $|X| = m \leq n = |Y|$. If $e(G) \geq n(m-1) + 2$, then G contains cycles $C_{2\ell}$ for all $2 \leq \ell \leq m$.*

In 2009, Adamus generalized the previous two results with the following theorem.

Theorem 2.17 [3] *Let G be a bipartite graph with partite sets X and Y , $|X| = m \leq n = |Y|$, where $m \geq \frac{1}{2}k^2 + \frac{3}{2}k + 4$, $k \in \mathbb{N}$. If*

$$e(G) \geq n(m-k-1) + k + 1,$$

then either G contains a cycle of length $2m - 2k$, or else $E(G) = n(m-k-1) + k + 1$ and G is isomorphic to a graph in the family $\mathcal{G}_{m,n,k}$ (see below). In both cases, G contains $C_{2\ell}$ for all $2 \leq \ell \leq m - k - 1$.

The family $\mathcal{G}_{m,n,k}$ is defined as the set of bipartite graphs with partite sets X and Y , $|X| = m$, $|Y| = n$, such that $X = A \cup B$ where $|A| = m - k - 1$, $|B| = k + 1$, and $\deg(v) = n$ for every $v \in A$, and $\deg(w) = 1$ for every $w \in B$.

Researchers have also found edge conditions for Hamiltonicity and large circumference in k -partite graphs with $k \geq 3$. Adamus proved a result about Hamiltonicity of balanced tripartite graphs.

Theorem 2.18 [1] *Let G be a balanced tripartite graph of order $3n$, $n \geq 2$. If G has at least $3n^2 - 2n + 2$ edges, then G is Hamiltonian.*

Ferrero and Lesniak [11] later generalized this result to tripartite graphs which may not be balanced. The following theorem, which we will use in proving our main result, is a combination of their Theorem 18, Corollary 21, Theorem 22, and Theorem 23, respectively.

Theorem 2.19 [11] *Let $k \geq 3$ and let $n_1 \geq n_2 \geq \dots \geq n_k$ be positive integers with $p = \sum_{i=1}^k n_i$. Let G be a k -partite graph with part sizes $n_1, n_2, \dots, n_k$. Then G is Hamiltonian provided that $e(G) \geq e(K_{n_1, n_2, \dots, n_k}) - (p - n_1 - 2)$ and $n_1 \leq p/2$.*

3 Main Result

In order to prove the main result, we will use Theorems 2.6, 2.12, and 2.19 (above) and a classic extremal result by Turán. For ease of reference the theorem is restated here.

Theorem 3.1 [18] *Let G be a graph on n vertices that does not contain K_{r+1} as a subgraph. Then $e(G) \leq \left(\frac{n}{r}\right)^2 \binom{r}{2}$. Moreover, the complete r -partite graph such that all partite sets have cardinality $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$ is the unique graph that contains the maximum number of edges for a given n and r .*

Theorem 1.1 *Let G be a balanced tripartite graph on $3n$ vertices such that $n = 1, 2$ or $n \geq 14$. If $e(G) \geq 3n^2 - 4n + 5$, then G contains a cycle of length at least $3n - 1$.*

Proof For $n = 1$ the theorem holds vacuously. Let $n = 2$ or $n \geq 14$. Let G be a balanced tripartite graph on $3n$ vertices such that $e(G) \geq 3n^2 - 4n + 5$. Observe that if G is viewed as a subgraph of $K_{n,n,n}$, then its construction involves deleting at most $4n - 5$ edges from $K_{n,n,n}$. Thus, subgraphs of $K_{n,n,n}$ that would require the deletion of more than $4n - 5$ edges from $K_{n,n,n}$ are impossible. This idea will be used frequently in the proof.

Claim If $\delta(G) \leq 2$, then G contains a cycle of length $3n - 1$.

Suppose G has a vertex v with $\deg(v) \leq 2$. Observe that $G - v$ is a tripartite graph with part sizes $n, n, n - 1$ such that a largest partite set contains at most half of the vertices. Moreover,

$$e(G - v) \geq e(G) - 2 \geq (3n^2 - 4n + 5) - 2 = 3n^2 - 4n + 3$$

and

$$e(K_{n,n,n-1}) - (3n - 1 - n - 2) = 3n^2 - 4n + 3.$$

Thus, Theorem 2.19 implies $G - v$ is Hamiltonian. Thus, G contains a cycle of length $3n - 1$ and the claim holds. $\square$

Now we can assume $\delta(G) \geq 3$. If $n = 2$, then Theorem 2.1 implies G is Hamiltonian, so we can assume that $n \geq 3$.

Claim G is 2-connected.

Suppose G has at least two components. Let H be a smallest component of G with partite sets H_1 , H_2 , and H_3 . Let J_1 , J_2 , and J_3 be the partite sets of $G - H$ such that $H_i \cup J_i$ is a partite set of G for each i .

Observe that no tripartite graph on 4 or fewer vertices can have minimum degree 3; thus, we can assume $|H| \geq 5$.

Observe that G is missing all the edges between H_i and $J_{i+1} \cup J_{i+2}$, where indices are taken modulo 3. Let $h = |H|$ and $h_i = |H_i|$. Thus, construction of G from $K_{n,n,n}$ would require deleting

$$h_1(2n - h_2 - h_3) + h_2(2n - h_1 - h_3) + h_3(2n - h_1 - h_2) \quad (1)$$

$$= 2nh - 2e(K_{h_1,h_2,h_3}) \geq 2nh - \frac{2h^2}{3} \quad (2)$$

edges. Define $f(h) = 2nh - \frac{2h^2}{3}$. Note that the inequality follows from an application of Turán's theorem (Theorem 3.1).

Since H is a smallest component, it follows that $5 \leq h \leq 3n/2$ which implies that $n \geq 4$. Observe that $f(h) \geq 0$ on $[5, 3n]$, and $f(h)$ is increasing for $h \leq 3n/2$. Thus, $f(h)$ is minimized when $h = 5$ and, consequently, the number of edges deleted from $K_{n,n,n}$ to construct G is at least $f(5) = 10n - 50/3$, which is greater than $4n - 5$ for all $n \geq 3$.

Thus, G must be connected.

Suppose there exists a cut-vertex v in G . Let H be a smallest component of $G - v$, and label the partite sets of H and $G - H$ as in the previous case. Without loss of generality, we can assume $v \in J_1$. We will apply the same argument from the previous disconnected case with slightly different counts resulting from possible adjacencies to v .

As before, the observation that no tripartite graph on 4 or fewer vertices can have minimum degree 3 implies that the component H along with the cut vertex v must consist of at least 5 vertices. Thus, $|H| \geq 4$ which implies that $n \geq 3$. If $|H| = 4$, then $e(G) < 8 + (n - 1)^2 + (n - 2)(2n - 2) = 3n^2 - 8n + 13$, which is too few edges when $n \geq 3$.

Thus, we can assume that $|H| \geq 5$. As before, we know that G is missing all the edges between H_i and $J_{i+1} \cup J_{i+2}$ except possibly those to v , where indices are taken modulo 3. Thus, construction of G from $K_{n,n,n}$ would require deleting at least

$$h_1(2n - h_2 - h_3) + h_2(2n - h_1 - 1 - h_3) + h_3(2n - h_1 - 1 - h_2) \quad (3)$$

$$= (2n - 1)h - 2e(K_{h_1, h_2, h_3}) \geq (2n - 1)h - \frac{2h^2}{3} \quad (4)$$

edges. Define $g(h) = (2n - 1)h - \frac{2h^2}{3}$. Again, the function $g(h)$ is minimized when $h = 5$. Thus the minimum number of edges deleted from $K_{n,n,n}$ to construct G is $g(5) = 10n - 65/3$, which is larger than $4n - 5$ for all $n \geq 3$.

Thus, G must be 2-connected and the claim is proven. $\square$

Now that G is 2-connected, Pósa's Theorem (Theorem 2.12) can be applied. If its hypotheses are satisfied for all c such that $c \leq 3n - 1$, then G contains a cycle of length at least $3n - 1$.

If the hypotheses of Pósa's Theorem fail, then there exists a value of j such that $3 \leq j \leq \frac{3n-2}{2}$ and $|\{v : \deg(v) \leq j\}| \geq j$. Let r be such a value of j .

Claim $r > \frac{3n-5}{2}$.

Since G has at least r vertices of degree at most r , in constructing G from $K_{n,n,n}$, we must delete at least $2n - r$ edges from at least r vertices. Furthermore, at most $3(r/3)^2$ of the deleted edges may have both endpoints among the r low-degree vertices. Therefore,

$$4n - 5 \geq e(K_{n,n,n}) - e(G) \geq r(2n - r) - 3(r/3)^2 = 2nr - \frac{4}{3}r^2.$$

Solving for n , we obtain

$$n \leq \frac{(4/3)r^2 - 5}{2r - 4} = \frac{2}{3}r + \frac{4}{3} + \frac{1}{6}(r - 2)^{-1}.$$

Since $3 \leq r$, we obtain

$$n \leq \frac{2}{3}r + \frac{4}{3} + \frac{1}{6} = \frac{4r + 9}{6}.$$

Solving for r , we obtain

$$r \geq \frac{3n - 5}{2} + \frac{1}{4}$$

and the claim has been proven. $\square$

At this point, we jump over values $n = 3, 4, \dots, 13$ and assume $n \geq 14$.

Observe that if G is Hamiltonian then we have the desired long cycle. If G is not Hamiltonian then by Theorem 2.6, there must exist two nonadjacent vertices in different partite sets such that their degree sum is at most $5n/2$. Thus, the two vertices of smallest degree must have degree sum at most $5n/2$.

We will again use Theorem 2.12 and recall that if the hypotheses of this Theorem fail, they must do so for a value of r such that $r > (3n - 5)/2$. Since $n \geq 14$, it follows that

$$r > (3 \cdot 14 - 5)/2 = 17.5 > 10.$$

Thus, if the hypotheses of Theorem 2.12 fail, there must exist at least 10 vertices all with degrees bounded above by $(3n - 2)/2$.

Let $R = \{v_1, \dots, v_{10}\}$ be the ten lowest-degree vertices of G , each of which necessarily has degree at most $(3n - 2)/2$. Let v_1 and v_2 be two vertices of R of smallest degree. Considering only degrees of vertices of R , constructing G from $K_{n,n,n}$ requires deleting at least

$$\left(4n - \frac{5n}{2}\right) + 8 \left(2n - \frac{3n-2}{2}\right) - 33 \quad (5)$$

where $4n - \frac{5n}{2}$ is the number of edges deleted from v_1 and v_2 , $8 \left(2n - \frac{3n-2}{2}\right)$ is the number of edges deleted from the remaining 8 vertices of R , and at most 33 edges were counted twice in the previous two terms. Simplifying Equation 5, we conclude that failing the hypotheses of Theorem 2.12 results in the deletion of $5.5n - 25$ edges, which is more than $4n - 5$ provided $n \geq 14$. Thus, Theorem 2.12 holds for $c = 3n - 1$, and it follows that G contains a cycle of length at least $3n - 1$. $\square$

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