On the H-Property for Step-Graphons and Edge Polytopes

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***Abstract*—Graphons W can be used as stochastic models to sample graphs G_n on n nodes for n arbitrarily large. A graphon W is said to have the H-property if G_n admits a decomposition into disjoint cycles with probability one as n goes to infinity. Such a decomposition is known as a Hamiltonian decomposition. In this letter, we provide necessary conditions for the H-property to hold. The proof builds upon a hereby established connection between the so-called edge polytope of a finite undirected graph associated with W and the H-property. Building on its properties, we provide a purely geometric solution to a random graph problem. More precisely, we assign two natural objects to W, which we term concentration vector and skeleton graph, denoted by x^* and S, respectively. We then establish two necessary conditions for the H-property to hold: (1) the edge-polytope of S, denoted by $\mathcal{X}(S)$, is of maximal rank, and (2) x^* belongs to $\mathcal{X}(S)$.**

***Index Terms*—Graphon, network analysis and control.**

I. Introduction

Graphons, a portmanteau of graph and functions, have been recently introduced [1], [2] to study very large graphs. A graphon can be understood as both the limit object of a convergent sequence, where convergence is in the cut-norm [3], of graphs of increasing size, and as a statistical model from which to sample random graphs. Taking this latter point of view, we investigate in this letter the so-called H-property (see Definition 1 below) for graphons.

A graphon is a symmetric, measurable function $W : [0, 1]^2 \to [0, 1]$. It gives rise to a stochastic model for undirected graphs on n nodes, denoted by $G_n \sim W$:

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Sampling Procedure: Let Uni$[0, 1]$ be the uniform distribution on $[0, 1]$. Given a graphon W, a graph $G_n = (V, E)$ W on n nodes sampled from W is obtained as follows:

1) Sample $y_1, \ldots, y_n \sim$ Uni$[0, 1]$ independently. We call y_i the *coordinate of node* $v_i \in V$.
2) For any two distinct nodes v_i and v_j, place an edge $(v_i, v_j) \in E$ with probability $W(y_i, y_j)$.

Note that if $0 \le p \le 1$ is a *constant* and $W(s, t) = p$ for all $(s, t) \in [0, 1]^2$, then $G_n \sim W$ is nothing but an Erdös-Rényi random graph with parameter p. Thus, graphons can be seen, in a sense, as a way to introduce *inhomogeneity* of edge densities between different pairs of nodes, and thus increase greatly the type of random graphs one can model. However, all large graphs sampled from (non-zero) graphons have the property of being dense [4].

H-Property: Let W be a graphon and $G_n \sim W$. In the sequel, we use the notation $\vec{G}_n = (V, \vec{E})$ to denote the *directed* version of G_n, defined by the edge set

$$\vec{E} := \{v_i v_j, v_j v_i | (v_i, v_j) \in E\}.$$

In words, we replace an undirected edge (v_i, v_j) with two directed edges $v_i v_j$ and $v_j v_i$. The directed graph $\vec{G}_n$ is said to have a *Hamiltonian decomposition* if it contains a subgraph $\vec{H} = (V, \vec{E}')$, with the same node set, such that $\vec{H}$ is a disjoint union of directed cycles. With the preliminaries above, we now have the following definition:

Definition 1 (H-Property): Let W be a graphon and $G_n \sim W$. Then, W has the H-property if

$$\lim_{n\to\infty} \mathbb{P}(\vec{G}_n \text{ has a Hamiltonian decomposition}) = 1.$$

We let $\mathcal{E}_n$ be the event that $\vec{G}_n$ has a Hamiltonian decomposition. The above definition implicitly requires the sequence $\mathbb{P}(\mathcal{E}_n)$ to converge. We mention here that for almost all graphons, this sequence converges and, moreover, it converges to either 1 or 0. In other words, the H-property is a "zero-one" property. This fact is, however, beyond the scope of this letter and will be proven in a forthcoming publication.

The H-property is central in the study of structural stability of linear systems [5], [6] and structural controllability of linear ensemble systems [7]. Indeed, in [5], [6], the question of structural stability of linear systems, i.e., of whether a sparsity pattern of matrices contains a stable (Hurwitz) matrix was considered. Using the standard isomorphism between sparsity patterns of square matrices and directed graphs (stemming from interpreting the sparsity pattern as an adjacency matrix), necessary and sufficient conditions were derived on the associated graphs. These conditions required the existence of subgraphs containing Hamiltonian decompositions. In [7], the author considered continuum ensembles of sparse linear control systems where the individual systems share a common sparsity pattern, represented by a digraph as above, and characterized the digraphs that can sustain ensemble controllability. A complete solution was provided for the case where the parameterization spaces of the ensembles are closed intervals. In particular, it was shown that the subgraph of the state-nodes needs to have a Hamiltonian decomposition.

In this letter, we take the first step in our investigation of the H-property by focusing on a special class of graphons, which we term step-graphons (the same objects have also been investigated in [8]). Roughly speaking, W is a step-graphon W if one can divide the interval [0, 1] into subintervals $\mathcal{R}_1, \dots, \mathcal{R}_q$ so that W is constant when restricted to every rectangle $\mathcal{R}_i \times \mathcal{R}_j$. A more precise definition can be found in Definition 2. Step-graphons are a particular case of the class of step-function graphons introduced in [9], where the partitioning is into measurable subsets of [0, 1]. The main contribution of this letter is to obtain *necessary* conditions, formulated in Theorem 1, for an arbitrary step-graphon to have the H-property.

The key observation underlying the proof is a connection between the H-property and polytopes. The study of graphs obtained from polytopes has a long tradition in discrete geometry [10] but, later, insights into graph theoretic notions have been obtained from polytopes derived from graphs [11]. Our contribution in this letter falls closer to the latter category: we draw a conclusion about a graphon W from a polytope associated with it. The polytope of interest here is the so-called edge polytope [11], defined as the convex hull spanned by the columns of the incidence matrix of an undirected graph; see Definition 6.

This edge polytope appears naturally when seeking characteristics of a step-graphon relevant to whether it has the H-property or not. The first object we exhibit in this vein is the *concentration vector* of a step-graphon W, denoted by x^*, the entries of which are the lengths of the subintervals $\mathcal{R}_i$. These entries are also the probabilities that a random variable $y \sim \text{Uni}\,[0, 1]$ belongs to $\mathcal{R}_i$ (see the first item of the sampling procedure). The second object assigned to a step-graphon is its *skeleton graph* S, which can be construed as representing the adjacency relations between rectangles $\mathcal{R}_i \times \mathcal{R}_j$ where the step-graphon is non-zero (see Definition 4). The above mentioned polytope is then the edge-polytope of the skeleton graph; we denote it by $\mathcal{X}(S)$. The two necessary conditions we exhibit in Theorem 1 are as follows: (1) S has an odd cycle (i.e., a cycle with an odd number of nodes/edges) or, equivalently, $\mathcal{X}(S)$ has maximal rank, and (2) $x^* \in \mathcal{X}(S)$.

Literature Review: In recent years, graphons have been used as models for large networks in control and game theory. For control, we mention [8]; there, the authors consider infinite-dimensional linear control systems $\dot{x} = Ax + Bu$, where x and u are elements in $\mathrm{L}^2([0, 1], \mathbb{R})$ and A and B are bounded linear operators on $\mathrm{L}^2([0, 1], \mathbb{R})$, obtained by adding scalar

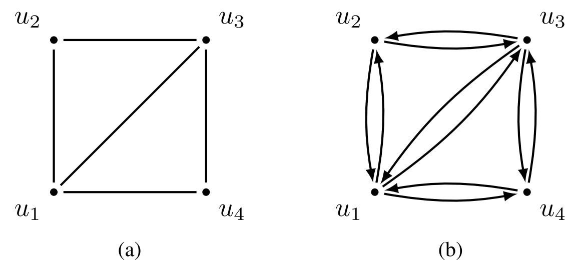


Fig. 1. *Left:* An undirected graph G on 4 nodes with self-loops on u_2 and u_4. *Right:* The directed graph $\vec{G}$ obtained from G. There are several different Hamiltonian decompositions in $\vec{G}$. For example, the cycle C_1 with edge set $\{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ forms a Hamiltonian decomposition of $\vec{G}$. Similarly, the two cycles C_2 and C_3 with edge sets $\{u_1u_2, u_2u_1\}$ and $\{u_3u_4, u_4u_3\}$, respectively, also form a Hamiltonian decomposition of $\vec{G}$.

multiples of identity operators to graphons. For this class of systems they investigate, among others, the associated controllability properties and finite-dimensional approximations. For game theory, we mention [12], [13] where the authors introduce different types of graphon games; broadly speaking, these are the games that comprise a continuum of agents (over the closed interval [0, 1]) with relations between these agents described by a graphon. They then proceed to investigate, among others, the existence of Nash equilibria and properties of finite-dimensional approximations. Finally, the prevalence of the Hamiltonian decompositions was also investigated for Erdös-Rényi random graphs in [14].

Notations and Terminology: For $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we let $\text{Diag}(v)$ be the $n \times n$ diagonal matrix whose iith entry is v_i. We use $\mathbf{1}$ to denote the vector whose entries are all ones, and with dimension appropriate for the context.

For $S = (U, F)$, an undirected graph, without multi-edges but possibly with self-loops, we let $\vec{S}$ be the *directed* version of S, as defined above, but if (u_i, u_i) is a self-loop on node $u_i \in U$, then we replace it with a single self-loop u_iu_i.

Given a directed graph $\vec{G} = (V, \vec{E})$ on n nodes without self-loops, the Laplacian matrix $L = [L_{ij}]$ associated with $\vec{G}$ is the $n \times n$ infinitesimally row stochastic matrix with off-diagonal entries L_{ij} given by $L_{ij} = 1$ if v_iv_j is an edge of $\vec{G}$ and $L_{ij} = 0$ otherwise, and with diagonal entries picked so that the row sums of L are all 0, i.e., $L\mathbf{1} = 0$.

For positive integers ℓ, q, and a set of vectors $z_1, \dots, z_\ell \in \mathbb{R}^q$, we denote by $\text{conv}\{z_1, \dots, z_\ell\}$ their *convex hull*:

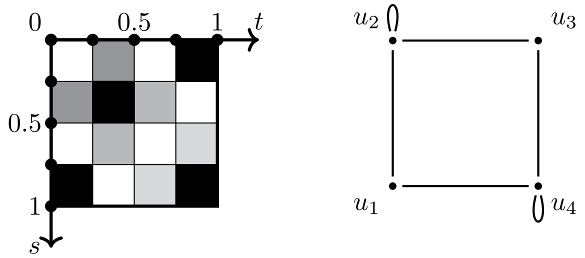
$$\text{conv}\{z_1, \dots, z_\ell\} := \left\{ \sum_{i=1}^{\ell} \lambda_i z_i \Big| \sum_{i=1}^{\ell} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\}.$$

II. Preliminaries and Main Result

In this section, we start by defining step-graphons, in Section II-A, and then their associated skeleton graphs and concentration vectors, in Section II-B. Then, in Section II-C, we present the main result of this letter.

A. Step-Graphons

We have the following definition:



(a) (b)

Fig. 2. *Left:* A step-graphon W with the partition $\sigma = (0, 0.25, 0.5, 0.75, 1)$. *Right:* The associated skeleton graph $S(W)$.

Definition 2 (Step-Graphon and Its Partition): We call a graphon W a **step-graphon** if there exists an increasing sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_q = 1$ such that W is constant over each rectangle $[\sigma_i, \sigma_{i+1}) \times [\sigma_j, \sigma_{j+1})$ for all $0 \le i, j \le q-1$ (there are q^2 rectangles in total). The sequence $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_q)$ is called a **partition for** W.

Remark 1: If W is a step-graphon, then there exists an infinite number of compatible partitions for W. Indeed, given any partition σ for the step-graphon W, the partition σ' obtained from σ by inserting σ_i', for any $\sigma_i < \sigma_i' < \sigma_{i+1}$, is also a partition for W.

We provide an example of a step-graphon in Fig. 2. Note that a graph G_n sampled from a step-graphon could be seen as a graph sampled from the so-called stochastic block-model [15], but with a random assignment of the nodes to the q communities with a multinomial distribution determined by the partition.

Throughout this letter, we let $n_i(G_n)$ be the number of nodes v_j of G_n whose coordinates $y_j \in [\sigma_{i-1}, \sigma_i)$ (see item 1 of the sampling procedure in Section I). When G_n is clear from the context, we simply write n_i.

B. Concentration Vectors and Skeleton Graphs

In this subsection, we introduce three key objects associated with a step-graphon; namely, its concentration vector, skeleton graph, and the so-called edge polytope of the skeleton graph.

Concentration Vector: We have the following definition:

Definition 3 (Concentration Vector): Let W be a step-graphon with partition $\sigma = (\sigma_0, \ldots, \sigma_q)$. The associated **concentration vector** $x^* = (x_1^*, \ldots, x_q^*)$ has entries defined as follows: $x_i^* := \sigma_i - \sigma_{i-1}$, for all $i = 1, \ldots, q$.

There is a one-to-one correspondence between concentration vectors and partitions for W. We further define the *empirical concentration vector* of a graph $G_n \sim W$:

$$x(G_n) := \frac{1}{n}(n_1(G_n), \ldots, n_q(G_n)). \tag{1}$$

whose name is justified by the following observation: $nx(G_n) = (n_1(G_n), \ldots, n_q(G_n))$ is a multinomial random variable with n trials and q events with probabilities x_i^*, for $1 \le i \le q$. A straightforward application of Chebyshev's inequality yields that for any $\epsilon > 0$,

$$\mathbb{P}(\|x(G_n) - x^*\| > \epsilon) \le \frac{c}{n^2\epsilon^2}, \tag{2}$$

where c is some constant independent of ϵ and n. When G_n is clear from the context, we will suppress it and simply write x.

Skeleton Graph: A partition σ for a step-graphon W induces a partition of the node set of any $G_n \sim W$ according to which of the intervals $[\sigma_{i-1}, \sigma_i)$ the coordinate y_j (of the sampling procedure) of v_j belongs. Elaborating on this, we can in fact construct a graph which encompasses most of the relevant characteristics of a step-graphon.

Definition 4 (Skeleton Graph): To a step-graphon W with a partition $\sigma = (\sigma_0, \ldots, \sigma_q)$, we assign the undirected graph $S = (U, F)$ on q nodes, with $U = \{u_1, \ldots, u_q\}$ and edge set F defined as follows: there is an edge between u_i and u_j if and only if W is non-zero over $[\sigma_{i-1}, \sigma_i) \times [\sigma_{j-1}, \sigma_j)$. We call S the **skeleton graph** of W for the partition σ.

We decompose the edge set of S as $F = F_0 \cup F_1$, where elements of F_0 are self-loops, and elements of F_1 are edges between distinct nodes.

Let $\mathcal{I} := \{1, \ldots, |F|\}$ be the index set for F. Let $\mathcal{I}_0$ (resp. $\mathcal{I}_1$) index the self-loops (resp. edges between distinct nodes) of S: $f_i \in F_0$ for $i \in \mathcal{I}_0$ (resp. $f_i \in F_1$ for $i \in \mathcal{I}_1$).

Given a step-graphon W and a skeleton graph S, there is a graph homomorphism which assigns the nodes of an arbitrary $G_n = (V, E) \sim W$ to their corresponding nodes in S:

$$\pi : v_j \in V \mapsto \pi(v_j) = u_i \in U, \tag{3}$$

where u_i is such that $\sigma_{i-1} \le y_j < \sigma_i$, with y_j the coordinate of v_j. It should be clear that $n_i(G_n) = |\pi^{-1}(u_i)|$ for all $i = 1, \ldots, q$.

Edge Polytope of a Skeleton Graph: To introduce the polytope, we start with the following definition:

Definition 5 (Incidence Matrix): Let $S = (U, F)$ be a skeleton graph. Given an arbitrary ordering of its edges and self-loops, we let $Z = [z_{ij}]$ be the associated *incidence matrix*, defined as the $|U| \times |F|$ matrix with entries:

$$z_{ij} := \frac{1}{2}\begin{cases} 2, & \text{if } f_j \in F_0 \text{ is a loop on node } u_i, \\ 1, & \text{if node } u_i \text{ is incident to } f_j \in F_1, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Owing to the factor $\frac{1}{2}$ in (4), all columns of Z are probability vectors, i.e., all entries are nonnegative and sum to one. The edge polytope of S was introduced in [11] and is reproduced below (with slight difference in inclusion of the factor $\frac{1}{2}$ of the generators z_j).

Definition 6 (Edge Polytope): Let $S = (U, F)$ be a skeleton graph and Z be the associated incidence matrix. Let z_j, for $1 \le j \le |F|$, be the columns of Z. The *edge polytope* of S, denoted by $\mathcal{X}(S)$, is the finitely generated convex hull:

$$\mathcal{X}(S) := \operatorname{conv}\{z_j | j = 1, \ldots, |F|\}. \tag{5}$$

Since each z_j is a probability vector and $\mathcal{X}(S)$ is a convex hull spanned by these vectors, $\mathcal{X}(S)$ is a subset of the standard simplex in $\mathbb{R}^q$. We provide below relevant properties of $\mathcal{X}(S)$.

We first describe $\mathcal{X}(S)$ by characterizing its extremal generators. Recall that x is an extremal point of $\mathcal{X}(S)$ if there is no line segment in $\mathcal{X}(S)$ that contains x in its interior. Then, the maximal set of extremal points is the set of extremal generators for $\mathcal{X}(S)$. Because $\mathcal{X}(S)$ is generated by the columns of Z, the set of extremal generators is necessarily a subset of the set of these column vectors. To characterize it further, we let $\mathcal{I}_2 \subseteq \mathcal{I}_1$ index the edges of S that are *not* incident to two self-loops. We then have the following result.

Proposition 1: The set of extremal generators of $\mathcal{X}(S)$ is $\{z_i | i \in \mathcal{I}_0 \cup \mathcal{I}_2\}$.

Proof: It should be clear from (4) that every z_i, for $i \in \mathcal{I}_0$, is an extremal point. Next, note that if f_i, for $i \in \mathcal{I}_1$, is incident to two self-loops, say f_j and f_k, then $z_i = \frac{1}{2}(z_j + z_k)$ and, hence, z_i is not extremal. It now remains to show that if $i \in \mathcal{I}_2$, then z_i is an extremal point. Suppose not; then, one can write $z_i = \sum_{j \neq i} c_j z_j$, with $c_j \geq 0$. Since z_i only has two non-zero entries and since the c_j's are non-negative, if the support of z_j is not included in the support of z_i, then $c_j = 0$. It has two implications: (*i*) For any $j \in \mathcal{I}_1 - \{i\}$, $c_j = 0$; (*ii*) If $j \in \mathcal{I}_0$, then the self-loop f_j has to be incident to f_i. Thus, the expression $z_i = \sum_{\ell \neq i} c_\ell z_\ell$ reduces to $z_i = c_j z_j$, where f_j is the self-loop incident to z_i (if it exists), which clearly cannot hold. ■

We conclude this subsection with a known result [11] on the rank of $\mathcal{X}(S)$ (or, similarly, a result [16] on the rank of Z introduced in Definition 5), where the rank of $\mathcal{X}(S)$ is the dimension of its relative interior.

Proposition 2: Let $S = (U, F)$ be a connected, undirected graph on q nodes, possibly with loops. Then,

$$\operatorname{rank} \mathcal{X}(S) = \begin{cases} q-1 & \text{if } S \text{ has an odd cycle,} \\ q-2 & \text{otherwise.} \end{cases} \tag{6}$$

C. Main Result

For ease of exposition, we assume from now on that the step-graphons W are such that their corresponding skeleton graphs S are connected. However, all the results below hold for step-graphons W whose skeleton graphs have several connected components by requiring that the conditions exhibited for S hold for each connected component of S.

Theorem 1: Let W be a step-graphon with σ a partition. Let S and x^* be the associated (connected) skeleton graph and concentration vector, respectively. Let $G_n \sim W$ and $\vec{G}_n$ be the directed version of G_n. If S has no odd cycle or if $x^* \notin \mathcal{X}(S)$, then

$$\lim_{n\to\infty} \mathbb{P}(\vec{G}_n \text{ has a Hamiltonian decomposition}) = 0. \tag{7}$$

The proof goes by showing that if one of the two conditions holds, then the edge polytope $\mathcal{X}(S)$ contains *at most* a zero-measure subset of the support of x^*. In particular, if S does not have an odd cycle, then the codimension of $\mathcal{X}(S)$ in the standard simplex is 1 (see Proposition 2) and thus the probability that the vector x^* belongs to $\mathcal{X}(S)$ is negligible in the asymptotic regime.

The conditions exhibited in Theorem 1 almost completely determine whether W has the H-property: we can show that if S has an odd cycle and x^* is in the *interior* of $\mathcal{X}(S)$, then $\lim_{n\to\infty} \mathbb{P}(\vec{G}_n$ has a Hamiltonian decomposition$) = 1$. The proof of this statement is much more involved than the proof of Theorem 1, and will be presented on another occasion.

Remark 2: It may seem at first that the main result depends on a certain partition σ, which defines S and x^*. We have established in [17, Proposition 3] the following fact: The condition that S has an odd cycle and the condition that $x^* \in \mathcal{X}(S)$ are independent of the choice of σ. More specifically, for any two partitions σ and σ' for W, let x^*, x'^* be the corresponding concentration vectors and let S, S' be the corresponding skeleton graphs. Then, it holds that (1) S has an odd cycle if and only if S' does, and (2) $x^* \in \mathcal{X}(S)$ if and only if $x'^* \in \mathcal{X}(S')$.

III. Analysis and Proof of Theorem 1

A. On the Edge Polytope of S

Let W be a step-graphon with partition sequence σ and corresponding skeleton graph S on q nodes. In this subsection, we introduce in Definition 7 the set $\mathcal{A}(S)$ of sparse infinitesimally stochastic matrices whose sparsity pattern is determined by the skeleton graph S. We then show that the edge polytope $\mathcal{X}(S)$, defined by (5), is exactly the set of row sums of these matrices. This expression of $\mathcal{X}(S)$ will be used in the proof of Theorem 1.

We start with the following result.

Lemma 1: Assume that $\vec{G}_n$ has a Hamiltonian decomposition, denoted by $\vec{H}$, and let $n_{ij}(\vec{H})$ be the number of edges of $\vec{H}$ from a node in $\pi^{-1}(u_i)$ to a node in $\pi^{-1}(u_j)$. Then, for all $u_i \in U$,

$$n_i(G_n) = \sum_{u_j \in N(u_i)} n_{ij}(\vec{H}) = \sum_{u_j \in N(u_i)} n_{ji}(\vec{H}). \tag{8}$$

Proof: Each node of $\vec{H}$ has exactly one incoming edge and one outgoing edge. The result then follows from the fact that $\sum_{u_j \in N(u_i)} n_{ij}(\vec{H})$ counts the number of outgoing edges from the nodes of $\pi^{-1}(u_i)$ while $\sum_{u_j \in N(u_i)} n_{ji}(\vec{H})$ counts the number of incoming edges to the nodes of $\pi^{-1}(u_i)$, and the fact that $\vec{H}$ has the same node set as $\vec{G}_n$. ■

Following Lemma 1, we now assign to the skeleton graph S a convex set that will be instrumental in the study of Hamiltonian decompositions of $\vec{G}_n$:

Definition 7: To an arbitrary undirected graph $S = (U, F)$ on q nodes, possibly with self-loops, we assign the set $\mathcal{A}(S)$ of $q \times q$ nonnegative matrices $A = [a_{ij}]$ that satisfy the following two conditions:

1) if $(u_i, u_j) \notin F$, then $a_{ij} = 0$;
2) $A\mathbf{1} = A^\top \mathbf{1}$, and $\mathbf{1}^\top A \mathbf{1} = 1$.

Note that $\mathbf{1}^\top A \mathbf{1}$ is nothing but the sum of all the entries of A. Because every defining condition for $\mathcal{A}(S)$ is affine, the set $\mathcal{A}(S)$ is a convex set.

Now, to each Hamiltonian decomposition $\vec{H}$ of $\vec{G}_n$, we assign the following $q \times q$ matrix:

$$\rho(\vec{H}) := \frac{1}{n}[n_{ij}(\vec{H})]_{1 \leq i,j \leq q}. \tag{9}$$

The next lemma then follows immediately from Lemma 1:

Lemma 2: If $\vec{H}$ is a Hamiltonian decomposition of $\vec{G}_n$, then $\rho(\vec{H}) \in \mathcal{A}(S)$ and $\rho(\vec{H})\mathbf{1} = x$, where x is the empirical concentration vector of G_n.

The relation $\rho(\vec{H})\mathbf{1} = x$ in the above lemma leads us to investigate the set of the possible row sums of $A \in \mathcal{A}(S)$. The main result of this subsection is that this set is equal to $\mathcal{X}(S)$ introduced in (5).

Proposition 3: The following holds:

$$\mathcal{X}(S) = \{x \in \mathbb{R}^q | x = A\mathbf{1} \text{ for some } A \in \mathcal{A}(S)\}. \tag{10}$$

Proof: We prove the result by using double-inclusion:

1. *Proof that $\mathcal{X}(S) \subseteq \mathcal{A}(S)\mathbf{1}$:* We show that for each generator z_j of $\mathcal{X}(S)$ as in (4), there exists an $A \in \mathcal{A}(S)$ such that $z_j = A\mathbf{1}$. If $j \in \mathcal{I}_0$, then f_j is a loop on some node u_i. Let $A_j := e_i e_i^\top \in \mathcal{A}(S)$; then, $A_j\mathbf{1} = z_j$. If $j \in \mathcal{I}_1$, then $f_j = (u_k, u_\ell)$ is an edge between two distinct nodes. Let $A_j := \frac{1}{2}(e_k e_\ell^\top + e_\ell e_k^\top) \in \mathcal{A}(S)$; then, $A_j\mathbf{1} = z_j$.

2. *Proof that $\mathcal{X}(S) \supseteq \mathcal{A}(S)\mathbf{1}$:* Let $A \in \mathcal{A}(S)$, and we show that $A\mathbf{1} \in \mathcal{X}(S)$. By Definition 7, $A\mathbf{1}$ belongs to the standard simplex. Thus, it suffices to show that $A\mathbf{1}$ can be written as a nonnegative combination of the z_j's; indeed, if this holds, then it has to be a convex combination of the z_j's and, hence, $A\mathbf{1} \in \mathcal{X}(S)$.

Decompose $A =: A_0 + A_1$ where A_0 (resp. A_1) is the diagonal (resp. off-diagonal) part of A. Then, $A\mathbf{1} = A_0\mathbf{1} + A_1\mathbf{1}$. We show that both $A_0\mathbf{1}$ and $A_1\mathbf{1}$ can be written as nonnegative combinations of z_j's.

For $A_0\mathbf{1}$, note that if the iith entry of A_0 is not 0, then u_i has a self-loop, say f_j. Thus, we obtain that $A_0\mathbf{1}$ can be expressed as a nonnegative combination of z_j's, for $j \in \mathcal{I}_0$.

For $A_1\mathbf{1}$, we translate the problem into a problem about decompositions of infinitesimally doubly stochastic matrices into Laplacian matrices of cycles. First, since $A_1\mathbf{1} = A_1^\top\mathbf{1}$, *replacing* the diagonal entries of A_1 with the entries of $-A_1\mathbf{1}$ results in an infinitesimally *doubly* stochastic matrix. We denote it by A_1' (i.e., $A_1' := A_1 - \mathrm{Diag}(A_1\mathbf{1})$).

Now, consider the directed version of S, denoted by $\vec{S}$. For each directed cycle $\vec{C}_k$ of $\vec{S}$, other than self-loops, we let L_k' be the associated Laplacian matrix. It is known that A_1' can be expressed as a nonnegative combination of these L_k' [18, Proposition 3] (the statement can be viewed as an infinitesimal version of the Birkhoff Theorem [19] for doubly stochastic matrices). In particular, the diagonal of A_1' (which is $-A_1\mathbf{1}$) is a nonnegative combination of the diagonals of L_k'. Hence, it remains to show that the diagonal of each L_k' can be written as a nonpositive combination of the z_j's, for $j \in \mathcal{I}_1$. Let $j_1, \ldots, j_m$ be the indices in $\mathcal{I}_1$ that correspond to the undirected versions of the edges of $\vec{C}$. Then, $-\mathrm{diag}(L_k') = \sum_{\ell=1}^m z_{j_\ell}$. This completes the proof. ■

B. Proof of Theorem 1

Let $G_n \sim W$ and x be the associated empirical concentration vector. We will address subsequently the two conditions (1) $x^* \notin \mathcal{X}(S)$, and (2) S having no odd cycle:

Condition (1) ($x^* \notin \mathcal{X}(S)$): Since $\mathcal{X}(S)$ is closed, if $x^* \notin \mathcal{X}(S)$, then there is an open neighborhood $\mathcal{U}$ of x^* in the standard simplex such that $\mathcal{U} \cap \mathcal{X}(S) = \varnothing$. On the one hand, by (2), the probability that x belongs to $\mathcal{U}$ tends to 1 as n goes to infinity. On the other hand, if $\vec{G}_n$ admits a Hamiltonian decomposition $\vec{H}$, then by Lemma 2, $A := \rho(\vec{H}) \in \mathcal{A}(S)$ and $x = A\mathbf{1} \in \mathcal{X}(S)$. The above arguments imply that if $x^* \notin \mathcal{X}(S)$, then (7) holds.

Condition (2) (S has no odd Cycle): In this case, by the definition of $\mathcal{X}(S)$ in (5) and Proposition 2, the co-dimension

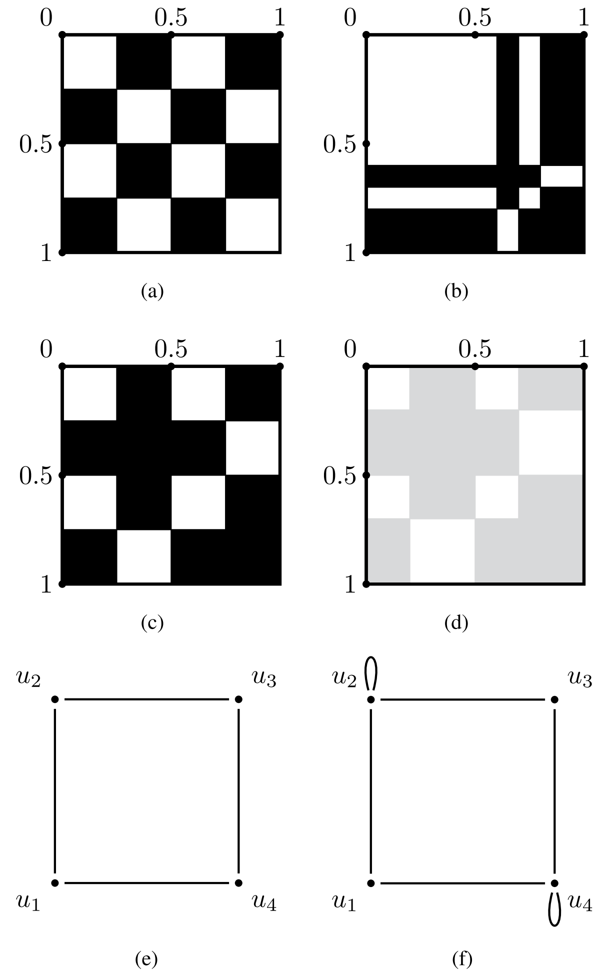


Fig. 3. The step-graphon depicted in Fig. 3(a) has the skeleton graph shown in Fig. 3(e); the step-graphons depicted in Figs. 3(b), 3(c), 3(d) have the skeleton graph shown in Fig. 3(f). The value of $W(s,t)$ is color-coded, with black being 1 and white being 0. The gray value in step-graphon 3(d) is 0.2. For each of these step-graphons, we sampled $N = 2 \cdot 10^4$ graphs G_n for various n and evaluated whether $\vec{G}_n$ have a Hamiltonian decomposition. The results are shown in Fig. 4.

of $\mathcal{X}(S)$ is 1 in the standard simplex. We introduce the random variable $\omega_n := \sqrt{n}(x - x^*) + x^*$. Since $\mathbb{E}x = x^*$, it is known [20] that ω_n converges in law to a Gaussian random variable ω with mean x^* and covariance $\Sigma := \mathrm{Diag}(x^*) - x^* x^{*\top}$. A short calculation yields that $\Sigma\mathbf{1} = 0$ and that Σ has rank $(q-1)$ (one could see this by, e.g., relating it to a weighted Laplacian matrix of a complete graph). Hence, the support of ω is the affine hyperplane $Q := \{x^* + v | v^\top\mathbf{1} = 0\}$. Next, let $Q' \subsetneq Q$ be the smallest affine hyperplane containing $\mathcal{X}(S)$, i.e., $Q' := \{\sum_{i=1}^{|F|} \lambda_i z_i | \sum_{i=1}^{|F|} \lambda_i = 1\}$. Its co-dimension in Q is 1, so $\mathbb{P}(\omega \in Q') = 0$. Since ω_n converges in law to ω, $\lim_{n\to\infty} \mathbb{P}(\omega_n \in Q') = 0$. We conclude the proof by noting that the event $\omega_n \in Q'$ is necessary for $x \in \mathcal{X}(S)$ and, hence, by Lemma 2, necessary for G_n to have a Hamiltonian decomposition.

IV. Numerical Validations

We performed numerical studies to understand how rapidly the asymptotic regime appears as n grows larger. The simulation results can also be understood as a validation of our

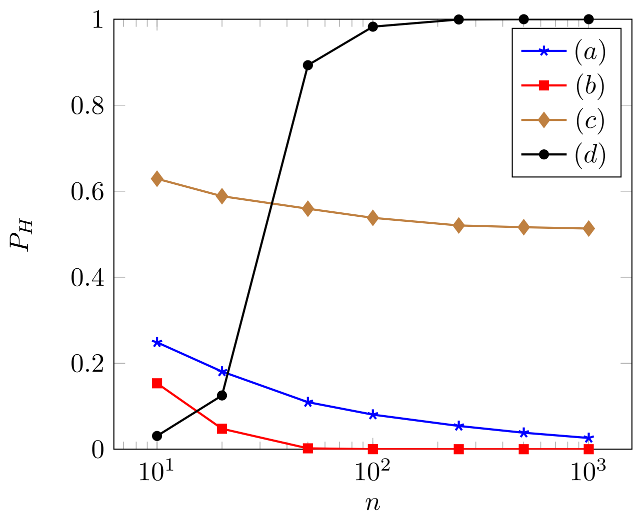


Fig. 4. Plot of the proportion P_H of $\vec{G}_n$, with $G_n \sim W$, that have a Hamiltonian decomposition, for the four step-graphons depicted in Fig. 3.

main theorem and the claims made in this letter. The set-up is the following: we consider the four step-graphons depicted in Fig. 3. For each step-graphon, we sampled sets of $N = 2 \cdot 10^4$ graphs G_n for each $n \in \{10, 20, 50, 100, 250, 500, 1000\}$ and evaluated the proportion of $\vec{G}_n$ that have a Hamiltonian decomposition. Namely, we evaluated

$$P_H := \frac{\#\vec{G}_n \text{ with Hamiltonian decomposition}}{2 \cdot 10^4}.$$

Below are the observations from the experiments:

Experiment (a): The step-graphon shown in Fig. 3(a) has associated concentration vector $x^* = [0.25, 0.25, 0.25, 0.25]$. Its skeleton graph S, shown in Fig. 3(e), does not have an odd cycle, but $x^* \in \mathcal{X}(S)$. We observe in Fig. 4 that the proportion of $\vec{G}_n \sim W$ that contains a Hamiltonian decomposition goes to zero as $n \to \infty$.

Experiment (b): The step-graphon shown in Fig. 3(b) has associated concentration vector $x^* = [0.6, 0.1, 0.1, 0.2]$. The skeleton graph S, shown in Fig. 3(f), has an odd cycle. However, $x^* \notin \mathcal{X}(S)$. We observe in Fig. 4 that the proportion of $\vec{G}_n \sim W$ that contains a Hamiltonian decomposition goes to zero as $n \to \infty$.

Experiment (c): The step-graphon shown in Fig. 3(c) has associated concentration vector $x^* = [0.25, 0.25, 0.25, 0.25]$. The skeleton graph S, shown in Fig. 3(f), has an odd cycle. One can check that $x^* \in \partial\mathcal{X}(S)$, i.e., the *boundary* of $\mathcal{X}(S)$. We observe in Fig. 4 that the proportion of $\vec{G}_n \sim W$ that contains a Hamiltonian decomposition does not vanish as $n \to \infty$ nor goes to 1. Note that the class of step-graphons such that $x^* \in \partial\mathcal{X}(S)$ is not generic.

Experiment (d): The step-graphon shown in Fig. 3(d) has associated concentration vector $x^* = [0.25, 0.25, 0.25, 0.25]$. The skeleton graph S, shown in Fig. 3(f), has an odd cycle. One can check that $x^* \in \operatorname{int}\mathcal{X}(S)$, the interior of $\mathcal{X}(S)$. We observe in Fig. 4 that the proportion of $\vec{G}_n \sim W$ that contain a Hamiltonian decomposition converges to 1 as $n \to \infty$.

V. Conclusion

We have exhibited two necessary conditions for the H-property to hold for the class of step-graphons W. The starting point of our analysis was the introduction of two novel objects associated with W: its concentration vector x^* and its skeleton graph S. We have then highlighted a novel connection between the edge polytope of S, denoted by $\mathcal{X}(S)$, and the H-property for the underlying graphon W: it requires that $x^* \in \mathcal{X}(S)$ and $\mathcal{X}(S)$ is of maximal rank. We also validated our results via numerical studies in Section IV. As was claimed after Theorem 1 and shown in Figure 4, the two conditions that x^* belongs to the interior of $\mathcal{X}(S)$ and that $\mathcal{X}(S)$ is of maximal rank are sufficient for a step-graphon W to have the H-property.

References

[1] L. Lovász and B. Szegedy, "Limits of dense graph sequences," *J. Comb. Theory Ser. B*, vol. 96, no. 6, pp. 933–957, 2006.
[2] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi, "Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing," *Adv. Math.*, vol. 219, no. 6, pp. 1801–1851, 2008.
[3] A. Frieze and R. Kannan, "Quick approximation to matrices and applications," *Combinatorica*, vol. 19, no. 2, pp. 175–220, 1999.
[4] L. Lovász, *Large Networks and Graph Limits*, vol. 60. Providence, RI, USA: Amer. Math. Soc., 2012.
[5] M.-A. Belabbas, "Algorithms for sparse stable systems," in *Proc. 52nd IEEE Conf. Decis. Control*, Firenze, Italy, 2013, pp. 3457–3462.
[6] M.-A. Belabbas, "Sparse stable systems," *Syst. Control Lett.*, vol. 62, no. 10, pp. 981–987, 2013.
[7] X. Chen, "Sparse linear ensemble systems and structural controllability," *IEEE Trans. Autom. Control*, early access, Jul. 14, 2021, doi: 10.1109/TAC.2021.3097289.
[8] S. Gao and P. E. Caines, "Graphon control of large-scale networks of linear systems," *IEEE Trans. Autom. Control*, vol. 65, no. 10, pp. 4090–4105, Oct. 2019.
[9] L. Lovász and B. Szegedy, "Finitely forcible graphons," *J. Comb. Theory Ser. B*, vol. 101, no. 5, pp. 269–301, 2011.
[10] E. Steinitz, *Über Isoperimetrische Probleme Bei Konvexen Polyedern*. Berlin, Germany: Walter de Gruyter, 1928.
[11] H. Ohsugi and T. Hibi, "Normal polytopes arising from finite graphs," *J. Algebra*, vol. 207, no. 2, pp. 409–426, 1998.
[12] S. Gao, R. F. Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," *Commun. Inf. Syst.*, vol. 21, no. 3, pp. 341–369, 2021.
[13] F. Parise and A. Ozdaglar, "Analysis and interventions in large network games," *Annu. Rev. Control Robot. Auton. Syst.*, vol. 4, pp. 455–486, May 2021.
[14] M.-A. Belabbas and A. Kirkoryan, "On the structural stability of random systems," 2020, *arXiv:2003.04139*.
[15] P. W. Holland, K. B. Laskey, and S. Leinhardt, "Stochastic blockmodels: First steps," *Soc. Netw.*, vol. 5, no. 2, pp. 109–137, 1983.
[16] C. Van Nuffelen, "On the incidence matrix of a graph," *IEEE Trans. Circuits Syst.*, vol. 23, no. 9, p. 572, Sep. 1976.
[17] M.-A. Belabbas, X. Chen, and T. Başar, "On the H-property for step-graphons and edge polytopes," 2021, *arXiv:2109.08340*.
[18] X. Chen, M.-A. Belabbas, and T. Başar, "Distributed averaging with linear objective maps," *Automatica*, vol. 70, pp. 179–188, Aug. 2016.
[19] G. Birkhoff, "Tres observaciones sobre el algebra lineal," *Universidad Nacional Tucumán Ser. A*, vol. 5, pp. 147–154, 1946.
[20] N. Arenbaev, "Asymptotic behavior of the multinomial distribution," *Theory Probab. Appl.*, vol. 21, no. 4, pp. 805–810, 1977.