arXiv:1809.09831v1 [math.AP] 26 Sep 2018

LARGE GLOBAL SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS I, MASS-SUBCRITICAL CASES

MARIUS BECEANU, QINGQUAN DENG, AVY SOFFER, AND YIFEI WU

Abstract. In this paper, we consider the nonlinear Schrödinger equation,

$$i\partial_t u + \Delta u = \mu |u|^p u, \quad (t,x) \in \mathbb{R}^{d+1},$$

with $\mu = \pm 1, p > 0$.

In this work, we consider the mass-subcritical cases, that is, $p \in (0, \frac{4}{d})$. We prove that under some restrictions on d, p, any radial initial data in the critical space $\dot{H}^{s_c}(\mathbb{R}^d)$ with compact support, implies global well-posedness.

Contents

2010 *Mathematics Subject Classification.* Primary 35Q55.
Key words and phrases. Nonlinear Schrödinger equation, global well-posedness, critical regularity.

1. Introduction

We study the Cauchy problem for the following nonlinear Schrödinger equation (NLS) on $\mathbb{R}\times\mathbb{R}^d$:

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^p u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with $\mu = \pm 1, p > 0$. Here $u(t,x) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ is a complex-valued function. The case $\mu = 1$ is referred to the defocusing case, and the case $\mu = -1$ is the focusing case. The class of solutions to equation (1.1) is invariant under the scaling

$$u(t,x) \to u_\lambda(t,x) = \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x) \quad \text{for} \quad \lambda > 0, \tag{1.2}$$

which maps the initial data as

$$u(0) \to u_\lambda(0) := \lambda^{\frac{2}{p}} u_0(\lambda x) \quad \text{for} \quad \lambda > 0.$$

Denote

$$s_c = \frac{d}{2} - \frac{2}{p}.$$

Then the scaling leaves $\dot H^{s_c}$ norm invariant, that is,

$$\|u\|_{\dot H^{s_c}} = \|u_\lambda\|_{\dot H^{s_c}},$$

which is called *critical regularity* s_c. It is also considered as the lowest regularity for which the problem (1.1) is well-posed for general $H^s(\mathbb{R}^d)$-data. Indeed, one can find some special initial datum belonging to $H^s(\mathbb{R}^d), s < s_c$ such that the problem (1.1) is ill-posed.

The H^1-solution of equation (1.1) also enjoys mass, momentum and energy conservation laws, which read

$$\begin{aligned} M(u(t)) &:= \int |u(t,x)|^2\,dx = M(u_0), \\ P(u(t)) &:= \operatorname{Im}\int \overline{u(t,x)}\nabla u(t,x)\,dx = P(u_0), \\ E(u(t)) &:= \int |\nabla u(t,x)|^2\,dx + \frac{2\mu}{p+2}\int |u(t,x)|^{p+2}\,dx = E(u_0). \end{aligned} \tag{1.3}$$

The well-posedness and scattering theory for Cauchy problem (1.1) with initial data in $H^s(\mathbb{R}^d)$ were extensively studied, which we here briefly review. The local well-posedness theory follows from a standard fixed point argument, implying that for all $u_0 \in H^s(\mathbb{R}^d)$, there exists $T_0 > 0$ such that its corresponding solution $u \in C([0,T_0),\ H^s(\mathbb{R}^d))$. In fact, the above T_0 depends on $\|u_0\|_{H^s(\mathbb{R}^d)}$ when $s > s_c$ and also the profile of u_0 when $s = s_c$. Some of the results can be found in Cazenave and Weissler [10].

Such argument can be applied directly to prove the global well-posedness for solutions to equation (1.1) with small initial data in $H^s(\mathbb{R}^d)$ with $s \ge s_c$. In the mass-subcritical cases, that is, $p < \frac{4}{d}$, if we consider the solution in $L^2(\mathbb{R}^d)$ space, the local theory above, together with the mass conservation laws (1.3), yields the global well-posedness for any initial data $u_0 \in L^2(\mathbb{R}^d)$. In the mass-supercritical, energy-subcritical cases, that is, $\frac{4}{d} < p < \frac{4}{d-2}$, if we consider the solution in energy space $H^1(\mathbb{R}^d)$, the local theory above together with conservation laws (1.3) yields the global well-posedness for all initial data $u_0 \in H^1(\mathbb{R}^d)$ in the defocusing case $\mu = 1$, and for any initial data $u_0 \in H^1(\mathbb{R}^d)$ with some restrictions in the focusing case. Furthermore, the scattering under the same conditions were also obtained by Ginibre, Velo [27] in the defocusing case and [24] in the focusing case. In the mass-critical and energy-critical cases, since the conservation laws do not imply directly the global existence of the solutions, the problem becomes much more complicated. In the energy-critical case, the global well-posedenss and scattering in the defocusing case was first proved by Bourgain [3] in the radial data case and then by Colliander, Keel, Takaoka, Staffilani and Tao [13] in the non-radial data case; the global well-posedenss and scattering in the focusing case was proved by Kenig and Merle [30] in the radial data case, then by Killip, Visan [42] in the non-radial case when the dimensions are five and higher, and by Dodson [21] in four dimensions, see also [43, 60, 61, 63, 64] for some previous works and simplified proofs. In the mass-critical case, the global well-posedenss and scattering was first proved by Killip, Tao, Visan [39] in the radial data case in dimension two, and Killip, Visan, Zhang [45] in dimensions higher than two, then in the non-radial data case, the problem was solved in a series of papers of Dodson [17, 18, 19, 20].

More complicated situation appears if one considers the general nonlinear Schrödinger equations in the critical space $\dot{H}^{s_c}(\mathbb{R}^d)$. Recently, conditional global and scattering results with the assumption of $u \in L_t^\infty(I, \dot{H}_x^{s_c}(\mathbb{R}^d))$ (here I is the maximal lifespan) were considered by many authors, which was started from [31, 32], and then developed by [5, 23, 25, 26, 33, 36, 37, 40, 41, 42, 48, 49, 50, 51, 65] and cited references. That is, if the initial data $u_0 \in \dot{H}^{s_c}(\mathbb{R}^d)$ and the solution has priori estimate

$$\sup_{0<t<T_{out}(u_0)} \|u\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} < +\infty, \tag{1.4}$$

then $T_{out}(u_0) = +\infty$ and the solution scatters in $\dot{H}^{s_c}(\mathbb{R}^d)$, here $[0, T_{out}(u_0))$ is the maximal interval in positive direction for existence of the solution. Consequently, these results give the blowup criterion which the lifetime depends only on the critical norm $\|u\|_{L_t^\infty \dot{H}_x^{s_c}(I\times\mathbb{R}^d)}$. However, it seems that no such large data global results are known, if only the initial data $u_0 \in \dot{H}^{s_c}(\mathbb{R}^d)$. Furthermore, many authors considered the large global solutions for rough data from a probabilistic point of view, that is, one may construct a large sets of initial data of super-critical regularity which leads to global solutions, see [1, 2, 6, 7, 8, 14, 15, 16, 22, 38, 52, 53, 54, 55, 56, 57, 58, 62].

In the first part of our series of works, we consider the global solution for the mass-subcritical nonlinear Schödinger equation in the critical space $\dot{H}^{s_c}(\mathbb{R}^d)$. Due to the mass conservation law, L^2-initial datum lead to the global solutions. It is known from Christ, Colliander and Tao [12] and Kenig, Ponce, Vega [35] that the problem is ill-posed in some sense for the non-radial datum in $\dot{H}^s(\mathbb{R}^d), s < 0$. However, for the radial data, due to the better radial Strichartz estimates, one may establish the local well-posedness result in negative regularity Sobolev spaces. Indeed, it was proved by Guo and Wang [28] that there exists $p_0(d) < \frac{4}{d}$, such that for any $p \in (p_0(d), \frac{4}{d})$, if the initial datum are radial and small in the critical space $\dot{H}^{s_c}(\mathbb{R}^d)$, then the nonlinear solutions of (1.1) are global and scatter. Very recently, Killip, Masaki, Murphy and Visan [36, 37] proved a conditional result; that in the defocusing case, there exists $p_0(d) < \frac{4}{d}$, such that for any $p \in (p_0(d), \frac{4}{d})$, if the radial solution $u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^d)$, then $I = \mathbb{R}$ and the solution scatters, by using concentration-compactness arguments. This is the first global result for large data theory in the critical spaces for the mass-subcritical NLS.

In this paper, we prove unconditional global well-posedness. We prove that for radial initial data with compact support in space, and is in the critical space, there exists solution global in time.

Theorem 1.1. *Let $d \geq 4$, and $\mu = \pm 1$. Then there exists $p_0(d) \in (0, \frac{4}{d})$, such that for any $p \in [p_0(d), \frac{4}{d})$, the following is true. Suppose that $u_0 \in \dot{H}^{s_c}(\mathbb{R}^d)$ is a radial function satisfying*

$$\text{supp } u_0 \subset \{x : |x| \leq 1\}.$$

Then the solution u to the equation (1.1) with the initial data u_0 exists globally in time, and $u \in C(\mathbb{R}^+; \dot{H}^{s_c}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+; \dot{H}^{s_c}(\mathbb{R}^d) + L^2(\mathbb{R}^d))$. Moreover, for any $t \in \mathbb{R}$,

$$\big\| u(t) \big\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \lesssim 1 + |t|.$$

Remark 1.2. We make several remarks regarding the above statements.

(1) Our conclusions are valid for both of the focusing and the defocusing cases. Further, by scaling, one can extend the size of the radius 1 to an arbitrary large number. Moreover, the compact support assumption on initial data are not necessary and can be replaced by some weighted assumption.

(2) In the present paper, we are not going to give the sharp conditions on $p_0(d)$ and d.

In the mass-subcritical cases, there is a new difficulty when we consider the global solution in the negative Sobolev space. It is worth noting that in this case, we can not use the mass, energy conservation laws, and Morawetz estimates. Moreover, the pseudo-conformal conservation law has no good sign.

Further, because all of the conservation laws are beyond the critical scaling regularity, we believe that analogous scattering result in $\dot{H}^{s_c}(\mathbb{R}^d)$ is very hard to pursue in the mass-subcritical case (it is similar to the energy-supercritical case in which all the conservation laws are below the critical scaling regularity), even if the initial data is smooth enough.

Sketch of the proof:

First, in step 1, we show an improved (supercritical) Strichartz estimates for the initial data localized in space under the linear flow. More precisely, we prove that for all $N \geq 1$, there exist $\alpha_0 > 1, \beta_0 > 0$, such that

$$\Big\| \langle t^{\alpha_0} |\nabla| \rangle^{\beta_0} |\nabla|^{s_c} \big(e^{it\Delta} \chi_{\leq 10}(P_{\geq N} g) \big) \Big\|_{L^2_t L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \| P_{\geq N} g \|_{\dot{H}^{s_c}(\mathbb{R}^d)}$$

(a slight stronger estimate is needed, see Section 5 below). From this estimate, we gain the regularity and time decay for $t \gtrsim 1$.

In step 2, given small constant $\delta_0 > 0$, we break the initial data into two parts, $u_0 = v_0 + w_0$, with

$$v_0 = \chi_{\leq 10}(P_{\geq N} u_0) \text{ with } \|v_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \leq \delta_0, \quad \text{and} \quad w_0 \in L^2(\mathbb{R}^d).$$

Now, let v be the solution of the following *time cut-off* equation,

$$\begin{cases} i\partial_t v + \Delta v = \chi_{\leq 1}(t)|v|^p v, \\ v(0, x) = v_0(x). \end{cases}$$

In this step, we prove that the analogous estimates in Step 1 hold true for the nonlinear solution v. That is,

$$\Big\| \langle t^{\alpha_0}|\nabla|\rangle^{\beta_0}|\nabla|^{s_c}P_M v\Big\|_{l^\infty_M L^2_t L_x^{\frac{2d}{d-2}}(\{M\geq 1\}\times\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)},$$

which we use later with $t \gtrsim 1$.

In step 3, we prove the uniform in time boundedness of $\|w(t)\|_{L^2_x(\mathbb{R}^d)}$. Note that w obeys the equation of

$$i\partial_t w + \Delta w = |u|^p u - \chi_{\leq 1}(t)|v|^p v.$$

We find that the nonlinearity obeys

$$\big||u|^p u - \chi_{\leq 1}(t)|v|^p v\big| \lesssim \big(|u|^p + |\chi_{\lesssim 1}(t)v|^p\big)\big(|w| + |\chi_{\gtrsim 1}(t)v|\big).$$

Due to the good estimates on $\chi_{\gtrsim 1}(t)v$ obtained in Step 2, we can prove the desired estimate by the almost mass conservation of w.

2. Preliminary

2.1. **Notation.** We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_a Y$ denotes the assertion that $X \leq C(a)Y$ for some $C(a)$ depending on a. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$.

The notation $|\nabla|^\alpha = (-\partial_x^2)^{\alpha/2}$. We denote $\mathcal{S}(\mathbb{R}^d)$ to be the Schwartz Space in $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ to be the topological dual of $\mathcal{S}(\mathbb{R}^d)$. Let $h \in \mathcal{S}'(\mathbb{R}^{d+1})$, we use $\|h\|_{L^q_t L^p_x}$ to denote the mixed norm $\Big(\displaystyle\int \|h(\cdot,t)\|^q_{L^p}\,dt\Big)^{\frac{1}{q}}$, and $\|h\|_{L^q_{xt}} := \|h\|_{L^q_x L^q_t}$. Sometimes, we use the notation $q' = \frac{q}{q-1}$.

Throughout this paper, we use $\chi_{\leq a}$ for $a \in \mathbb{R}^+$ to be the smooth function

$$\chi_{\leq a}(x) = \begin{cases} 1, \ |x| \leq a, \\ 0, \ |x| \geq \dfrac{11}{10}a. \end{cases}$$

Moreover, we denote $\chi_{\geq a} = 1 - \chi_{\leq a}$ and $\chi_{a\leq\cdot\leq b} = \chi_{\leq b} - \chi_{\leq a}$. We denote $\chi_a = \chi_{\leq 2a} - \chi_{\leq a}$ for short.

Also, we need some Fourier operators. For each number $N > 0$, we define the Fourier multipliers $P_{\leq N}, P_{>N}, P_N$ as

$$\widehat{P_{\leq N}f}(\xi) := \chi_{\leq N}(\xi)\hat f(\xi),$$
$$\widehat{P_{>N}f}(\xi) := \chi_{>N}(\xi)\hat f(\xi),$$
$$\widehat{P_N f}(\xi) := \chi_N(\xi)\hat f(\xi),$$

and similarly $P_{<N}$ and $P_{\geq N}$. We will usually use these multipliers when N are *dyadic numbers* (that is, of the form 2^k for some integer k).

2.2. **Some basic lemmas.** First, we need the following radial Sobolev embedding, see [61] for example.

Lemma 2.1. *Let α, q, p, s be the parameters which satisfy*

$$\alpha > -\frac{d}{q}; \quad \frac{1}{q} \le \frac{1}{p} \le \frac{1}{q} + s; \quad 1 \le p, q \le \infty; \quad 0 < s < d$$

with

$$\alpha + s = d(\frac{1}{p} - \frac{1}{q}).$$

Moreover, at most one of the equalities hold:

$$p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} = \frac{1}{q} + s.$$

Then

$$\big\| |x|^\alpha u \big\|_{L^q(\mathbb{R}^d)} \lesssim \big\| |\nabla|^s u \big\|_{L^p(\mathbb{R}^d)}.$$

The second is the following fractional Leibniz rule, see [34, 4, 46] and the references therein.

Lemma 2.2. *Let $0 < s < 1$, $\frac{1}{2} < p \le \infty$, and $1 < p_1, p_2, p_3, p_4 \le \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{p} = \frac{1}{p_3} + \frac{1}{p_4}$, and let $f, g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\big\| |\nabla|^s (fg) \big\|_{L^p} \lesssim \big\| |\nabla|^s f \big\|_{L^{p_1}} \|g\|_{L^{p_2}} + \big\| |\nabla|^s g \big\|_{L^{p_3}} \|f\|_{L^{p_4}}.$$

A simple consequence is the following elementary inequality.

Lemma 2.3. *For any $a > 0, 1 \le p \le \infty, 0 \le \gamma < \frac{d}{p}$, and $|\nabla|^\gamma g \in L^p(\mathbb{R}^d)$,*

$$\big\| |\nabla|^\gamma \big(\chi_{\le a} g\big) \big\|_{L^p(\mathbb{R}^d)} \lesssim \big\| |\nabla|^\gamma g \big\|_{L^p(\mathbb{R}^d)}. \tag{2.1}$$

Here the implicit constant is independent on a. The same estimate holds for $\chi_{\ge a} g$.

Proof. The case $\gamma = 0$ is trivial. Further, we may assume that $0 < \gamma < 1$. Otherwise, we can use the standard Leibniz rule and the Hölder inequality to reduce the derivatives.

From Lemma 2.2, the Hölder and Sobolev inequalities, we have

$$\begin{aligned} \big\| |\nabla|^\gamma \big(\chi_{\le a} g\big) \big\|_{L^p(\mathbb{R}^d)} \lesssim & \big\| |\nabla|^\gamma \chi_{\le a} \big\|_{L^{\frac{d}{\gamma}}(\mathbb{R}^d)} \big\| g \big\|_{L^{\frac{dp}{d-p\gamma}}(\mathbb{R}^d)} + \big\| \chi_{\le a} \big\|_{L^\infty(\mathbb{R}^d)} \big\| |\nabla|^\gamma g \big\|_{L^2(\mathbb{R}^d)} \\ \lesssim & \Big(\big\| |\nabla|^\gamma \chi_{\le a} \big\|_{L^{\frac{d}{\gamma}}(\mathbb{R}^d)} + \big\| \chi_{\le a} \big\|_{L^\infty(\mathbb{R}^d)} \Big) \big\| g \big\|_{L^{\frac{dp}{d-p\gamma}}(\mathbb{R}^d)}. \end{aligned}$$

Note that $\|\chi_{\le a}\|_{L^\infty(\mathbb{R}^d)} \lesssim 1$ and

$$\big\| |\nabla|^\gamma \chi_{\le a} \big\|_{L^{\frac{d}{\gamma}}(\mathbb{R}^d)} = a^{-\gamma} \big\| |\nabla|^\gamma \chi_{\le 1}(\frac{\cdot}{a}) \big\|_{L^{\frac{d}{\gamma}}(\mathbb{R}^d)} = \big\| |\nabla|^\gamma \chi_{\le 1} \big\|_{L^{\frac{d}{\gamma}}(\mathbb{R}^d)} \lesssim 1.$$

Hence we obtain (2.1).

Note that

$$\chi_{\ge a} g = 1 - \chi_{\le a} g,$$

then by (2.1), we have

$$\big\| \chi_{\ge a} g \big\|_{\dot{H}^\gamma(\mathbb{R}^d)} \lesssim \big\| g \big\|_{\dot{H}^\gamma(\mathbb{R}^d)} + \big\| \chi_{\le a} g \big\|_{\dot{H}^\gamma(\mathbb{R}^d)} \lesssim \big\| g \big\|_{\dot{H}^\gamma(\mathbb{R}^d)}.$$

Hence, the same estimate holds for $\chi_{\le a} g$. Thus we finish the proof of the lemma. □

Moreover, we need the following mismatch result, which is helpful in commuting the spatial and the frequency cutoffs.

Lemma 2.4 (Mismatch estimates, see [47]). *Let ϕ_1 and ϕ_2 be smooth functions obeying*

$$|\phi_j| \le 1 \quad \text{and} \quad dist(\mathrm{supp}\phi_1, \mathrm{supp}\phi_2) \ge A,$$

for some large constant A. Then for $\sigma > 0$, $M \le 1$ and $1 \le r \le q \le \infty$,

$$\big\|\phi_1 |\nabla|^\sigma P_{\le M}(\phi_2 f)\big\|_{L^q_x(\mathbb{R}^d)} + \big\|\phi_1 \nabla |\nabla|^{\sigma-1} P_{\le M}(\phi_2 f)\big\|_{L^q_x(\mathbb{R}^d)} \lesssim A^{-\sigma-\frac{d}{r}+\frac{d}{q}} \|\phi_2 f\|_{L^r_x(\mathbb{R}^d)}; \quad (2.2)$$

$$\big\|\phi_1 \nabla P_{\le M}(\phi_2 f)\big\|_{L^q_x(\mathbb{R}^d)} \lesssim_m M^{1-m} A^{-m} \|f\|_{L^q_x(\mathbb{R}^d)}, \text{ for any } m \ge 0. \quad (2.3)$$

Furthermore, we need the following elementary formulas.

Lemma 2.5. *Let the vector function $f \in (\mathcal{S}(\mathbb{R}^d))^d$ and the scale function $g \in \mathcal{S}(\mathbb{R}^d)$, then for any integer N,*

$$\nabla_\xi \cdot \big(f\, \nabla_\xi\big)^{N-1} \cdot (fg) = \sum_{\substack{l_1,\cdots,l_N \in \mathbb{R}^d, l' \in \mathbb{R}^d; \\ |l_j| \le j; |l_1|+\cdots+|l_N|+|l'| = N}} C_{l_1,\cdots,l_N,l'} \partial_\xi^{l_1} f \cdots \partial_\xi^{l_N} f \, \partial_\xi^{l'} g,$$

where we have used the notation

$$\nabla_\xi = \{\partial_{\xi_1}, \cdots, \partial_{\xi_d}\}; \quad \partial_\xi^l = \partial_{\xi_1}^{l^1} \cdots \partial_{\xi_N}^{l^N}, \text{ for any } l = \{l^1, \cdots, l^N\} \in \mathbb{R}^N.$$

Proof. When $N = 1$, it is directly followed from the Leibniz rule. Denote that

$$\Lambda_N(f, g) = \nabla_\xi \cdot \big(f\, \nabla_\xi\big)^{N-1} \cdot (fg),$$

then we have

$$\Lambda_N(f, g) = \nabla_\xi \cdot \big(f \Lambda_{N-1}(f, g)\big).$$

The identity is then followed from the induction. □

2.3. **Linear Schrödinger operator.** Let the operator $S(t) = e^{it\Delta}$ be the linear Schrödinger flow, that is,

$$(i\partial_t + \Delta) S(t) \equiv 0.$$

The following are some fundamental properties of the operator $e^{it\Delta}$. The first is the explicit formula, see for example Cazenave [9].

Lemma 2.6. *For all $\phi \in \mathcal{S}(\mathbb{R}^d)$, $t \ne 0$,*

$$S(t)\phi(x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} \phi(y)\, dy.$$

Moreover, for any $r \ge 2$,

$$\|S(t)\phi\|_{L^r_x(\mathbb{R}^d)} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{r})} \|\phi\|_{L^{r'}(\mathbb{R}^d)}.$$

The following is the standard Strichartz estimates, see for example [29].

Lemma 2.7. *Let I be a compact time interval and let $u : I \times \mathbb{R}^d \to \mathbb{R}$ be a solution to the inhomogeneous Schrödinger equation*

$$iu_t - \Delta u + F = 0.$$

Then for any $t_0 \in I$, any pairs $(q_j, r_j), j = 1, 2$ satisfying

$$q_j \geq 2, \ r_j \geq 2, \ \text{and} \ \frac{2}{q_j} + \frac{d}{r_j} = \frac{d}{2},$$

the following estimates hold,

$$\big\|u\big\|_{C(I;L^2(\mathbb{R}^d))} + \big\|u\big\|_{L_t^{q_1}L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \big\|u(t_0)\big\|_{L_x^2(\mathbb{R}^d)} + \big\|F\big\|_{L_t^{q_2'}L_x^{r_2'}(I\times\mathbb{R}^d)}.$$

We also need the special Strichartz estimates for radial data, which was firstly proved by Shao [59], and then developed in [11, 28].

Lemma 2.8 (Radial Strichartz estimates)**.** *Let $g \in L^2(\mathbb{R}^d)$ be a radial function, k be an integer, then for any triple (q, r, γ) satisfying*

$$\gamma \in \mathbb{R}, \ q \geq 2, \ r > 2, \ \frac{2}{q} + \frac{2d-1}{r} < \frac{2d-1}{2}, \ \text{and} \ \frac{2}{q} + \frac{d}{r} = \frac{d}{2} + \gamma, \tag{2.4}$$

we have that

$$\big\||\nabla|^{\gamma}e^{it\Delta}g\big\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \big\|g\big\|_{L^2(\mathbb{R}^d)}.$$

Furthermore, let $F \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}(\mathbb{R}^{d+1})$ be a radial function in x, then

$$\Big\|\int_0^t e^{i(t-s)\Delta}F(s)\,ds\Big\|_{L_t^qL_x^r(\mathbb{R}^{d+1})} + \Big\||\nabla|^{-\gamma}\int_0^t e^{i(t-s)\Delta}F(s)\,ds\Big\|_{L_t^\infty L_x^2(\mathbb{R}^{d+1})} \lesssim \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}(\mathbb{R}^{d+1})},$$

where the triples (q, r, γ), $(\tilde{q}, \tilde{r}, -\gamma)$ satisfy (2.4).

The following is a remark regarding the lemma above.

Remark 2.9. One may ask about the optimal smoothing effect one can gain from the radial Strichartz estimates, corresponding to the supremum of γ as above. In fact, from Lemma 2.8, fixing $q \geq 2$, then we find

$$\gamma < \frac{2}{q} \cdot \frac{d-1}{2d-1}.$$

On the other hand, for any $q \geq 2$ and any $\gamma < \frac{2}{q} \cdot \frac{d-1}{2d-1}$, there exists r such that (q, r, γ) verifies (2.4), and thus the radial Strichartz estimates hold.

3. Linear flow estimates on localized functions

We begin with preliminary linear estimates we need. In this section, we give the following estimates.

Proposition 3.1. *Let $r \geq 2$, then for any $t : |t| \geq \frac{100}{N}$, and any s satisfying*

$$0 \leq s < (d-2)\Big(\frac{1}{2} - \frac{1}{r}\Big) + s_c,$$

the following estimate holds,

$$\Big\||\nabla|^s\big(e^{it\Delta}\big(\chi_{\leq 10}P_{\geq N}g\big)\big)\Big\|_{L_x^r(\mathbb{R}^d)} \lesssim N^{-(d-2)\left(\frac{1}{2}-\frac{1}{r}\right)+s-s_c}|t|^{-(d-1)\left(\frac{1}{2}-\frac{1}{r}\right)}\|P_{\geq N}g\|_{\dot{H}^{s_c}(\mathbb{R}^d)}. \tag{3.1}$$

Moreover, let $d \geq 3$, (q, r, γ) be the triple satisfying (2.4) and

$$\frac{1}{q} < (d-1)\Big(\frac{1}{2} - \frac{1}{r}\Big). \tag{3.2}$$

Then there exist $s_* = s_*(q) < 0, \alpha_* = \alpha_*(q,r,\gamma) \geq 1$ and $\beta_* = \beta_*(q,r,\gamma) > 0$, such that for any $\alpha \geq \alpha_*, \beta \leq \beta_*$, and any $s_c \in [s_*, 0)$, the following estimate holds,

$$\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(e^{it\Delta} \chi_{\leq 10}(P_{\geq N} g)\big)\Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|P_{\geq N} g\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.3}$$

Remark 3.2. From the proof of Proposition 3.1 below, it also follows that $s_*(q)$ can be chosen to be a decreasing function with $s_*(+\infty) = 0$.

The proof of the proposition is based on the following two lemmas. First of all, we show the estimate in the local domain.

Lemma 3.3. *Let $M \geq 1$, $r \geq 2$ and $s \geq 0$, then for any $t : |t| \geq \frac{100}{M}$, any $K \in \mathbb{Z}^+$,*

$$\Big\| |\nabla|^s \big(\chi_{\leq \frac{1}{10}M|t|} e^{it\Delta} \big(\chi_{\leq 10} P_M g\big)\big)\Big\|_{L^r_x(\mathbb{R}^d)} \lesssim_{s,K} |t|^{-d(\frac12 - \frac1r)} M^{-K} \|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.4}$$

Moreover, let (q, r, γ) be the triple satisfying the same conditions as in Proposition 3.1, and let α, β be the constants satisfying

$$\alpha \geq 1, \qquad d(\frac12 - \frac1r) > \alpha\beta + \frac1q. \tag{3.5}$$

Then there exists $s_{*,1} = s_{*,1}(q) < 0$ such that for any $s_c \in [s_{*,1}, 0)$, the following estimate holds,

$$\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(\chi_{\leq \frac{1}{10}M|t|} e^{it\Delta} \big(\chi_{\leq 10} P_M g\big)\big)\Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.6}$$

Proof. First, we show that for any $M \geq 1$, $t : |t| \geq \frac{100}{M}$, $s \in \mathbb{Z}^+ \cup \{0\}$ and $K \in \mathbb{Z}^+$,

$$\Big| |\nabla|^s \Big(\chi_{\leq \frac{1}{10}M|t|}\, e^{it\Delta} \big(\chi_{\leq 10} P_M g\big)\Big)\Big| \lesssim_K |t|^{-\frac d2} M^{-K} \|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.7}$$

To show this, we use the formula in Lemma 2.6 to obtain

$$e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)(x) = \frac{1}{(4\pi i t)^{\frac d2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t} + iy\cdot\xi} \chi_{\leq 10}(y)\chi_M(\xi)\hat g(\xi)\, dy d\xi. \tag{3.8}$$

Fix x, ξ, and define the phase as

$$\phi(y) = -\frac{x\cdot y}{2t} + \frac{|y|^2}{4t} + y\cdot\xi,$$

then from (3.8),

$$e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)(x) = \frac{e^{\frac{i|x|^2}{4t}}}{(4\pi i t)^{\frac d2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\phi(y)} \chi_{\leq 10}(y)\chi_M(\xi)\hat g(\xi)\, dy d\xi. \tag{3.9}$$

Moreover, we have

$$\nabla_y \phi(y) = \frac{y-x}{2t} + \xi, \tag{3.10}$$

and

$$\partial_{y_j y_k}\phi(y) = \frac{\delta_{jk}}{2t}, \quad \partial_{y_j y_k y_h}\phi(y) = 0, \text{ for any } \quad j,k,h \in \{1,\cdots,d\}. \tag{3.11}$$

Note that

$$|t|M \geq 100,\ |\xi| \geq \frac{9}{10}M,\ |y| \leq \frac{101}{10}, \ \text{ and } |x| \leq \frac{11}{100}M|t|,$$

from (3.10) we have

$$|\nabla\phi(y)| \gtrsim |\xi|. \tag{3.12}$$

Then using the formula

$$e^{i\phi} = \nabla_y e^{i\phi} \cdot \frac{\nabla_y \phi}{i|\nabla_y\phi|^2},$$

and integration by parts K times on right-hand side of (3.9), we obtain

$$\begin{aligned}\chi_{\le\frac{1}{10}M|t|}\, e^{it\Delta}\big(\chi_{\le10}P_M g\big) = \frac{C_K e^{\frac{i|x|^2}{4t}}}{t^{\frac{d}{2}}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\phi(y)} \\ \cdot \nabla_y \cdot \Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\nabla_y\Big)^{K-1} \cdot \Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\chi_{\le1}(y)\Big)\, dy \cdot \chi_M(\xi)\hat{g}(\xi)\, d\xi. \end{aligned}\tag{3.13}$$

We claim that

$$\Big|\nabla_y \cdot \Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\nabla_y\Big)^{K-1} \cdot \Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\chi_{\le1}(y)\Big)\Big| \lesssim |\xi|^{-K}\chi_{\lesssim1}(\cdot). \tag{3.14}$$

Indeed, from Lemma 2.5, we expand the left-hand side of (3.14) as

$$\sum_{\substack{l_1,\cdots,l_K\in\mathbb{R}^d, l'\in\mathbb{R}^d;\\ |l_j|\le j; |l_1|+\cdots+|l_K|+|l'|=K}} C_{l_1,\cdots,l_K,l'}\,\partial_y^{l_1}\Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\Big)\cdots\partial_y^{l_K}\Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\Big)\,\partial_y^{l'}\big(\chi_{\le10}(\cdot)\big). \tag{3.15}$$

Note that from (3.11) and (3.12), we have that for any non-negative integer vectors l, l',

$$\Big|\partial_y^l\Big(\frac{\nabla_y\phi}{i|\nabla_y\phi|^2}\Big)\Big| \lesssim \frac{1}{|\xi|}\frac{1}{|t\xi|^{|l|}},$$

and

$$\Big|\partial_y^{l'}\big(\chi_{\le1}(\cdot)\big)\Big| \lesssim \chi_{\lesssim1}(\cdot).$$

Hence, using these two estimates,

$$|(3.15)| \lesssim \sum_{\substack{l_1,\cdots,l_K\in\mathbb{R}^d, l'\in\mathbb{R}^d;\\ |l_j|\le j; |l_1|+\cdots+|l_K|+|l'|=K}} \frac{1}{|\xi|}\frac{1}{|t\xi|^{|l_1|}}\cdots\frac{1}{|\xi|}\frac{1}{|t\xi|^{|l_K|}}\cdot\chi_{\lesssim1}(y) \lesssim |\xi|^{-K}\chi_{\lesssim1}(y).$$

Therefore, we obtain (3.14).

Inserting (3.14) into (3.13), we obtain

$$\begin{aligned}\Big|\chi_{\le\frac{1}{10}M|t|}\, e^{it\Delta}\big(\chi_{\le10}P_M g\big)\Big| &\lesssim |t|^{-\frac{d}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\chi_{\lesssim1}(y)|\xi|^{-K}\chi_M(\xi)\hat{g}(\xi)\,dy\,d\xi \\ &\lesssim |t|^{-\frac{d}{2}}M^{-K-s_c+d}\|P_M g\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.\end{aligned}$$

Note that when the derivatives hit the cut-off functions $\chi_{\le\frac{1}{10}M|t|}$ and $\chi_{\le10}$, the estimates on

$$\chi_{\le\frac{1}{10}M|t|}\, e^{it\Delta}\big(\chi_{\le10}P_M g\big)$$

become better, hence by choosing K suitable large, we obtain that for any $s\in\mathbb{Z}^+$,

$$\begin{aligned}\Big||\nabla|^s\Big(\chi_{\le\frac{1}{10}M|t|}\, e^{it\Delta}\big(\chi_{\le10}P_M g\big)\Big)\Big| &\lesssim |t|^{-\frac{d}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\chi_{\lesssim1}(y)|\xi|^{-K}\chi_M(\xi)\hat{g}(\xi)\,dy\,d\xi \\ &\lesssim |t|^{-\frac{d}{2}}M^{-K-s_c+d}\|P_M g\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.\end{aligned}$$

Replacing K by $K - s_c + d$, we obtain (3.7).

Further, using (3.7), Lemma 2.4, Hölder's inequality and interpolation when s is not an integer, we obtain (3.4).

Now we prove (3.6). We write

$$\begin{aligned}
&\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\ge \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\quad + \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)}.
\end{aligned}$$

For the first term, using (3.4), we have

$$\begin{aligned}
&\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\ge \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\lesssim \Big\| |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\ge \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\quad + \Big\| |t|^{\alpha\beta} |\nabla|^{s_c+\beta+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\ge \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\lesssim \Big(\big\| t^{-d(\frac{1}{2}-\frac{1}{r})} \big\|_{L_t^q(\{|t|\ge \frac{100}{M}\})} + \big\| t^{-d(\frac{1}{2}-\frac{1}{r})+\alpha\beta} \big\|_{L_t^q(\{|t|\ge \frac{100}{M}\})} \Big) M^{-K} \|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)} \lesssim \|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)}.
\end{aligned}$$

Here we have used the condition of $d(\frac{1}{2} - \frac{1}{r}) > \alpha\beta + \frac{1}{q}$.

For the second term, we have

$$\begin{aligned}
&\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\lesssim \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} P_{\le \frac{3}{2}M} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\quad + \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} P_{\ge \frac{3}{2}M} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)} \\
&\lesssim \Big\| |\nabla|^{s_c+\gamma} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)} \qquad (3.16) \\
&\quad + \Big\| |\nabla|^{s_c+\gamma} P_{\ge \frac{3}{2}M} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \qquad (3.17) \\
&\quad + M^{-\alpha\beta M} \Big\| |\nabla|^{s_c+\beta+\gamma} P_{\ge \frac{3}{2}M} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)}. \qquad (3.18)
\end{aligned}$$

Here we have used the condition $\alpha \ge 1$ in the second step. Now we consider (3.16)–(3.18) term by term.

For (3.16), if $s_c + \gamma \ge 0$, then using Lemma 2.3 twice and Lemma 2.8, it is controlled by $\|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)}$. If $s_c + \gamma < 0$, we further decompose it into the following two parts,

$$\Big\| |\nabla|^{s_c+\gamma} P_{\le \frac{1}{2}M} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)} \qquad (3.19)$$

$$+ \Big\| |\nabla|^{s_c+\gamma} P_{\ge \frac{1}{2}M} \big(\chi_{\le \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\le 10} P_M g\big)\big) \Big\|_{L_t^q L_x^r(\{|t|\le \frac{100}{M}\}\times\mathbb{R}^d)}. \qquad (3.20)$$

For (3.19), using the Hölder and Sobolev inequalities, it is bounded by

$$M^{-\frac{1}{q}}\Big\||\nabla|^{s_c+\frac{2}{q}}P_{\le\frac12 M}\big(\chi_{\le\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\le 10}P_Mg\big)\big)\Big\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)}.$$

Now we set $s_*\ge-\frac{2}{q}$ such that $s_c+\frac{2}{q}\ge 0$, then using Lemma 2.4 twice, we obtain

$$(3.19)\lesssim M^{-10}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.21}$$

For (3.20), since $s_c+\gamma<0$, using Bernstein's inequality, it is controlled by

$$M^{s_c+\gamma}\Big\|\chi_{\le\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\le 10}P_Mg\big)\Big\|_{L_t^qL_x^r(\{|t|\le\frac{100}{M}\}\times\mathbb{R}^d)}.$$

Then we decompose it into following two subparts again,

$$\begin{aligned}&M^{s_c+\gamma}\Big\|\chi_{\le\frac{1}{10}M|t|}e^{it\Delta}P_{\le\frac12 M}\big(\chi_{\le 10}P_Mg\big)\Big\|_{L_t^qL_x^r(\{|t|\le\frac{100}{M}\}\times\mathbb{R}^d)}\\&+M^{s_c+\gamma}\Big\|\chi_{\le\frac{1}{10}M|t|}e^{it\Delta}P_{\ge\frac12 M}\big(\chi_{\le 10}P_Mg\big)\Big\|_{L_t^qL_x^r(\{|t|\le\frac{100}{M}\}\times\mathbb{R}^d)}.\end{aligned}$$

The first subpart, we treat similarly as (3.19), and conclude that it is bounded by $M^{-10}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}$; the second subpart, we use Bernstein's inequality and Lemma 2.8, and conclude that it is bounded by $\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}$. Hence, we obtain

$$(3.20)\lesssim\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.22}$$

Combining (3.21) and (3.22), we get

$$(3.16)\lesssim\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.23}$$

For (3.17) and (3.18), using Lemma 2.4 twice and Lemma 2.8, we obtain

$$(3.17)+(3.18)\lesssim M^{-10}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

This last estimate combined with (3.23) yields

$$\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}\big(\chi_{\le\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\le 10}P_Mg\big)\big)\Big\|_{L_t^qL_x^r(\{|t|\le\frac{100}{M}\}\times\mathbb{R}^d)}\lesssim\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Together with the estimates of the first and the second terms, we get (3.6). □

The second lemma shows the estimates of the linear flow in the domain far away from the origin.

Lemma 3.4. *Let $M\ge1$, $r\ge2$ and $s\ge0$, then for any $t:|t|\gtrsim\frac{1}{M}$,*

$$\Big\||\nabla|^s\Big(\chi_{\ge\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\le 10}P_Mg\big)\Big)\Big\|_{L_x^r(\mathbb{R}^d)}\lesssim M^{-(d-2)\left(\frac12-\frac1r\right)+s}|t|^{-(d-1)\left(\frac12-\frac1r\right)}\|P_Mg\|_{L_x^2(\mathbb{R}^d)}. \tag{3.24}$$

Moreover, let (q,r,γ) be the triple satisfying the same conditions as in Proposition 3.1, and let α,β be the constants satisfying $\alpha\ge1,\beta>0$ and

$$(d-1)(\frac12-\frac1r)>\max\{\alpha\beta,-\alpha(s_c+\gamma)\}+\frac1q, \tag{3.25}$$

Then there exists $s_{*,2}=s_{*,2}(q)<0$ such that for any $s_c\in[s_{*,2},0)$, the following estimate holds,

$$\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}\big(\chi_{\ge\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\le 10}P_Mg\big)\big)\Big\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)}\lesssim\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{3.26}$$

Proof. From the radial Sobolev embedding, we have

$$\begin{aligned}\Big\|\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\Big\|_{L^r_x(\mathbb{R}^d)} \lesssim&\big(M|t|\big)^{-(d-1)\left(\frac12-\frac1r\right)}\Big\||\nabla|^{\frac12-\frac1r}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\Big\|_{L^2_x(\mathbb{R}^d)}\\ \lesssim&\big(M|t|\big)^{-(d-1)\left(\frac12-\frac1r\right)}\Big\||\nabla|^{\frac12-\frac1r}\big(\chi_{\leq 10}P_Mg\big)\Big\|_{L^2_x(\mathbb{R}^d)}. \end{aligned}\tag{3.27}$$

Using Lemma 2.3, we have

$$\Big\||\nabla|^{\frac12-\frac1r}\big(\chi_{\leq 10}P_Mg\big)\Big\|_{L^2_x(\mathbb{R}^d)} \lesssim M^{\frac12-\frac1r}\left\|P_Mg\right\|_{L^2_x(\mathbb{R}^d)}.$$

This last estimate combined with (3.27) yields

$$\Big\|\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\Big\|_{L^r_x(\mathbb{R}^d)} \lesssim M^{-(d-2)\left(\frac12-\frac1r\right)}|t|^{-(d-1)\left(\frac12-\frac1r\right)}\left\|P_Mg\right\|_{L^2_x(\mathbb{R}^d)}. \tag{3.28}$$

Similarly, we also obtain that for any $s\geq 0$, we have (3.24). Indeed, if the derivatives hit the cut-off functions $\chi_{\geq\frac{1}{10}M|t|}$ (since $M|t|\gtrsim 1$) and $\chi_{\leq 10}$, the analogous estimates become better. Hence by the same way as (3.28), we obtain the estimates above.

Now we prove (3.26). We decompose it into the following three terms.

$$\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}$$

$$\lesssim \Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\leq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)} \tag{3.29}$$

$$+\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\leq(3M)^{-\frac1\alpha}\}\times\mathbb{R}^d)} \tag{3.30}$$

$$+\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\geq(3M)^{-\frac1\alpha}\}\times\mathbb{R}^d)}. \tag{3.31}$$

For the term (3.29), if $s_c+\gamma\geq 0$, then using Lemma 2.3 and Lemma 2.8, we have

$$\begin{aligned}&\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\leq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}\\ &\quad\lesssim\Big\||\nabla|^{s_c+\gamma}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}\lesssim\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.\end{aligned}$$

If $s_c+\gamma<0$, then setting $s_*\geq-\frac1q$, using the homogeneous Littlewood-Paley decomposition and treating similarly as (3.16), we also get the bound of $\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}$.

For the term (3.30), since

$$|t|^{-\alpha}\geq 3M,$$

we have

$$\begin{aligned}&\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\leq(3M)^{-\frac1\alpha}\}\times\mathbb{R}^d)}\\ &\quad\lesssim\Big\||t|^{\alpha\beta}|\nabla|^{s_c+\beta+\gamma}P_{\geq 3M}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\leq(3M)^{-\frac1\alpha}\}\times\mathbb{R}^d)}\\ &\quad\lesssim M^{-\beta}\Big\||\nabla|^{s_c+\beta+\gamma}P_{\geq 3M}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}.\end{aligned}$$

Then using Lemma 2.4 twice, Lemma 2.3 and Lemma 2.8, it is bounded by

$$M^{-10}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Therefore, we obtain

$$\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\leq(3M)^{-\frac{1}{\alpha}}\}\times\mathbb{R}^d)}\lesssim M^{-10}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

For the term (3.31), we have

$$\begin{aligned}&\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\geq(3M)^{-\frac{1}{\alpha}}\}\times\mathbb{R}^d)}\\&\lesssim\Big\|t^{\alpha\beta}|\nabla|^{s_c+\beta+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\geq(3M)^{-\frac{1}{\alpha}}\}\times\mathbb{R}^d)}.\end{aligned}\tag{3.32}$$

If $s_c+\beta+\gamma\geq 0$, using (3.24), (3.32) is bounded by

$$M^{-(d-2)(\frac{1}{2}-\frac{1}{r})+\beta+\gamma}\Big\|t^{\alpha\beta-(d-1)(\frac{1}{2}-\frac{1}{r})}\Big\|_{L^q_t(\{|t|\geq(3M)^{-\frac{1}{\alpha}}\})}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Using the condition of $(d-1)(\frac{1}{2}-\frac{1}{r})>\alpha\beta+\frac{1}{q}$, it is dominated by

$$M^{-(d-2)(\frac{1}{2}-\frac{1}{r})+(d-1)(\frac{1}{2}-\frac{1}{r})\cdot\frac{1}{\alpha}-\frac{1}{q\alpha}+\gamma}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.\tag{3.33}$$

Now we claim that

$$-(d-2)(\frac{1}{2}-\frac{1}{r})+(d-1)(\frac{1}{2}-\frac{1}{r})\cdot\frac{1}{\alpha}-\frac{1}{q\alpha}+\gamma\leq 0.\tag{3.34}$$

Indeed, using (2.4), the left-hand side of (3.34) is equal to

$$\frac{2\alpha-1}{\alpha}\cdot\Big(\frac{1}{q}-(d-1)\Big(\frac{1}{2}-\frac{1}{r}\Big)\Big).$$

Note that $\alpha\geq 1$, and from (3.25): $\frac{1}{q}<(d-1)\Big(\frac{1}{2}-\frac{1}{r}\Big)$, the last quantity above is negative. Hence, (3.34) is valid. Using (3.34), we have that (3.33) and then (3.32) are bounded by $\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}$.

If $s_c+\beta+\gamma<0$, using the Bernstein inequality and then using (3.24), (3.32) is bounded by

$$\begin{aligned}&\Big\|t^{-\alpha(s_c+\gamma)}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\geq(3M)^{-\frac{1}{\alpha}}\}\times\mathbb{R}^d)}\\&\lesssim M^{-(d-2)(\frac{1}{2}-\frac{1}{r})-s_c}\Big\|t^{-\alpha(s_c+\gamma)-(d-1)(\frac{1}{2}-\frac{1}{r})}\Big\|_{L^q_t(\{|t|\geq(3M)^{-\frac{1}{\alpha}}\})}\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.\end{aligned}$$

Then similarly as above, and using the condition of $(d-1)(\frac{1}{2}-\frac{1}{r})+\alpha(s_c+\gamma)>\frac{1}{q}$, it is also bounded by $\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}$. Hence, we obtain that

$$\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}P_{\geq|t|^{-\alpha}}\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^q_tL^r_x(\{|t|\geq(3M)^{-\frac{1}{\alpha}}\}\times\mathbb{R}^d)}\lesssim\|P_Mg\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Combining the three estimates above, we get (3.26). □

Together with Lemma 3.3 and Lemma 3.4, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Using Littlewood-Paley's decomposition, we have

$$\begin{aligned}&\Big\||\nabla|^s\big(e^{it\Delta}\big(\chi_{\leq 10}P_{\geq N}g\big)\big)\Big\|_{L^r_x(\mathbb{R}^d)}\lesssim\sum_{M\geq N}\Big\||\nabla|^s\big(e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^r_x(\mathbb{R}^d)}\\&\lesssim\sum_{M\geq N}\Big\||\nabla|^s\big(\chi_{\leq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^r_x(\mathbb{R}^d)}+\sum_{M\geq N}\Big\||\nabla|^s\big(\chi_{\geq\frac{1}{10}M|t|}e^{it\Delta}\big(\chi_{\leq 10}P_Mg\big)\big)\Big\|_{L^r_x(\mathbb{R}^d)}.\end{aligned}$$

Using Lemma 3.3, we get that for any $K \in \mathbb{Z}^+$,

$$\sum_{M\geq N} \Big\| |\nabla|^s \big(\chi_{\leq \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)\big) \Big\|_{L^r_x(\mathbb{R}^d)} \lesssim |t|^{-d(\frac12-\frac1r)} N^{-K} \|P_M g\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Using Lemma 3.4,

$$\sum_{M\geq N} \Big\| |\nabla|^s \big(\chi_{\geq \frac{1}{10}M|t|} e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)\big) \Big\|_{L^r_x(\mathbb{R}^d)} \lesssim N^{-(d-2)\left(\frac12-\frac1r\right)+s-s_c} |t|^{-(d-1)\left(\frac12-\frac1r\right)} \|P_M g\|_{\dot H^{s_c}_x(\mathbb{R}^d)}.$$

Combining these estimates, we obtain (3.1).

Now we prove (3.3). Firstly, we give a reduction as following. Fix $q \geq 2$, and let $\varepsilon_0 = \frac{1}{100q}$. Then to prove (3.3), we only need to consider the estimates on the triples (q, r, γ) when $\gamma \geq \varepsilon_0$. Indeed, if $\gamma \leq \varepsilon_0$, then we only need to consider the case when r and γ satisfy

$$\frac{1}{r} = \frac{1}{2} - \frac{2}{dq} + \frac{\varepsilon_0}{d}; \quad \gamma = \varepsilon_0. \tag{3.35}$$

The reason is that, the triple (q, r, γ) above satisfies the conditions in Proposition 3.1 when $d \geq 3$, moreover, the estimates on the general cases can be followed by the estimate on this case and the Sobolev inequality. Then using Littlewood-Paley's decomposition,

$$\begin{aligned}
&\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(e^{it\Delta}\big(\chi_{\leq 10} P_{\geq N} g\big)\big) \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \Big(e^{it\Delta}\big(\chi_{\leq 10} \sum_{M\geq N} P_M g\big)\Big) \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \Big\| \Big(\sum_{M\geq N} \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)\big) \Big\|^r_{L^r_x(\mathbb{R}^d)} \Big)^{\frac1r} \Big\|_{L^q_t(\mathbb{R})}.
\end{aligned}$$

Since $q \geq 2, r \geq 2$, it is dominated by

$$\Big(\sum_{M\geq N} \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)\big) \Big\|^2_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \Big)^{\frac12}.$$

Therefore, we obtain

$$\begin{aligned}
&\Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(e^{it\Delta}\big(\chi_{\leq 10} P_{\geq N} g\big)\big) \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \Big(\sum_{M\geq N} \Big\| \langle t^\alpha |\nabla|\rangle^\beta |\nabla|^{s_c+\gamma} \big(e^{it\Delta}\big(\chi_{\leq 10} P_M g\big)\big) \Big\|^2_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \Big)^{\frac12}.
\end{aligned}$$

Now we check the conditions (3.5) and (3.25). Setting

$$s_* = \max\{-\varepsilon_0, s_{*,1}, s_{*,2}\},$$

then $s_c + \gamma \geq 0$. Hence, the conditions (3.5) and (3.25) reduce to

$$(d-1)(\frac12 - \frac1r) > \alpha\beta + \frac1q,$$

which is valid by choosing $\alpha\beta$ small enough. Then by Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}\Big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c+\gamma}\big(e^{it\Delta}\big(\chi_{\le 10}P_{\ge N}g\big)\big)\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}\\ \lesssim\Big(\sum_{M\ge N}\|P_Mg\|^2_{\dot H^{s_c}(\mathbb{R}^d)}\Big)^{\frac12}\lesssim\|P_{\ge N}g\|_{\dot H^{s_c}(\mathbb{R}^d)}.\end{aligned}$$

This proves the proposition. □

4. Nonlinear flow estimates on localized initial data

In this section, we give some nonlinear estimates. Firstly, we give some local time and small data estimates.

4.1. **Local theory.** Since $u_0\in\dot H^{s_c}(\mathbb{R}^d)$, we have the following local and small data results, the proofs are standard. However, we give the details for the sake of the completeness. The first is essentially proved by Guo, Wang [28].

Lemma 4.1. *Let $s_0=-\min\{\frac{d-1}{2d-1},\frac{2(d-1)}{(2d-1)(p+1)}\}$, then for any $s_c>s_0$, the following result holds. There exists $\delta>0$, such that for radial function $u_0\in\dot H^{s_c}(\mathbb{R}^d)$, there exists $t_0=t_0(u_0,\delta)>0$, such that the Cauchy problem (1.1) is well-posed on the time interval $[0,t_0]$. Moreover the solution u satisfies*

$$\|u\|_{L^\infty_t\dot H^{s_c}_x([0,t_0]\times\mathbb{R}^d)}\lesssim 1;\quad \big\||\nabla|^{s_c+\gamma}u\big\|_{L^q_tL^r_x([0,t_0]\times\mathbb{R}^d)}\lesssim\delta. \tag{4.1}$$

Here the triple (q,r,γ) verifies (2.4) and $\gamma\in[-s_c,-s_0)$.

Remark 4.2. The result in this lemma improves the index obtained by Guo, Wang [28], who proved the local well-posedness in $\dot H^{s_c}(\mathbb{R}^d)$ when $s_c>-\frac{d-1}{2d+1}$ for radial datum. In particular, in this lemma, when $d\ge4$, the restriction is $s_c>-\frac{d-1}{2d-1}$ ($s_c>-0.275,-0.388$ when $d=2,3$ respectively).

Proof of Lemma 4.1. We only show (4.1) for some $t_0=t_0(u_0)>0$. Then the local well-posedness with the lifespan $[0,t_0)$ is followed by the standard fixed point argument. In the following, we prove (4.1) by two cases: $p\le1$ and $p>1$ separately.

If $p\le1$, we denote the parameter r_1 as

$$\frac{1}{r_1}=\frac12-\frac1d+\frac{\gamma}{d}.$$

Then for any $s_c>-\frac{d-1}{2d-1}$ and $\gamma\ge-s_c$, by the Duhamel formula and Lemma 2.8, we have

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\lesssim\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}(|u|^pu)\big\|_{L^{\tilde q'}_tL^{\tilde r'}_x([0,t_0]\times\mathbb{R}^d)},$$

where $(\tilde q,\tilde r)$ satisfies

$$\tilde q=\frac{2}{1-p},\quad \frac{2}{\tilde q}+\frac{d}{\tilde r}=\frac d2-\gamma.$$

Hence, by Lemma 2.2, we get

$$\begin{aligned}\||\nabla|^{s_c+\gamma}u\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\lesssim&\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\\&+\big\||\nabla|^{s_c+\gamma}u\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\big\|u\big\|^p_{L^2_tL^{r_2}_x([0,t_0]\times\mathbb{R}^d)},\end{aligned}$$

where the parameter r_2 satisfies

$$\frac{1}{r_2}=\frac{1}{2}-\frac{1}{d}-\frac{s_c}{d}.$$

Then by the Sobolev inequality, we obtain that

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\lesssim\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}u\big\|^{p+1}_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}.$$

Therefore, there exists $\delta>0$, if

$$\big\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\le\delta, \tag{4.2}$$

then by the continuity argument,

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\lesssim\delta. \tag{4.3}$$

Note that

$$\big\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\big\|_{L^2_tL^{r_1}_x(\mathbb{R}\times\mathbb{R}^d)}\lesssim\|u_0\|_{\dot H^{s_c}(\mathbb{R}^d)},$$

(4.2) is verified when $t_0=t_0(u_0)$ is small enough, and thus we have (4.3).

Similarly,

$$\begin{aligned}\|u\|_{L^\infty_t\dot H^{s_c}_x([0,t_0]\times\mathbb{R}^d)}\lesssim&\|e^{it\Delta}u_0\|_{L^\infty_t\dot H^{s_c}_x([0,t_0]\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}(|u|^pu)\big\|_{L^{\tilde q'}_tL^{\tilde r'}_x([0,t_0]\times\mathbb{R}^d)}\\ \lesssim&\|u_0\|_{\dot H^{s_c}(\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}u\big\|^{p+1}_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}.\end{aligned}$$

Then by (4.3), we obtain that

$$\|u\|_{L^\infty_t\dot H^{s_c}_x([0,t_0]\times\mathbb{R}^d)}\lesssim 1.$$

Further, for general triple (q,r,γ) verifying (2.4) and $\gamma\in[-s_c,\frac{d-1}{2d-1})$,

$$\begin{aligned}\big\||\nabla|^{s_c+\gamma}u\big\|_{L^q_tL^r_x([0,t_0]\times\mathbb{R}^d)}\lesssim&\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\|_{L^q_tL^r_x([0,t_0]\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}(|u|^pu)\big\|_{L^{\tilde q'}_tL^{\tilde r'}_x([0,t_0]\times\mathbb{R}^d)}\\ \lesssim&\|u_0\|_{\dot H^{s_c}(\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}u\big\|^{p+1}_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}.\end{aligned}$$

Hence, we get

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^q_tL^r_x([0,t_0]\times\mathbb{R}^d)}\lesssim\delta.$$

If $p>1$, we denote the parameter r_3 as

$$\frac{1}{r_3}=\frac{1}{2}-\frac{2}{d(p+1)}+\frac{\gamma}{d}.$$

Then similarly as above, we obtain that for any $s_c>-\frac{2(d-1)}{(2d-1)(p+1)}$ and $\gamma\ge-s_c$,

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}\lesssim\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\|_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}(|u|^pu)\big\|_{L^1_tL^{r_4'}_x([0,t_0]\times\mathbb{R}^d)},$$

where r_4 satisfies

$$\frac{d}{r_4}=\frac{d}{2}-\gamma.$$

Hence, by Lemma 2.2 and Sobolev's inequality, we get

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}\lesssim\||\nabla|^{s_c+\gamma}e^{it\Delta}u_0\|_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}u\big\|^{p+1}_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}.$$

Treating similarly as above, by choosing $t_0=t_0(u_0)$ small enough, we obtain that

$$\big\||\nabla|^{s_c+\gamma}u\big\|_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}\lesssim\delta,$$

and thus obtain (4.1). □

For simplicity, we set $t_0(u_0)=2$. Moreover, let δ_0 be some positive small constant decided later, we set a number $N=N(\delta_0)$ such that

$$\|P_{\geq N}u_0\|_{\dot H^{s_c}(\mathbb{R}^d)}\leq \delta_0. \tag{4.4}$$

To prove Theorem 1.1, we split the initial data u_0 into three parts as

$$u_0=\chi_{\leq 10}\big(P_{\geq N}u_0\big)+P_{\leq N}u_0+\chi_{\geq 10}\big(P_{\geq N}u_0\big).$$

Accordingly, let

$$v_0=\chi_{\leq 10}\big(P_{\geq N}u_0\big),$$

and v be the solution of the following equation,

$$\begin{cases} i\partial_t v+\Delta v=\chi_{\leq 1}(t)|v|^p v,\\ v(0,x)=v_0. \end{cases} \tag{4.5}$$

Moreover, let

$$w_0=\chi_{\geq 10}\big(P_{\geq N}u_0\big)+P_{\leq N}u_0,$$

and $w=u-v$. Then w is the solution of the following equation,

$$\begin{cases} i\partial_t w+\Delta w=|u|^p u-\chi_{\leq 1}(t)|v|^p v,\\ w(0,x)=w_0. \end{cases} \tag{4.6}$$

Then the second result is a global result with small data.

Lemma 4.3. *For any $s_c>s_0$, the following result holds. Let $u_0\in \dot H^{s_c}(\mathbb{R}^d)$ be radial, then there exist a small constant δ_0 and a large constant N verifying (4.4), such that the Cauchy problem (4.5) is globally well-posed. In particular, the solution v satisfies*

$$\|v\|_{L^\infty_t \dot H^{s_c}_x(\mathbb{R}\times\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}v\big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}\lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Here the triple (q,r,γ) verifies (2.4) and $\gamma\in[-s_c,-s_0)$.

Proof. We adopt the same notation and argue similarly as in the proof of Lemma 4.1. In the case of $p\leq 1$, for any $s_c>-\frac{d-1}{2d-1}$,

$$\big\||\nabla|^{s_c+\gamma}v\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}v\big\|^{p+1}_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}.$$

Hence, by the continuity argument and choosing δ_0 small enough, we obtain

$$\big\||\nabla|^{s_c+\gamma}v\big\|_{L^2_tL^{r_1}_x([0,t_0]\times\mathbb{R}^d)}\lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Using the estimate above, we have the desired results. In the case of $p>1$, for any $s_c>-\frac{2(d-1)}{(2d-1)(p+1)}$,

$$\big\||\nabla|^{s_c+\gamma}v\big\|_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}\lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}+\big\||\nabla|^{s_c+\gamma}v\big\|^{p+1}_{L^{p+1}_tL^{r_3}_x([0,t_0]\times\mathbb{R}^d)}.$$

Hence, arguing similarly as above, we obtain the desired estimates again. □

4.2. **Nonlinear estimates on v.** In this subsection, we give the estimates on v. For convenience, we introduce some notation. We denote $X(\alpha,\beta)$ be the space with the norm:

$$\|f\|_{X(\alpha,\beta)} = \big\|\langle t^\alpha|\nabla|\rangle^\beta|\nabla|^{s_c}P_M f\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\mathbb{R}\times\mathbb{R}^d)}.$$

Then the main result in this subsection is

Proposition 4.4. *Let v be the solution of (4.5), then there exist $\alpha_0\ge 1,\beta_0>0$ and $s_*<0$, such that for any $s_c\in[s_*,0)$,*

$$\|v\|_{X(\alpha_0,\beta_0)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Proof. We write

$$\begin{aligned}
&\big\|\langle t^{\alpha_0}|\nabla|\rangle^{\beta_0}|\nabla|^{s_c}P_M v\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\mathbb{R}\times\mathbb{R}^d)}\\
&\quad= \big\|\langle t^{\alpha_0}|\nabla|\rangle^{\beta_0}|\nabla|^{s_c}P_M v\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\le M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \qquad (4.7)\\
&\qquad+ \big\|\langle t^{\alpha_0}|\nabla|\rangle^{\beta_0}|\nabla|^{s_c}P_M v\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\ge M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}. \qquad (4.8)
\end{aligned}$$

Estimates on (4.7). Note that

$$\big\|\langle t^{\alpha_0}|\nabla|\rangle^{\beta_0}|\nabla|^{s_c}P_M v\big\|_{L_t^2 L_x^{\frac{2d}{d-2}}(\{|t|\le M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c}P_M v\big\|_{L_t^2 L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)}.$$

Then by Lemma 4.3 (where we choose the triple $(q,r,\gamma)=(2,\frac{2d}{d-2},0)$), it is further controlled by $\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}$. Therefore, we have the bound of (4.7) as

$$\big\|\langle t^{\alpha_0}|\nabla|\rangle^{\beta_0}|\nabla|^{s_c}P_M v\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\le M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}. \qquad (4.9)$$

Estimates on (4.8). It is controlled by

$$\big\|t^{\alpha_0\beta_0}|\nabla|^{\beta_0+s_c}P_M v\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\ge M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}.$$

We only consider the positive time, that is, $t\ge 0$, the negative time being obtained in the same way. Now we write

$$\begin{aligned}
P_M v = e^{it\Delta}P_M v_0 &+ \int_0^{\frac12 t} e^{i(t-s)\Delta}\chi_{\le1}(s)P_M(|v|^p v)\,ds\\
&+ \int_{\frac12 t}^{t} e^{i(t-s)\Delta}\chi_{\le1}(s)P_M(|v|^p v)\,ds,
\end{aligned}$$

then we need to consider the following three parts,

$$\big\|t^{\alpha_0\beta_0}|\nabla|^{\beta_0+s_c}e^{it\Delta}P_M v_0\big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\ge M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}; \qquad (4.10)$$

$$\Big\|t^{\alpha_0\beta_0}|\nabla|^{\beta_0+s_c}\int_0^{\frac12 t} e^{i(t-s)\Delta}\chi_{\le1}(s)P_M(|v|^p v)\,ds\Big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\ge M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}; \qquad (4.11)$$

and

$$\Big\|t^{\alpha_0\beta_0}|\nabla|^{\beta_0+s_c}\int_{\frac12 t}^{t} e^{i(t-s)\Delta}\chi_{\le1}(s)P_M(|v|^p v)\,ds\Big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\ge1\}\times\{|t|\ge M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}. \qquad (4.12)$$

Estimates on (4.10). Here we choose $s_* = s_*(2)$, $\alpha_0 \geq \alpha_*(2, \frac{2d}{d-2}, 0)$ and $\beta_0 \leq \beta_*(2, \frac{2d}{d-2}, 0)$, where s_*, α_*, β_* are the parameters obtained in Proposition 3.1 (we may narrow s_* suitably in the following if necessary). Then by Proposition 3.1, we obtain that for any $s_c \geq s_*$,

$$\big\| t^{\alpha_0\beta_0} |\nabla|^{\beta_0+s_c} e^{it\Delta} P_M v_0 \big\|_{l^\infty_M L^2_t L_x^{\frac{2d}{d-2}}(\{M\geq 1\}\times\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Estimates on (4.11). From Lemma 2.6,

$$\begin{aligned}
&\Big\| t^{\alpha_0\beta_0} |\nabla|^{\beta_0+s_c} \int_0^{\frac12 t} e^{i(t-s)\Delta} \chi_{\leq 1}(s) P_M(|v|^p v)\, ds \Big\|_{L^2_t L_x^{\frac{2d}{d-2}}(\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \\
&\lesssim \Big\| t^{\alpha_0\beta_0} \int_0^{\frac12 t} |t-s|^{-1} \chi_{\leq 1}(s) \big\| |\nabla|^{\beta_0+s_c} P_M(|v|^p v) \big\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}\, ds \Big\|_{L^2_t(\{|t|\geq M^{-\frac{1}{\alpha_0}}\})} \\
&\lesssim \Big\| t^{\alpha_0\beta_0-1} \int_0^2 \big\| |\nabla|^{\beta_0+s_c} P_M(|v|^p v) \big\|_{L_x^{\frac{2d}{d+2}}(\mathbb{R}^d)}\, ds \Big\|_{L^2_t(\{|t|\geq M^{-\frac{1}{\alpha_0}}\})},
\end{aligned}$$

where we have used the relationship $|t-s| \sim |t|$. We can choose $\alpha_0\beta_0$ small enough, such that $\alpha_0\beta_0 < \frac12$. Then taking L^2_t first and using Bernstein's inequality, the inequality above is bounded by

$$\int_0^2 \big\| P_M(|v|^p v) \big\|_{L_x^{\frac{2d}{d+2}}(\mathbb{R}^d)}\, ds. \tag{4.13}$$

Now we consider the following two cases. The first case is $s_c + \frac{1}{2\alpha_0} \leq 0$. Then (4.13) is dominated by

$$\int_0^2 \big\| |v|^p v \big\|_{L_x^{\frac{2d}{d+2}}(\mathbb{R}^d)}\, ds.$$

Using the Hölder inequality, it is further controlled by

$$\int_0^2 \|v\|^{p+1}_{L_x^{\frac{2d(p+1)}{d+2}}(\mathbb{R}^d)}\, ds. \tag{4.14}$$

Let q_1 verify

$$\frac{1}{q_1} = \frac{1}{p} - \frac{d+2}{4(p+1)}.$$

Then $(q_1, \frac{2d(p+1)}{d+2}, -s_c)$ verifies (2.4) (decreasing the distance between $p_0(d)$ and $\frac4d$ to satisfy the conditions in (2.4) if necessary).

Note that $q_1 \geq p+1$ when s_* is close enough to zero (indeed, if $s_c = 0$, then $q_1 = 2(p+1)$), and thus (4.14) is bounded by

$$\|v\|^{p+1}_{L_t^{q_1} L_x^{\frac{2d(p+1)}{d+2}}(\mathbb{R}\times\mathbb{R}^d)}.$$

Using Lemma 4.3, it is bounded again by $\|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}$. Hence, we obtain

$$\Big\| t^{\alpha_0\beta_0} |\nabla|^{\beta_0+s_c} \int_0^{\frac12 t} e^{i(t-s)\Delta} \chi_{\leq 1}(s) P_M(|v|^p v)\, ds \Big\|_{L^2_t L_x^{\frac{2d}{d-2}}(\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \lesssim \|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}.$$

The second case is $s_c + \frac{1}{2\alpha_0} > 0$. Then (4.13) is bounded by

$$\int_0^2 \big\| |\nabla|^{s_c+\frac{1}{2\alpha_0}} (|v|^p v) \big\|_{L_x^{\frac{2d}{d+2}}(\mathbb{R}^d)}\, ds.$$

Then using Lemma 2.2 and the Hölder inequality, it is further controlled by

$$\int_0^2 \big\| |\nabla|^{s_c+\frac{1}{2\alpha_0}} v \big\|_{L_x^{\frac{2d(p+1)}{d+2}}(\mathbb{R}^d)} \|v\|^p_{L_x^{\frac{2d(p+1)}{d+2}}(\mathbb{R}^d)} \, ds. \tag{4.15}$$

Let q_2 verify

$$\frac{1}{q_2} = \frac{d}{4} + \frac{1}{4\alpha_0} - \frac{d+2}{4(p+1)},$$

then for suitable large α_0 and small $|s_*|$, $(q_2, \frac{2d(p+1)}{d+2}, -\frac{1}{2\alpha_0})$ verifies (2.4). Moreover, we have

$$\frac{1}{q_2} + \frac{p}{q_1} \leq 1.$$

(In particular, if $s_c = 0, \alpha_0 = +\infty$, then $q_1 = q_2 = 2(p+1)$, hence the conclusions verify when we choose $|s_*|$ small enough and α_0 large enough). Hence, (4.15) is bounded by

$$\big\| |\nabla|^{s_c+\frac{1}{2\alpha_0}} v \big\|_{L_t^{q_2} L_x^{\frac{2d(p+1)}{d+2}}(\mathbb{R}\times\mathbb{R}^d)} \big\| v \big\|^p_{L_t^{q_1} L_x^{\frac{2d(p+1)}{d+2}}(\mathbb{R}\times\mathbb{R}^d)}.$$

Using Lemma 4.3 again, it is bounded by $\|v_0\|^{p+1}_{\dot{H}^{s_c}(\mathbb{R}^d)}$. Hence, we also obtain

$$\Big\| t^{\alpha_0\beta_0} |\nabla|^{\beta_0+s_c} \int_0^{\frac{1}{2}t} e^{i(t-s)\Delta} \chi_{\leq 1}(s) P_M(|v|^p v) \, ds \Big\|_{L_t^2 L_x^{\frac{2d}{d-2}}(\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \lesssim \|v_0\|^{p+1}_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

Therefore, we get

$$\Big\| t^{\alpha_0\beta_0} |\nabla|^{\beta_0+s_c} \int_0^{\frac{1}{2}t} e^{i(t-s)\Delta} \chi_{\leq 1}(s) P_M(|v|^p v) \, ds \Big\|_{l_M^\infty L_t^2 L_x^{\frac{2d}{d-2}}(\{M\geq 1\}\times\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \lesssim \|v_0\|^{p+1}_{\dot{H}^{s_c}(\mathbb{R}^d)}. \tag{4.16}$$

Estimates on (4.12). By the Sobolev and the Bernstein inequalities, we have

$$\begin{aligned} &\Big\| t^{\alpha_0\beta_0} |\nabla|^{\beta_0+s_c} \int_{\frac{1}{2}t}^{t} e^{i(t-s)\Delta} \chi_{\leq 1}(s) P_M(|v|^p v) \, ds \Big\|_{L_t^2 L_x^{\frac{2d}{d-2}}(\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)} \\ &\lesssim M^{\beta_0+s_c} \Big\| \int_{\frac{1}{2}t}^{t} e^{i(t-s)\Delta} \chi_{\leq 1}(s) s^{\alpha_0\beta_0} P_M(|v|^p v) \, ds \Big\|_{L_t^2 L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)}. \end{aligned} \tag{4.17}$$

Now we split it into two cases: $p \leq 1$ and $p > 1$.

If $p \leq 1$, using Lemma 2.7 and (4.17), (4.12) is further bounded by

$$M^{\beta_0+s_c} \Big\| \chi_{\leq 1}(t) t^{\alpha_0\beta_0} P_M(|v|^p v) \Big\|_{L_t^{\frac{2}{p+1}} L_x^{r_1'}(\mathbb{R}\times\mathbb{R}^d)}, \tag{4.18}$$

where r_1 is the parameter satisfying

$$\frac{1}{r_1} = \frac{1}{2} - \frac{1-p}{d}.$$

Now we consider the term

$$\big\| P_M(|v|^p v) \big\|_{L_x^{r_1'}(\mathbb{R}^d)}.$$

We write

$$\big\| P_M(|v|^p v) \big\|_{L_x^{r_1'}(\mathbb{R}^d)} \leq \big\| P_M\big(|P_{\leq M} v|^p P_{\leq M} v \big) \big\|_{L_x^{r_1'}(\mathbb{R}^d)} \tag{4.19}$$

$$+ \big\| P_M\big(|v|^p v - |P_{\leq M} v|^p P_{\leq M} v \big) \big\|_{L_x^{r_1'}(\mathbb{R}^d)}. \tag{4.20}$$

We choose $s_* < 0$ suitably close to 0 such that for any $s_c \in (s_*, 0)$,

$$s_c + \beta_0 > 0.$$

Then for (4.19), by Bernstein's inequality, we have

$$\begin{aligned}&\big\|P_M\big(|P_{\le M}v|^p P_{\le M}v\big)\big\|_{L_x^{r_1'}(\mathbb{R}^d)}\\ &\quad\lesssim M^{-(s_c+\beta_0+\epsilon)}\big\||\nabla|^{s_c+\beta_0+\epsilon}P_M\big(|P_{\le M}v|^p P_{\le M}v\big)\big\|_{L_x^{r_1'}(\mathbb{R}^d)}.\end{aligned}$$

where ϵ is a small positive constant such that $s_c + \beta_0 + \epsilon < p + 1$. Then by Lemma 2.2, we further obtain

$$\begin{aligned}&\big\|P_M\big(|P_{\le M}v|^p P_{\le M}v\big)\big\|_{L_x^{r_1'}(\mathbb{R}^d)}\\ &\quad\lesssim M^{-(s_c+\beta_0+\epsilon)}\big\||\nabla|^{s_c+\beta_0+\epsilon}P_{\le M}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}\big\|P_{\le M}v\big\|^p_{L_x^{\frac{dp}{2-p}}(\mathbb{R}^d)}.\end{aligned}$$

Now by Littlewood-Paley's decomposition, we write

$$\big\||\nabla|^{s_c+\beta_0+\epsilon}P_{\le M}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c+\beta_0+\epsilon}P_{\le 1}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \sum_{1\le M_1\le M} M_1^{\epsilon}\big\||\nabla|^{s_c+\beta_0}P_{M_1}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}.$$

Note that by Lemma 4.3,

$$\big\||\nabla|^{s_c+\beta_0+\epsilon}P_{\le 1}v\big\|_{L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c}P_{\le 1}v\big\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Hence, we obtain that

$$\begin{aligned}&M^{\beta_0+s_c}\Big\|\chi_{\le 1}(t)t^{\alpha_0\beta_0}P_M\big(|P_{\le M}v|^p P_{\le M}v\big)\Big\|_{L_t^{\frac{2}{p+1}}L_x^{r_1'}(\mathbb{R}\times\mathbb{R}^d)}\\
&\lesssim M^{-\epsilon}\Big\|\chi_{\le 1}(t)t^{\alpha_0\beta_0}\Big(\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)} + \sum_{1\le M_1\le M} M_1^{\epsilon}\big\||\nabla|^{s_c+\beta_0}P_{M_1}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}\Big)\\
&\qquad\cdot \big\|P_{\le M}v\big\|^p_{L_x^{\frac{dp}{2-p}}(\mathbb{R}^d)}\Big\|_{L_t^{\frac{2}{p+1}}(\mathbb{R})}\\
&\lesssim M^{-\epsilon}\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}\|v\|^p_{L_t^2L_x^{\frac{dp}{2-p}}(\mathbb{R}\times\mathbb{R}^d)}\\
&\qquad + M^{-\epsilon}\Big(\sum_{1\le M_1\le M} M_1^{\epsilon}\big\|t^{\alpha_0\beta_0}|\nabla|^{s_c+\beta_0}P_{M_1}v\big\|_{L_t^2L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)}\Big)\cdot\|v\|^p_{L_t^2L_x^{\frac{dp}{2-p}}(\mathbb{R}\times\mathbb{R}^d)}. \quad (4.21)\end{aligned}$$

Now by Lemma 4.3 and the definition of $X(\alpha_0, \beta_0)$, we have

$$\|v\|_{L_t^2L_x^{\frac{dp}{2-p}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)},$$

and

$$\big\|t^{\alpha_0\beta_0}|\nabla|^{s_c+\beta_0}P_{M_1}v\big\|_{L_t^2L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v\|_{X(\alpha_0,\beta_0)}.$$

Inserting these two estimates into (4.21), then (4.21) is controlled by

$$M^{-\epsilon}\|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)} + M^{-\epsilon}\sum_{1\le M_1\le M} M_1^{\epsilon}\|v\|_{X(\alpha_0,\beta_0)}\cdot\|v_0\|^p_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Taking summation, we obtain

$$\begin{aligned}&M^{\beta_0+s_c}\Big\|\chi_{\le 1}(t)t^{\alpha_0\beta_0}P_M\big(|P_{\le M}v|^p P_{\le M}v\big)\Big\|_{L_t^{\frac{2}{p+1}}L_x^{r_1'}(\mathbb{R}\times\mathbb{R}^d)}\\
&\qquad\lesssim \|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)} + \|v_0\|^p_{\dot H^{s_c}(\mathbb{R}^d)}\|v\|_{X(\alpha_0,\beta_0)}. \quad (4.22)\end{aligned}$$

For (4.20), by Bernstein's inequality, we have

$$\begin{aligned}\big\|P_M\big(|v|^p v-|P_{\le M}v|^pP_{\le M}v\big)\big\|_{L_x^{r_1'}(\mathbb{R}^d)}&\\ \lesssim \big\|P_{\ge M}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}\big\|v\big\|^p_{L_x^{\frac{dp}{2-p}}(\mathbb{R}^d)}&\\ \lesssim \sum_{M_1\ge M}\big\|P_{M_1}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}\big\|v\big\|^p_{L_x^{\frac{dp}{2-p}}(\mathbb{R}^d)}&\\ \lesssim \sum_{M_1\ge M}M_1^{-(s_c+\beta_0)}\big\||\nabla|^{s_c+\beta_0}P_{M_1}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}\big\|v\big\|^p_{L_x^{\frac{dp}{2-p}}(\mathbb{R}^d)}&.\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}&M^{\beta_0+s_c}\Big\|\chi_{\le1}(t)t^{\alpha_0\beta_0}P_M\big(|v|^pv-|P_{\le M}v|^pP_{\le M}v\big)\Big\|_{L_t^{\frac{2}{p+1}}L_x^{r_1'}(\mathbb{R}\times\mathbb{R}^d)}\\ &\lesssim M^{\beta_0+s_c}\sum_{M_1\ge M}M_1^{-(s_c+\beta_0)}\Big\|\chi_{\le1}(t)t^{\alpha_0\beta_0}\big\||\nabla|^{s_c+\beta_0}P_{M_1}v\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}\big\|v\big\|^p_{L_x^{\frac{dp}{2-p}}(\mathbb{R}^d)}\Big\|_{L_t^{\frac{2}{p+1}}(\mathbb{R})}\\ &\lesssim M^{\beta_0+s_c}\sum_{M_1\ge M}M_1^{-(s_c+\beta_0)}\Big\|t^{\alpha_0\beta_0}|\nabla|^{s_c+\beta_0}P_{M_1}v\Big\|_{L_t^2L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)}\big\|v\big\|^p_{L_t^2L_x^{\frac{dp}{2-p}}(\mathbb{R}\times\mathbb{R}^d)}.\end{aligned}$$

Similar as above, it is further bounded by

$$M^{\beta_0+s_c}\sum_{M_1\ge M}M_1^{-(s_c+\beta_0)}\|v\|_{X(\alpha_0,\beta_0)}\cdot\|v_0\|^p_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Taking summation, we obtain that

$$\begin{aligned}M^{\beta_0+s_c}\Big\|\chi_{\le1}(t)t^{\alpha_0\beta_0}P_M\big(|v|^pv-|P_{\le M}v|^pP_{\le M}v\big)\Big\|_{L_t^{\frac{2}{p+1}}L_x^{r_1'}(\mathbb{R}\times\mathbb{R}^d)}&\\ \lesssim\|v_0\|^p_{\dot H^{s_c}(\mathbb{R}^d)}\|v\|_{X(\alpha_0,\beta_0)}.&\end{aligned}\tag{4.23}$$

Now, together with (4.18), (4.19), (4.20), (4.22) and (4.23), we obtain the estimates on (4.12) in the case of $p\le1$ as

$$\begin{aligned}\Big\|t^{\alpha_0\beta_0}|\nabla|^{\beta_0+s_c}\int_{\frac12t}^t e^{i(t-s)\Delta}\chi_{\le1}(s)P_M(|v|^pv)\,ds\Big\|_{L_t^2L_x^{\frac{2d}{d-2}}(\{|t|\ge M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}&\\ \lesssim\|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}+\|v_0\|^p_{\dot H^{s_c}(\mathbb{R}^d)}\|v\|_{X(\alpha_0,\beta_0)}.&\end{aligned}\tag{4.24}$$

Next, we consider the case when $p>1$ (now $d=2,3$), which can be treated similarly as above. Then using Lemma 2.7 and (4.17), (4.12) is bounded by

$$M^{\beta_0+s_c}\Big\|\chi_{\le1}(t)t^{\alpha_0\beta_0}P_M(|v|^pv)\Big\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}^d)}.\tag{4.25}$$

Arguing similarly as the case of $p\le1$, and based on the Hölder inequality,

$$\big\||f|^pf\big\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}^d)}\le\|f\|_{L_t^2L_x^{\frac{2d}{d-2}}(\mathbb{R}\times\mathbb{R}^d)}\|f\|^p_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)},$$

we also obtain (4.24) when $p>1$.

Now collecting the estimates in (4.10), (4.16) and (4.24), we get the estimates on (4.8) that

$$\big\|t^{\alpha_0\beta_0}|\nabla|^{\beta_0+s_c}P_M v\big\|_{l^\infty_M L^2_t L_x^{\frac{2d}{d-2}}(\{M\geq 1\}\times\{|t|\geq M^{-\frac{1}{\alpha_0}}\}\times\mathbb{R}^d)}$$
$$\lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)} + \|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)} + \|v_0\|^{p}_{\dot H^{s_c}(\mathbb{R}^d)}\|v\|_{X(\alpha_0,\beta_0)}.$$

Combining this estimate with (4.9), we obtain that

$$\|v\|_{X(\alpha_0,\beta_0)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)} + \|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)} + \|v_0\|^{p}_{\dot H^{s_c}(\mathbb{R}^d)}\|v\|_{X(\alpha_0,\beta_0)}.$$

Using (4.4) and choosing δ_0 suitably small, we give the proof of the proposition. □

As a consequence, we have

Corollary 4.5. *There exists $s_* < 0$, such that for any $s_c \in [s_*, 0)$, the following result holds. Let (q, r, s) be the triple satisfying*

$$s \in [s_c, 0],\ q \geq 2,\ r \geq 2,\ \frac{2}{q} + \frac{2d-1}{r} < \frac{2d-1}{2},\ \text{ and }\ \frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \tag{4.26}$$

and $q \leq q_0$ for some $q_0 = q_0(s_c)$, then it holds that

$$\|v\|_{L^q_t L^r_x([\frac{1}{2},+\infty)\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Remark 4.6. This corollary implies that v has the smoothing effect when the time is away from zero. Moreover, one may note that $q_0(s_c) \to +\infty$ if $s_c \to 0$.

Proof of Corollary 4.5. For the low frequency part, by Lemma 4.3 we have

$$\|P_{\leq 1} v\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|P_{\leq 1}|\nabla|^{-s+s_c} v\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

For the high frequency part, from Proposition 4.4 and Sobolev's inequality, we have that for any $M \geq 1$,

$$\big\|t^{\alpha_1\beta_0}|\nabla|^{\beta_0+s_c}P_M v\big\|_{L^2_t L_x^{\frac{2d}{d-2}}([\frac{1}{2},+\infty)\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

This implies that

$$\big\|P_M v\big\|_{L^2_t L_x^{\frac{2d}{d-2}}([\frac{1}{2},+\infty)\times\mathbb{R}^d)} \lesssim M^{-(\beta_0+s_c)}\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Choosing s_* small enough such that $\beta_0 + s_c > 0$, then the last estimate interpolating with the following estimates from Lemma 4.3: for any the triple (q, r, γ) verifies (2.4),

$$\|v\|_{L^\infty_t \dot H^{s_c}_x(\mathbb{R}\times\mathbb{R}^d)} + \big\||\nabla|^{s_c+\gamma} v\big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)},$$

gives that

$$\|P_{\geq 1} v\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Hence, we obtain the desired estimates. □

5. The proof of Theorem 1.1

In this section, we prove Theorem 1.1.

5.1. **Nonlinear estimates on w.** In this subsection, we give some nonlinear estimates of the solution with the low frequency initial data.

The first we need is the following local estimates of w in more regular spaces. First, for $u_0 \in \dot{H}^{s_c}(\mathbb{R}^d)$, $s_c < 0$ with supp $u_0 \subset \{x : |x| \le 1\}$, we claim that

$$w_0 \in L^2(\mathbb{R}^d) \quad \textit{and} \quad \|w_0\|_{L^2(\mathbb{R}^d)} \lesssim N^{-s_c}\big\|u_0\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)}. \tag{5.1}$$

Indeed, by the mismatch estimate in Lemma 2.4,

$$\big\|\chi_{\ge 10}\big(P_{\ge N}u_0\big)\big\|_{L^2(\mathbb{R}^d)} \lesssim N^{-10}\big\|u_0\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)},$$

and by the Bernstein estimate,

$$\big\|P_{\le N}u_0\big\|_{L^2(\mathbb{R}^d)} \lesssim N^{-s_c}\big\|u_0\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

This gives (5.1).

Lemma 5.1. *Under the same assumption in Lemma 4.1, the Cauchy problem (4.6) is locally well-posed in $L^2(\mathbb{R}^d)$ in the time interval $[0,2]$. In particular, the solution w satisfies that*

$$\big\|w\big\|_{L_t^qL_x^r([0,2]\times\mathbb{R}^d)} + \big\||\nabla|^s w\big\|_{L_t^qL_x^{\tilde r}([0,2]\times\mathbb{R}^d)} \lesssim 1 + \|w_0\|_{L^2(\mathbb{R}^d)}.$$

Here the triples $(q,r,s), (q,\tilde r,0)$ verify (4.26) or $(q,r,s) = (\infty,2,0)$.

Proof. Note that when $s = s_c$, the estimates follow directly from Lemmas 4.1 and 4.3. Now we consider the case when $s = 0$, then the general cases $s \in (s_c, 0)$ can be obtained by interpolation. By Lemma 2.8, we have

$$\|w\|_{L_t^qL_x^{\tilde r}([0,2]\times\mathbb{R}^d)} \lesssim \|w_0\|_{L^2(\mathbb{R}^d)} + \big\||u|^pu - \chi_{\le 1}(t)|v|^pv\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([0,2]\times\mathbb{R}^d)},$$

where $(q,\tilde r,0)$ verifies (2.4) or $(q,\tilde r,0) = (\infty,2,0)$. Next, we consider

$$\big\||u|^pu - \chi_{\le 1}(t)|v|^pv\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([0,2]\times\mathbb{R}^d)}.$$

Note that

$$\big||u|^pu - \chi_{\le 1}(t)|v|^pv\big| \lesssim \big(|u|^p + |\tilde\chi_{\le 1}(t)v|^p\big)\big(|w| + |\tilde\chi_{\ge 1}(t)v|\big).$$

Here we denote the time-dependent functions $\tilde\chi_{\le 1}(t) = \chi_{\le 1}^{\frac{1}{p+1}}(t)$ and $\tilde\chi_{\ge 1}(t) = 1 - \tilde\chi_{\le 1}(t)$. Hence,

$$\begin{aligned}
&\big\||u|^pu - \chi_{\le 1}(t)|v|^pv\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([0,2]\times\mathbb{R}^d)}\\
&\quad\lesssim\Big(\|u\|^p_{L_t^{q_1}L_x^{r_1}([0,2]\times\mathbb{R}^d)} + \big\|\tilde\chi_{\le 1}(t)v\big\|^p_{L_t^{q_1}L_x^{r_1}([0,2]\times\mathbb{R}^d)}\Big)\\
&\qquad\cdot\big(\|w\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)} + \|\tilde\chi_{\ge 1}(t)v\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}\big),
\end{aligned} \tag{5.2}$$

where

$$q_1 = \frac{2(d+2)}{d+4}(p+1),$$

and r_1, r_2 satisfy

$$\frac{1}{r_1} = \frac{2}{dp} - \frac{d+4}{d(d+2)(p+1)}, \qquad \frac{1}{r_2} = \frac{1}{2} - \frac{d+4}{d(d+2)(p+1)}.$$

Narrowing suitably the distance between $p_0(d)$ and $\frac{4}{d}$ such that

$$(q_1, r_2, 0) \text{ and } (q_1, r_1, -s_c) \text{ verify (2.4)}, \quad q_1 \le q_0, \quad r_2 \ge 2, \tag{5.3}$$

where q_0 is the parameter obtained in Corollary 4.5 (thanks to the radial Strichartz estimates in Lemma 2.8, the conditions (5.3) verify and are not attained the borderline when $p=\frac{4}{d}$, hence the conditions (5.3) also verify when p is around $\frac{4}{d}$), then from Lemma 4.1,

$$\|u\|_{L_t^{q_1}L_x^{r_1}([0,2]\times\mathbb{R}^d)}\lesssim 1.$$

Without loss of generality, we may assume that

$$\|u\|_{L_t^{q_1}L_x^{r_1}([0,2]\times\mathbb{R}^d)}\lesssim \delta_0. \tag{5.4}$$

Otherwise, we may split the time interval $[0,2]$ into $K(\delta_0)\sim\delta_0^{-q_1}$ parts such that each part verifies (5.4), and then consider each part separately.

Moreover, from Lemma 4.3 and Corollary 4.5,

$$\big\|\tilde{\chi}_{\le 1}(t)v\big\|_{L_t^{q_1}L_x^{r_1}([0,2]\times\mathbb{R}^d)}+\|\tilde{\chi}_{\ge 1}(t)v\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}\lesssim\|v_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}\lesssim\delta_0.$$

Hence, combining this estimate with (5.4) and (5.2), we obtain

$$\big\||u|^pu-\chi_{\le 1}(t)|v|^pv\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([0,2]\times\mathbb{R}^d)}\lesssim\delta_0^p\big(\|w\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}+\delta_0\big).$$

Therefore, we obtain that for any triple $(q,\tilde{r},0)$ verifies (2.4),

$$\|w\|_{L_t^{q}L_x^{\tilde{r}}([0,2]\times\mathbb{R}^d)}\lesssim\|w_0\|_{L^2(\mathbb{R}^d)}+\delta_0^p\|w\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}. \tag{5.5}$$

Note that $(q_1,r_2,0)$ verifies (2.4), thus a consequence of (5.5) is

$$\|w\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}\lesssim 1+\|w_0\|_{L^2(\mathbb{R}^d)}+\delta_0^p\|w\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}.$$

Choosing δ_0 suitably small, we obtain

$$\|w\|_{L_t^{q_1}L_x^{r_2}([0,2]\times\mathbb{R}^d)}\lesssim 1+\|w_0\|_{L^2(\mathbb{R}^d)}.$$

Inserting this estimate into (5.5), we get that

$$\|w\|_{L_t^{q}L_x^{\tilde{r}}([0,2]\times\mathbb{R}^d)}\lesssim 1+\|w_0\|_{L^2(\mathbb{R}^d)}.$$

This finishes the proof of the lemma. □

Next, we give the global estimates of w. The following is a modified mass estimate for $\dot{H}^{s_c}(\mathbb{R}^d)$-datum.

Proposition 5.2. *Let $u_0\in\dot{H}^{s_c}(\mathbb{R}^d)$ and I be the lifespan of the solution u, then there exists $s_*<0$, such that for any $s_c\in(s_*,0)$, the following estimate holds,*

$$\|w\|^2_{L_t^\infty L_x^2(I\times\mathbb{R}^d)}\lesssim N^{-2s_c}\big\|u_0\big\|^2_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

Proof. For simplicity, we denote

$$F(v,w)=|w+v|^p(w+v)-\chi_{\le 1}(t)|v|^pv.$$

Then from the equation (4.6), we have

$$\partial_t\|w\|^2_{L_x^2}=2\mathrm{Im}\int F(v,w)\bar{w}\,dx.$$

Integrating in time, we obtain that for any $t\in I$,

$$\|w(t)\|^2_{L_x^2}=\|w_0\|^2_{L^2}+2\mathrm{Im}\int_0^t\int_{\mathbb{R}^d}F(v,w)\bar{w}\,dxds.$$

We may assume $t \geq 2$, otherwise, the estimate has been included in Lemma 5.1. Then we further write

$$\|w(t)\|_{L^2_x}^2 = \|w_0\|_{L^2}^2 + 2\mathrm{Im}\int_0^2\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds + 2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds. \tag{5.6}$$

First, we consider

$$2\mathrm{Im}\int_0^2\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds,$$

which is bounded by

$$\big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([0,2]\times\mathbb{R}^d)}\big\|w\big\|_{L_{tx}^{\frac{2d+4}{d}}([0,2]\times\mathbb{R}^d)}.$$

Then from the proof of Lemma 5.1, we have

$$\big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([0,2]\times\mathbb{R}^d)} \lesssim N^{-s_c}\big\|u_0\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

Moreover, by Lemma 5.1, we have

$$\big\|w\big\|_{L_{tx}^{\frac{2d+4}{d}}([0,2]\times\mathbb{R}^d)} \lesssim N^{-s_c}\big\|u_0\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

Hence, we get

$$\left|2\mathrm{Im}\int_0^2\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds\right| \lesssim N^{-2s_c}\big\|u_0\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)}^2. \tag{5.7}$$

Second, we consider

$$2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds.$$

To do this, we denote the time-dependent functions $\tilde{\chi}_{\leq 1}(t) = \chi_{\leq 1}^{\frac{1}{p+1}}(t)$ and $\tilde{\chi}_{\geq 1}(t) = 1-\tilde{\chi}_{\leq 1}(t)$ as before, and write

$$F(v,w) = |u|^p w + |u|^p v - |\tilde{\chi}_{\leq 1}(t)v|^p\tilde{\chi}_{\leq 1}(t)v.$$

Note that

$$2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} |u|^p w\bar{w}\,dxds = 0.$$

Moreover, because of the time support, we find

$$2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} |\tilde{\chi}_{\leq 1}(t)v|^p\tilde{\chi}_{\leq 1}(t)v\,\bar{w}\,dxds = 0.$$

Hence,

$$2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds = 2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} |u|^p v\,\bar{w}\,dxds,$$

and thus

$$\begin{aligned}&\left|2\mathrm{Im}\int_2^t\int_{\mathbb{R}^d} F(v,w)\bar{w}\,dxds\right|\\ &\quad\lesssim \int_2^t\int_{\mathbb{R}^d} |w|^{p+1}|v|\,dxds + \int_2^t\int_{\mathbb{R}^d} |w||v|^{p+1}\,dxds.\end{aligned} \tag{5.8}$$

For the first term in (5.8), we write

$$\int_2^t \int_{\mathbb{R}^d} |w|^{p+1}|v|\,dxds = \sum_{j=j_0}^{J} \int_{I_j}\int_{\mathbb{R}^d} |w|^{p+1}|v|\,dxds, \tag{5.9}$$

where j_0, J are some numbers with $\mu_0 j_0 N^{-2} \sim 1, \mu_0 J N^{-2} \sim t$, μ_0 is a small constant decided later, and

$$I_j = [j\mu_0 N^{-2}, (j+1)\mu_0 N^{-2}].$$

Since $d \geq 4$, we find $p < 1$. Now we denote some parameters $a, \varrho_1, \sigma_1, \varrho_2, \sigma_2$, which satisfy

$$\frac{1}{\varrho_1} = \frac{a}{2}; \quad \frac{1}{\sigma_1} = \frac{1}{2} - \frac{a}{d}; \quad \frac{1}{\varrho_2} = 1 - \frac{a}{2}; \quad \frac{1}{\sigma_2} = \frac{1-p}{2} + \frac{a}{d},$$

and $a \leq 1$ is a positive parameter decided later. Then by Hölder's inequality, we get

$$\int_2^t \int_{\mathbb{R}^d} |w|^{p+1}|v|\,dxds \lesssim \sum_{j=j_0}^{J} \|w\|^p_{L_t^\infty L_x^2(I_j\times\mathbb{R}^d)} \|w\|_{L_t^{\varrho_1} L_x^{\sigma_1}(I_j\times\mathbb{R}^d)} \|v\|_{L_t^{\varrho_2} L_x^{\sigma_2}(I_j\times\mathbb{R}^d)}. \tag{5.10}$$

For the second term in (5.8), we have

$$\int_2^t \int_{\mathbb{R}^d} |w||v|^{p+1}\,dxds \lesssim \|w\|_{L_t^\infty L_x^2([2,t)\times\mathbb{R}^d)} \|v\|^{p+1}_{L_t^{p+1} L_x^{2(p+1)}([2,t)\times\mathbb{R}^d)}. \tag{5.11}$$

To continue, we need the following three lemmas. The first is related to some fixed length spacetime estimates.

Lemma 5.3. *Suppose that w exists on $[0,T)$ with $T \in I$ and $T > 2$, and*

$$\|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)} \lesssim N^{-s_c}.$$

Then there exists an absolute constant $\mu_0 \sim 1$, such that for any $t_0 \in [2,T)$,

$$\|w\|_{L_t^2 L_x^{\frac{2d}{d-2}}(\{[t_0,t_0+\mu_0 N^{-2}]\cap[0,T)\}\times\mathbb{R}^d)} \lesssim 1 + \|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}. \tag{5.12}$$

Proof. By the Duhamel formula,

$$w(t) = e^{i(t-t_0)\Delta} w(t_0) + \int_{t_0}^t e^{i(t-s)\Delta} F(v,w)\,ds,$$

and by Lemma 2.7, we have for any $t \leq t_0 + \mu_0 N^{-2}$,

$$\|w\|_{L_t^2 L_x^{\frac{2d}{d-2}}([t_0,t]\times\mathbb{R}^d)} \lesssim \big\|w(t_0)\big\|_{L_x^2(\mathbb{R}^d)} + \big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t]\times\mathbb{R}^d)}.$$

Treating similarly as (5.2) (using the same notations there), we obtain

$$\begin{aligned}\big\|F(v,w)\big\|&_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t]\times\mathbb{R}^d)}\\ &\lesssim \Big(\|w\|^p_{L_t^{q_1} L_x^{r_1}([t_0,t]\times\mathbb{R}^d)} + \|v\|^p_{L_t^{q_1} L_x^{r_1}([t_0,t]\times\mathbb{R}^d)}\Big)\\ &\qquad \cdot \big(\|w\|_{L_t^{q_1} L_x^{r_2}([t_0,t]\times\mathbb{R}^d)} + \|\tilde\chi_{\geq 1}(t) v\|_{L_t^{q_1} L_x^{r_2}([t_0,t]\times\mathbb{R}^d)}\big);\end{aligned}$$

Hence, by Lemma 4.3 and Corollary 4.5, we further obtain that

$$\begin{aligned}\big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t]\times\mathbb{R}^d)} &\lesssim \Big(\|w\|^p_{L_t^{q_1} L_x^{r_1}([t_0,t]\times\mathbb{R}^d)} + \delta_0^p\Big)\\ &\qquad \cdot \big(\|w\|_{L_t^{q_1} L_x^{r_2}([t_0,t]\times\mathbb{R}^d)} + \delta_0\big).\end{aligned}$$

Let q_2 be the parameter satisfying

$$\frac{1}{q_1}+\frac{s_c}{2}=\frac{1}{q_2},$$

then $\frac{2}{q_2}+\frac{d}{r_1}=\frac{d}{2}$. Hence, by Hölder's inequality, we get

$$\begin{aligned}\big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t]\times\mathbb{R}^d)} \lesssim&\Big(\mu_0^p N^{s_c p}\|w\|^p_{L_t^{q_2}L_x^{r_1}([t_0,t]\times\mathbb{R}^d)}+\delta_0^p\Big)\\ &\cdot\big(\|w\|_{L_t^{q_1}L_x^{r_2}([t_0,t]\times\mathbb{R}^d)}+\delta_0\big).\end{aligned}$$

Note that by interpolation, there exist some $\theta_1\in(0,1),\theta_2\in(0,1)$, such that

$$\|w\|_{L_t^{q_2}L_x^{r_1}([t_0,t]\times\mathbb{R}^d)}\lesssim\|w\|^{\theta_1}_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\|w\|^{1-\theta_1}_{L_t^2L_x^{\frac{2d}{d-2}}([t_0,t]\times\mathbb{R}^d)};$$

and

$$\|w\|_{L_t^{q_1}L_x^{r_2}([t_0,t]\times\mathbb{R}^d)}\lesssim\|w\|^{\theta_2}_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\|w\|^{1-\theta_2}_{L_t^2L_x^{\frac{2d}{d-2}}([t_0,t]\times\mathbb{R}^d)}.$$

Then by Cauchy-Schwartz's inequality we obtain that there exist $0<p_1<p+1$ and $\tilde{\mu_0}\sim\mu_0^p+\delta_0^p$, such that

$$\begin{aligned}\big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t]\times\mathbb{R}^d)}\lesssim&1+\|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}+\delta_0^p\|w\|_{L_t^2L_x^{\frac{2d}{d-2}}([t_0,t]\times\mathbb{R}^d)}\\ &+\widetilde{\mu_0}N^{s_cp}\|w\|^{p+1-p_1}_{L_t^\infty L^2([0,T)\times\mathbb{R}^d)}\|w\|^{p_1}_{L_t^2L_x^{\frac{2d}{d-2}}([t_0,t]\times\mathbb{R}^d)}.\end{aligned}$$

For convenience, we denote

$$\Lambda(t_0;t)=N^{s_c}\|w\|_{L_t^2L_x^{\frac{2d}{d-2}}([t_0,t]\times\mathbb{R}^d)};\quad \Lambda_0=N^{s_c}\|w\|_{L_t^\infty L^2([0,T)\times\mathbb{R}^d)}.$$

Then

$$\big\|F(v,w)\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t]\times\mathbb{R}^d)}\lesssim N^{-s_c}\big(N^{s_c}+\Lambda_0+\delta_0^p\Lambda(t_0;t)+\widetilde{\mu_0}\Lambda_0^{p+1-p_1}\Lambda(t_0;t)^{p_1}\big),$$

and thus for any $t\le\min\{t_0+\mu_0N^{-2},T\}$,

$$\Lambda(t_0;t)\lesssim N^{s_c}+\Lambda_0+\delta_0^p\Lambda(t_0;t)+\widetilde{\mu_0}\Lambda_0^{p+1-p_1}\Lambda(t_0;t)^{p_1}.$$

Choosing μ_0 and δ_0 suitably small and using the continuity argument, we have that

$$\Lambda\big(t_0;\min\{t_0+\mu_0N^{-2},T\}\big)\lesssim N^{s_c}+\Lambda_0.$$

This finishes the proof of the lemma. □

Lemma 5.4. *There exist $s_*<0$ and $\mu(a)>1$, such that for any $s_c\in(s_*,0)$, the following estimate holds.*

$$\|v\|_{L_t^{\varrho_2}L_x^{\sigma_2}(I_j\times\mathbb{R}^d)}\lesssim\delta_0\big(\mu_0jN^{-2}\big)^{-\mu(a)}.$$

Proof. By the Duhamel formula and the Sobolev inequality, we have

$$\|v\|_{L_x^{\sigma_2}(\mathbb{R}^d)}\lesssim\big\|e^{it\Delta}v_0\big\|_{L_x^{\sigma_2}(\mathbb{R}^d)}+\int_0^t\big\|e^{i(t-t')\Delta}\chi_{\le1}(t')|v|^pv\big\|_{L_x^{\sigma_2}(\mathbb{R}^d)}\,dt'. \tag{5.13}$$

From Proposition 3.1, we get

$$\big\|e^{it\Delta}v_0\big\|_{L_x^{\sigma_2}(\mathbb{R}^d)}\lesssim N^{-(d-2)(\frac{p}{2}-\frac{a}{d})-s_c}|t|^{-(d-1)(\frac{p}{2}-\frac{a}{d})}\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{5.14}$$

From Lemma 2.6, we have

$$\int_0^t \left\| e^{i(t-t')\Delta}\chi_{\le 1}(t')|v|^p v \right\|_{L_x^{\sigma_2}(\mathbb{R}^d)} dt' \lesssim \int_0^t |t-t'|^{-d(\frac{p}{2}-\frac{a}{d})}\chi_{\le 1}(t')\left\||v|^p v\right\|_{L_x^{\sigma_2'}(\mathbb{R}^d)} dt'.$$

Since $t \ge 2$, it is further bounded by

$$\begin{aligned} &|t|^{-d(\frac{p}{2}-\frac{a}{d})}\int_0^t \chi_{\le 1}(t')\left\||v|^p v\right\|_{L_x^{\sigma_2'}(\mathbb{R}^d)} dt' \\ &\lesssim |t|^{-\frac{dp}{2}+a}\int_0^2 \|v\|^{p+1}_{L_x^{(p+1)\sigma_2'}(\mathbb{R}^d)} dt'. \end{aligned}$$

Let q be the parameter satisfying

$$\frac{1}{q} = \frac{a}{2(1+p)} - s_c,$$

such that

$$\frac{2}{q} + \frac{d}{(p+1)\sigma_2'} = \frac{d}{2} - s_c.$$

Note that if

$$-s_c < \frac{(d-1)a}{2d(p+1)}, \tag{5.15}$$

then $(q,(p+1)r',-s_c)$ satisfies (2.4) and thus by Lemma 4.3,

$$\|v\|_{L_t^q L_x^{(p+1)r'}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Therefore, choosing a suitable small such that $q \ge p+1$ ($q=+\infty$ when $a=0$), we have

$$\int_0^2 \|v\|^{p+1}_{L_x^{(p+1)\sigma_2'}(\mathbb{R}^d)} dt' \lesssim \|v\|^{p+1}_{L_t^q L_x^{(p+1)\sigma_2'}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Hence, we obtain that

$$\int_0^t \left\| e^{i(t-t')\Delta}\chi_{\le 1}(t')|v|^p v \right\|_{L_x^{\sigma_2}(\mathbb{R}^d)} dt' \lesssim |t|^{-\frac{dp}{2}+a}\|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}.$$

This together with (5.14) yields

$$\|v\|_{L_x^{\sigma_2}(\mathbb{R}^d)} \lesssim N^{-(d-2)(\frac{p}{2}-\frac{a}{d})-s_c}|t|^{-(d-1)(\frac{p}{2}-\frac{a}{d})}\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)} + |t|^{-\frac{dp}{2}+a}\|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}. \tag{5.16}$$

Note that when p is suitable close to $\frac{4}{d}$ and a is suitable small, $-(d-2)(\frac{p}{2}-\frac{a}{d})-s_c \le 0$ and the right-hand side of the inequality above is integral in time from 2 to ∞. Moreover, by (4.4),

$$\begin{aligned} \|v\|_{L_t^{\varrho_2}L_x^{\sigma_2}(I_j\times\mathbb{R}^d)} &\lesssim \delta_0\Big(\Big\||t|^{-(d-1)(\frac{p}{2}-\frac{a}{d})}\Big\|_{L_t^{\varrho_2}(I_j)} + \Big\||t|^{-\frac{dp}{2}+a}\Big\|_{L_t^{\varrho_2}(I_j)}\Big) \\ &\lesssim \delta_0\Big(\big(\mu_0 jN^{-2}\big)^{-(d-1)(\frac{p}{2}-\frac{a}{d})} + \big(\mu_0 jN^{-2}\big)^{-\frac{dp}{2}+a}\Big). \end{aligned}$$

Let $\mu(a) = \min\{(d-1)(\frac{p}{2}-\frac{a}{d}), \frac{dp}{2}-a\}$. Then we can choose s_* and a suitable small, such that (5.15) is valid and $\mu(a) > 1$. This finishes the proof of the lemma. ☐

The second lemma we need is the following.

Lemma 5.5. *There exists $s_* < 0$, such that for any $s_c \in (s_*,0)$, the following estimate holds.*

$$\|v\|_{L_t^{p+1}L_x^{2(p+1)}([2,t)\times\mathbb{R}^d)} \lesssim \delta_0.$$

Proof. By the Duhamel formula, we have

$$\begin{aligned}\|v\|_{L_t^{p+1}L_x^{2(p+1)}([2,t)\times\mathbb{R}^d)} \lesssim & \big\|e^{it\Delta}v_0\big\|_{L_t^{p+1}L_x^{2(p+1)}([2,t)\times\mathbb{R}^d)}\\ &+\Big\|\int_0^t \big\|e^{i(t-s)\Delta}\chi_{\le 1}(s)|v|^p v\big\|_{L_x^{2(p+1)}(\mathbb{R}^d)}\,ds\Big\|_{L_t^{p+1}([2,t))}.\end{aligned}$$

From Proposition 3.1, we get

$$\big\|e^{it\Delta}v_0\big\|_{L_t^{p+1}L_x^{2(p+1)}([2,t)\times\mathbb{R}^d)} \lesssim N^{-\frac{(d-2)p}{2(p+1)}-s_c}\Big\||t|^{-\frac{(d-1)p}{2(p+1)}}\Big\|_{L_t^{p+1}([2,+\infty))}\|v_0\|_{\dot H^{s_c}(\mathbb{R}^d)} \lesssim \delta_0, \tag{5.17}$$

when s_* is suitable close to 0.

From Lemma 2.6, we have

$$\begin{aligned}&\Big\|\int_0^t \big\|e^{i(t-s)\Delta}\chi_{\le 1}(s)|v|^p v\big\|_{L_x^{2(p+1)}(\mathbb{R}^d)}\,ds\Big\|_{L_t^{p+1}([2,t))}\\ &\quad\lesssim \Big\|\int_0^t |t-s|^{-\frac{dp}{2(p+1)}}\chi_{\le 1}(s)\big\||v|^p v\big\|_{L_x^{\frac{2(p+1)}{2p+1}}(\mathbb{R}^d)}\,ds\Big\|_{L_t^{p+1}([2,t))}\\ &\quad\lesssim \Big\||t|^{-\frac{dp}{2(p+1)}}\Big\|_{L_t^{p+1}([2,+\infty))}\int_0^2 \|v\|^{p+1}_{L_x^{\frac{2(p+1)^2}{2p+1}}(\mathbb{R}^d)}\,ds \lesssim \int_0^2 \|v\|^{p+1}_{L_x^{\frac{2(p+1)^2}{2p+1}}(\mathbb{R}^d)}\,ds.\end{aligned}$$

Setting q such that

$$\frac{1}{q}=\frac{1}{p}-\frac{d(2p+1)}{4(p+1)^2},$$

then if s_* is close to 0, then $(q,\frac{2(p+1)^2}{2p+1},-s_c)$ satisfies (2.4) and $q\ge p+1$. Hence, by Hölder's inequality and Lemma 4.3, we have

$$\int_0^2 \|v\|^{p+1}_{L_x^{\frac{2(p+1)^2}{2p+1}}(\mathbb{R}^d)}\,ds \lesssim \|v\|^{p+1}_{L_t^q L_x^{\frac{2(p+1)^2}{2p+1}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|v_0\|^{p+1}_{\dot H^{s_c}(\mathbb{R}^d)}.$$

Hence, we obtain that

$$\Big\|\int_0^t \big\|e^{i(t-s)\Delta}\chi_{\le 1}(s)|v|^p v\big\|_{L_x^{2(p+1)}(\mathbb{R}^d)}\,ds\Big\|_{L_t^{p+1}([2,t))} \lesssim \delta_0.$$

This last estimate combined with (5.17), gives the proof of the lemma. $\square$

Now we continue to prove the proposition.

From Lemma 5.1, we have that there exists some absolute constant $C>0$, such that

$$\|w\|_{L_t^\infty L_x^2([0,2)\times\mathbb{R}^d)} \le C\big(1+\|w_0\|_{L^2}\big).$$

Suppose that there exists a time T with $2<T\in I$, such that

$$\|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)} \le 2C\big(1+\|w_0\|_{L^2}\big)$$

(note that by continuity, it is valid when T is suitable close to 2). Then using Lemma 5.3 and interpolation, we have

$$\|w\|_{L_t^{\varrho_1}L_x^{\sigma_1}(I_j\times\mathbb{R}^d)} \lesssim 1+\|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)},$$

for any $j = j_0, \cdots, J$ with $\mu_0 j_0 N^{-2} \sim 1, \mu_0 J N^{-2} \sim T$. Hence, using (5.10), the last estimate above and Lemma 5.4, we obtain

$$\begin{aligned}\int_2^T \int_{\mathbb{R}^d} |w|^{p+1}|v| \, dxds \lesssim& \delta_0 \big(1 + \|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\big)\|w\|^p_{L_t^\infty L_x^2([2,T)\times\mathbb{R}^d)} \sum_{j=j_0}^{J} \big(\mu_0 j N^{-2}\big)^{-\mu(a)}\\ =&\delta_0 \big(1 + \|w\|_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\big)\|w\|^p_{L_t^\infty L_x^2([2,T)\times\mathbb{R}^d)} \big(\mu_0 j_0 N^{-2}\big)^{-\mu(a)}\\ \lesssim&\delta_0 \big(1 + \|w\|^{p+1}_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\big). \end{aligned} \tag{5.18}$$

Moreover, using Lemma 5.5 and (5.11), we have

$$\int_2^T \int_{\mathbb{R}^d} |w||v|^{p+1} \, dxds \lesssim \delta_0 \|w\|_{L_t^\infty L_x^2([2,T)\times\mathbb{R}^d)}. \tag{5.19}$$

Now together with the estimates (5.18), (5.19) and (5.8), we obtain that

$$\begin{aligned}&\Big|2\mathrm{Im} \int_2^T \int_{\mathbb{R}^d} F(v,w)\bar{w} \, dxds\Big|\\ &\qquad \lesssim \delta_0 \Big(1 + \|w\|_{L_t^\infty L_x^2([2,T)\times\mathbb{R}^d)} + \|w\|^{p+1}_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\Big).\end{aligned}$$

This together with (5.6) and (5.7) yields that for any $t \le T$,

$$\begin{aligned}\|w(t)\|^2_{L_x^2} \le& \|w_0\|^2_{L^2} + CN^{-2s_c}\big\|u_0\big\|^2_{\dot{H}^{s_c}(\mathbb{R}^d)}\\ &+ C\delta_0 \Big(1 + \|w\|_{L_t^\infty L_x^2([2,T)\times\mathbb{R}^d)} + \|w\|^{p+1}_{L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)}\Big).\end{aligned}$$

Note that $p < 1$, then choosing δ_0 suitable small and N suitable large, by the Cauchy-Schwartz inequality, we obtain

$$\sup_{t\in[0,T]} \|w(t)\|^2_{L_x^2} \le 2\big(1 + \|w_0\|^2_{L^2}\big).$$

Hence, by the bootstrap argument, we can extend $T = \sup I$. By (5.1), we obtain the desired estimate. This finishes the proof of the proposition. □

5.2. **Global existence.** Now we prove $I = \mathbb{R}$. It follows from the standard bootstrap argument. Fixing any $2 \le t_0 \in I$ and $\delta > 0$, then by Lemma 2.8, we have

$$\begin{aligned}&\|w\|_{L_t^q L_x^{\tilde{r}}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \big\||\nabla|^\gamma w\big\|_{L_t^q L_x^r([t_0,t_0+\delta]\times\mathbb{R}^d)}\\ &\qquad \lesssim \|w(t_0)\|_{L^2(\mathbb{R}^d)} + \big\||u|^p u - \chi_{\le 1}(t)|v|^p v\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t_0+\delta]\times\mathbb{R}^d)},\end{aligned}$$

where $(q,\tilde{r},0), (q,r,\gamma)$ verify (2.4). Similar as the estimation in (5.2), we have

$$\begin{aligned}&\big\||u|^p u - \chi_{\le 1}(t)|v|^p v\big\|_{L_{tx}^{\frac{2d+4}{d+4}}([t_0,t_0+\delta]\times\mathbb{R}^d)}\\ &\qquad\lesssim \Big(\|w\|^p_{L_t^{q_1} L_x^{r_1}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \|v\|^p_{L_t^{q_1} L_x^{r_1}(\mathbb{R}\times\mathbb{R}^d)}\Big)\\ &\qquad\quad \cdot \big(\|w\|_{L_t^{q_1} L_x^{r_2}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \|\tilde{\chi}_{\ge 1}(t)v\|_{L_t^{q_1} L_x^{r_2}([2,+\infty)\times\mathbb{R}^d)}\big).\end{aligned} \tag{5.20}$$

Here q_1, r_1, r_2 are the parameters defined in the proof of Lemma 5.1. From Lemma 4.3 and Corollary 4.5,

$$\|v\|_{L_t^{q_1} L_x^{r_1}(\mathbb{R}\times\mathbb{R}^d)} + \|\tilde{\chi}_{\ge 1}(t)v\|_{L_t^{q_1} L_x^{r_2}([2,+\infty)\times\mathbb{R}^d)} \lesssim \|v_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \lesssim \delta_0.$$

Setting q_2 such that

$$\frac{1}{q_1} - \frac{1}{q_2} = \frac{-s_c}{2},$$

then the triples $(q_2, r_1, 0), (q_1, r_2, 0)$ verify (2.4). Hence, from (5.20) and by the Hölder inequality, we obtain

$$\begin{aligned}&\|w\|_{L_t^{q_2}L_x^{r_1}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \|w\|_{L_t^{q_1}L_x^{r_2}([t_0,t_0+\delta]\times\mathbb{R}^d)}\\ &\quad\lesssim \|w(t_0)\|_{L^2(\mathbb{R}^d)} + \Big(\delta^{-\frac{s_c p}{2}}\|w\|^p_{L_t^{q_2}L_x^{r_1}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \delta_0^p\Big)\big(\|w\|_{L_t^{q_1}L_x^{r_2}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \delta_0\big).\end{aligned}$$

Hence, there exists $\delta = \delta(\|w(t_0)\|_{L^2(\mathbb{R}^d)})$ such that

$$\|w\|_{L_t^{q_2}L_x^{r_1}([t_0,t_0+\delta]\times\mathbb{R}^d)} + \|w\|_{L_t^{q_1}L_x^{r_2}([t_0,t_0+\delta]\times\mathbb{R}^d)} \lesssim \delta_0^{p+1} + \|w(t_0)\|_{L^2(\mathbb{R}^d)}.$$

Using Proposition 5.2, $\|w(t_0)\|_{L^2(\mathbb{R}^d)}$ is only dependent on N, but not dependent on t_0. Hence $\delta = \delta(N)$. This extends the lifespan to $\mathbb{R}$ and thus proves the global well-posedness.

Lastly, we prove that $w(t) \in \dot{H}^{s_c}(\mathbb{R}^d)$ for any $t \in \mathbb{R}$. Suppose that for some $t_0 \in \mathbb{R}$, $w(t_0) \in \dot{H}^{s_c}(\mathbb{R}^d)$, then arguing similarly as the proof of Lemma 4.3, we obtain that for $T > t_0$,

$$\big\|w\big\|_{L_t^\infty \dot{H}_x^{s_c}\cap L_t^2 L_x^{r_1}([t_0,T]\times\mathbb{R}^d)} \lesssim \|w(t_0)\|_{\dot{H}^{s_c}(\mathbb{R}^d)} + \big\|u\big\|^{p+1}_{L_t^2 L_x^{r_1}([t_0,T]\times\mathbb{R}^d)}, \tag{5.21}$$

where we denote the parameter r_1 as

$$\frac{1}{r_1} = \frac{1}{2} - \frac{1}{d} - \frac{s_c}{d}.$$

Note that from Proposition 5.2 and Lemma 5.3, we have

$$\|w\|_{L_t^{q_1}L_x^{r_1}(([t_0,t_0+\mu_0 N^{-2}])\times\mathbb{R}^d)} \lesssim N^{-s_c},$$

where $q_1 = \frac{2}{1+s_c}$. Hence, combining the last estimates and Lemma 4.3, for $T = t_0 + \mu_0 N^{-2}$,

$$\begin{aligned}\big\|u\big\|_{L_t^2 L_x^{r_1}([t_0,T]\times\mathbb{R}^d)} \lesssim&\big\|v\big\|_{L_t^2 L_x^{r_1}([t_0,T]\times\mathbb{R}^d)} + \big\|w\big\|_{L_t^2 L_x^{r_1}([t_0,T]\times\mathbb{R}^d)}\\ \lesssim&\delta_0 + \big(T - t_0\big)^{-\frac{s_c}{2}} N^{-s_c}\\ \lesssim&\delta_0 + \mu_0^{-\frac{s_c}{2}}.\end{aligned}$$

This last estimate combining with (5.21), yields that if $w(t_0) \in \dot{H}^{s_c}(\mathbb{R}^d)$, then for any $t \in [t_0, t_0 + \mu_0 N^{-2}]$, $w(t) \in \dot{H}^{s_c}(\mathbb{R}^d)$, and

$$\big\|w(t_0 + \mu_0 N^{-2})\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \le \|w(t_0)\|_{\dot{H}^{s_c}(\mathbb{R}^d)} + C\big(\delta_0 + \mu_0^{-\frac{s_c}{2}}\big)^{p+1}.$$

Since $t_0 = 0$, $w(t_0) \in \dot{H}^{s_c}(\mathbb{R}^d)$, the inductive sequence $t_k = \mu_0 N^{-2} 2^k$ can extend from 0 to any $t \in \mathbb{R}^+$, and thus proves that $w(t) \in \dot{H}^{s_c}(\mathbb{R}^d)$ for any $t \in \mathbb{R}^+$. Further,

$$\big\|w(t)\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \lesssim_{\delta_0,\mu_0,N} 1 + t.$$

The negative direction can be treated similarly. This finishes the proof of Theorem 1.1.

Acknowledgements

Part of this work was done while M. Beceanu, A. Soffer and Y. Wu were visiting CCNU (C.C.Normal University) Wuhan, China. The authors thank the institutions for their hospitality and the support. M.B. is partially supported by NSF grant DMS 1700293. Q.D.is partially supported by NSFC 11661061 and 11771165, and the Fundamental Research Funds for the Central Universities (CCNU18QN030, CCNU18CXTD04). A.S is partially supported by NSF grant DMS 01600749 and NSFC 11671163. Y.W. is partially supported by NSFC 11771325 and 11571118. A.S. and Y.W. would like to thank Ch. Miao for useful discussions.

References

[1] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures.* Comm. Math. Phys. 166 (1994), no. 1, 1–26.

[2] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation.* Comm. Math. Phys. 176 (1996), no. 2, 421–445.

[3] J. Bourgain, *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case.* J. Amer. Math. Soc. **12** (1999), 145–171.

[4] J. Bourgain and D. Li. *On an endpoint Kato-Ponce inequality.* Differential Integral Equations 27 (2014), no. 11-12, 1037–1072.

[5] A. Bulut, *The defousing energy-supercritical cubic nonlinear wave equation in dimension five.* Preprint, arXiv: 1112.0629.

[6] N. Burq, and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory.* Invent. Math. 173 (2008), no. 3, 449–475.

[7] N. Burq, and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. II. A global existence result.* Invent. Math. 173 (2008), 477–496.

[8] N. Burq, and N. Tzvetkov, *Probabilistic well-posedness for the cubic wave equation.* J. Eur. Math. Soc. 16 (2014), no. 1, 1–30.

[9] T. Cazenave, *Semilinear Schrödinger Equations.* Courant Lecture Notes in Mathematics, 10, American Mathematical Society, 2003.

[10] T. Cazenave and F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s.* Nonlinear Anal., Theory, Methods and Applications, **14** (1990), 807–836.

[11] Y. Cho, S. Lee, *Strichartz estimates in spherical coordinates.* Indiana Univ. Math. J., 62 (2013), no.3, 991–1020.

[12] M. Christ, J. Colliander, and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations.* Preprint arXiv:math/0311048.

[13] J. Colliander, M. Keel, G. Staffilani, T. Takaoka, and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$,* Ann. of Math. (2) 167 (2008), no. 3, 767–865.

[14] J. Colliander, and T. Oh, *Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(T)$.* Duke Math. J. 161 (2012), no. 3, 367–414.

[15] Y. Deng, *Two-dimensional nonlinear Schrödinger equation with random radial data.* Anal. PDE 5 (2012), no. 5, 913–960.

[16] Y. Deng, N. Tzvetkov, and N. Visciglia, *Invariant measures and long time behaviour for the Benjamin-Ono equation III.* Comm. Math. Phys. 339 (2015), no. 3, 815–857.

[17] B. Dodson, *Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d \geq 3$.* J. Amer. Math. Soc. 25 (2012), no. 2, 429–463.

[18] B. Dodson, *Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 1$.* Amer. J. of Math. 138 (2016), no. 2, 531–569.

[19] B. Dodson, *Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$.* Duke Math. J. 165 (2016), no. 18, 3435–3516.

[20] B. Dodson, *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state.* Adv. Math. 285 (2015), 1589–1618.

[21] B. Dodson, *Global well-posedness and scattering for the focusing, energy-critical nonlinear Schrödinger problem in dimension $d = 4$ for initial data below a ground state threshold.* Preprint arXiv:1409.1950.

[22] B. Dodson, J. Lührmann, and D. Mendelson, *Almost sure scattering for the 4D energy-critical defocusing nonlinear wave equation with radial data*. to appear in Amer. J. Math.
[23] B. Dodson, C. Miao, J. Murphy, and J. Zheng, *The defocusing quintic NLS in four space dimensions*. Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), no. 3, 759–787.
[24] T. Duyckaerts, J. Holmer, S. Roudenko, *Scattering for the non-radial 3D cubic nonlinear Schrödinger equation*. Math. Res. Lett. 15 (2008), no. 6, 1233–1250.
[25] T. Duyckaerts, C. Kenig and F. Merle, *Scattering for radial, bounded solutions of focusing supercritical wave equations*. Int. Math. Res. Not., 2014 (2014), 224–258.
[26] T. Duyckaerts, and T. Roy. *Blow-up of the critical Sobolev norm for nonscattering radial solutions of supercritical wave equations on* $\mathbb{R}^3$. Preprint arXiv:1506.00788 (2015).
[27] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*. J. Math. Pures Appl., **64** (1985), 363–401.
[28] Z. Guo and Y. Wang, *Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations*. J. Anal. Math., 124 (2014), 1–38.
[29] M. Keel and T. Tao, *Endpoint Strichartz Estimates*. Amer. J. Math., **120** (1998), 955–980.
[30] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energycritical, focusing, non-linear Schrödinger equation in the radial case*. Invent. Math. 166 (2006), 645–675.
[31] C. Kenig, F. Merle, *Scattering for* $\dot{H}^{\frac{1}{2}}$ *bounded solutions to the cubic, defocusing NLS in 3 dimensions*. Trans. Amer. Math. Soc. **362** (2010), 1937–1962.
[32] C. Kenig, F. Merle, *Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications*. Amer. J. Math., 133 (2011), 1029–1065.
[33] C. Kenig, F. Merle, *Radial solutions to energy supercritical wave equations in odd dimensions*. Discrete Continuous Dynamical Systems-A, 31 (4) (2011), 1365–1381.
[34] C. Kenig, G. Ponce and L. Vega. *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*. Comm. Pure Appl. Math., 46 (1993), no. 4, 527–620.
[35] C. E. Kenig, G. Ponce, and L. Vega, *On the ill-posedness of some canonical dispersive equations*. Duke Math. J., 106 (2001), no. 3, 617–633.
[36] R. Killip, S. Masaki, J. Murphy, and M. Visan. *Large data mass-subcritical NLS: critical weighted bounds imply scattering*. NoDEA Nonlinear Differential Equations Appl., 24 (2017), no. 4, Art. 38, 33pp.
[37] R. Killip, S. Masaki, J. Murphy, and M. Visan., *The radial mass-subcritical NLS in negative order Sobolev spaces*. Preprint, arXiv:1804.06753v1.
[38] R. Killip, J. Murphy, and M. Visan., *Almost sure scattering for the energy-critical NLS with radial data below* $H^1(\mathbb{R}^4)$. Prepint, arXiv:1707.09051.
[39] R. Killip, T. Tao, and M. Visan, *The cubic nonlinear Schrödinger equation in two dimensions with radial data*. J. Eur. Math. Soc. (JEMS) 11 (2009), no. 6, 1203–1258.
[40] R. Killip and M. Visan, *Energy-supercritical NLS: critical* $\dot{H}^s$*-bounds imply scattering*. Comm. Partial Differential Equations **35** (2010), no. 6, 945–987.
[41] R. Killip and M. Visan, *The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions*. Prerint, arXiv:1002.1756.
[42] R. Killip and M. Visan, *The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher*. Amer. J. Math., 132 (2010), no. 2, 361–424.
[43] R. Killip and M. Visan, *Global well-posedness and scattering for the defocusing quintic NLS in three dimensions*. Anal. PDE, 5 (2012), no. 4, 855–885.
[44] R. Killip and M. Visan, *Nonlinear Schrödinger equations at critical regularity*. Clay Math. Proc., 17 (2013), 325–437.
[45] R. Killip, M. Visan, and X. Zhang, *The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher*. Anal. PDE, 1 (2008), no. 2, 229–266.
[46] D. Li, *On Kato-Ponce and fractional Leibniz*. To appear in Rev. Mat. Iberoam., 2017.
[47] D. Li, X. Zhang, *Regularity of almost periodic modulo scaling solutions for mass-critical NLS and applications*. Anal. PDE, **3** (2010), no 2, 175–195.
[48] C. Miao, J. Murphy and J. Zheng, *The defocusing energy-supercritical NLS in four space dimensions*. J. Funct. Anal., **267** (2014), no. 6, 1662–1724.
[49] C. Miao, Y. Wu and X. Zhang, *The defocusing energy-supercritical nonlinear wave equation when* $d = 4$. Preprint.

[50] J. Murphy, *Intercritical NLS: critical $\dot{H}^s$-bounds imply scattering*. SIAM J. Math. Anal., **46** (2014), no. 1, 939–997.
[51] J. Murphy, *The radial defocusing nonlinear Schrödinger equation in three space dimensions.* Comm. Partial Differential Equations, 40 (2015), no. 2, 265–308.
[52] J. Lürmann, and D. Mendelson, *Random data Cauchy theory for nonlinear wave equations of power-type on* $\mathbb{R}^3$. Comm. Partial Differential Equations, 39 (2014), no. 12, 2262–2283.
[53] A. Nahmod, T. Oh, L. Rey-Bellet, and G. Staffilani, *Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS*. J. Eur. Math. Soc., 14 (2012), no. 4, 1275–1330.
[54] A. Nahmod, and G. Staffilani, *Almost sure well-posedness for the periodic 3D quintic nonlinear Schröinger equation below the energy space*. J. Eur. Math. Soc., 17 (2015), no. 7, 1687–1759.
[55] T. Oh, M. Okamoto, and O. Pocovnicu, *On the probabilistic well-posedness of the nonlinear Schröinger equations with non-algebraic nonlinearities*. Preprint, arXiv:1708.01568.
[56] T. Oh, and O. Pocovnicu, *Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on* $\mathbb{R}^3$. J. Math. Pures Appl., 105 (2016), 342–366.
[57] O. Pocovnicu, *Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on* $\mathbb{R}^d$*,* $d = 4$ *and* 5. J. Eur. Math. Soc., 19 (2017), no. 8, 2521–2575.
[58] A. Poiret, D. Robert, and L. Thomann, *Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator*. Anal. PDE, 7 (2014), no. 4, 997–1026.
[59] S. Shao, *Sharp linear and bilinear restriction estimates for paraboloids in the cylindrically symmetric case*. Rev. Mat. Iberoam., 25 (2009), 1127–1168.
[60] T. Tao, *Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data*. New York J. Math., 11 (2005), 57–80.
[61] T. Tao, M. Visan, X. Zhang, *Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions.* Duke Math. J., 140 (2007), no. 1, 165–202.
[62] L. Thomann, *Random data Cauchy problem for supercritical Schrödinger equations*. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 6, 2385–2402.
[63] M. Visan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions*. Duke Math. J. 138 (2007), no. 2, 281–374.
[64] M. Visan, *Global well-posedness and scattering for the defocusing cubic nonlinear Schrödinger equation in four dimensions*. Int. Math. Res. Not., 2012, no. 5, 1037– 1067.
[65] J. Xie and D. Fang, *Global well-posedness and scattering for the defocusing $\dot{H}^s$-critical NLS*. Chin. Ann. Math. Ser. B, 34 (2013), no. 6, 801–842.

Department of Mathematics and Statistics, University at Albany SUNY, Earth Science 110, Albany, NY, 12222, USA,

E-mail address: mbeceanu@albany.edu

Department of Mathematics, Hubei Key Laboratory of Mathematical Science, Central China Normal University, Wuhan 430079, China.,

E-mail address: dengq@mail.ccnu.edu.cn

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ, 08854, USA,

E-mail address: soffer@math.rutgers.edu

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

E-mail address: yerfmath@gmail.com