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Co-designed sampled-data output consensus for multi-agent systems

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Abstract

Sampled-data output consensus problem has been firstly considered within co-design framework where control gains and sampling periods are incorporated together. In construction of co-designed controllers, we firstly introduce a scaled coordinate transformation for the original system in the continuous-time domain. Then design of sampled-data controllers with absolute/relative damping and stability analysis of closed-loop system have been conducted directly in discrete-time domain. Main novelty of the co-design method lies in that the sampling period can be arbitrarily large due to the compensation of control gains. Compared with previous work on sampled-data consensus protocols, conservatism on allowable sampling periods has been greatly relaxed and more flexibility has been provided for digital implementation of the obtained sampled-data controllers. Simulation studies with different sampling periods are also provided to illustrate the effectiveness of the co-designed sampled-data output controllers.

KEYWORDS

co-design, multi-agent systems, output consensus, sample-data consensus

1 | INTRODUCTION

Multi-agent systems (MAS) has drawn a lot of attention in the past decades due to its wide applications in cooperative control of multiple vehicles such as formation control, search and rescue in disaster sites, military surveillance and reconnaissance, exploration and exploitation in unknown environments.[1] The fundamental problem behind these applications is the consensus problem which means all agents (or dynamic systems) reach an agreement regarding a certain quantity of interest that depends on the state of all agents. Thus a lot of consensus algorithms or interaction rules that specify the way of exchanging information between one agent and its neighbors have been intensively studied for different systems,[2,3] such as consensus of integrator,[4,5] consensus of linear MAS,[6,7] and even consensus of nonlinear MAS.[8-10]

With the revolutionary advances in microelectronics, almost all of the control systems constructed today are based on microprocessor and micro-controller which renders that it is impossible and unnecessary to exchange information continuously or control each agent continuously. Sampled-data controller design has received a lot of attention of many researchers in recent years.[11] Parallel to consensus problem in continuous-time domain, sampled-data consensus problem has also received a lot of attention. To the best of our knowledge, cooperative control of MAS with sampled information has been firstly reported by Tomohisa[12] where energy-based controllers have been inspired by the energy dissipation of mass-spring-damper system. Then necessary and sufficient conditions[5] depending on sampling period and communication topology have been achieved for integrator of second-order under sampled-data consensus protocols with absolute or relative damping. Using method of emulation, sampled-data consensus protocol for second-order integrator has also been obtained[13] by directly discretizing its consensus protocols in continuous-time domain. Due to the important role of second-order integrator in modeling dynamical system in physical world, some other types of sampled-data consensus protocols have also been considered such as consensus when only relative position is available[14] or consensus under dynamic communication topology.[15] Due to inherent restrictions on controller design methods, the finally allowable sampling period is inevitably small which is not easy for hardware implementation of obtained controllers.

Based on the above research on consensus of second-order integrator, mean square consensus of linear-time-invariant system can be achieved[6] as long as the sampling period is sufficiently small. Sampled-data consensus of nonlinear system such as fully actuated ships has also been considered[16] with the help of some state and input transformations so that sampled-data consensus protocols for integrator can be directly extended. As far as intrinsic general nonlinear system is concerned such as those restricted by Lipschitz condition, the upper bound of allowable sampling intervals has been presented in terms of some complex LMIs[17] after transforming the sampled-data consensus of nonlinear MAS into stabilization problem of an equivalent nonlinear system with a time-varying delay. For nonlinear system of special geometrical structure such as strict-feedback system[18] or upper-triangular system,[19] method of directly design in the discrete-time domain and method of emulation have also been utilized in construction of sampled-data consensus protocols. Although sampled-data controllers have been constructed for MAS of integrator, linear-time-invariant system and even nonlinear system, requirement of small sampling period has been imposed in most of the obtained results or conditions on sampling period have been implicitly expressed in some complex LMIs which is hard to use in real applications.

As shown above, the requirement of small sampling period has been widely imposed on sampled-data consensus problem no matter the MAS is composed of integrator system, linear-time-invariant system or nonlinear system. This small requirement on small sampling period has made the authors to consider the necessity of this requirement and the possibility of removing it so that hardware requirements can be further relaxed especially in their digital implementations. The main reason of imposing this requirement lie in that there is an explicit or implicit "two-step design process" in construction of sampled-data controllers in the previous research. In other words, the control gain has always been specified firstly and then the sampling period is tuned to guarantee stability of the closed-loop system. Main drawbacks of the independent design between control gain and sampling period are that only small sampling period is allowable in the end. Although this kind of disadvantages has been noticed in only a few works,[12,17] the sampled-data consensus protocols have not been explicitly considered within the framework of co-design. Based on the above consideration, sampled-data controllers for MAS of second-order integrator have been reconstructed in this article where control gain and sampling period have been co-designed directly in the discrete-time domain. With the co-designed output feedback controller, consensus of second-order integrator can be guaranteed under any arbitrary large sampling period due to the compensation of control gain. Besides the flexibility of choosing arbitrary sampling period endowed by the framework of co-design, results obtained in this article will also contribute to explain some numeric phenomena that are contrary to the common view[17] such as the larger the coupling strengths, the easier the consensus.

The main aim of this article is to propose co-designed consensus protcocols that is directly constructed in discrete-time domain, where

- Sampled-data feedback controllers have been constructed directly in the discrete-time domain without resorting to the emulation method where conservative of the obtained controllers has been reduced;
- Arbitrary large sampling period can be selected in consensus of MAS of second-integrator under the sampled-data controllers proposed which means that much more flexibility has been provided in the digital implementation of obtained controllers;
- Sampled-data output consensus of all states have been successfully realized without constructing any dynamic observer for each agent's velocity where backward difference has been employed to estimate them. Due to relaxation of requirements on sampling period, the time-step adopted in the difference approximation is neither required to be sufficient small where requirements on hardware has been further reduced.

We believe that the main result of this article is a good compensation to the previous work on sampled-data control of linear and nonlinear system and there are still a lot of research can be considered within the co-designed framework such as co-design of sampling period and control gain for some special nonlinear system such as feedforwrd systems[20] or nonlinear systems in the p-normal form[21] which have uncontrollable and unobservable linearization around the origin, or even combination of the co-designed time-regulation method of this article with those event-trigger strategies to find a minimal dwell-time and prevent the Zeno phenomenon where a lot of work[22-26] has been done.

The contents of this article are organized as follows. In Section 2, we provide some definitions and lemmas that will be used in the following sections. Co-design of sampled-data consensus protocols with absolute or relative damping and their convergence analysis have been presented in Section 3, respectively. In Section 4, simulation studies have been presented to illustrate effectiveness of the main results obtained in Section 3. Finally, Section 5 contains the conclusions of this article.

2 | PROBLEM STATEMENT

2.1 | Problem statement

Sampled-data consensus of the following double-integrator systems

$$\begin{cases} \dot{r}_{i1} = v_{i2}, \\ \dot{v}_{i2} = u_i, \\ y_i = r_{i1}, \end{cases} \tag{1}$$

will be considered in this article, where $i = 1, 2, \ldots, N$ is the index of each agent. These N agents can be represented as N vertices and their communication topology is represented by graph $G_N = (\mathcal{V}_{\mathcal{N}}, E_N)$, where $\mathcal{V}_{\mathcal{N}} = \{1, \ldots, N\}$ is the node set and $E_N \subseteq \mathcal{V}_{\mathcal{N}} \times \mathcal{V}_{\mathcal{N}}$ is the edge set. Denote $A_N \in R^{N\times N}$ as the adjacency matrix of the directed graph G_N where $a_{ii} = 0$, and $a_{ij} = 0$ if node i cannot receive information from node j otherwise $a_{ij} > 0$. For undirected graph, A_N will be symmetrical matrix.

Definition 1. Consensus of system (1) is globally reached if under some appropriate sampled-data consensus protocols and for any initial position and initial velocity, the followings

$$\lim_{k\to\infty} r_i[k] = \lim_{k\to\infty} r_j[k] \quad \text{and} \quad \lim_{k\to\infty} v_i[k] = 0$$

or

$$\lim_{k\to\infty} r_i[k] = \lim_{k\to\infty} r_j[k] \quad \text{and} \quad \lim_{k\to\infty} v_i[k] = \lim_{k\to\infty} v_j[k]$$

are satisfied, where $i, j \in \mathcal{V}_{\mathcal{N}}$, k is the discrete-time index and T is the sampling period.

The main aim of this article is to find some sampled-data output feedback consensus protocols

$$u_i(t) = \alpha(y_i(t_k), y_i(t_{k-1}), y_j(t_k), y_j(t_{k-1})), \tag{2}$$

where $t \in [t_k, t_{k+1})$, $j \in \mathcal{N}_i$, $k = 1, 2, \ldots$, such that consensus defined in Definition 1 is satisfied where the sampled-data controller is digitally implemented through a zero-order-holder (ZOH), that is, $u(t) = u(t_k)$ over the time interval $[t_k, t_{k+1})$.

Lemma 1 (Schur's formula[27]). *Assume that*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a partitioned matrix, where $A \in R^{n\times n}, B \in R^{n\times n}, C \in R^{n\times n}, D \in R^{n\times n}$. *Then*

$$\det(M) = \det(AD - BC)$$

if A, B, C, D *are pairwise commute.*

Lemma 2 (7). *Let $A = [a_{ij}] \in R^{n\times n}$ be positive ($a_{ij} > 0$). The algebraic multiplicity of $\rho(A)$ as an eigenvalue of A is 1. If x and y are the right and left Perron vectors of A, then*

$$\lim_{m\to\infty} (\rho(A)^{-1}A)^m = xy^T,$$

which is a positive rank-one matrix, where $\rho(A)$ is the spectral radius of A.

3 | CODESIGNED CONSENSUS WITH ABSOLUTE DAMPING

3.1 | Codesigned sampled-data consensus protocol

Consensus of system (1) has been realized by the following continuous proportional-derivative-like consensus protocol[4]

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\left(r_{j1}(t) - r_{i1}(t)\right) - \alpha v_{i2}(t), \tag{3}$$

under undirected and connected communication topology, where $\alpha > 0$ is constant. Corresponding to the continuous protocol (3), its sampled-data version

$$u_i[k] = \sum_{j\in\mathcal{N}_i} a_{ij}[k]\left(r_{j1}[k] - r_{i1}[k]\right) - \alpha v_{i2}[k], \tag{4}$$

has also been proposed.[7] However, the implicit conditions imposed on allowable sampling period T and control parameter α is not easy to use in engineering applications. Different to the previous results, the following co-designed sampled-data consensus protocol will be considered in this article

$$u_i[k] = \sum_{j\in\mathcal{N}_i} a_{ij}k_1\left(r_{j1}[k] - r_{i1}[k]\right) - k_2\frac{r_{i1}[k] - r_{i1}[k-1]}{T}, \tag{5}$$

where the velocity term has been replaced with its backward difference approximation and k_i are constants to be co-designed.

After applying the following coordinate transformation

$$\begin{cases} z_{i1} = r_{i1}, \\ z_{i2} = v_{i2}/L, \\ \tau_i = u_i/L^2, \end{cases} \tag{6}$$

where $L > 0$ is constant to be determined, system (1) can be written as

$$\begin{cases} \dot{z}_{i1} = Lz_{i2}, \\ \dot{z}_{i2} = L\tau_i, \end{cases} \tag{7}$$

whose equivalent system in the discrete-time domain can be described by

$$\begin{cases} z_{i1}\left[k+1\right] = z_{i1}\left[k\right] + LTz_{i2}\left[k\right] + \frac{L^2T^2}{2}\tau_i\left[k\right], \\ z_{i2}\left[k+1\right] = z_{i2}\left[k\right] + LT\cdot\tau_i\left[k\right]. \end{cases} \tag{8}$$

Denote $z_{i0}[k] = z_{i1}[k-1]$ and rearrange the order of each agent's state, the closed-loop system (8) under the sampled-data controller $\tau_i[k] = u_i[k]/L^2$ is

$$\begin{bmatrix} Z_0[k+1] \\ Z_1[k+1] \\ Z_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & I_N & 0 \\ \frac{Tk_2}{2} I_N & \left(1-\frac{Tk_2}{2}\right) I_N - \frac{T^2}{2} k_1 \mathcal{L}_N & LTI_N \\ \frac{k_2}{L} I_N & -\frac{k_2}{L} I_N - \frac{T}{L} k_1 \mathcal{L}_N & I_N \end{bmatrix} \begin{bmatrix} Z_0[k] \\ Z_1[k] \\ Z_2[k] \end{bmatrix}, \tag{9}$$

where $\mathcal{L}_N$ is the Laplacian matrix of the communication topology among these N agents, $Z_0[k] = \left[z_{10}[k], z_{20}[k], \dots, z_{N0}[k]\right]^T$, $Z_1[k] = \left[z_{11}[k], z_{21}[k], \dots, z_{N1}[k]\right]^T$ and $Z_2[k] = \left[z_{12}[k], z_{22}[k], \dots, z_{N2}[k]\right]^T$. For purpose of co-design between control gain and sampling period. Then we select $k_1 = L^2$ and $k_2 = L$ and then the closed-loop system (9) can be rewritten as

$$Z[k+1] = \Omega Z[k], \tag{10}$$

where $Z[k] = \left[Z_0[k], Z_1[k], Z_2[k]\right]^T$ and

$$\Omega = \begin{bmatrix} 0 & I_N & 0 \\ \frac{LT}{2} I_N & \left(1-\frac{LT}{2}\right) I_N - \frac{(LT)^2}{2} \mathcal{L}_N & LTI_N \\ I_N & -I_N - LT\mathcal{L}_N & I_N \end{bmatrix}.$$

Remark 1. Compared with previous consensus results on MAS of second-order integrator where control gain and sampling period have been selected independently, much more flexibility have been provided by the co-designed consensus protocols of this article. Within the framework of co-design, it can be obtained that consensus of the closed-loop system (10) is totally determined by the communication topology $\mathcal{L}_N$ and the value of LT. Thus the selection of control gain L and the choosing of sampling period are incorporated together T which means that some large and unallowable sampling period now is possible due to the compensation of control gain.

3.2 | Convergence analysis

Theorem 1. *Suppose that the undirected graph $\mathcal{G}$ is connected. Consensus of system (1) can be guaranteed by the following co-designed sampled-data consensus protocol*

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} L^2 \left(r_{j1}[k] - r_{i1}[k]\right) - \frac{L}{T} r_{i1}[k] + \frac{L}{T} r_{i1}[k-1], \tag{11}$$

where $t \in [t_k, t_{k+1})$, $L > 0$ is the scaling gain, T is the sampling period and for each eigenvalue $\mu_i < 0 (i = 1, 2, \dots, N)$ of the Laplacian matrix $\mathcal{L}_N$ of undirected graph $\mathcal{G}$, the following is satisfied

$$LT \in \left(0, \quad min\left\{2, \frac{1-\mu_i-\sqrt{(\mu_i-1)^2-4\mu_i}}{\mu_i}\right\}\right).$$

In particular, $r_i[k] \to p^T r(0) + p^T v(0)$ and $v_i[k] \to 0$ as $k \to \infty$.

Proof. The characteristic polynomial of system matrix Ω is given by

$$\begin{aligned} |zI_{3N} - \Omega| &= det \begin{bmatrix} zI_N & -I_N & 0 \\ -\frac{LT}{2} I_N & \left(z - 1 + \frac{LT}{2}\right) I_N + \frac{(LT)^2}{2} \mathcal{L}_N & -LTI_N \\ -I_N & I_N + LT\mathcal{L}_N & (z-1)I_N \end{bmatrix} \\ &= det \begin{bmatrix} \left(z^2 - z + \frac{LT}{2} z - \frac{LT}{2}\right) I_N + \frac{(LT)^2}{2} z\mathcal{L}_N & -LTI_N \\ (z-1) I_N + zLT\mathcal{L}_N & (z-1)I_N \end{bmatrix}. \end{aligned}$$

Using the Schur's formula in Lemma 1, the above characteristic polynomial can be written as

$$|zI_{3N} - \Omega| = det\left[\left(z^3 - \left(2 - \frac{LT}{2}\right)z^2 + \left(1 - \frac{LT}{2}\right)z + \frac{LT}{2}\right)I_N + (z^2 - z)\frac{(LT)^2}{2}\mathcal{L}_N + (z-1)\,LTI_n + z(LT)^2\mathcal{L}_N\right]$$
$$= det\left[\left(z^3 - \left(2 - \frac{LT}{2}\right)z^2 + z - \frac{LT}{2}\right)I_N + (z^2 + z)\frac{(LT)^2}{2}\mathcal{L}_N\right].$$

Thus the relationship between eigenvalues of Ω and that of $-\mathcal{L}_N$ is

$$\left(z^3 - \left(2 - \frac{LT}{2}\right)z^2 + z - \frac{LT}{2}\right)I_N - (z^2 + z)\frac{(LT)^2}{2}\mu_i = 0, \tag{12}$$

where μ_i is the ith eigenvalue of $-\mathcal{L}_N$.

Based on (12), there are three eigenvalues z_i of Ω corresponding to each eigenvalue μ_i of $-\mathcal{L}_N$. Since the Laplacian matrix $\mathcal{L}_N$ has a simple zero eigenvalue and all the other eigenvalues are positive under undirected and connected communication topology, then we will consider two different cases in the followings.

- $\mu_1 = 0$;

  Denote $\frac{LT}{2} = c$, the characteristic polynomial (12) can be written as

  $$z^3 - (2-c)z^2 + z - c = (z-1)\left(z^2 - (1-c)z + c\right),$$

  where 1 is obviously an eigenvalue. After applying the Tustin's transformation $z = \frac{s+1}{s-1}$ to the remaining part, we can verify that the eigenvalues of

  $$2cs^2 + 2(1-c)s + 2 = 0 \tag{13}$$

  have negative real part if and only if $c < 1$. In a word, for the eigenvalue $\mu_1 = 0$ of $-\mathcal{L}_N$, one of the three eigenvalue of Ω is 1 and the other two lie within the unit circle.
- $\mu_i < 0\ (i = 2, 3, \ldots, N)$;

  The characteristic polynomial (12) in this case is

  $$z^3 - (2 - c + 2c^2\mu_i)z^2 + (1 - 2c^2\mu_i)z - c = 0.$$

  Applying the Tustin's transformation again, the characteristic polynomial in continuous-time domain is

  $$-4c^2\mu_i s^3 + 4cs^2 + \left(4 - 4c + 4c^2\mu_i\right)s + 4 = 0. \tag{14}$$

  Based on the Hurwitz criterion, it can be verified that all s_i of (14) have negative real parts if the following condition is satisfied

  $$\mu_i c^2 + (\mu_i - 1)c + 1 > 0, \tag{15}$$

  where $c > 0$ is constant and $\mu_i < 0(i = 2, \ldots, N)$ is the eigenvalue of $-\mathcal{L}_N$. Thus under appropriate selection of c, all eigenvalues of Ω, corresponding to $\mu_i < 0(i = 2, \ldots, N)$ of $-\mathcal{L}_N$, lie within the unit circle.

Note that $\left[1_N^T \quad 1_N \quad 0_N^T\right]^T$ and $\left[p_N^T \quad 0_N \quad p_N^T\right]$ are right eigenvector and left eigenvector of Ω associate with the eigenvalue 1, where the nonnegative $p \in R^N$ satisfying $p^T\mathcal{L}_N = 0_N$ and $p^T 1_N = 1$. Since 1 is the unique eigenvalue with maximum modulus, then it follows that

$$\lim_{k\to\infty}\Omega^k = \left[1_N^T \quad 1_N \quad 0_N^T\right]^T\left[p_N^T \quad 0_N \quad p_N^T\right].$$

Therefore, consensus of system (1) can be realized by the co-designed consensus protocol (11) as

$$\lim_{k\to\infty}\begin{bmatrix} Z_0[k] \\ Z_1[k] \\ Z_2[k] \end{bmatrix} = \begin{bmatrix} p^T & 0 & p^T \\ p^T & 0 & p^T \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} p^T x_1(0) + p^T x_3(0) \\ p^T x_1(0) + p^T x_3(0) \\ 0 \end{bmatrix}.$$

■

Remark 2. Based on the analysis of (13) and (15) shown in the above proof, the interval for LT is $(0, \quad T_0)$, where $T_0 = min\left\{2, \frac{1-\mu_i-\sqrt{(\mu_i-1)^2-4\mu_i}}{\mu_i}\right\}$, and $\mu_i < 0 (i = 1, 2, \dots, N)$ is an eigenvalue of Laplacian matrix $\mathcal{L}_N$. Thus existence of LT is guaranteed and once the value of LT is specified, the selection of sampling period T and that of control gain L are incorporated together. Due to the co-design between sampling period and control gain, arbitrary large sampling period can be selected with appropriate control gain which is much preferred especially in digital implementation of obtained controllers.

4 | CODESIGNED CONSENSUS WITH RELATIVE DAMPING

4.1 | Codesigned sampled-data consensus protocol

Parallel to co-designed consensus protocol with absolute damping, we will consider co-designed consensus protocol with relative damping in this section. Based on the following continuous-time consensus protocol[4] with relative damping

$$u_i(t) = \sum a_{ij}\left[k_1\left(x_{j1}(t) - x_{i1}(t)\right) + k_2\left(x_{j2}(t) - x_{i2}(t)\right)\right],$$

its sampled-data version

$$u_i[k] = \sum_{j\in\mathcal{N}_i} a_{ij}\left\{\left[(k_1 + \frac{k_2}{T})x_{j1}[k] - \frac{k_2}{T}x_{j1}[k-1]\right] - \left[(k_1 + \frac{k_2}{T})x_{i1}[k] - \frac{k_2}{T}x_{i1}[k-1]\right]\right\} \tag{16}$$

will be co-designed in the followings, where $x_{j2}(t)$ has been approximated by $\frac{x_{j1}[k]-x_{j1}[k-1]}{T}$, k_1 and k_2 are constants to be co-designed.

Similar to what has been done in the previous section, the closed-loop system composed by (1) and (16) can be described by

$$\begin{bmatrix} Z_0[k+1] \\ Z_1[k+1] \\ Z_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & I_N & 0 \\ \frac{T^2}{2}\frac{k_2}{T}\mathcal{L}_N & I_N - \frac{T^2}{2}(k_1 + \frac{k_2}{T})\mathcal{L}_N & LTI_N \\ \frac{T}{L}\frac{k_2}{T}\mathcal{L}_N & -\frac{T}{L}(k_1 + \frac{k_2}{T})\mathcal{L}_N & I_N \end{bmatrix}\begin{bmatrix} Z_0[k] \\ Z_1[k] \\ Z_2[k] \end{bmatrix}.$$

Then under the selection of $k_1 = L^2$ and $k_2 = L$, the above closed-loop system can be rewritten as

$$Z[k+1] = \Xi Z[k], \tag{17}$$

where $Z[k] = \left[Z_0[k], Z_1[k], Z_2[k]\right]^T$ and

$$\Xi = \begin{bmatrix} 0 & I_N & 0 \\ \frac{LT}{2}\mathcal{L}_N & I_N - \left(\frac{(LT)^2}{2} + \frac{LT}{2}\right)\mathcal{L}_N & LTI_N \\ \mathcal{L}_N & (-LT-1)\mathcal{L}_N & I_N \end{bmatrix}.$$

4.2 | Convergence analysis

Theorem 2. *Suppose that the undirected graph $\mathcal{G}$ is connected. Consensus of system (1) can be guaranteed by the following sampled-data co-designed consensus protocol*

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\left\{\left[\left(L^2+\frac{L}{T}\right)x_{j1}[k]-\frac{L}{T}x_{j1}[k-1]\right]-\left[\left(L^2+\frac{L}{T}\right)x_{i1}[k]-\frac{L}{T}x_{i1}[k-1]\right]\right\}, \tag{18}$$

where $t\in[t_k,t_{k+1})$, $L>0$ is the scaling gain, T is the sampling period and for each eigenvalue $\mu_i<0 (i=1,2,\dots,N)$ of the Laplacian matrix $\mathcal{L}_N$ of undirected graph $\mathcal{G}$, the following is satisfied

$$LT\in\left(0,\quad min\left\{2,\frac{1-\mu_i-\sqrt{(\mu_i-1)^2-4\mu_i}}{\mu_i}\right\}\right).$$

In particular, $r_i[k]\to\frac{p^T}{c}r(0)+(k+3)p^Tv(0)$ and $v_i[k]\to\frac{p^T}{c}v(0)$ as $k\to\infty$.

Proof. Denote $c=\frac{LT}{2}$, the characteristic polynomial of closed-loop system (17) is

$$\begin{aligned}|zI_{3N}-\Xi| &= \det\begin{bmatrix} zI_N & -I_N & 0\\ -\frac{LT}{2}\mathcal{L}_N & zI_N-\left(I_N-\left(\frac{(LT)^2}{2}+\frac{LT}{2}\right)\mathcal{L}_N\right) & -LTI_N\\ -\mathcal{L}_N & (LT+1)\mathcal{L}_N & zI_N-I_N\end{bmatrix}\\ &= (-1)^{n+1}\det\begin{bmatrix}(z^2-z)I_n+\left(2c^2+c\right)z\mathcal{L}_n-c\mathcal{L}_n & -\frac{c}{2}I_n\\ (2c+1)z\mathcal{L}_n-\mathcal{L}_n & (z-1)I_n\end{bmatrix}\\ &= (-1)^{n+1}\det\left[(z^3-2z^2+z)I_n+\left((2c^2+c)z^2+2c^2z-c\right)\mathcal{L}_n\right].\end{aligned}$$

Similar to the proof of Theorem 1, the following analysis are also based on the relationship between eigenvalues of Ξ and that of $-\mathcal{L}_N$

$$(z^3-2z^2+z)I_n-\left((2c^2+c)z^2+2c^2z-c\right)\mu_i=0. \tag{19}$$

In the followings, two cases are also considered respectively according to different eigenvalues of $-\mathcal{L}_N$ when the communication topology is undirected and connected.

- $\mu_1=0$;
  The characteristic polynomial (19) in this case is simplified to

  $$z^3-2z^2+z=z(z-1)^2,$$

  where 0 (algebraic multiplicity is 1) and 1 (algebraic multiplicity is 2) are obviously eigenvalues of Ξ, corresponding to $\mu_1=0$ of $-\mathcal{L}_N$.
- $\mu_i<0\,(i=2,3,\dots,N)$;
  After applying the Tustin's transformation, the characteristic polynomial (19) in discrete-time domain will be transformed into the following one in continuous-time domain

  $$(s+1)^3-(2+\mu_i(2c^2+c))(s+1)^2(s-1)+(1-2\mu_ic^2)(s+1)(s-1)^2+\mu_ic(s-1)^3=0,$$

  which can be rewritten as

  $$-4\mu_ic^2s^3-4\mu_ics^2+[4+4\mu_ic^2+4\mu_ic]s+4=0. \tag{20}$$

Similar to derivation of Equation (15), all s_i of (20) have negative real parts based on the Hurwitz criterion if the following sufficient conditions are satisfied

$$\begin{cases} 1 + \mu_i c^2 + \mu_i c > 0 \\ 1 + \mu_i c^2 + \mu_i c > c \end{cases} \Rightarrow \mu_i c^2 + (\mu_i - 1)c + 1 > 0 \tag{21}$$

where $c > 0$ is constant. Thus for each $\mu_i < 0 (i = 2, \dots, N)$ of $-\mathcal{L}_N$, there exist one common interval that derived from Equation (21) such that all s_i of (20) have negative real parts.

After some lengthy computation, it follows that $\begin{bmatrix} 1_N^T & 1_N^T & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1_N^T & 2_N^T & \frac{1}{c}1_N^T \end{bmatrix}^T$ are, respectively, a right eigenvector and a generalized right eigenvector associate with the eigenvalue 1, and $\begin{bmatrix} 1_N^T & 0 & 0 \end{bmatrix}^T$ is a right eigenvector associate with the eigenvalue 0. Similarly, $\begin{bmatrix} 0 & 0 & p^T \end{bmatrix}$ and $\begin{bmatrix} 0 & \frac{1}{c}p^T & p^T \end{bmatrix}$ are, respectively, a left eigenvector and a generalized left eigenvector associate with the eigenvalue 1, and $\begin{bmatrix} \frac{1}{c}p^T & -\frac{1}{c}p^T & 0 \end{bmatrix}$ is a left eigenvector associate with the eigenvalue 0 where p^T is a left eigenvector of $\mathscr{L}_{\mathcal{N}}$ satisfying $p^T \mathscr{L}_{\mathcal{N}} = 0$ and $p^T 1_N = 1$. Note that Ξ can be written in the Jordan canonical form as $\Xi = PJP^{-1}$, where J is the Jordan block diagonal matrix whose diagonal entries are eigenvalues of Ξ, and columns of P are the right eigenvector or generalized right eigenvector, and rows of P^{-1} are the left eigenvector or generalized left eigenvector. Since Ξ has a eigenvalue 1 with algebraic multiplicity of two, a simple 0 and all the other eigenvalues lie within the unit circle, then the following can be verified

$$\begin{aligned} &\lim_{k\to\infty} \left\| PJ^kP^{-1} - \begin{bmatrix} 1_N & 1_N & 1_N \\ 1_N & 2_N & 0 \\ 0 & \frac{1_N}{c} & 0 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{p^T}{c} & p^T \\ 0 & 0 & p^T \\ \frac{p^T}{c} & -\frac{p^T}{c} & 0 \end{bmatrix} \right\| \\ &= \lim_{k\to\infty} \left\| PJ^kP^{-1} - \begin{bmatrix} 0 & \frac{p^T}{c} & (k+2)p^T \\ 0 & \frac{p^T}{c} & (k+3)p^T \\ 0 & 0 & \frac{p^T}{c} \end{bmatrix} \right\| = 0. \end{aligned}$$

Since the solution of the closed-loop system (17) is

$$\begin{bmatrix} Z_0[k] \\ Z_1[k] \\ Z_2[k] \end{bmatrix} = \Xi^k \begin{bmatrix} Z_0[0] \\ Z_1[0] \\ Z_2[0] \end{bmatrix} = PJ^kP^{-1} \begin{bmatrix} Z_0[0] \\ Z_1[0] \\ Z_2[0] \end{bmatrix},$$

consensus of system (1) can be guaranteed by the co-designed sample-data controller (18) as $Z_1[k] \to \frac{p^T}{c} Z_1[0] + (k+3) p^T Z_2[0]$ and $Z_2[k] \to \frac{p^T}{c} Z_3[0]$ when $k \to \infty$. ■

Remark 3. Similar to the computation of LT in Theorem 1, the possible interval for LT is also $(0, \ T_0)$, where $T_0 = min\left\{2, \frac{1-\mu_i-\sqrt{(\mu_i-1)^2-4\mu_i}}{\mu_i}\right\}$, and $\mu_i < 0 (i = 1, 2, \dots, N)$ is an eigenvalue of Laplacian matrix $\mathcal{L}_N$.

Remark 4. Although consensus of all states have been realized, only information of x_{i1} has been used in the sampled-data controllers obtained in Theorems 1 and 2. Without resorting to dynamic observer, information of x_{i2} has been approximated through its backward difference. Thanks to the co-design between sampling period and control gain, the approximation step in the backward difference is neither required to be small which is easy to be implemented using hardware.

5 | SIMULATION STUDIES

In this section, numerical studies are conducted to illustrate the effectiveness of obtained sampled-data controllers under different sampling periods.

5.1 | Consensus under sampled-data consensus protocol with absolute damping

Two different communication topologies shown in Figure 1 have been employed in consensus of MAS composed by four second-order integrator in the following simulation studies. Firstly, the following sampled-data consensus protocol with absolute damping is tested under ring topology

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\left\{\left[\left(L^2+\frac{L}{T}\right)x_{j1}[k]-\frac{L}{T}x_{j1}[k-1]\right]-\left[\left(L^2+\frac{L}{T}\right)x_{i1}[k]-\frac{L}{T}x_{i1}[k-1]\right]\right\}, \tag{22}$$

where the control gain L and the sampling period T can be selected freely under the only requirement that $LT \in (0.1, 0.4)$ for guaranteeing the conditions on eigenvalues in analysis of (12). With $L = 0.15$, $T = 1$, trajectories and control inputs of all agents have been shown in Figures 2 and 3 under ring topology.

For easy of comparison, we have selected $L = 0.1$, $T = 2$ in Figure 4 under chained topology. From the simulation results presented in Figures 2 to 4, it is clear that the sampled-data consensus problem is still solvable even the sampling period is increased to 2 under our co-designed controller (22). Compared with some previous results,[5,7] consensus have been successfully realized by the co-designed controller under different sampling periods and flexibility of choosing different sampling periods has been greatly enhanced which is much preferred in digital implementation of sampled-data controllers.

5.2 | Consensus under sampled-data consensus protocol with relative damping

In this section, we will show the effectiveness of the following sampled-data controller

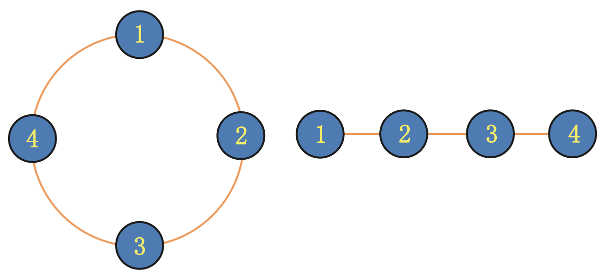

FIGURE 1 Communication topology: Ring topology and chained topology [Colour figure can be viewed at wileyonlinelibrary.com]

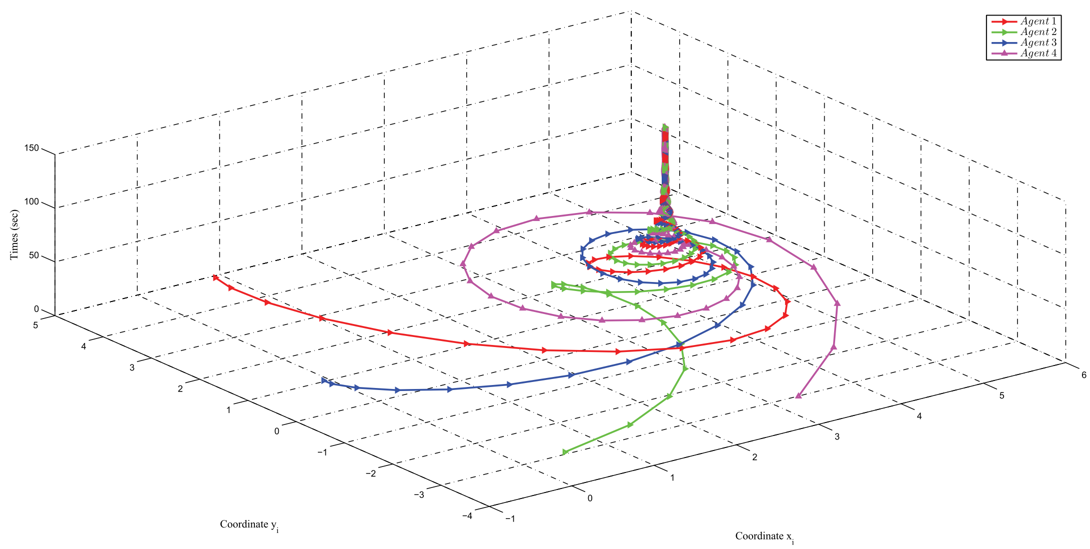

FIGURE 2 All the agents' state trajectories on simulation interval $[0, 120)$ based on Algorithm (11) with $L = 0.15$, $T = 1$ (ring topology) [Colour figure can be viewed at wileyonlinelibrary.com]

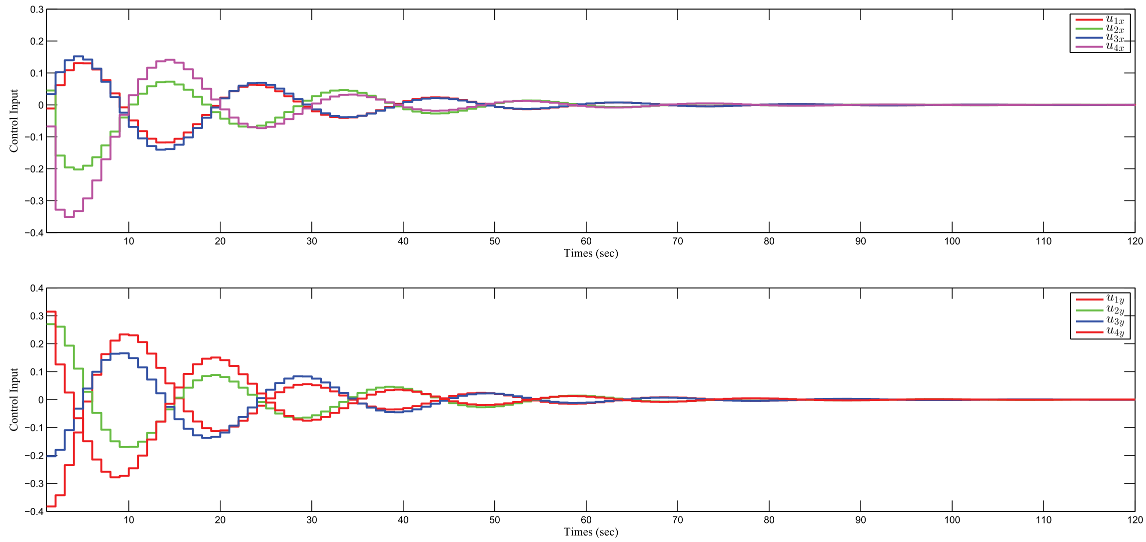

FIGURE 3 All the agents' control input on simulation interval $[0, 120)$ based on Algorithm (11) with $L = 0.15$, $T = 1$ (ring topology) [Colour figure can be viewed at wileyonlinelibrary.com]

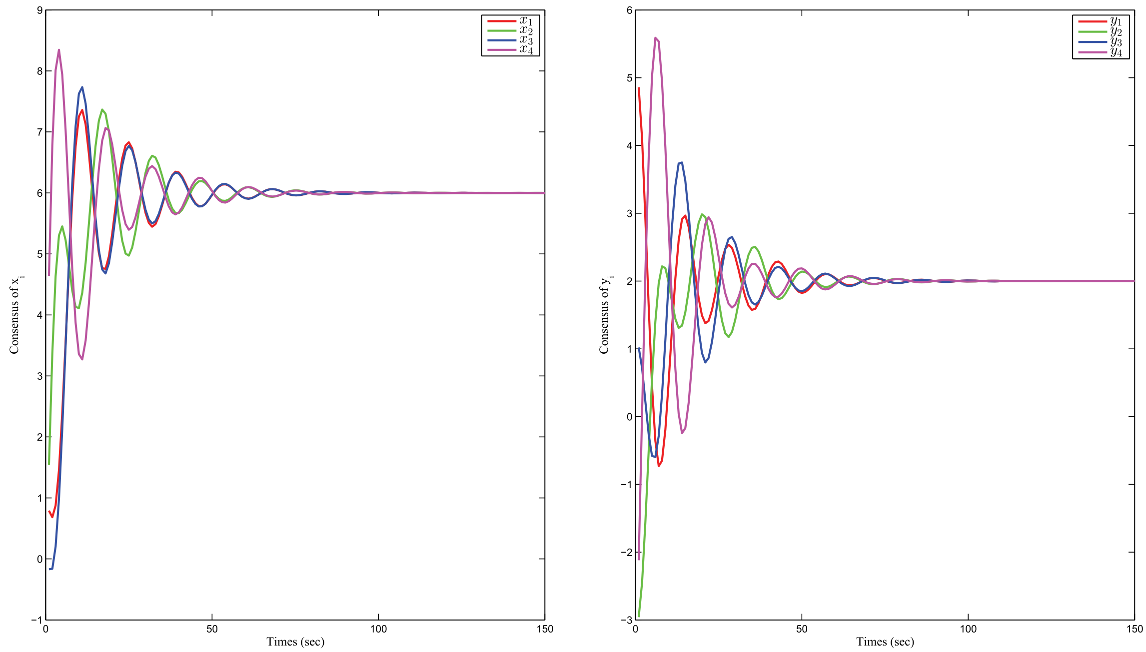

FIGURE 4 Differences between the state of each agent based on Algorithm (11) with $L = 0.1$, $T = 2$ (chained topology) [Colour figure can be viewed at wileyonlinelibrary.com]

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \left\{ \left[\left(L^2 + \frac{L}{T} \right) x_{j1}[k] - \frac{L}{T} x_{j1}[k-1] \right] - \left[\left(L^2 + \frac{L}{T} \right) x_{i1}[k] - \frac{L}{T} x_{i1}[k-1] \right] \right\} \tag{23}$$

in consensus of MAS of second-order integrator under different sampling periods. Similar to simulations in the previous section, we have selected $L = 0.15$, $T = 1$ (ring topology) and $L = 0.1$, $T = 2$ (chained topology) in the simulation studies as shown in Figures 5, 6, and 7. Based on the results obtained, it is clear that consensus under different sampling periods can also be successfully guaranteed by the co-designed controller (23) even under large sampling period. Thus much more flexibility has been provided using the co-designed consensus protocols which is much preferred in their digital implementation.

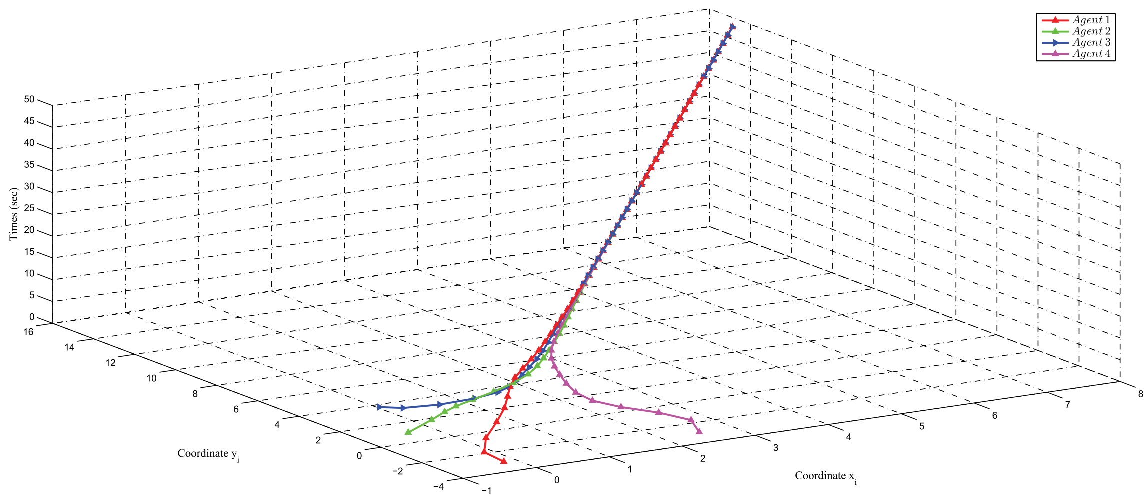

FIGURE 5 All the agents' state trajectories on simulation interval [0, 50) based on Algorithm (18) with $L = 0.15$, $T = 1$ (ring topology) [Colour figure can be viewed at wileyonlinelibrary.com]

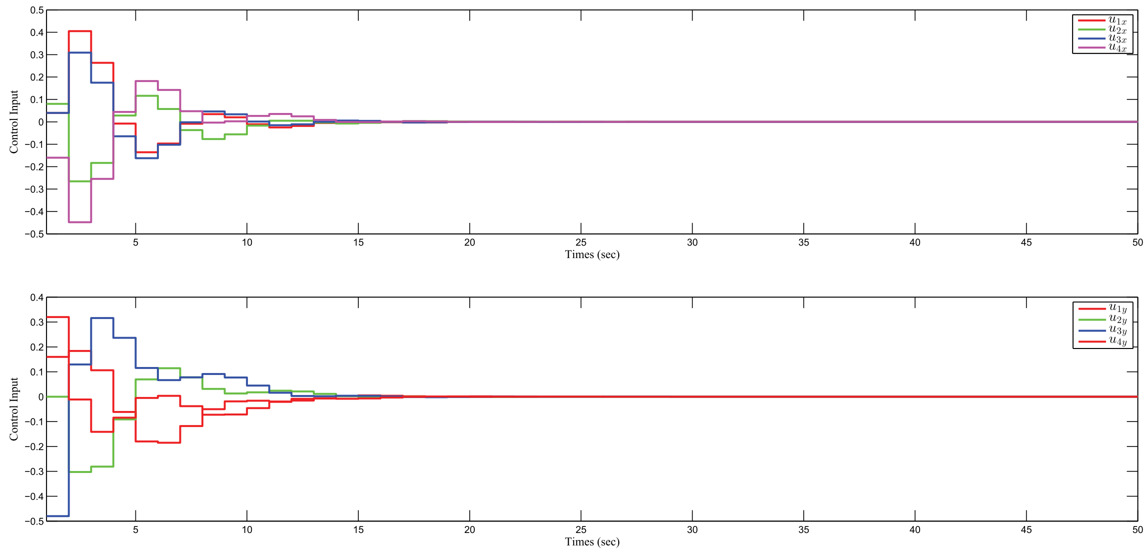

FIGURE 6 All the agents' control input on simulation interval [0, 50) based on Algorithm (18) with $L = 0.15$, $T = 1$ (ring topology) [Colour figure can be viewed at wileyonlinelibrary.com]

Remark 5. Results obtained have shown that the most important feature of the co-design between the control gain L and the sampling period T lie in the flexibility of choosing different sampling periods which is much preferred in digital implementation. From the viewpoint of theoretical study, arbitrary large sampling period T is allowable due to the compensation of control gain L. However, this not true in reality since uncertainties and/or disturbances exist in almost all control systems. For consensus of system with uncertainties and/or disturbances, feedback domination in terms of the scaling gain L can be utilized as done in our previous work.[28] Thus extensions of the co-designed consensus protocols obtained in this article to other uncertain nonlinear systems are currently under the author's consideration.

6 | CONCLUSIONS

In this article, sampled-data consensus problem for MAS of second-order integrator has been considered using within the framework of co-design. Both sampled-data controller with absolute damping and sampled-data controller with relative

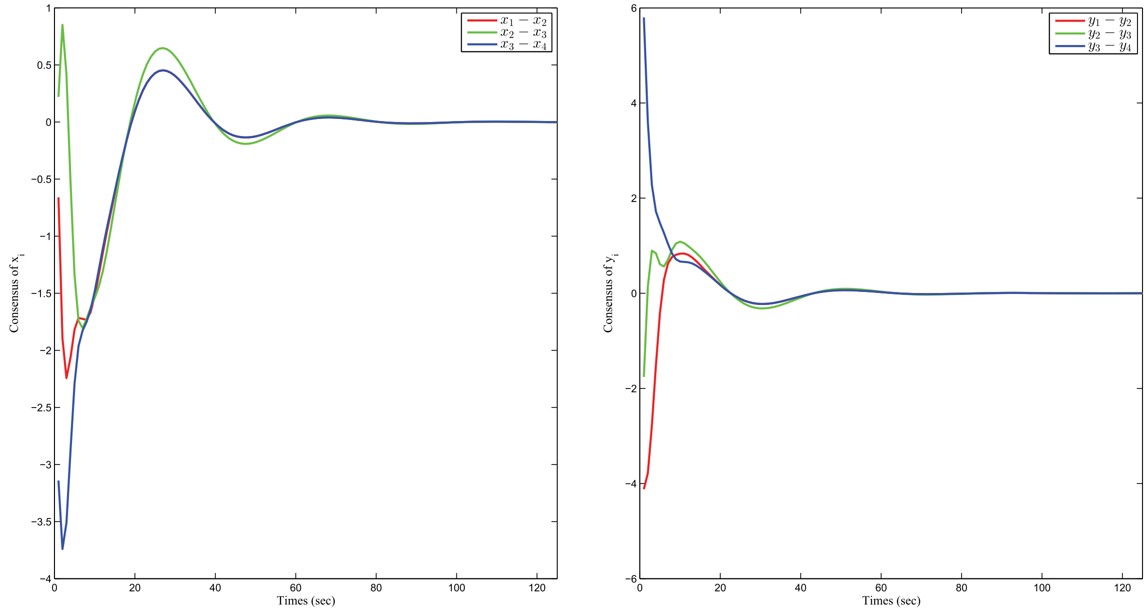

FIGURE 7 Differences between the state of each agent based on Algorithm (18) with $L = 0.1$, $T = 2$ (chained topology) [Colour figure can be viewed at wileyonlinelibrary.com]

damping have been co-designed directly in the discrete-time domain. Compared with the previous work on sampled-data consensus problem, flexibility of choosing different sampling periods has been greatly enhanced by the co-designed consensus protocols which means less hardware requirements are imposed in the digital implementation of the co-designed consensus protocols.

Currently, co-designed sampled-data control of nonholonomic system are under the author's consideration using theory of nonlinear cascaded system. Results presented in the article can also be further extended to nonlinear system with upper-triangular nonlinearity or nonlinear system in general p-normal form using the linear domination approach[28] that was proposed in our previous work.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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