Clustered Variants of Hajós' Conjecture*

Chun-Hung Liu† David R. Wood‡

September 13, 2021

Abstract

Hajós conjectured that every graph containing no subdivision of the complete graph K_{s+1} is properly s-colorable. This conjecture was disproved by Catlin. Indeed, the maximum chromatic number of such graphs is $\Omega(s^2/\log s)$. We prove that $O(s)$ colors are enough for a weakening of this conjecture that only requires every monochromatic component to have bounded size (so-called *clustered* coloring).

Our approach leads to more results, many of which only require a much weaker assumption that forbids an 'almost $(\leqslant 1)$-subdivision' (where at most one edge is subdivided more than once). This assumption is best possible, since no bound on the number of colors exists unless we allow at least one edge to be subdivided arbitrarily many times. We prove the following (where $s \geqslant 2$):

1. Graphs of bounded treewidth and with no almost $(\leqslant 1)$-subdivision of K_{s+1} are s-choosable with bounded clustering.
2. For every graph H, graphs with no H-minor and no almost $(\leqslant 1)$-subdivision of K_{s+1} are $(s+1)$-colorable with bounded clustering.
3. For every graph H of maximum degree at most d, graphs with no H-subdivision and no almost $(\leqslant 1)$-subdivision of K_{s+1} are $\max\{s+3d-5, 2\}$-colorable with bounded clustering.
4. For every graph H of maximum degree d, graphs with no $K_{s,t}$ subgraph and no H-subdivision are $\max\{s+3d-4, 2\}$-colorable with bounded clustering.
5. Graphs with no K_{s+1}-subdivision are $(4s-5)$-colorable with bounded clustering.

The first result is tight and shows that the clustered analogue of Hajós' conjecture is true for graphs of bounded treewidth. The second result implies an upper bound for the clustered version of Hadwiger's conjecture that is only one color away from the known lower bound, and shows that the number of colors is independent of the forbidden minor. The final result is the first $O(s)$ bound on the clustered chromatic number of graphs with no K_{s+1}-subdivision.

*This material is based upon work supported by the National Science Foundation under Grant No. DMS-1929851 and DMS-1954054.

†Department of Mathematics, Texas A&M University, College Station, Texas, USA, chliu@math.tamu.edu. Partially supported by NSF under Grant No. DMS-1929851 and DMS-1954054.

‡School of Mathematics, Monash University, Melbourne, Australia, david.wood@monash.edu. Supported by the Australian Research Council.

Keywords: Graph Coloring, clustered coloring, topological minors

1 Introduction

In the 1940s, Hajós conjectured that every graph containing no subdivision of the complete graph K_{s+1} is s-colorable; see [25, 30, 31]. Dirac [5] proved the conjecture for $s \leqslant 3$. It is open for $s \in \{4, 5\}$, which would imply the Four Color Theorem. Catlin [3] presented counterexamples for all $s \geqslant 6$, and Erdős and Fajtlowicz [8] proved that the conjecture is false for almost all graphs. Indeed, there are graphs with no K_{s+1}-subdivision and with chromatic number $\Omega(s^2/\log s)$. The best upper bound on the number of colors is $O(s^2)$, independently due to Bollobás and Thomason [2] and Komlós and Szemerédi [19]; see [12] for a related result. See [25, 31] for more explicit counterexamples and further discussion of connections to other areas of graph theory.

The purpose of this paper is to prove several positive results in the direction of weakenings of Hajós' conjecture. Define a *coloring* of a graph G to simply be a function that assigns one color to each vertex of G. For a coloring c of a graph G, a *monochromatic c-component* of G is a connected component of a subgraph of G induced by all the vertices assigned the same color by c. When c is clear, we simply write *monochromatic component*. A coloring has *clustering* η if every monochromatic component has at most η vertices. Our focus is on minimizing the number of colors, with small clustering as a secondary goal. The *clustered chromatic number* of a graph class $\mathcal{F}$ is the minimum integer k for which there exists an integer c such that every graph in $\mathcal{F}$ has a k-coloring with clustering c. There have been several recent papers on this topic [1, 4, 6, 10, 11, 13–18, 20, 21, 26, 28]; see [33] for a survey.

Most of our results actually hold (in some sense) for more general classes of graphs than those with no K_{s+1}-subdivision, as we now explain. Say a graph H' is an *almost* $(\leqslant 1)$*-subdivision* of a graph H if H' can be obtained from H by subdividing edges, where at most one edge is subdivided more than once. Most of our results say that all graphs containing no almost $(\leqslant 1)$-subdivision of K_{s+1}, plus some other properties, are s-colorable with bounded clustering.

The following is our first main result. It provides a Hajós-type result for clustered coloring of graphs with bounded treewidth.

Theorem 1. *For all $s, w \in \mathbb{N}$, there exists $\eta \in \mathbb{N}$ such that every graph with treewidth at most w and with no almost $(\leqslant 1)$-subdivision of K_{s+1} is s-choosable with clustering η.*

The notion of s-choosable with bounded clustering is defined in Section 2.1. Note that every graph that is s-choosable with bounded clustering is also s-colorable with bounded clustering. This shows that the number of colors in Theorem 1 is best possible in the following strong sense: for all $s \in \mathbb{N}$ and $\eta \in \mathbb{N}$ there is a graph G with treewidth at most $s-1$ (and thus with no subdivision of K_{s+1}), such that every $(s-1)$-coloring of G has a monochromatic component with at least η vertices; see [33]. In particular, at least s colors are required even for this weakening of Hajós' conjecture.

The assumption of bounded treewidth in Theorem 1 is equivalent to saying that the graph excludes a planar graph as a minor by Robertson and Seymour's Grid Minor The-

orem [29]. What if we exclude a general graph as a minor? Our next result answers this question (with one more color).

Theorem 2. *For every $s \in \mathbb{N}$ and every graph H, there exists $\eta \in \mathbb{N}$ such that every graph containing no H-minor and containing no almost $(\leqslant 1)$-subdivision of K_{s+1} is $(s+1)$-colorable with clustering η.*

Theorem 2 (with $H = K_{s+1}$) has the following interesting corollary for graphs excluding a minor.

Corollary 3. *For every $s \in \mathbb{N}$ there exists $\eta \in \mathbb{N}$ such that every graph containing no K_{s+1}-minor is $(s+1)$-colorable with clustering η.*

Kawarabayashi and Mohar [18] first proved that graphs containing no K_{s+1}-minor are $O(s)$-colorable with bounded clustering. The bound on the number of colors has since been steadily improved [7, 15, 21, 27, 32]. Prior to the present work, the best bound was $s+2$, which followed from a general result by the authors [23]. Corollary 3 improves this bound to $s+1$, although it should be noted that results from [23] are essential for the proof of Theorem 2 and Corollary 3. Dvořák and Norin [6] have announced that a forthcoming paper will prove that s colors suffice (which is the clustered analogue of Hadwiger's Conjecture, and would be best possible). Their result is incomparable with Theorem 2 and the aforementioned general result in [23].

Our next result relaxes the assumption that the graph contains no H-minor, and instead assumes that it contains no H-subdivision. The price paid is an increase in the number of colors, depending only on the maximum degree of H.

Theorem 4. *For every $s \in \mathbb{N}$ and every graph H with maximum degree $d \in \mathbb{N}$, there exists $\eta \in \mathbb{N}$ such that every graph with no H-subdivision and no almost $(\leqslant 1)$-subdivision of K_{s+1} is $\max\{s+3d-5, 2\}$-colorable with clustering η.*

The next theorem relaxes the assumption of no almost $(\leqslant 1)$-subdivision of K_{s+1}, and instead assumes the graph contains no $K_{s,t}$-subgraph. Interestingly the number of colors does not depend on t. Note that $K_{s,t}$ contains a K_{s+1}-subdivision where every edge is subdivided at most once, when t is sufficiently large.

Theorem 5. *For $s, t, d \in \mathbb{N}$ and every graph H of maximum degree d, there exists $\eta \in \mathbb{N}$ such that every graph with no $K_{s,t}$-subgraph and no H-subdivision is $\max\{s+3d-4, 2\}$-colorable with clustering η.*

We remark that all of the above theorems forbid $(\leqslant 1)$-subdivisions of K_{s+1} or subdivisions of H. That is, we forbid a subdivision of a graph where some edge is allowed to be subdivided arbitrarily many times. This condition is required since there are graphs of arbitrarily high girth and arbitrarily high chromatic number [9], which therefore require arbitrarily many colors for any fixed clustering value; this shows that excluding finitely many graphs as subgraphs cannot ensure any upper bound on the number of colors.

Our final theorem simply excludes a K_{s+1}-subdivision. This is the first $O(s)$ bound on the clustered chromatic number of the class of graphs excluding a K_{s+1}-subdivision.

Theorem 6. *For each $s \in \mathbb{N}$, there exists $\eta \in \mathbb{N}$ such that every graph containing no K_{s+1}-subdivision is $\max\{4s-5, 1\}$-colorable with clustering η.*

We now compare the above theorems with Hajós' conjecture. First note that Theorems 1–4 are stronger than Hajós' conjecture in the sense that they only exclude an almost ($\leqslant 1$)-subdivision of K_{s+1}, whereas Hajós' conjecture excludes all subdivisions of K_{s+1}. Moreover, Theorem 1 also holds in the stronger setting of choosability. On the other hand, Theorems 1–6 are weaker than Hajós' conjecture in the sense that they have bounded clustering rather than a proper coloring. However, such a weakening is unavoidable since Hajós' conjecture is false. Indeed, the proof of the theorem of Erdős and Fajtlowicz [8] mentioned above shows that, for a suitable constant c, almost every graph on cs^2 vertices contains no subdivision of K_{s+1} and has chromatic number $\Omega(s^2/\log s)$. Trivially, such a graph has treewidth at most cs^2 and contains no K_{cs^2}-minor. Thus the clustering function in all of the above theorems is at least $\Omega(s/\log s)$.

The paper is organized as follows. Section 2 introduces preliminary definitions and results from our companion papers [23, 24] that are used in the present paper. Section 3 introduces a structure theorem of the first author and Thomas [22] for graphs excluding a fixed subdivision, and uses it to prove Theorem 5. Building on this work, Section 4 proves the remaining theorems mentioned above.

2 Preliminaries

We use the following notation. Let $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{N} := \{1, 2, \dots\}$. For $m, n \in \mathbb{N}_0$, let $[m, n] := \{m, m+1, \dots, n\}$ and $[n] := [1, n]$.

Let G be a graph (allowing loops and parallel edges). For $v \in V(G)$, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ be the neighborhood of v, and let $N_G[v] := N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, let $N_G(X) := \bigcup_{v \in X}(N_G(v) - X)$ and $N_G[X] := N_G(X) \cup X$. Denote the subgraph of G induced by X by $G[X]$.

For a graph G, a subset X of $V(G)$, and an integer $s \geqslant 1$, let

$$N_G^{\geqslant s}(X) := \{v \in V(G) - X : |N_G(v) \cap X| \geqslant s\} \text{ and}$$
$$N_G^{<s}(X) := \{v \in V(G) - X : 1 \leqslant |N_G(v) \cap X| < s\}.$$

When the graph G is clear from the context we write $N^{\geqslant s}(X)$ instead of $N_G^{\geqslant s}(X)$, and similarly for $N^{<s}(X)$.

Lemma 7 ([24, Lemma 12])**.** *For all $s, t \in \mathbb{N}$, there exists a function $f_{s,t} : \mathbb{N}_0 \to \mathbb{N}_0$ such that for every graph G with no $K_{s,t}$ subgraph, if $X \subseteq V(G)$ then $|N^{\geqslant s}(X)| \leqslant f_{s,t}(|X|)$.*

Lemma 7 is sufficient to prove the theorems in this paper. But when G excludes a fixed minor or subdivision of a fixed graph, the function $f_{s,t}$ in Lemma 7 can be made linear; see [24]. This improves the clustering function in all our results, although to simplify the presentation, we choose not to explicitly evaluate our clustering functions.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{X} = (X_x : x \in V(T)))$, where T is a tree and for each node $x \in V(T)$, X_x is a subset of $V(G)$ called a *bag*, such that for each

vertex $v \in V(G)$, the set $\{x \in V(T) : v \in X_x\}$ induces a non-empty (connected) subtree of T, and for each edge $vw \in E(G)$ there is a node $x \in V(T)$ such that $\{v, w\} \subseteq X_x$. The *width* of a tree-decomposition $(T, \mathcal{X})$ is $\max\{|X_x| - 1 : x \in V(T)\}$. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G.

Let H be a graph. An *H-minor* of a graph G is a map α with domain $V(H) \cup E(H)$ such that:

- For every $h \in V(H)$, $\alpha(h)$ is a nonempty connected subgraph of G.
- If h_1 and h_2 are different vertices of H, then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint.
- For each edge e of H with endpoints h_1, h_2, $\alpha(e)$ is an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$; furthermore, if $h_1 = h_2$, then $\alpha(e) \in E(G) - E(\alpha(h_1))$.
- If e_1, e_2 are two different edges of H, then $\alpha(e_1) \neq \alpha(e_2)$.

2.1 List Coloring

For our purposes, a *color* is an element of $\mathbb{Z}$. Let G be a graph. A *list-assignment* of G is a function L with domain containing $V(G)$, such that $L(v)$ is a non-empty set of colors for each vertex $v \in V(G)$. For a list-assignment L of $V(G)$, an *L-coloring* of G is a coloring c of G such that $c(v) \in L(v)$ for every $v \in V(G)$. An L-coloring has *clustering* η if every monochromatic component has at most η vertices. A list-assignment L of G is an *ℓ-list-assignment* if $|L(v)| \geqslant \ell$ for every vertex $v \in V(G)$. A graph G is *ℓ-choosable with clustering η* if G is L-colorable with clustering η for every ℓ-list-assignment L of G.

For a graph G, a subset $Y_1 \subseteq V(G)$, and $s, r \in \mathbb{N}$, a list-assignment L of G is an *(s, r, Y_1)-list-assignment* if:

(L1) $|L(v)| \in [s + r]$ for every $v \in V(G)$.
(L2) $Y_1 = \{v \in V(G) : |L(v)| = 1\}$.
(L3) For every $y \in N^{<s}(Y_1)$,

$$|L(y)| = s + r - |N_G(y) \cap Y_1| \geqslant r + 1$$

and $L(y) \cap L(u) = \emptyset$ for every $u \in N_G(y) \cap Y_1$.
(L4) For every $v \in V(G) - N_G[Y_1]$, we have $|L(v)| = s + r$.
(L5) For every $v \in V(G) - Y_1$, we have $|L(v)| \geqslant r + 1$.

We say that an (s, r, Y_1)-list-assignment L of G is *restricted* if:

(L1') $L(v) \subseteq [s + r]$ for every $v \in V(G)$.

Note that a restricted $(s, 2, Y_1)$-list-assignment is called a $(s, Y_1, 0, 0)$-list-assignment in our companion paper [23].

For a list-assignment L of a graph G with $Y_1 = \{v \in V(G) : |L(v)| = 1\}$, for $\eta \in \mathbb{N}$ and a nondecreasing function $g : \mathbb{N} \to \mathbb{N}$, an L-coloring c of G is *(η, g)-bounded* if:

- the union of the monochromatic components intersecting Y_1 contains at most $|Y_1|^2 g(|Y_1|)$ vertices, and
- every monochromatic component contains at most $\eta^2 g(\eta)$ vertices.

2.2 Companion Results

Our companion paper proves the following results for graphs with no $K_{s,t}$ subgraph. The first assumes bounded treewidth, the second assumes an excluded minor.

Theorem 8 ([23, Theorem 17]). *For all $s, t, w \in \mathbb{N}$, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that if G is a graph of treewidth at most w and with no $K_{s,t}$ subgraph, Y_1 is a subset of $V(G)$ with $|Y_1| \leqslant \eta$, and L is an $(s, 1, Y_1)$-list-assignment of G, then there exists an (η, g)-bounded L-coloring of G.*

Theorem 9 ([23, Theorem 24]). *For all $s, t \in \mathbb{N}$ and for every graph H, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that if G is a graph with no $K_{s,t}$ subgraph and no H-minor, Y_1 is a subset of $V(G)$ with $|Y_1| \leqslant \eta$, and L is a restricted $(s, 2, Y_1)$-list-assignment of G, then there exists an (η, g)-bounded L-coloring.*

2.3 Progress

The concept of "progress" from the proofs of the above two theorems are re-used in the present paper. Let $s, r \in \mathbb{N}$ and L be an (s, r, Y_1)-list-assignment of a graph G. For $W \subseteq V(G)$, a *W-progress* of L is a list-assignment L' of G defined as follows:

- Let $Y_1' := Y_1 \cup W$.
- For every $y \in Y_1$, let $L'(y) := L(y)$.
- For every $y \in Y_1' - Y_1$, let $L'(y)$ be a 1-element subset of $L(y)$ (which exists by (L2)–(L5)).
- For each $v \in N^{<s}(Y_1')$, let $L'(v)$ be a subset of $L(v) - \{L'(w) : w \in N_G(v) \cap (W - Y_1)\}$ of size $|L(v)| - |N_G(v) \cap (W - Y_1)|$.
- For every $v \in V(G) - (Y_1' \cup N^{<s}(Y_1'))$, let $L'(v) := L(v)$.

Lemma 10 ([23, Lemma 12 with $F = \emptyset$]). *Let $s, r \in \mathbb{N}$ and L be an (s, r, Y_1)-list-assignment of a graph G. Let $W \subseteq V(G)$. Then every W-progress L' of L is an $(s, r, Y_1 \cup W)$-list-assignment of G, and $L'(v) \subseteq L(v)$ for every $v \in V(G)$.*

Lemma 11 ([23, Lemma 13 with $F = \emptyset$]). *For all $s, t, k \in \mathbb{N}$, there exist a number $\eta > k$ and a nondecreasing function g with domain $\mathbb{N}_0$ and with $g(0) \geqslant \eta$ such that if G is a graph with no $K_{s,t}$ subgraph, $r \in \mathbb{N}$, Y_1 is a subset of $V(G)$ with $|Y_1| \leqslant \eta$, and L is an (s, r, Y_1)-list-assignment of G, then at least one of the following holds:*

1. *There exists an (η, g)-bounded L-coloring of G.*
2. *$|Y_1| > k$.*
3. *For every color ℓ, there exist a subset Y_1' of $V(G)$ with $\eta \geqslant |Y_1'| > |Y_1|$ and an (s, r, Y_1')-list-assignment L' of G with $L'(v) \subseteq L(v)$ for every $v \in V(G)$, such that:*
   (a) *there does not exist an (η, g)-bounded L'-coloring c' of G,*
   (b) *for every L'-coloring of G, every monochromatic component intersecting Y_1 is contained in $G[Y_1']$, and*

(c) for every $y \in Y_1'$ with $\ell \in L'(y)$, we have $\{v \in N_G(y) - Y_1' : \ell \in L'(v)\} = \emptyset$.

4. *$Y_1 \neq \emptyset$, $N_G(Y_1) = \emptyset$, and there does not exist an (η, g)-bounded $L|_{G-Y_1}$-coloring of $G - Y_1$.*

2.4 Separations and Tangles

A *separation* of a graph G is an ordered pair (A, B) of edge-disjoint subgraphs of G with $A \cup B = G$. The *order* of (A, B) is $|V(A \cap B)|$. A *tangle* $\mathcal{T}$ in a graph G of order θ is a set of separations of G of order less than θ such that:

(T1) For every separation (A, B) of G of order less than θ, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$.

(T2) If $(A_i, B_i) \in \mathcal{T}$ for $i \in [3]$, then $A_1 \cup A_2 \cup A_3 \neq G$.

(T3) If $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

Lemma 12 ([23, Lemma 16 with $F = \emptyset$]). *For all $s, t, \theta, \eta, r \in \mathbb{N}$ with $\eta \geqslant 9\theta + 1$, for every nondecreasing function g with domain $\mathbb{N}_0$, if G is a graph with no $K_{s,t}$ subgraph, Y_1 is a subset of $V(G)$ with $9\theta + 1 \leqslant |Y_1| \leqslant \eta$, and L is an (s, r, Y_1)-list-assignment of G, then at least one of the following holds:*

1. *There exists an (η, g)-bounded L-coloring of G.*
2. *There exist an induced subgraph G' of G with $|V(G')| < |V(G)|$, a subset Y_1' of $V(G')$ with $|Y_1'| \leqslant \eta$ and an (s, r, Y_1')-list-assignment L' of G' such that:*
   (a) $L'(v) \subseteq L(v)$ for every $v \in V(G')$.
   (b) There does not exist an (η, g)-bounded L'-coloring of G'.
3. *$\mathcal{T} := \{(A, B) : |V(A \cap B)| < \theta, |V(A) \cap Y_1| \leqslant 3\theta\}$ is a tangle of order θ in G.*

A tangle $\mathcal{T}$ in G *controls* an H-minor α if there does not exist $(A, B) \in \mathcal{T}$ of order less than $|V(H)|$ such that $V(\alpha(h)) \subseteq V(A)$ for some $h \in V(H)$.

Lemma 13 ([23, Lemma 23 with $\ell = r = 0$]). *For all $s, t, t' \in \mathbb{N}$, there exist $\theta^* \in \mathbb{N}$ and nondecreasing functions g^*, η^* with domain $\mathbb{N}_0$ such that if G is a graph with no $K_{s,t}$ subgraph, $\theta \in \mathbb{N}$ with $\theta \geqslant \theta^*$, $\eta \in \mathbb{N}$ with $\eta \geqslant \eta^*(\theta)$, $Y_1 \subseteq V(G)$ with $3\theta < |Y_1| \leqslant \eta$, L is a restricted $(s, 2, Y_1)$-list-assignment of G, g is a nondecreasing function with domain $\mathbb{N}_0$ with $g \geqslant g^*$, and $\mathcal{T} := \{(A, B) : |V(A \cap B)| < \theta, |V(A) \cap Y_1| \leqslant 3\theta\}$ is a tangle in G of order θ that does not control a $K_{t'}$-minor, then either:*

1. *there exists an (η, g)-bounded L-coloring of G, or*
2. *there exist $(A^*, B^*) \in \mathcal{T}$, a set Y_{A^*} with $|Y_{A^*}| \leqslant \eta^*(\theta)$ and $Y_1 \cap V(A^*) \subseteq Y_{A^*} \subseteq V(A^*)$, and a restricted $(s, 2, Y_{A^*})$-list-assignment L_{A^*} of $G[V(A^*)]$ such that there exists no (η, g)-bounded L_{A^*}-coloring of $G[V(A^*)]$.*

3 Excluding Subdivisions

The following theorem is a special case of a theorem by the first author and Thomas [22].

Theorem 14 ([22, Theorem 6.8])**.** *For any integers d, h and graph H on h vertices with maximum degree at most d, there exist integers θ, ξ such that if G is a graph containing no H-subdivision, and if $\mathcal{T}$ is a tangle in G of order at least θ controlling a $K_{\lfloor \frac{3}{2}dh \rfloor}$-minor, then there exists $Z \subseteq V(G)$ with $|Z| \leqslant \xi$ such that for every vertex $v \in V(G) - Z$, there exists $(A, B) \in \mathcal{T} - Z$ of order less than d such that $v \in V(A) - V(B)$.*

The next two lemmas imply Theorem 5, since if $s, d \in \mathbb{N}$ and $3d + s < 7$, then $d = 1$.

Lemma 15. *If H is a graph of maximum degree at most 1, then every graph with no H-subdivision is 2-colorable with clustering* $\max\{2|V(H)| - 2, 1\}$.

Proof. Since H is of maximum degree at most one, G has no H-subdivision implies that G does not contain a matching of size $|V(H)|$, and hence G contains a vertex-cover S of size at most $2|V(H)| - 2$. By coloring every vertex in S with 1 and coloring every vertex in $V(G) - S$ with 2, we obtain a 2-coloring of G with clustering $\max\{|S|, 1\} \leqslant \max\{2|V(H)| - 2, 1\}$. □

Lemma 16. *For any $s, t, d \in \mathbb{N}$ and graph H of maximum degree d with $3d + s \geqslant 7$, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that if G is a graph with no $K_{s,t}$ subgraph and no H-subdivision, $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta$ and L is a restricted $(s', 2, Y_1)$-list-assignment of G, then there exists an (η, g)-bounded L-coloring, where $s' = 3d + s - 6$.*

Proof. Define the following:

- Let f be the function $f_{s,t}$ mentioned in Lemma 7.
- Let θ_0 be the number θ^* and g_0, η_0 be the functions g^*, η^*, respectively, mentioned in Lemma 13 by taking $s = s', t = t$ and $t' = \lfloor \frac{3}{2}d|V(H)| \rfloor$.
- Let θ_1 and ξ be the numbers θ and ξ mentioned in Theorem 14, respectively, by taking $d = d$, $h = |V(H)|$ and $H = H$.
- Let $a_0 := f(\xi)d^2 + \xi + 1$, and let $a_i := da_{i-1} + 1$ for $i \in \mathbb{N}$.
- Let $\theta := \theta_0 + \theta_1 + (d-1)a_{(d-1)a_0}$.
- Let η_1 be the number η and let g_1 be the function g mentioned in Lemma 11 by taking $s = s'$, $t = t$ and $k = 9\theta$. Note that $g(0) \geqslant \eta_1 > 9\theta$ by Lemma 11.
- Let $\eta := \eta_0(\theta) + \eta_1 + (d-1)a_{(d-1)a_0}$.
- Let $g : \mathbb{N} \to \mathbb{N}$ be the function defined by $g(0) := g_0(0) + g_1(0)$ and $g(x+1) := g_0(x+1) + g_1(x+1) + \sum_{i=0}^{x} i^2 g(i)$ for $x \in \mathbb{N}$.

Let G be a graph with no $K_{s,t}$ subgraph and with no subdivision of H, let $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta$, and let L be a restricted $(s', 2, Y_1)$-list-assignment of G. Suppose to the contrary that there exists no (η, g)-bounded L-coloring of G. We further assume that $|V(G)|$ is minimum, and subject to this, $|Y_1|$ is maximum.

Claim 16.1. *$Y_1 \neq \emptyset$ and $N_G(Y_1) \neq \emptyset$.*

Proof. First suppose that $Y_1 = \emptyset$. Let v be a vertex of G, and let L' be a $\{v\}$-progress of L. Let $Y_1' = \{v\}$. By Lemma 10, L' is an $(s', 2, Y_1')$-list-assignment of G. Since $|Y_1'| \leqslant \eta$, the maximality of $|Y_1|$ implies that there exists an (η, g)-bounded L'-coloring c' of G. But c' is an (η, g)-bounded L-coloring c of G, a contradiction.

So $Y_1 \neq \emptyset$. Suppose that $N_G(Y_1) = \emptyset$. Let $G' := G - Y_1$. Then $L|_{G'}$ is an $(s', 2, \emptyset)$-list-assignment of $G - Y_1$. By the minimality of $|V(G)|$, there exists an (η, g)-bounded $L|_{G'}$-coloring c of G'. Color each vertex y in Y_1 with the unique element in $L(y)$. Since $|Y_1| \leqslant |Y_1|^2 g(|Y_1|)$, we obtain an (η, g)-bounded L-coloring of G, a contradiction. □

Claim 16.2. $|Y_1| \geqslant 9\theta + 1$.

Proof. Suppose $|Y_1| \leqslant 9\theta$. So $|Y_1| < \eta_1$. Since G has no $K_{s,t}$ subgraph, G has no $K_{s',t}$ subgraph. Applying Lemma 11 and Claim 16.1, either there exists an (η_1, g_1)-bounded L-coloring of G, or there exist $Y_1' \subseteq V(G)$ with $\eta_1 \geqslant |Y_1'| > |Y_1|$ and an $(s', 2, Y_1')$-list-assignment L' of G with $L'(v) \subseteq L(v)$ for every $v \in V(G)$ such that for every L'-coloring of G, every monochromatic component intersecting Y_1 is contained in $G[Y_1']$. Since $\eta_1 \leqslant \eta$ and $g_1 \leqslant g$, every (η_1, g_1)-bounded L-coloring of G is an (η, g)-bounded L-coloring of G, so the former does not hold. Hence there exist $Y_1' \subseteq V(G)$ with $\eta_1 \geqslant |Y_1'| > |Y_1|$ and a restricted $(s', 2, Y_1')$-list-assignment L' of G with $L'(v) \subseteq L(v)$ for every $v \in V(G)$ such that for every L'-coloring of G, every monochromatic component intersecting Y_1 is contained in $G[Y_1']$. Since $|Y_1'| \leqslant \eta_1 \leqslant \eta$, the maximality of $|Y_1|$ implies that there exists an (η, g)-bounded L'-coloring c' of G. So every monochromatic component respect to c' contains at most $\eta^2 g(\eta)$ vertices. Since $L'(v) \subseteq L(v)$ for every $v \in V(G)$, c' is also an L-coloring of G. Every monochromatic c'-component intersecting Y_1 is contained in $G[Y_1']$ and hence contains at most $|Y_1'| \leqslant \eta_1 \leqslant g_1(0) \leqslant g(0) \leqslant |Y_1|^2 g(|Y_1|)$ vertices. So c' is an (η, g)-bounded L-coloring of G, a contradiction. □

Let $\mathcal{T}$ be the set of separations (A, B) of G such that $|V(A \cap B)| < \theta$ and $|V(A) \cap Y_1| \leqslant 3\theta$.

Claim 16.3. *$\mathcal{T}$ is a tangle in G of order θ.*

Proof. Suppose that $\mathcal{T}$ is not a tangle in G of order θ. Note that G has no $K_{s',t}$ subgraph and L is an $(s', 2, Y_1)$-list-assignment of G with $\eta \geqslant |Y_1| \geqslant 9\theta + 1$ by Claim 16.2. Applying Lemma 12 by taking $s = s'$, $t = t$, $\theta = \theta$, $\eta = \eta$, $r = 2$ and $g = g$, there exists an induced subgraph G' of G with $|V(G')| < |V(G)|$, a subset $Y_1' \subseteq V(G')$ with $|Y_1'| \leqslant \eta$, and an $(s', 2, Y_1')$-list-assignment L' of G' with $L'(v) \subseteq L(v)$ for every $v \in V(G)$ such that there exists no (η, g)-bounded L'-coloring of G'. This contradicts the minimality of $|V(G)|$. □

Claim 16.4. *$\mathcal{T}$ controls a $K_{\lfloor \frac{3}{2} d|V(H)| \rfloor}$-minor.*

Proof. Suppose to the contrary that $\mathcal{T}$ does not control a $K_{\lfloor \frac{3}{2} d|V(H)| \rfloor}$-minor. Note that $\theta \geqslant \theta_0$, $\eta \geqslant \eta_0(\theta)$ and $g \geqslant g_0$. Apply Lemma 13 with $s = s'$, $t = t$ and $t' = \lfloor \frac{3}{2} d|V(H)| \rfloor$. Since there does not exist an (η, g)-bounded L-coloring of G, we know there exist $(A^*, B^*) \in \mathcal{T}$, a set Y_{A^*} with $|Y_{A^*}| \leqslant \eta_0(\theta) \leqslant \eta$ and $Y_1 \cap V(A^*) \subseteq Y_{A^*} \subseteq V(A^*)$, and a restricted

$(s', 2, Y_{A^*})$-list-assignment L_{A^*} of $G[V(A^*)]$ such that there exists no (η, g)-bounded L_{A^*}-coloring of $G[V(A^*)]$. But $|V(A^*)| < |V(G)|$ since $|V(A^*) \cap Y_1| < 3\theta < |Y_1|$. This contradicts the minimality of $|V(G)|$. □

Since G contains no subdivision of H, by Theorem 14 and Claim 16.4, there exists $Z \subseteq V(G)$ with $|Z| \leqslant \xi$ such that for every $v \in V(G) - Z$, there exists $(A_v, B_v) \in \mathcal{T} - Z$ of order at most $d-1$ such that $v \in V(A_v) - V(B_v)$.

We may assume that for every $v \in V(G) - Z$,

(i) $(A_v, B_v) \in \mathcal{T} - Z$ has order at most $d-1$ and $v \in V(A_v) - V(B_v)$,
(ii) subject to (i), $A_v - V(A_v \cap B_v)$ is connected,
(iii) subject to (i) and (ii), every vertex in $V(A_v \cap B_v)$ is adjacent to some vertex in $V(A_v) - V(B_v)$,
(iv) subject to (i)–(iii), $V(A_v)$ is maximal,
(v) subject to (i)–(iv), $|V(A_v \cap B_v)|$ is minimal, and
(vi) subject to (i)–(v), A_v is maximal.

Note that for every $v \in V(G) - Z$, A_v is connected and for every two vertices $x, y \in V(A_v)$, there exists a path in A_v from x to y internally disjoint from $V(A_v \cap B_v)$ since $A_v - V(A_v \cap B_v)$ is connected and every vertex in $V(A_v \cap B_v)$ is adjacent to some vertex in $V(A_v) - V(B_v)$

For any subset $\mathcal{C} \subseteq \mathcal{T} - Z$, let $(A_{\mathcal{C}}, B_{\mathcal{C}})$ be the separation $(\bigcup_{(A,B)\in\mathcal{C}} A, \bigcap_{(A,B)\in\mathcal{C}} B)$. Note that $V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq \bigcup_{(A,B)\in\mathcal{C}} V(A \cap B)$, so $|V(A_{\mathcal{C}} \cap B_{\mathcal{C}})| \leqslant |\mathcal{C}|(d-1)$.

Claim 16.5. *Let $\mathcal{C} = \{(A_w, B_w) : w \in W\}$ for some $W \subseteq V(G) - Z$. If x is a vertex in $V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, then $V(A_x \cap B_x) - V(B_w) \neq \emptyset$ for some $w \in V(G) - Z$ with $(A_w, B_w) \in \mathcal{C}$.*

Proof. Since $x \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, there exists $w \in W \subseteq V(G) - Z$ such that $(A_w, B_w) \in \mathcal{C}$ and $x \in V(A_w \cap B_w)$. Suppose to the contrary that $V(A_x \cap B_x) \subseteq V(B_w)$.

First suppose that there exists $v \in V(A_w) - (V(B_w) \cup V(A_x))$. Since $A_w - V(B_w)$ is connected by (ii) and every vertex in $V(A_w \cap B_w)$ is adjacent to a vertex in $V(A_w) - V(B_w)$ by (iii), there exists a path P in $G[(V(A_w) - V(B_w)) \cup \{x\}]$ from x to v. Since $x \in V(A_x) - V(B_x)$ and $v \in V(G) - (Z \cup V(A_x))$, $P - x$ intersects $V(A_x \cap B_x) \subseteq V(B_w)$. But $V(P - x) \subseteq V(A_w) - V(B_w)$, a contradiction. So $V(A_w) - V(B_w) \subseteq V(A_x)$.

Suppose that there exists a vertex $u \in V(A_w \cap B_w) - V(A_x)$. Since $u \in V(A_w \cap B_w)$, there exists $u' \in N_G(u) \cap V(A_w) - V(B_w)$ by (iii). So $u' \in N_G(u) \cap V(A_w) - V(B_w) \subseteq N_G(u) \cap V(A_x)$. Since $u \notin V(A_x)$, $u' \in V(A_x \cap B_x) \cap V(A_w) - V(B_w)$, contradicting the assumption $V(A_x \cap B_x) \subseteq V(B_w)$. Hence $V(A_w \cap B_w) \subseteq V(A_x)$.

Therefore, $V(A_w) \subseteq V(A_x)$. By (v), every vertex in $V(A_x \cap B_x)$ is adjacent to some vertex in $V(B_x) - V(A_x)$. So if $V(A_x) = V(A_w)$, then $V(A_x \cap B_x) \subseteq V(A_w \cap B_w)$, and since (A_w, B_w) satisfies (v), $V(B_w) = V(B_x)$. Hence if $V(A_x) = V(A_w)$, then $(A_x, B_x) = (A_w, B_w)$ by (vi). Since $x \in V(A_x) - V(B_x)$ and $x \in V(A_w \cap B_w)$, $(A_x, B_x) \neq (A_w, B_w)$. So $V(A_w) \subset V(A_x)$. Since (A_w, B_w) satisfies (iv), $w \in V(B_x)$. Since $V(A_x \cap B_x) \subseteq V(B_w)$ and $w \notin V(B_w)$, $w \in V(B_x) - V(A_x)$. So $V(A_w) \not\subseteq V(A_x)$, a contradiction. □

Let $Z' := \{v \in V(G) - (Y_1 \cup Z) : |N_G(v) \cap Z| \geqslant s\}$. Note that $|Z'| \leqslant f(|Z|) \leqslant f(\xi)$ by Lemma 7.

We say that a triple $(\mathcal{C}, S, T)$ is *useful* if the following hold:

(U1) There exists $W \subseteq V(G) - Z$ such that $\mathcal{C} = \{(A_v, B_v) : v \in W\}$.

(U2) $N_G[N_G[Z']] \cap V(B_{\mathcal{C}}) = \emptyset$.

(U3) S is a subset of $N_G[V(A_{\mathcal{C}} \cap B_{\mathcal{C}})] \cap V(A_{\mathcal{C}})$ and T is a subset of $Y_1 \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$ such that there exists a bijection ι from a subset of $Y_1 \cap V(A_{\mathcal{C}})$ to S such that:

- $|S|+|T|+|Z|+1 \leqslant |Y_1 \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})| + |\{y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) : \iota(y) = y\}|$, and
- for every vertex y in the domain of ι,
  - if $y \in V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$ and there exists a vertex $v \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - S$, then $\iota(y) \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, and
  - if $y \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, then $\iota(y) = y$.

(U4) T is disjoint from Z' and the domain of ι.

Claim 16.6. *There exists a collection $\mathcal{C}$ of members of $\mathcal{T} - Z$ with $|\mathcal{C}| \leqslant |Z'|d^2 + |Z| + 1$ such that $(\mathcal{C}, \emptyset, \emptyset)$ is useful.*

Proof. For every $u \in V(G) - Z$, let $\mathcal{C}_u := \{(A_u, B_u), (A_v, B_v) : v \in N_G(u) \cap V(B_u)\}$. Note that $|N_G(u) \cap V(B_u)| \leqslant |V(A_u \cap B_u)| \leqslant d - 1$ since $u \in V(A_u) - V(B_u)$. So $|\mathcal{C}_u| \leqslant d$. Note that $N_G[\{u\}] \cap V(B_{\mathcal{C}_u}) = \emptyset$.

For every $u \in V(G) - Z$, let $\mathcal{C}'_u := \mathcal{C}_u \cup \{(A_v, B_v) : v \in N_G(N_G[\{u\}]) \cap V(B_{\mathcal{C}_u})\}$. Note that $|N_G(N_G[\{u\}]) \cap V(B_{\mathcal{C}_u})| \leqslant |V(A_{\mathcal{C}_u} \cap B_{\mathcal{C}_u})| \leqslant (d-1)|\mathcal{C}_u| \leqslant (d-1)d$. So $|\mathcal{C}'_u| \leqslant |\mathcal{C}_u| + (d-1)d \leqslant d^2$. Note that $N_G[N_G[\{u\}]] \cap V(B_{\mathcal{C}'_u}) = \emptyset$.

Let $\mathcal{C}' := \bigcup_{z \in Z'} \mathcal{C}'_z$. Then $N_G[N_G[Z']] \cap V(B_{\mathcal{C}'}) = \emptyset$. And $|\mathcal{C}'| \leqslant |Z'|d^2$. Since $|Y_1 - Z| \geqslant |Y_1| - |Z| \geqslant 9\theta - \xi \geqslant 8\theta > |Z|$, there exists a subset Y of $Y_1 - Z$ with $|Y| = |Z| + 1$. Let $\mathcal{C} := \mathcal{C}' \cup \{(A_y, B_y) : y \in Y\}$. Clearly $(\mathcal{C}, \emptyset, \emptyset)$ satisfies (U1) and (U4). Since $B_{\mathcal{C}'} \supseteq B_{\mathcal{C}}$, $(\mathcal{C}, \emptyset, \emptyset)$ satisfies (U2). Since $Y \subseteq V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$, $|Z| + 1 = |Y| \leqslant |Y_1 \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})|$, so $(\mathcal{C}, \emptyset, \emptyset)$ satisfies (U3). Note that $|C| \leqslant |\mathcal{C}'| + |Y| \leqslant |Z'|d^2 + |Z| + 1$. □

For a useful triple $(\mathcal{C}, S, T)$, a vertex v of $V(G) - Z$ is:

- $(\mathcal{C}, S, T)$-*dangerous* if $v \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - S$ and there exists $v' \in N_G(v) \cap V(A_{\mathcal{C}}) - (V(B_{\mathcal{C}}) \cup S)$ such that either:
  - $v' \notin Y_1$ and $|((Y_1 \cap V(A_{\mathcal{C}})) \cup (S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}}))) \cap N_G(v')| \geqslant 2d - 4$, or
  - $v' \in Y_1 - T$,
- $(\mathcal{C}, S, T)$-*heavy* if $v \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - S$ and

$$|N_G(v) \cap ((Y_1 \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})) \cup (S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})))| \geqslant d - 1.$$

Claim 16.7. *Let $(\mathcal{C}, S, T)$ be a useful triple and let $x \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$ be a $(\mathcal{C}, S, T)$-heavy vertex. Then there exists a useful triple $(\mathcal{C}', S', T')$ with $\mathcal{C}' = \mathcal{C} \cup \{(A_x, B_x)\}$, such that:*

- $V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S' \subseteq V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - S$,
- *the set of* $(\mathcal{C}', S', T')$*-heavy vertices is strictly contained in the set of* $(\mathcal{C}, S, T)$*-heavy vertices, and*
- *the set of* $(\mathcal{C}', S', T')$*-dangerous vertices is a subset of the set of* $(\mathcal{C}, S, T)$*-dangerous vertices.*

Proof. Let $\mathcal{C}' := \mathcal{C} \cup \{(A_x, B_x)\}$. Let $X := N_G(x) \cap (Y_1 \cup S) \cap V(A_x \cap A_{\mathcal{C}}) - V(B_x \cup B_{\mathcal{C}})$. Since $x \in V(A_x) - V(B_x)$, $N_G(x) \subseteq V(A_x)$. So $|X| \geqslant |N_G(x) \cap (Y_1 \cup S) \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})| - |V(A_x \cap B_x \cap A_{\mathcal{C}}) - V(B_{\mathcal{C}})| \geqslant d - 1 - |V(A_x \cap B_x \cap A_{\mathcal{C}}) - V(B_{\mathcal{C}})|$ since x is $(\mathcal{C}, S, T)$-heavy. That is, $|V(A_x \cap B_x) - V(B_{\mathcal{C}})| = |V(A_x \cap B_x \cap A_{\mathcal{C}}) - V(B_{\mathcal{C}})| \geqslant d - 1 - |X|$. Since x is $(\mathcal{C}, S, T)$-heavy, $x \notin S$. Let ι be a bijection mentioned in (U3) witnessing that $(\mathcal{C}, S, T)$ is useful. Let X' be the intersection of X and the domain of ι.

For each $y \in X'$, since $x \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - S$, $\iota(y) \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$ by (U3). Since $X' \subseteq X \subseteq V(A_x \cap A_{\mathcal{C}}) - V(B_x \cup B_{\mathcal{C}})$, for each $y \in X'$, $N_G(y) \subseteq V(A_x)$, so $\iota(y) \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq V(A_x) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq (V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})) \cup V(A_{\mathcal{C}} \cap B_{\mathcal{C}} \cap A_x \cap B_x)$. Let

$$\begin{aligned} Z_1 &:= V(A_x \cap B_x) - V(A_{\mathcal{C}}) \\ Z_2 &:= V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}}) - \{\iota(y) : y \in X'\} \\ Z_3 &:= V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cap \{\iota(y) : y \in X'\}. \end{aligned}$$

So $\{Z_1, Z_2, Z_3\}$ is a partition of $V(A_x \cap B_x \cap B_{\mathcal{C}})$, and hence

$$\begin{aligned} |Z_1 \cup Z_2 \cup Z_3| &= |V(A_x \cap B_x \cap B_{\mathcal{C}})| \\ &= |V(A_x \cap B_x)| - |V(A_x \cap B_x) - V(B_{\mathcal{C}})| \\ &\leqslant d - 1 - |V(A_x \cap B_x) - V(B_{\mathcal{C}})|. \end{aligned}$$

Recall that $|V(A_x \cap B_x) - V(B_{\mathcal{C}})| \geqslant d - 1 - |X|$. So $|Z_1 \cup Z_2 \cup Z_3| \leqslant (d-1) - (d-1-|X|) = |X|$.

Let

$$S' := \big(S \cap V(B_x) - V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}})\big) \cup \big(V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})\big) \cup V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}}).$$

Note that $(V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cup V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq V(A_x \cap B_x \cap B_{\mathcal{C}}) = Z_1 \cup Z_2 \cup Z_3$. So

$$\begin{aligned} |S'| &\leqslant (|S \cap V(B_x)| - |S \cap V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}})|) + |Z_1 \cup Z_2 \cup Z_3| \\ &\leqslant |S \cap V(B_x)| - |\{y \in X' : \iota(y) \in Z_3\}| + |X| \\ &= |S \cap V(B_x)| + |X - \{y \in X' : \iota(y) \in Z_3\}| \\ &= |\{y \in Y_1 \cap V(A_{\mathcal{C}}) : \iota(y) \in S \cap V(B_x)\}| + |X - X'| + |X' - \{y \in X' : \iota(y) \in Z_3\}|. \end{aligned}$$

Recall that for every $y \in X'$, $\iota(y) \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cap V(A_x)$. So if $y \in X' - \{y \in X' : \iota(y) \in Z_3\}$, then

$$\iota(y) \in N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cap V(A_x) - V(A_x \cap B_x \cap A_{\mathcal{C}} \cap B_{\mathcal{C}}) = N_G(y) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}} \cap A_x) - V(B_x),$$

so $\iota(y) \notin S \cap V(B_x)$. That is, $\{y \in Y_1 \cap V(A_{\mathcal{C}}) : \iota(y) \in S \cap V(B_x)\}$ and $X' - \{y \in X' : \iota(y) \in Z_3\}$ are disjoint. Note that $X - X'$ is disjoint from the domain of ι. So $\{y \in Y_1 \cap V(A_{\mathcal{C}}) : \iota(y) \in S \cap V(B_x)\}$, $X - X'$ and $X' - \{y \in X' : \iota(y) \in Z_3\}$ are pairwise disjoint sets. Therefore,

$$|S'| \leqslant |\{y \in Y_1 \cap V(A_{\mathcal{C}}) : \iota(y) \in S \cap V(B_x)\} \cup (X - \{y \in X' : \iota(y) \in Z_3\})|.$$

Since $X \subseteq Y_1 \cup S$ and $X \cap V(B_x) = \emptyset$, for every $x \in X - \{y \in X' : \iota(y) \in Z_3\}$, if $x \notin X \cap Y_1 - \{y \in X' : \iota(y) \in Z_3\}$, then $x \in X \cap S - Y_1$ and $\iota(y) = x$ for some $y \in Y_1 \cap V(A_{\mathcal{C}})$ such that $\iota(y) \notin S \cap V(B_x)$. In addition, if y is a vertex in $Y_1 \cap V(A_{\mathcal{C}})$ such that $\iota(y) \in X \cap S - Y_1$, then $\iota(y) \notin S \cap V(B_x)$.

Since $|S'| \leqslant |\{y \in Y_1 \cap V(A_{\mathcal{C}}) : \iota(y) \in S \cap V(B_x)\} \cup (X - \{y \in X' : \iota(y) \in Z_3\})|$, there exists an injection ι' such that

- $\iota'(y) = \iota(y)$ if y is in the domain of ι and $\iota(y) \in S \cap V(B_x)$,
- for each $v \in S' - (S \cap V(B_x))$, there exists exactly one element $y \in (X \cap Y_1 - \{y \in X' : \iota(y) \in Z_3\}) \cup \{y \in Y_1 \cap V(A_{\mathcal{C}}) : \iota(y) \in X \cap S - Y_1\}$ such that $\iota'(y) = v$, and
- if $\iota(y_1) = \iota'(y_2)$ for some y_1, y_2, then $y_1 = y_2$.

Recall that $\iota(y) \notin S \cap V(B_x)$ for every $y \in (X \cap Y_1 - \{y \in X' : \iota(y) \in Z_3\}) \cup \{y \in Y_1 : \iota(y) \in X \cap S - Y_1\}$. Then ι' is a bijection from a subset of $Y_1 \cap V(A_{\mathcal{C}'})$ to S'. We further modify ι' and S' by applying the following operations for some vertex $y \in V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})$ in the domain of ι' with $\iota'(y) \notin N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$ and $N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S' \neq \emptyset$, and then repeating until no such vertex y exists:

- add a vertex $v \in N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S'$ into S',
- delete $\iota'(y)$ from S', and
- redefine $\iota'(y)$ to be v."

Now, further modify ι' and S' by applying the following operations for some vertex $z \in S' - N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})]$, and repeating until no such vertex z exists:

- remove z from S', and
- if y is the element in the domain of ι' with $\iota'(y) = z$, then remove y from the domain of ι'.

Notice that for each vertex z removed from S' in the above procedure, $z \in S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$ and $N_G(z) \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(B_x)$. Note that ι' remains a bijection from a subset of $Y_1 \cap V(A_{\mathcal{C}'})$ to S'.

Observe that for every y in the domain of ι' with $y \in V(A'_{\mathcal{C}}) - V(B'_{\mathcal{C}})$ and $N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S' \neq \emptyset$, $\iota'(y) \in N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$ due to the above modification. In addition, if y is in the domain of ι' and $y \in V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$, then $y \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$ and y is in the domain of ι such that $\iota(y) = \iota'(y)$, so $\iota'(y) = \iota(y) = y$.

Let T' be the set obtained from T by deleting the domain of ι'. So T' is disjoint from the domain of ι'. Since T is disjoint from Z', T' is disjoint from Z'. So $(\mathcal{C}', S', T')$ satisfies

(U4). In addition, $|S'| - |S|$ is at most the number of vertices in X and in the domain of ι' but not in the domain of ι. So $|S'| - |S| \leqslant |T| - |T'|$. Hence $|S'| + |T'| \leqslant |S| + |T|$. Since $\{y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) : \iota(y) = y\} - \{y \in Y_1 \cap S' \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) : \iota'(y) = y\} \subseteq (V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})) - (V(A_{\mathcal{C}}) - V(B_{\mathcal{C}}))$, $(\mathcal{C}', S', T')$ satisfies (U3) and is useful.

It is easy to see that $V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S' \subseteq V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - S$. Note that each vertex $v \in V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S'$ belongs to $V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(A_x)$, so $N_G(v) \cap V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'}) = N_G(v) \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$. Furthermore, $S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) \subseteq S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$. Hence every $(\mathcal{C}', S', T')$-heavy vertex is $(\mathcal{C}, S, T)$-heavy. Since x is $(\mathcal{C}, S, T)$-heavy but not $(\mathcal{C}', S', T')$-heavy, the set of $(\mathcal{C}', S', T')$-heavy vertices is strictly contained in the set of $(\mathcal{C}, S, T)$-heavy vertices.

Let v be a $(\mathcal{C}', S', T')$-dangerous vertex and let v' be a vertex in $N_G(v) \cap V(A_{\mathcal{C}'}) - (V(B_{\mathcal{C}'}) \cap S')$ witnessing the definition of being dangerous. Since $v \notin S'$, $v \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(A_x)$, so $v' \in N_G[V(A_{\mathcal{C}} \cap B_{\mathcal{C}})] \cap V(A_{\mathcal{C}}) \cap V(B_x)$. So $v' \in V(B_x) - (V(B_{\mathcal{C}}) \cup S')$. Since $v \in V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$, $v' \in N_G(v) \cap N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})]$. Since $v' \notin S'$ and $v' \notin V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$ and $v' \in V(B_x)$ and $N_G(v') \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(A_x) \neq \emptyset$, we know $v' \notin S$ by the procedure of modifying S. So $v' \in (N_G(v) \cap V(A_{\mathcal{C}}) - (V(B_{\mathcal{C}}) \cup S)) \cap V(B_x)$. Note that $T - T' \subseteq V(A_x) - V(B_x)$. So if $v' \in Y_1 - T'$, then $v' \in Y_1 - T$ and v is $(\mathcal{C}, S, T)$-dangerous. Furthermore, $Y_1 \cap V(A_{\mathcal{C}}) \cap N_G(v') = Y_1 \cap V(A_{\mathcal{C}'}) \cap N_G(v')$ and $S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) \subseteq S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, so v is $(\mathcal{C}, S, T)$-dangerous. Therefore, the set of $(\mathcal{C}', S', T')$-dangerous vertices is a subset of the set of $(\mathcal{C}, S, T)$-dangerous vertices. This proves the claim. □

Claim 16.8. *Let $(\mathcal{C}, S, T)$ be a useful triple. Then there exists a set S' with $S \cup (Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})) \subseteq S' \subseteq N_G[V(A_{\mathcal{C}} \cap B_{\mathcal{C}})] \cap V(A_{\mathcal{C}})$ such that $(\mathcal{C}, S', T)$ is a useful triple and:*

- *If ι' is the bijection witnessing that $(\mathcal{C}, S', T)$ satisfies (U3), then for every $y \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, the unique element of the domain of ι' mapped to y by ι' is y.*
- *The set of $(\mathcal{C}, S', T)$-dangerous vertices is contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices.*
- *The set of $(\mathcal{C}, S', T)$-heavy vertices is contained in the set of $(\mathcal{C}, S, T)$-heavy vertices.*

Proof. Let ι be a function mentioned in (U3) witnessing that $(\mathcal{C}, S, T)$ is a useful triple. We may assume that $Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq S$, since if some vertex $y \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$ does not belong to S, then y is not in the domain of ι, and we can define $\iota(y) = y$ without violating (U3) and (U4) such that the set of dangerous vertices and the set of heavy vertices remain the same.

Since ι is a bijection, we write the element mapped to y by ι as $\iota^{(-1)}(y)$. Modify ι and S by applying the following operations to some vertex $y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$ with $\iota^{(-1)}(y) \neq y$, and repeat until no such y exists:

- remove $\iota^{(-1)}(y)$ from the domain of ι,
- define $\iota(y) := y$,

Then define S' and ι' to be the modified S and ι, respectively. Clearly, $(\mathcal{C}, S', T)$ satisfies (U3), $S \subseteq S' \subseteq N_G[V(A_{\mathcal{C}} \cap B_{\mathcal{C}})] \cap V(A_{\mathcal{C}})$, and $\iota'(y) = y$ for every $y \in Y_1 \cap S' \cap$

$V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$. Since we assume that $Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq S$, we have $S \cup (Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})) \subseteq S' \subseteq N_G[V(A_{\mathcal{C}} \cap B_{\mathcal{C}})] \cap V(A_{\mathcal{C}})$ and $\iota'(y) = y$ for every $y \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$. Since $T \subseteq V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$, $(\mathcal{C}, S', T)$ satisfies (U4). Since $S' - S \subseteq V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, the set of $(\mathcal{C}, S', T)$-dangerous vertices is contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices, and the set of $(\mathcal{C}, S', T)$-heavy vertices is contained in the set of $(\mathcal{C}, S, T)$-heavy vertices. □

Claim 16.9. *Let $(\mathcal{C}, S, T)$ be a useful triple, and let x be a $(\mathcal{C}, S, T)$-dangerous vertex. If there exists no $(\mathcal{C}, S, T)$-heavy vertex, then there exists a useful triple $(\mathcal{C}', S', T')$ with $\mathcal{C}' = \mathcal{C} \cup \{(A_x, B_x)\}$ such that the set of $(\mathcal{C}', S', T')$-dangerous vertices is strictly contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices.*

Proof. By Claim 16.8, we may assume that $Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq S$ and the function ι mentioned in (U3) witnessing that $(\mathcal{C}, S, T)$ is useful satisfies $\iota(y) = y$ for every $y \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$. Let $\mathcal{C}' := \mathcal{C} \cup \{(A_x, B_x)\}$.

We first assume that $|Y_1 \cap V(A_x \cap B_{\mathcal{C}})| \geqslant d - 2$. So $|Y_1 \cap V(A_x \cap B_{\mathcal{C}})| \geqslant |V(A_x \cap B_x) \cap V(B_{\mathcal{C}})|$ by Claim 16.5. Hence there exists a function ι' whose domain is a subset of $Y_1 \cap V(A_{\mathcal{C}} \cup A_x)$ such that:

- $\iota'(y) = \iota(y)$ for every $y \in Y_1 \cap V(A_{\mathcal{C}}) - V(A_x \cap B_{\mathcal{C}})$ belonging to the domain of ι with $\iota(y) \in V(A_{\mathcal{C}}) \cap N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})] - V(A_x)$, and
- for each vertex v in $V(A_x \cap B_x) \cap V(B_{\mathcal{C}})$, there exists exactly one element $y \in Y_1 \cap V(A_x \cap B_{\mathcal{C}})$ such that $\iota'(y) = v$ and if $v \in Y_1$, then $y = v$.

Let $S' := (S \cap N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})] - V(A_x)) \cup V(A_x \cap B_x \cap B_{\mathcal{C}})$. So ι' is a bijection from a subset of $Y_1 \cap V(A_{\mathcal{C}} \cup A_x)$ to S'. Note that every vertex in $S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$ is contained in $S - (V(A_x) \cup V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))$, so it is adjacent to some vertex in $V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})$.

Let $T' := T$. Since ι satisfies (U3) and $\iota(y) = y$ for every $y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, we know that ι' satisfies (U3). Since $T' = T \subseteq V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$, $(\mathcal{C}', S', T')$ satisfies (U4). So $(\mathcal{C}', S', T')$ is a useful triple.

Let v be a $(\mathcal{C}', S', T')$-dangerous vertex. So $v \in V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S' \subseteq V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(A_x)$. Let v' be a vertex witnessing that v is $(\mathcal{C}', S', T')$-dangerous. So $v' \in V(B_x) - (V(B_{\mathcal{C}}) \cup S')$ and $N_G(v') \subseteq V(A_{\mathcal{C}})$. Since $v \in V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - V(A_x)$, $v' \in N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})]$, so $v' \notin S$. Since $S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) \subseteq S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, if $v' \notin Y_1$ and $|((Y_1 \cap V(A_{\mathcal{C}'})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))) \cap N_G(v')| \geqslant 2d-4$, then $v' \notin Y_1$ and $|((Y_1 \cap V(A_{\mathcal{C}})) \cup (S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}}))) \cap N_G(v')| \geqslant 2d - 4$, so v is $(\mathcal{C}, S, T)$-dangerous. Since $T' = T$, if $v' \in Y_1 - T'$, then $v' \in Y_1 - T$ and v is $(\mathcal{C}, S, T)$-dangerous. So every $(\mathcal{C}', S', T')$-dangerous vertex is $(\mathcal{C}, S, T)$-dangerous. Since x is $(\mathcal{C}, S, T)$-dangerous but not $(\mathcal{C}', S', T')$-dangerous, the set of $(\mathcal{C}', S', T')$-dangerous vertices is strictly contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices. So the claim holds.

Hence we may assume that $|Y_1 \cap V(A_x \cap B_{\mathcal{C}})| \leqslant d - 3$.

Modify S and define ι' to be the function obtained from ι by applying the following operations to a vertex y in the domain of ι with $\iota(y) \notin N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})] \cap V(A_{\mathcal{C}'})$, and repeating until no such y exists:

- if $y \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(B_x)$ or $V(A_x \cap B_x) \cap V(B_{\mathcal{C}}) - S = \emptyset$, then remove y from the domain of ι and remove $\iota(y)$ from S,

- if $y \notin V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(B_x)$ and $N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S = \emptyset$ and $V(A_x \cap B_x) \cap V(B_{\mathcal{C}}) - S \neq \emptyset$, then redefine $\iota(y)$ to be an element in $V(A_x \cap B_x) \cap V(B_{\mathcal{C}}) - S$ and add this element into S,
- otherwise remove $\iota(y)$ from S, redefine $\iota(y)$ to be an element in $N_G(y) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) - S$ and add this element into S.

Let S' be the modified S, and let

$$T' := T \cup (Y_1 \cap V(B_{\mathcal{C}}) - V(A_{\mathcal{C}} \cup B_x)) \cup (Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - (V(B_x) \cup N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})])).$$

Clearly, T' is disjoint from the domain of ι'. By (U2), $Z' \cap V(B_{\mathcal{C}}) = \emptyset$. So T' is disjoint from Z' as T is disjoint from Z'. So $(\mathcal{C}', S', T')$ satisfies (U4).

Since $Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq S$, we know $Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - (V(B_x) \cup N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})]) \subseteq S$. For every $y \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - (V(B_x) \cup N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})])$, since $\iota(y) = y \notin N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})] \cap V(A_{\mathcal{C}'})$ and $y \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(B_x)$, $y \in S - S'$.

So $|S| \geqslant |S'| + |Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - (V(B_x) \cup N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})])|$. Hence $|S'| + |T'| \leqslant |S| + |T| + |Y_1 \cap V(B_{\mathcal{C}}) - V(A_{\mathcal{C}} \cup B_x)|$. Since $Y_1 \cap V(B_{\mathcal{C}}) - V(A_{\mathcal{C}} \cup B_x) \subseteq Y_1 \cap (V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})) - V(A_{\mathcal{C}})$ and $(\mathcal{C}, S, T)$ satisfies (U3), we know

$$\begin{aligned}
&|S'| + |T'| + |Z| + 1 \\
\leqslant\ &|S| + |T| + |Y_1 \cap V(B_{\mathcal{C}}) - V(A_{\mathcal{C}} \cup B_x)| + |Z| + 1 \\
\leqslant\ &|Y_1 \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})| + |\{y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) : \iota(y) = y\}| \\
&\quad + |Y_1 \cap (V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})) - V(A_{\mathcal{C}})| \\
\leqslant\ &|Y_1 \cap V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})| - |Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(B_x)| \\
&\quad + |\{y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) : \iota(y) = y\}| \\
\leqslant\ &|Y_1 \cap V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})| + |\{y \in Y_1 \cap S \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cap V(B_x) : \iota(y) = y\}| \\
\leqslant\ &|Y_1 \cap V(A_{\mathcal{C}'}) - V(B_{\mathcal{C}'})| + |\{y \in Y_1 \cap S' \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) : \iota'(y) = y\}|.
\end{aligned}$$

Hence $(\mathcal{C}', S', T')$ satisfies (U3). Therefore $(\mathcal{C}', S', T')$ is useful.

Suppose that the set of $(\mathcal{C}', S', T')$-dangerous vertices is not strictly contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices. Since x is $(\mathcal{C}, S, T)$-dangerous but not $(\mathcal{C}', S', T')$-dangerous, there exists a vertex v that is $(\mathcal{C}', S', T')$-dangerous but not $(\mathcal{C}, S, T)$-dangerous. So there exists a vertex $v' \in N_G(v) \cap V(A_{\mathcal{C}'}) - (V(B_{\mathcal{C}'}) \cup S')$ such that either $v' \in Y_1 - T'$, or $v' \notin Y_1$ and $|((Y_1 \cap V(A_{\mathcal{C}'})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})) \cap N_G(v')| \geqslant 2d - 4$. Since $v' \in N_G[V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})] \cap V(A_{\mathcal{C}'})$, if v' belongs to S at beginning, then v' is not removed from S during the process of modifying S, so $v' \in S'$, a contradiction. So $v' \notin S$.

Suppose that $v' \in V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$. So $v \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cap V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) \subseteq V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \cap V(B_x)$. Hence if v belongs to S at beginning, then v is not removed from S during the process of modifying S, so $v \in S'$. Since v is $(\mathcal{C}', S', T')$-dangerous, $v \notin S'$, so $v \notin S$. Since v is not $(\mathcal{C}, S, T)$-dangerous, $v' \notin Y_1 - T$, and either $v' \in Y_1$ or $|((Y_1 \cap V(A_{\mathcal{C}})) \cup (S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}}))) \cap N_G(v')| < 2d-4$. Since $T' \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}}) = T \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})$, $v' \notin Y_1 - T'$. Since v is $(\mathcal{C}', S', T')$-dangerous, $v' \notin Y_1$ and $|((Y_1 \cap V(A_{\mathcal{C}'})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))) \cap N_G(v')| \geqslant$

$2d-4$. Since $N_G(v') \subseteq V(A_{\mathcal{C}})$ and $S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}) \subseteq S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$,

$$\begin{aligned} 2d-4 &\leqslant |((Y_1 \cap V(A_{\mathcal{C}'})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))) \cap N_G(v')| \\ &= |((Y_1 \cap V(A_{\mathcal{C}})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))) \cap N_G(v')| \\ &\leqslant |((Y_1 \cap V(A_{\mathcal{C}})) \cup (S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}}))) \cap N_G(v')| \\ &< 2d-4, \end{aligned}$$

a contradiction.

Therefore, $v' \in V(B_{\mathcal{C}})$. So $v' \in (V(A_x) - V(B_x)) \cap V(B_{\mathcal{C}})$ and hence $N_G(v') \subseteq V(A_x)$. Since $Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) \subseteq S$, if $v' \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, then $v' \in S$, a contradiction. So $v' \notin Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$. Since $Y_1 \cap V(B_{\mathcal{C}}) - V(A_{\mathcal{C}} \cup B_x) \subseteq T'$, if $v' \in Y_1 - T'$, then $v' \in Y_1 \cap V(A_{\mathcal{C}} \cap B_{\mathcal{C}})$, a contradiction. So $v' \notin Y_1 - T'$. Since v is $(\mathcal{C}', S', T')$-dangerous, $|((Y_1 \cap V(A_{\mathcal{C}'})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))) \cap N_G(v')| \geqslant 2d-4$. Since $|Y_1 \cap V(A_x \cap B_{\mathcal{C}})| \leqslant d-3$ and $(S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'})) - (V(A_{\mathcal{C}}) \cup V(B_x)) = \emptyset$ and $N_G(v') \subseteq V(A_x)$,

$$\begin{aligned} &|N_G(v') \cap ((Y_1 \cap V(A_{\mathcal{C}}) - V(B_{\mathcal{C}})) \cup (S - V(A_{\mathcal{C}} \cap B_{\mathcal{C}}))| \\ \geqslant\ &|N_G(v') \cap ((Y_1 \cap V(A_x) - V(B_{\mathcal{C}})) \cup (S' - V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))| \\ \geqslant\ &|N_G(v') \cap ((Y_1 \cap V(A_x)) \cup (S' - (V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))))| - (d-3) \\ =\ &|N_G(v') \cap (Y_1 \cup (S' - (V(A_{\mathcal{C}'} \cap B_{\mathcal{C}'}))))| - (d-3) \\ \geqslant\ &(2d-4) - (d-3) \\ =\ &d-1. \end{aligned}$$

In particular, $v' \in V(A_{\mathcal{C}} \cap B_{\mathcal{C}}) - V(B_x)$. Since there exists no $(\mathcal{C}, S, T)$-heavy vertex, v' is not a $(\mathcal{C}, S, T)$-heavy vertex. So $v' \in S$, a contradiction. This proves the claim. □

Claim 16.10. *If $(\mathcal{C}, S, T)$ is a useful triple such that there exists a $(\mathcal{C}, S, T)$-dangerous vertex, then there exists a useful triple $(\mathcal{C}', S', T')$ with $\mathcal{C} \subseteq \mathcal{C}'$ and $|\mathcal{C}'| \leqslant |\mathcal{C}| + |V(A_{\mathcal{C}} \cap B_{\mathcal{C}})| + 1$ such that the set of $(\mathcal{C}', S', T')$-dangerous vertices is strictly contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices.*

Proof. Note that there are at most $|V(A_{\mathcal{C}} \cap B_{\mathcal{C}})|$ $(\mathcal{C}, S, T)$-heavy vertices. By repeatedly applying Claim 16.7 at most $|V(A_{\mathcal{C}} \cap B_{\mathcal{C}})|$ times, there exists a useful triple $(\mathcal{C}_1, S_1, T_1)$ with $\mathcal{C} \subseteq \mathcal{C}_1$ and $|\mathcal{C}_1| \leqslant |\mathcal{C}| + |V(A_{\mathcal{C}} \cap B_{\mathcal{C}})|$ such that there exists no $(\mathcal{C}_1, S_1, T_1)$-heavy vertices, and the set of $(\mathcal{C}_1, S_1, T_1)$-dangerous vertices is contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices. By Claim 16.9 applied to $\mathcal{C}_1$, there exists a useful triple $(\mathcal{C}', S', T')$ with $\mathcal{C}_1 \subseteq \mathcal{C}'$ and $|\mathcal{C}'| = |\mathcal{C}_1| + 1 \leqslant |\mathcal{C}| + |V(A_{\mathcal{C}} \cap B_{\mathcal{C}})| + 1$ such that the set of $(\mathcal{C}', S', T')$-dangerous vertices is strictly contained in the set of $(\mathcal{C}_1, S_1, T_1)$-dangerous vertices and hence is strictly contained in the set of $(\mathcal{C}, S, T)$-dangerous vertices. This proves the claim. □

Claim 16.11. *There exists a useful triple $(\mathcal{C}^*, S^*, T^*)$ with $|\mathcal{C}^*| \leqslant a_{(d-1)a_0}$ such that there exists no $(\mathcal{C}^*, S^*, T^*)$-dangerous vertex.*

Proof. By Claim 16.6, there exists a useful triple $(\mathcal{C}_0, \emptyset, \emptyset)$ with $|\mathcal{C}_0| \leqslant |Z'|d^2 + |Z| + 1$. Let $S_0 = \emptyset$ and $T_0 = \emptyset$. So $(\mathcal{C}_0, S_0, T_0)$ is a useful triple with $|\mathcal{C}_0| \leqslant f(\xi)d^2 + \xi + 1 = a_0$. For $i \geqslant 1$, if there exists a $(\mathcal{C}_{i-1}, S_{i-1}, T_{i-1})$-dangerous vertex, then by Claim 16.10, there exists a useful triple $(\mathcal{C}_i, S_i, T_i)$ such that $|\mathcal{C}_i| \leqslant |\mathcal{C}_{i-1}| + |V(A_{\mathcal{C}_{i-1}} \cap B_{\mathcal{C}_{i-1}})| + 1$ and the set of $(\mathcal{C}_i, S_i, T_i)$-dangerous vertices is strictly contained in the set of $(\mathcal{C}_{i-1}, S_{i-1}, T_{i-1})$-dangerous vertices. So $|\mathcal{C}_i| \leqslant a_{i-1} + (d-1)a_{i-1} + 1 \leqslant a_i$ for each $i \geqslant 1$ by induction on i. Since there are at most $|V(A_{\mathcal{C}_0} \cap B_{\mathcal{C}_0})| \leqslant |\mathcal{C}_0|(d-1) \leqslant (d-1)a_0$ $(\mathcal{C}_0, S_0, T_0)$-dangerous vertices. Hence there exists i^* with $0 \leqslant i^* \leqslant (d-1)a_0$ such that $(\mathcal{C}_{i^*}, S_{i^*}, T_{i^*})$ is a useful triple with no $(\mathcal{C}_{i^*}, S_{i^*}, T_{i^*})$-dangerous vertex. Note that $|\mathcal{C}_{i^*}| \leqslant a_{i^*} \leqslant a_{(d-1)a_0}$. □

Let ι^* be the function mentioned in (U3) witnessing that $(\mathcal{C}^*, S^*, T^*)$ is useful. By Claim 16.8, we may assume that $Y_1 \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) \subseteq S^*$ such that $\iota^*(y) = y$ for every $y \in Y_1 \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}*})$.

Define the following:

$$G_B := G[(\bigcap_{(A,B)\in\mathcal{C}^*} V(B)) \cup (Z \cup S^* \cup T^*)]$$
$$Y_B := (Y_1 \cap V(B_{\mathcal{C}^*})) \cup (Z \cup S^* \cup T^*).$$

Claim 16.12. *For every vertex $v \in V(G_B) - Y_B$, $N_G(v) \cap Y_1 \subseteq N_{G_B}(v) \cap Y_B$.*

Proof. Suppose to the contrary that there exist $v \in V(G_B) - Y_B$ and $y \in N_G(v) \cap Y_1 - (N_{G_B}(v) \cap Y_B)$. Since $y \in Y_1 - Y_B$, $y \in V(A_{\mathcal{C}^*}) - V(B_{\mathcal{C}^*})$. So $v \in V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) - S^*$. Since there exists no $(\mathcal{C}^*, S^*, T^*)$-dangerous vertex, v is not a $(\mathcal{C}^*, S^*, T^*)$-dangerous vertex. Since $y \in N_G(v) \cap V(A_{\mathcal{C}^*}) - (V(B_{\mathcal{C}^*}) \cup S^*)$, $y \notin Y_1 - T^*$. Since $y \in Y_1$, $y \in T^*$. So $y \in Y_B$, a contradiction. □

Define the following:

- For every $y \in Y_B$, let $L_B(y)$ be a 1-element subset of $L(y)$.
- For every $v \in V(G_B) - Y_B$ with $|N_{G_B}(v) \cap Y_B| \in [s'-1]$, let $L_B(v)$ be a subset of $L(v)$ with size $s' + 2 - |N_{G_B}(v) \cap Y_B|$ such that $L_B(v) \cap L_B(u) = \emptyset$ for every $u \in N_G(v) \cap Y_B$. (Note that such a subset of $L(v)$ exists by Claim 16.12.)
- For every other vertex v of G_B, let $L_B(v) := L(v)$.

Hence L_B is a restricted $(s', 2, Y_B)$-list-assignment by Claim 16.12. Since $(\mathcal{C}^*, S^*, T^*)$ is useful and $\iota^*(y) = y$ for every $y \in Y_1 \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}*})$,

$$\begin{aligned}|Y_B| &\leqslant |Y_1 \cap V(B_{\mathcal{C}^*})| + |S^*| - |Y_1 \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| + |Z| + |T^*| \\ &\leqslant (|Y_1| - |Y_1 \cap V(A_{\mathcal{C}^*}) - V(B_{\mathcal{C}^*})|) + |S^*| - |Y_1 \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| + |Z| + |T^*| \\ &\leqslant |Y_1| - 1\end{aligned}$$

by (U3).

Since $|Y_B| < |Y_1|$, we know $|V(G_B)| < |V(G)|$. By the minimality of $|V(G)|$, there exists an (η, g)-bounded L_B-coloring c_B of G_B. Define the following:

- Let $G_A := G[V(A_{\mathcal{C}^*}) \cup Z]$.
- Let $Y_A := (Y_1 \cap V(A_{\mathcal{C}^*})) \cup Z \cup S^* \cup T^* \cup V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})$.
- For every $y \in Y_A$, let $L_A(y)$ be a 1-element subset of $L(y)$ such that if $y \in V(G_B)$, then $L_A(y) = \{c_B(y)\}$.
- For every $v \in V(G_A) - Y_A$ with $1 \leqslant |N_{G_A}(v) \cap Y_A| \leqslant s' - 1$, let $L_A(v)$ be a subset of $L(v)$ with size $s' + 2 - |N_{G_A}(v) \cap Y_A|$ such that $L_A(v) \cap L_A(u) = \emptyset$ for every $u \in Y_A \cap N_{G_A}(v)$.
- For every other vertex v of G_A, let $L_A(v) := L(v)$.

Then L_A is a restricted $(s', 2, Y_A)$-list-assignment of G_A. Since $|\mathcal{C}^*| \leqslant a_{(d-1)a_0}$, $|V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| \leqslant (d-1)a_{(d-1)a_0} < \theta - \xi$. So $(A_{\mathcal{C}^*}, B_{\mathcal{C}^*}) \in \mathcal{T} - Z$ and hence $|Y_1 \cap V(A_{\mathcal{C}^*})| \leqslant 3\theta$. By (U3), $|S^*|+|T^*|+|Z| \leqslant |Y_1 \cap V(A_{\mathcal{C}^*}) - V(B_{\mathcal{C}^*})| + |Y_1 \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| \leqslant |Y_1 \cap V(A_{\mathcal{C}^*})| \leqslant 3\theta$. Hence $|Y_A| \leqslant |Y_1 \cap V(A_{\mathcal{C}^*})| + |S^*| + |T^*| + |Z| + |V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| \leqslant 3\theta + 3\theta + \theta \leqslant 7\theta$.

In particular, $|V(G_A)| < |V(G)|$. By minimality, there exists an (η, g)-bounded L_A-coloring c_A of G_A.

Claim 16.13. *For every $v \in V(G_A) - (V(G_B) \cup Y_1)$ with $N_G(v) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) - S^* \neq \emptyset$, $c_B(u) \notin L_A(v)$ for every $u \in N_G(v) \cap V(G_B)$.*

Proof. Since $v \in V(G_A) - (V(G_B) \cup Y_1)$, $v \in V(G) - (Z \cup V(B_{\mathcal{C}^*}))$. So there exists $(A, B) \in \mathcal{C}^*$ such that $v \in V(A) - V(B)$. Hence $N_G(v) \subseteq V(A)$ and $|N_G(v) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| \leqslant |N_G(v) \cap V(A \cap B)| \leqslant d - 1$.

Since $v \in N_G(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})$ and $\mathcal{C}^*$ satisfies (U2), $v \notin Z'$. So $|N_G(v) \cap Z| \leqslant s - 1$.

Since $N_G(v) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) - S^* \neq \emptyset$, there exists $w \in N_G(v) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) - S^*$. Since there exists no $(\mathcal{C}^*, S^*, T^*)$-dangerous vertex, w is not a $(\mathcal{C}^*, S^*, T^*)$-dangerous vertex. Since $S^* \subseteq V(G_B)$, $v \notin S^*$. Since $v \notin Y_1$, $v \notin Y_1 - T^*$. So $|N_G(v) \cap ((Y_1 \cap V(A_{\mathcal{C}^*})) \cup (S^* - V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})))| \leqslant 2d - 5$.

Since $T^* \subseteq Y_1 \cap V(A_{\mathcal{C}^*})$,

$$\begin{aligned}
&|N_G(v) \cap Y_A| \\
&\leqslant |N_G(v) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})| + |N_G(v) \cap Z| + |N_G(v) \cap ((Y_1 \cap V(A_{\mathcal{C}^*})) \cup (S^* - V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*})))| \\
&\leqslant (d-1) + (s-1) + (2d-5) \\
&= 3d + s - 7 = s' - 1.
\end{aligned}$$

So by the definition of L_A, $L_A(v) \cap \{c_B(u)\} = L_A(v) \cap L_A(u) = \emptyset$ for every $u \in Y_A \cap V(G_B) \cap N_G(v)$. Since $N_G(v) \cap V(G_B) \subseteq Y_A$, $c_B(u) \notin L_A(v)$ for every $u \in N_G(v) \cap V(G_B)$. □

Let c be the L-coloring of G defined by $c(v) := c_A(v)$ if $v \in V(G_A)$, and $c(v) := c_B(v)$ if $v \in V(G) - V(G_A)$.

Claim 16.14. *Let M be a monochromatic c-component intersecting both $V(G_A) - V(G_B)$ and $V(G_B) - V(G_A)$. Then every component of $M \cap G_A$ intersects Y_A, and every component of $M \cap G_B$ intersects Y_B.*

Proof. Since M intersects both $V(G_A) - V(G_B)$ and $V(G_B) - V(G_A)$, every component of $M \cap G_A$ intersects $V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) \cup Z \cup S^* \cup T^* \subseteq Y_A$.

Let M_B be a component of $M \cap G_B$. Suppose that M_B is disjoint from $Y_B = (Y_1 \cap V(B_{\mathcal{C}^*})) \cup Z \cup S^* \cup T^*$. Since M intersects both $V(G_A) - V(G_B)$ and $V(G_B) - V(G_A)$, there exist $u \in V(M_B) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) - S^*$ and $v \in N_{M_B}(v) \cap V(A_{\mathcal{C}^*}) - (V(B_{\mathcal{C}^*}) \cup S^* \cup T^*)$. Since u is not a $(\mathcal{C}^*, S^*, T^*)$-dangerous vertex, $v \notin Y_1 - T^*$. Since $v \notin T^*$, $v \notin Y_1$. So $v \in V(G_A) - (V(G_B) \cup Y_1)$ and $N_G(v) \cap V(A_{\mathcal{C}^*} \cap B_{\mathcal{C}^*}) - S^* \supseteq \{u\} \neq \emptyset$. By Claim 16.13, $c(v) = c_A(v) \neq c_B(u) = c(u)$. But M is a monochromatic c-component, a contradiction. Hence every component of $M \cap G_B$ intersects Y_B. □

Let U_A be the union of the monochromatic c_A-components of G_A intersecting Y_A. Let U_B be the union of the monochromatic c_B-components of G_B intersecting Y_B. Since c_A and c_B are (η, g)-bounded, $|V(U_A) \cup V(U_B)| \leqslant |Y_A|^2 g(|Y_A|) + |Y_B|^2 g(|Y_B|) \leqslant (7\theta)^2 g(7\theta) + (|Y_1| - 1)^2 g(|Y_1| - 1) \leqslant g(|Y_1|)$.

Since $V(G) \subseteq V(G_A) \cup V(G_B)$, by Claim 16.14, every monochromatic c-component intersecting both $V(G_A) - V(G_B)$ and $V(G_B) - V(G_A)$ is contained in $U_A \cup U_B$ and hence contains at most $g(|Y_1|) \leqslant \eta^2 g(\eta)$ vertices. Let M be a monochromatic c-component. If $V(M) \subseteq V(G_A)$, then M is a monochromatic c_A-component with at most $\eta^2 g(\eta)$ vertices since c_A is (η, g)-bounded. If $V(M) \subseteq V(G_B)$, then M is a monochromatic c_B-component with at most $\eta^2 g(\eta)$ vertices since c_B is (η, g)-bounded. Hence every monochromatic c-component contains at most $\eta^2 g(\eta)$ vertices.

Since $Y_1 \subseteq Y_A \cup Y_B$, by Claim 16.14, the union of the monochromatic c-components intersecting Y_1 is contained in $U_A \cup U_B$, so it contains at most $g(|Y_1|) \leqslant |Y_1|^2 g(|Y_1|)$ vertices. Therefore, c is an (η, g)-bounded L-coloring of G, a contradiction. This proves the theorem. □

4 Excluding Almost $(\leqslant 1)$-Subdivisions

Recall that an *almost* $(\leqslant 1)$*-subdivision* of a graph H is a graph obtained from H by subdividing edges such that at most one edge is subdivided more than once. The following simple observation is useful.

Lemma 17. *For $s \in \mathbb{N}$, let G be a graph and let H be a subgraph of G isomorphic to $K_{s-1,t}$ for some $t \geqslant \binom{s-1}{2} + 2$. Let (X, Z) be the bipartition of H with $|X| = s - 1$. If G does not contain an almost $(\leqslant 1)$-subdivision of K_{s+1}, then each component of $G - X$ contains at most one vertex in Z, and $G - X$ has at least two components.*

Proof. Let $C_1, C_2, \ldots, C_k$ be the components of $G - X$. For each $i \in [k]$, $|V(C_i) \cap Z| \leqslant 1$, as otherwise $G[X \cup Z]$ together with a path in C_i connecting two vertices in $V(C_i) \cap Z$ is an almost $(\leqslant 1)$-subdivision of K_{s+1}, a contradiction. Hence $k \geqslant |Z| \geqslant t \geqslant 2$. □

The following lemma shows that a result for graphs excluding a $K_{s,t}$ subgraph can be extended for graphs excluding an almost $(\leqslant 1)$-subdivision of K_{s+1}. Let $s, r \in \mathbb{N}$. Let G be a graph and $Y_1 \subseteq V(G)$. An (s, r, Y_1)-list-assignment of G is said to be *faithful* if for every $v \in V(G) - Y_1$ with $|N_G(v) \cap Y_1| = s$, we have $L(v) - \bigcup_{y \in Y_1 \cap N_G(v)} L(y) \neq \emptyset$.

Lemma 18. *Let $\mathcal{G}$ be a subgraph-closed family of graphs. Let β, r be functions with domain $\mathbb{N}$ such that $\beta(x) \geqslant x$ and $r(x) \in \mathbb{N}$ for every $x \in \mathbb{N}$.*

Assume that for every $s \in \mathbb{N}$, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that for every $G \in \mathcal{G}$ with no K_{s,t_s} subgraph, where $t_s := \max\{\binom{s}{2} + 2, s + 2\}$, for every $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta$, and for every $(\beta(s), r(s), Y_1)$-list-assignment L of G, there exists an (η, g)-bounded L-coloring of G.

Then for every $s \in \mathbb{N}$ with $s \geqslant 2$, there exist $\eta^* \in \mathbb{N}$ and a nondecreasing function g^* such that for every graph $G \in \mathcal{G}$ with no almost $(\leqslant 1)$-subdivision of K_{s+1}, for every $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta^*$, and for every faithful $(\beta(s-1), r(s-1), Y_1)$-list-assignment L of G, there exists an (η^*, g^*)-bounded L-coloring of G.

Proof. For every $s \in \mathbb{N}$, let η_s be the number and g_s be the function such that for every $G \in \mathcal{G}$ with no K_{s,t_s} subgraph, every $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta_s$ and every $(\beta(s), r(s), Y_1)$-list-assignment of G, there exists an (η_s, g_s)-bounded L-coloring of G. For every $s \in \mathbb{N}$ with $s \geqslant 2$, let $\eta_s^* := \eta_{s-1}$ and let g_s^* be the function defined by $g_s^*(0) := g_{s-1}(0)$ and $g_s^*(x) := g_{s-1}(x) + \eta_s^* \cdot g_s^*(x-1)$ for every $x \in \mathbb{N}$.

Fix $s \in \mathbb{N} - \{1\}$. Let $\beta' := \beta(s-1)$ and $r' := r(s-1)$. We shall prove that for every graph G in $\mathcal{G}$ with no almost $(\leqslant 1)$-subdivision of K_{s+1}, for every $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta_s^*$, and for every faithful (β', r', Y_1)-list-assignment L of G, there exists an (η_s^*, g_s^*)-bounded L-coloring of G.

Suppose to the contrary that G is a graph in $\mathcal{G}$ with no almost $(\leqslant 1)$-subdivision of K_{s+1}, Y_1 is a subset of $V(G)$ with $|Y_1| \leqslant \eta_s^*$, and L is a faithful (β', r', Y_1)-list-assignment of G such that there exists no (η_s^*, g_s^*)-bounded L-coloring of G. We further assume that $|V(G)|$ is as small as possible.

Since $\eta_s^* = \eta_{s-1}$ and $g_s^* \geqslant g_{s-1}$, there exists no (η_{s-1}, g_{s-1})-bounded L-coloring of G. Since $\eta_s^* = \eta_{s-1}$, by the definition of η_{s-1} and g_{s-1}, G contains a $K_{s-1,t_{s-1}}$ subgraph. Let t' be the maximum integer such that G contains a $K_{s-1,t'}$ subgraph. So $t' \geqslant t_{s-1}$. Let H be a subgraph of G isomorphic to $K_{s-1,t'}$. Let $\{P, Q\}$ be the bipartition of H such that $|P| = s-1$ and $|Q| = t'$. By the maximality of t', Q is the set of all vertices of $V(G) - P$ adjacent in G to all vertices in P.

Claim 18.1. *Every component of $G - P$ contains some vertex in Y_1.*

Proof. Suppose to the contrary that there exists a component C of $G - P$ disjoint from Y_1. By Lemma 17, $G - P$ contains at least two components and there exists at most one vertex in C adjacent in G to all vertices in P. By the minimality of G, there exists an (η_s^*, g_s^*)-bounded $L|_{V(G)-V(C)}$-coloring c of $G - V(C)$.

Since $\beta' \geqslant s-1$ and L is an (β', r', Y_1)-list-assignment and $V(C) \cap Y_1 = \emptyset$, for every

$v \in V(C)$ with $|N_G(v) \cap Y_1| \leqslant \beta' - 1$, we have $L(v) \cap \bigcup_{y \in N_G(v) \cap Y_1} L(y) = \emptyset$ and

$$\begin{aligned}
&|L(v) - \{c(y) : y \in N_G(v) \cap P\}| \\
&\geqslant |L(v)| - |\{c(y) : y \in N_G(v) \cap P - Y_1\}| \\
&= \beta' + r' - |N_G(v) \cap Y_1| - |\{c(y) : y \in N_G(v) \cap P - Y_1\}| \\
&= \beta' + r' - |N_G(v) \cap Y_1 \cap P| - |\{c(y) : y \in N_G(v) \cap P - Y_1\}| \\
&\geqslant \beta' + r' - |N_G(v) \cap P| \\
&\geqslant \beta' + r' - (s-1) \\
&\geqslant 1.
\end{aligned}$$

For every $v \in V(C)$ with $|N_G(v) \cap Y_1| \geqslant \beta'$,

$$|N_G(v) \cap P| \geqslant |N_G(v) \cap Y_1| \geqslant \beta' = \beta(s-1) \geqslant s - 1,$$

implying $P \subseteq Y_1$ and $|N_G(v) \cap Y_1| = \beta'$, so $L(v) - \{c(y) : y \in N_G(v) \cap P\} = L(v) - \bigcup_{y \in N_G(v) \cap Y_1} L(y) \neq \emptyset$ since L is faithful. So for every $v \in V(C)$, $L(v) - \{c(y) : y \in N_G(v) \cap P\} \neq \emptyset$.

Let L' be the following list-assignment of $G[V(C) \cup P]$:

- For every $v \in P$, let $L'(v) := \{c(v)\}$.
- For every $v \in V(C)$ with $|N_G(v) \cap P| \geqslant \beta'$, let $L'(v)$ be a 1-element subset of $L(v) - \bigcup_{u \in P} L'(u)$.
- Let $Y_1' := P \cup \{v \in V(C) : |N_G(v) \cap P| \geqslant \beta'\}$.
- For every $v \in V(C) \cap N^{<\beta'}(Y_1')$, let $L'(v)$ be a subset of $L(v) - \bigcup_{y \in N_G(v) \cap Y_1'} L'(y)$ of size $|L(v)| - |N_G(v) \cap Y_1' - Y_1| = \beta' + r' - |N_G(v) \cap Y_1'| \geqslant r' + 1$.
- For every $v \in V(C) \cap N^{<\beta'}(P) - N^{<\beta'}(Y_1')$, let $L'(v)$ be a subset of $L(v) - \bigcup_{y \in N_G(v) \cap P} L'(y)$ of size $|L(v)| - |N_G(v) \cap P - Y_1| = \beta' + r' - |N_G(v) \cap P| \geqslant r' + 1$.
- For every $v \in V(C) - (Y_1' \cup N^{<\beta'}(Y_1') \cup N^{<\beta'}(P))$, let $L'(v) := L(v)$.

Note that $Y_1' - P$ consists of the vertex in $V(C)$ adjacent in G to all vertices in P. Hence for every $v \in V(C) \cap N_G(P) - Y_1'$, $|N_G(v) \cap P| \in [\beta' - 1]$. That is, $V(C) \cap N_G(P) - Y_1' = V(C) \cap N^{<\beta'}(P)$. So for every $v \in V(C) \cap N_G(P) - Y_1'$, $L'(v) \cap L'(u) = \emptyset$ for every $u \in P \cap N_G(v)$.

Clearly, L' is an (β', r', Y_1')-list-assignment of G. If $v \in V(C) - Y_1'$ with $|N_G(v) \cap Y_1'| = \beta'$, then since $|Y_1' - P| = 1$, we know $v \in N^{<\beta'}(P) - N^{<\beta'}(Y_1')$, so $L'(v)$ is a set of size at least $r' + 1 \geqslant 2$ disjoint from $\bigcup_{y \in N_G(v) \cap P} L'(y)$. Hence if $v \in (V(C) \cup P) - Y_1'$ with $|N_G(v) \cap Y_1'| = \beta'$, then $L'(v) - \bigcup_{y \in N_G(v) \cap Y_1'} L'(y) = L'(v) - \bigcup_{y \in N_G(v) \cap Y_1' - P} L'(y)$ has size $|L'(y)| - 1 \geqslant 1$. Therefore, L' is a faithful (β', r', Y_1')-list-assignment of $G[V(C) \cup P]$.

Since $G - P$ contains at least two components, $|V(C) \cup P| < |V(G)|$. By the minimality of G, there exists an (η_s^*, g_s^*)-bounded L'-coloring c' of $G[V(C) \cup P]$.

For every $v \in V(C) \cap N_G(P)$, if $v \in N^{<\beta'}(P)$, then $L'(v)$ is disjoint from $\bigcup_{y \in N_G(v) \cap P} L'(y)$; if $v \in V(C)$ with $|N_G(v) \cap P| \geqslant \beta'$, then $v \in Y_1' - P$ and $L'(v)$ is disjoint from $\bigcup_{y \in P} L'(y)$. Hence every monochromatic c'-component intersecting P is contained in $G[P]$.

Let c^* be the L-coloring defined by $c^*(v) := c(v)$ if $v \in V(G)-V(C)$, and $c^*(v) := c'(v)$ if $v \in V(C)$. Hence every monochromatic c^*-component is either contained in $G-V(C)$ or contained in C, so it contains at most $\eta_s^{*2} g(\eta_s^*)$ vertices. Since $V(C) \cap Y_1 = \emptyset$, the union of the monochromatic c^*-components intersecting Y_1 equals the union of the monochromatic c-components intersecting Y_1, and contains at most $|Y_1|^2 g(|Y_1|^2)$ vertices. Therefore, c^* is an (η_s^*, g_s^*)-bounded L-coloring of G, a contradiction. □

Let $C_1, C_2, \ldots, C_k$ be the components of $G - P$. For $i \in [k]$, let $G_i := G[V(C_i) \cup P]$. By Lemma 17, $k \geqslant t_{s-1} \geqslant s+1$. By Claim 18.1, $Y_1 \cap V(C_i) \neq \emptyset$ for each $i \in [k]$, so $|V(G_i) \cap Y_1| \leqslant |Y_1| - (k-1) \leqslant |Y_1| - s$ for each $i \in [k]$, and $k \leqslant |Y_1| \leqslant \eta_s^*$. So $|(V(G_i) \cap Y_1) \cup P| \leqslant |Y_1| - s + |P| < |Y_1|$ for each $i \in [k]$.

Let L^* be the following list-assignment of G:

- Let $Y_1^* := Y_1 \cup P$.
- For each $v \in Y_1^*$, let $L^*(v)$ be a 1-element subset of $L(v)$.
- For each $v \in N^{<\beta'}(Y_1^*)$, let $L^*(v)$ be a subset of $L(v) - \bigcup_{y \in N_G(v) \cap Y_1^*} L^*(y)$ with size $|L(v)| - |N_G(v) \cap (Y_1^* - Y_1)| = \beta' + r' - |N_G(v) \cap Y_1^*|$.
- For each $v \in V(G) - (Y_1^* \cup N^{<\beta'}(Y_1^*))$, let $L^*(v) := L(v)$.

Clearly, L^* is an (β', r', Y_1^*)-list-assignment of G. Let $v \in V(G) - Y_1^*$ with $|N_G(v) \cap Y_1^*| = \beta'$. So $L^*(v) = L(v)$. If $|N_G(v) \cap Y_1| = \beta'$, then $N_G(v) \cap Y_1^* = N_G(v) \cap Y_1$, so $L^*(v) - \bigcup_{y \in N_G(v) \cap Y_1^*} L^*(y) = L(v) - \bigcup_{y \in N_G(v) \cap Y_1} L(y) \neq \emptyset$ since L is a faithful (β', r', Y_1)-list-assignment of G. If $|N_G(v) \cap Y_1| < \beta'$, then $|L^*(v)| = |L(v)| = \beta' + r' - |N_G(v) \cap Y_1|$ and $L(v)$ is disjoint from $\bigcup_{y \in N_G(v) \cap Y_1} L(y)$, so

$$\begin{aligned}
|L^*(v) - \bigcup_{y \in N_G(v) \cap Y_1^*} L^*(y)| &= |L^*(v) - \bigcup_{y \in N_G(v) \cap Y_1^* - Y_1} L^*(y)| \\
&\geqslant |L^*(v)| - |\bigcup_{y \in N_G(v) \cap Y_1^* - Y_1} L^*(y)| \\
&\geqslant \beta' + r' - |N_G(v) \cap Y_1| - |N_G(v) \cap Y_1^* - Y_1| \\
&= \beta' + r' - |N_G(v) \cap Y_1^*| \\
&= r \geqslant 1.
\end{aligned}$$

Therefore, L^* is a faithful (β', r', Y_1^*)-list-assignment of G.

Since $P \subseteq Y_1^*$, $L^*|_{V(G_i)}$ is a faithful $(\beta', r', Y_1^* \cap V(G_i))$-list-assignment of G_i. Recall that for each $i \in [k]$, $|Y_1^* \cap V(G_i)| \leqslant |Y_1| - 1 \leqslant \eta$. By the minimality of G, for each $i \in [k]$, there exists an (η_s^*, g_s^*)-bounded $L^*|_{V(G_i)}$-coloring c_i of G_i. Since $P \subseteq Y_1^* \cap V(G_i)$ for every $i \in [k]$, we know for every $v \in P$, $c_i(v) = c_j(v)$ for any $i, j \in [k]$. Let c^* be the L^*-coloring of G defined by $c(v) := c_1(v)$ if $v \in P$, and $c(v) := c_i(v)$ if $v \in V(C_i)$ for some $i \in [k]$.

Since $P \subseteq Y_1^* \cap V(G_i)$ for all $i \in [k]$, the number of vertices in the union of the

monochromatic c^*-components intersecting $Y_1 \cup P$ is at most

$$\begin{aligned}\sum_{i=1}^{k} |Y_1^* \cap V(G_i)|^2 g_s^*(|Y_1^* \cap V(G_i)|) &\leqslant \sum_{i=1}^{k} (|Y_1|-1)^2 g_s^*(|Y_1|-1) \\ &\leqslant \eta_s^* \cdot (|Y_1|-1)^2 g_s^*(|Y_1|-1) \\ &\leqslant |Y_1|^2 g_s^*(|Y_1|).\end{aligned}$$

Furthermore, every monochromatic c^*-component disjoint from $Y_1 \cup P$ is a monochromatic c_i-component for some $i \in [k]$, and hence contains at most $\eta_s^{*2} g_s^*(\eta)$ vertices. Therefore c^* is an (η_s^*, g_s^*)-bounded L-coloring of G. This proves the lemma. □

The following lemma is equivalent to Lemma 18 except it applies for restricted list assignments. The proof is identical, so we omit it.

Lemma 19. *Let $\mathcal{G}$ be a subgraph-closed family of graphs. Let β, r be functions with domain $\mathbb{N}$ such that $\beta(x) \geqslant x$ and $r(x) \in \mathbb{N}$ for every $x \in \mathbb{N}$.*

Assume that for every $s \in \mathbb{N}$, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that for every $G \in \mathcal{G}$ with no K_{s,t_s} subgraph, where $t_s := \max\{\binom{s}{2} + 2, s+2\}$, for every $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta$, and for every restricted $(\beta(s), r(s), Y_1)$-list-assignment L of G, there exists an (η, g)-bounded L-coloring of G.

Then for every $s \in \mathbb{N}$ with $s \geqslant 2$, there exist $\eta^* \in \mathbb{N}$ and a nondecreasing function g^* such that for every graph $G \in \mathcal{G}$ with no almost $(\leqslant 1)$-subdivision of K_{s+1}, for every $Y_1 \subseteq V(G)$ with $|Y_1| \leqslant \eta^*$, and for every restricted faithful $(\beta(s-1), r(s-1), Y_1)$-list-assignment L of G, there exists an (η^*, g^*)-bounded L-coloring of G.

Theorem 20. *If $s \in \mathbb{N}$ with $s \geqslant 2$, then the following hold:*

1. *For every $w \in \mathbb{N}$, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that for every graph G of treewidth at most w with no almost $(\leqslant 1)$-subdivision of K_{s+1}, every subset Y_1 of $V(G)$ with $|Y_1| \leqslant \eta$ and every faithful $(s-1, 1, Y_1)$-list-assignment of G, there exists an (η, g)-bounded L-coloring of G.*
2. *For every graph H, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that for every graph G with no H-minor and no almost $(\leqslant 1)$-subdivision of K_{s+1}, every subset Y_1 of $V(G)$ with $|Y_1| \leqslant \eta$ and every restricted faithful $(s-1, 2, Y_1)$-list-assignment L of G, there exists an (η, g)-bounded L-coloring of G.*
3. *For every $d \in \mathbb{N}$ with $d \geqslant 2$ and graph H of maximum degree at most d, there exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that for every graph G with no H-subdivision and no almost $(\leqslant 1)$-subdivision of K_{s+1}, every subset Y_1 of $V(G)$ with $|Y_1| \leqslant \eta$ and every restricted faithful $(s+3d-7, 2, Y_1)$-list-assignment L of G, there exists an (η, g)-bounded L-coloring of G.*
4. *There exist $\eta \in \mathbb{N}$ and a nondecreasing function g such that for every graph G with no K_{s+1}-subdivision, every subset Y_1 of $V(G)$ with $|Y_1| \leqslant \eta$ and every restricted faithful $(4s-7, 2, Y_1)$-list-assignment L of G, there exists an (η, g)-bounded L-coloring of G.*

Proof. Statement 1 follows from Theorem 8 and Lemma 18 by taking $\mathcal{G}$ to be the set of graphs of treewidth at most w, $\beta(s) = s$ and $r(s) = 1$.

Statement 2 follows from Theorem 9 and Lemma 19 by taking $\mathcal{G}$ to be the set of graphs with no H-minor, $\beta(s) = s$ and $r(s) = 2$.

Statement 3 follows from Lemmas 16 and 19 by taking $\mathcal{G}$ to be the set of graphs with no H-subdivision, $\beta(s) = 3d + s - 6$ and $r(s) = 2$. Note that $\beta(s) \geqslant s$ since $d \geqslant 2$. And $3d + s \geqslant 7$ since $d \geqslant 2$ and $s \geqslant 1$.

Statement 4 follows from Statement 3 by taking $H = K_{s+1}$. □

When $Y_1 = \emptyset$, every (s, r, Y_1)-list-assignment is faithful. Thus, Theorem 20 implies that for all $s, d, w \in \mathbb{N}$ with $s \geqslant 2$ and $d \geqslant 2$, for every graph H, there exists $\eta \in \mathbb{N}$ such that:

1. For every graph G with treewidth at most w and with no almost $(\leqslant 1)$-subdivision of K_{s+1}, and for every list-assignment L of G with $|L(v)| \geqslant s$ for every $v \in V(G)$, there exists an L-coloring with clustering η (Theorem 1 for $s \geqslant 2$).
2. For every graph G with no almost $(\leqslant 1)$-subdivision of K_{s+1} and with no H-minor, there exists an $(s + 1)$-coloring of G with clustering η (Theorem 2 for $s \geqslant 2$).
3. If the maximum degree of H is at most d, then for every graph G with no H-subdivision and no almost $(\leqslant 1)$-subdivision of K_{s+1}, there exists an $(s + 3d - 5)$-coloring of G with clustering η (Theorem 4 for $s \geqslant 2$ and $d \geqslant 2$).
4. For every graph G with no K_{s+1}-subdivision, there exists a $(4s - 5)$-coloring of G with clustering η (Theorem 6 for $s \geqslant 2$).

Note that when $s = 1$, graphs with no K_{s+1} subgraph have no edge, so they are 1-colorable with clustering 1. This together with Lemma 15 complete the proof of Theorems 1, 2, 4 and 6.

References

[1] Noga Alon, Guoli Ding, Bogdan Oporowski, and Dirk Vertigan. Partitioning into graphs with only small components. *J. Combin. Theory Ser. B*, 87(2):231–243, 2003.

[2] Béla Bollobás and Andrew Thomason. Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs. *European J. Combin.*, 19(8):883–887, 1998.

[3] Paul A. Catlin. Hajós' graph-coloring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B*, 26:268–274, 1979.

[4] Ilkyoo Choi and Louis Esperet. Improper coloring of graphs on surfaces. *J. Graph Theory*, 91(1):16–34, 2019.

[5] Gabriel A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.

[6] Zdeněk Dvořák and Sergey Norin. Islands in minor-closed classes. I. Bounded treewidth and separators. 2017, arXiv:1710.02727.

[7] Katherine Edwards, Dong Yeap Kang, Jaehoon Kim, Sang-il Oum, and Paul Seymour. A relative of Hadwiger's conjecture. *SIAM J. Discrete Math.*, 29(4):2385–2388, 2015.

[8] Paul Erdős and Siemion Fajtlowicz. On the conjecture of Hajós. *Combinatorica*, 1(2):141–143, 1981.

[9] Paul Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[10] Louis Esperet and Gwenaël Joret. Colouring planar graphs with three colours and no large monochromatic components. *Combinatorics, Probability & Computing*, 23(4):551–570, 2014.

[11] Louis Esperet and Pascal Ochem. Islands in graphs on surfaces. *SIAM J. Discrete Math.*, 30(1):206–219, 2016.

[12] Jacob Fox, Choongbum Lee, and Benny Sudakov. Chromatic number, clique subdivisions, and the conjectures of Hajós and Erdős-Fajtlowicz. *Combinatorica*, 33(2):181–197, 2013.

[13] Penny Haxell, Tibor Szabó, and Gábor Tardos. Bounded size components—partitions and transversals. *J. Combin. Theory Ser. B*, 88(2):281–297, 2003.

[14] Kevin Hendrey and David R. Wood. Defective and clustered colouring of sparse graphs. *Combin. Probab. Comput.*, 28(5):791–810, 2019.

[15] Jan van den Heuvel and David R. Wood. Improper colourings inspired by Hadwiger's conjecture. *J. London Math. Soc.*, 98:129–148, 2018.

[16] Dong Yeap Kang and Sang-il Oum. Improper coloring of graphs with no odd clique minor. *Combin. Probab. Comput.*, 28(5):740–754, 2019.

[17] Ken-ichi Kawarabayashi. A weakening of the odd Hadwiger's conjecture. *Combin. Probab. Comput.*, 17(6):815–821, 2008.

[18] Ken-ichi Kawarabayashi and Bojan Mohar. A relaxed Hadwiger's conjecture for list colorings. *J. Combin. Theory Ser. B*, 97(4):647–651, 2007.

[19] János Komlós and Endre Szemerédi. Topological cliques in graphs. II. *Combin. Probab. Comput.*, 5(1):79–90, 1996.

[20] Nathan Linial, Jiří Matoušek, Or Sheffet, and Gábor Tardos. Graph colouring with no large monochromatic components. *Combin. Probab. Comput.*, 17(4):577–589, 2008.

[21] Chun-Hung Liu and Sang-il Oum. Partitioning H-minor free graphs into three subgraphs with no large components. *J. Combin. Theory Ser. B*, 128:114–133, 2018.

[22] Chun-Hung Liu and Robin Thomas. Excluding subdivisions of bounded degree graphs. *J. Combin. Theory Ser. B*, 134:1–35, 2019.

[23] Chun-Hung Liu and David R. Wood. Clustered coloring of graphs excluding a subgraph and a minor. 2019, arXiv:1905.09495.

[24] Chun-Hung Liu and David R. Wood. Clustered graph coloring and layered treewidth. 2019, arXiv:1905.08969.

[25] Bojan Mohar. Triangulations and the Hajós conjecture. *Electron. J. Combin.*, 12:N15, 2005.

[26] Bojan Mohar, Bruce Reed, and David R. Wood. Colourings with bounded monochromatic components in graphs of given circumference. *Australas. J. Combin.*, 69(2):236–242, 2017.

[27] Sergey Norin. Conquering graphs of bounded treewidth. 2015. Unpublished manuscript.

[28] Sergey Norin, Alex Scott, Paul Seymour, and David R. Wood. Clustered colouring in minor-closed classes. *Combinatorica*, 39(6):1387–1412, 2019.

[29] Neil Robertson and Paul Seymour. Graph minors. V. Excluding a planar graph. *J. Combin. Theory Ser. B*, 41(1):92–114, 1986.

[30] Paul Seymour. Hadwiger's conjecture. In John Forbes Nash Jr. and Michael Th. Rassias, eds., *Open Problems in Mathematics*, pp. 417–437. Springer, 2015.

[31] Carsten Thomassen. Some remarks on Hajós' conjecture. *J. Combin. Theory Ser. B*, 93(1):95–105, 2005.

[32] David R. Wood. Contractibility and the Hadwiger conjecture. *European J. Combin.*, 31(8):2102–2109, 2010. arXiv:0811.2012.

[33] David R. Wood. Defective and clustered graph colouring. *Electron. J. Combin.*, DS23, 2018. Version 1.