ORIGINAL PAPER

Calderón-Zygmund theory for non-convolution type nonlocal equations with continuous coefficient

Mouhamed Moustapha Fall[1] · Tadele Mengesha[2] · Armin Schikorra[3] · Sasikarn Yeepo[4]

Abstract

Given $2 \le p < \infty$, $s \in (0, 1)$ and $t \in (1, 2s)$, we establish interior $W^{t,p}$ Calderón-Zygmund estimates for solutions of nonlocal equations of the form

$$\int_\Omega \int_\Omega K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dx dy = g[\varphi], \quad \forall \phi \in C_c^\infty(\Omega)$$

where $\Omega \subset \mathbb{R}^n$ is an open set. Here we assume K is bounded, nonnegative and continuous in the first entry – and ellipticity is ensured by assuming that K is strictly positive in a cone. The setup is chosen so that it is applicable for nonlocal equations on manifolds, but the structure of the equation is general enough that it also applies to the certain fractional p-Laplace equations around points where $u \in C^1$ and $|\nabla u| \ne 0$.

Mathematics Subject Classification 35R11 (35J99 46E35 47G30)

✉ Armin Schikorra
   armin@pitt.edu

   Mouhamed Moustapha Fall
   mouhamed.m.fall@aims-senegal.org

   Tadele Mengesha
   mengesha@utk.edu

   Sasikarn Yeepo
   6072857023@student.chula.ac.th

[1] African Institute for Mathematical Sciences in Senegal, AIMS-Senegal KM 2, Route de Joal, B.P. 14 18, Mbour, Senegal

[2] Department of Mathematics, The University of Tennessee, 204 Ayres Hall, 1403 Circle Drive Knoxville, Knoxville, TN 37996, USA

[3] Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA

[4] Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

---

🌲 Springer

Contents

1 Introduction and main results

Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be an open set. In this work, we study the interior $W^{t,p}$-regularity theory for functions $u \in W^{s,2}(\Omega)$ that satisfy the nonlocal equation

$$\int_\Omega \int_\Omega K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \, dx \, dy = f[\varphi], \quad \forall \varphi \in C_c^\infty(\Omega) \quad (1.1)$$

where $f[\varphi]$ represents the action of the distribution f on the smooth function φ. We take $K(\cdot, \cdot, \cdot) : \mathbb{R}^n \times [0, \infty) \times \mathbb{S}^{n-1} \to [0, \Lambda]$ as our coefficient kernel and is a nonnegative bounded and measurable function which is strictly positive on a substantial subset of $\mathbb{S}^{n-1}$. The model case is $K(x, r, h) = \frac{|\phi(x)-\phi(x+rh)|}{|r|}$ for a map ϕ with $|\nabla\phi(x)| > 0$. In particular, if ϕ is a diffeomorphism that parameterizes an n-dimension manifold $\mathcal{M}$, then (1.1) could be a nonlocal differential equation on $\mathcal{M}$. Other choices of naturally appearing kernels will be studied later. For simplicity, we always assume $K(\cdot, \cdot, h) = K(\cdot, \cdot, -h)$.

For $t \geq s$, local $W^{t,p}$-Calderón Zygmund theory has been studied in [11] for a different type of kernels by three of the authors. On the other hand the first author recently studied Hölder regularity theory for (1.1) in [13]. To some extent, with respect to motivation, techniques, and results, this paper is a combination of the ideas in [13] and [11], and we refer to those papers for a background and motivation of equations of the type we are considering.

The following theorem states the first main result of the paper.

Theorem 1.1 *Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $\Omega_1 \subset\subset \Omega$ is bounded. Assume $\Lambda > 1$, $\eta > 0$, $s_1 + s_2 = 2s$ and $s_1, s_2, s \in (0, 1)$, $p \in (1, \infty)$.*

Let $K : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}^{n-1} \to \mathbb{R}$ be in all entries bounded and measurable, assume that the kernel is nondegenerate in the following sense

$$0 \leq K(x, r, h) \leq \Lambda \quad \forall x \in \mathbb{R}^n, r \geq 0, \ h \in \mathbb{S}^{n-1},$$

and there exists $\Sigma \subset \mathbb{S}^{n-1}$ with $|\Sigma| > 0$ such that for every $x \in \mathbb{R}^n$,

$$\eta \leq \inf_{h \in \Sigma} K(x, 0, h). \quad (1.2)$$

Assume also the following continuity around points on Ω_1: for every $x_0 \in \Omega_1$, and some $\mathfrak{R} > 0$ such that $B(x_0, 5\mathfrak{R}) \subset\subset \Omega$, and each $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$\sup_{r < \lambda\mathfrak{R}} \sup_{x \in B(x_0, \lambda\mathfrak{R})} \sup_{|h|=1} |K(x, r, h) - K(x_0, 0, h)| \leq \varepsilon. \quad (1.3)$$

Now, for a given distribution g, let $u \in W^{s,2}(\Omega)$ satisfies

$$\int_\Omega \int_\Omega K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \, dx \, dy = g[\varphi], \quad \forall \varphi \in C_c^\infty(\Omega_1).$$

Then, if for some $\Lambda > 0$, we have the estimate

$$|g[\varphi]| \le \Lambda \left(\|\varphi\|_{L^{p'}(\mathbb{R}^n)} + [\varphi]_{W^{s_2,p'}(\mathbb{R}^n)} \right),$$

for all $\varphi \in C_c^\infty(\Omega_1)$, then $u \in W_{loc}^{s_1,p}(\Omega_1)$.

The regularity result comes with a corresponding estimate

$$\|u\|_{W^{s_1,p}(\Omega_1)} \le C(\|u\|_{W^{s,2}(\Omega)} + \|g\|)$$

where the constant C depends only on s, p, Ω_1, Ω, and ellipticity, continuity, and boundedness constants of K. See [11] for details. In the statement above, p' is the Hölder conjugate of p. Notice that the theorem is interesting when $s_2 < s$. For $p = 2$ the result of the theorem is only slightly better than [4] where Hölder continuity of K is assumed, but for $s_1 \ne s$ and $p \ne 2$ this is a new result. Recently, S. Nowak developed $W^{s,p}$-regularity theory under the weaker vanishing mean oscillation assumptions on the kernel, [14–18]. It is a natural question if for our main theorem the continuity condition (for $t < 1$) can be relaxed to a VMO-type condition, but we will leave this issue for a future work.

We conjecture that the condition $s_1 < 1$ in Theorem 1.1 to be sharp, although we do not have a counterexample. However, if we additionally assume Hölder continuity of the kernel, the result of Theorem 1.1 holds for $s_1 > 1$ as well. The following theorem states the second main result of the paper.

Theorem 1.2 *Assume all the conditions of Theorem 1.1 hold. Assume additionally α-Hölder continuity of K in the first variable, i.e.*

$$\sup_{x,y \in \mathbb{R}^n} \sup_{|h|=1, r \ge 0} |K(x, r, h) - K(y, r, h)| \le \Lambda |x - y|^\alpha. \tag{1.4}$$

Then the conclusion of Theorem 1.1 hold for $s_1 < \min\{2s, 1+\alpha\}$ (instead of $s_1 < \min\{2s, 1\}$).

We refer to [13] for applicability of our result to the fractional mean curvature, cf. [9]. The structure of the kernel K appears naturally in applications that leads to linear nonlocal differential equations on n-manifolds $\mathcal{M}$, say

$$\mathcal{L}_{\mathcal{M},s}(u, \varphi) := \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{d_{\mathcal{M}}(x, y)^{n+2s}} \, dx \, dy.$$

for smooth $\phi \in C_0^\infty(\mathcal{M})$. For simplicity, we may assume that all of $\mathcal{M}$ is parametrized by a bi-lipschitz diffeomorphism $\Phi : \Omega \to \mathcal{M}$ (otherwise one can work on a coordinate patch with the help of a decomposition of unity). Then for any $\psi \in C_c^\infty(\Omega)$, by change of variables we have

$$\mathcal{L}_{\mathcal{M},s}(u, \psi \circ \Phi^{-1}) = \int_\Omega \int_\Omega \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi(x) - \psi(y))}{d_{\mathcal{M}}(\Phi(x), \Phi(y))^{n+2s}} \operatorname{Jac}(D\Phi(x)) \operatorname{Jac}(D\Phi(y)) \, dx \, dy$$

where $\tilde{u} := u \circ \Phi$. Thus, if we set

$$K(x, r, h) := \left(\frac{r}{d_{\mathcal{M}}(\Phi(x), \Phi(x + rh))} \right)^{n+2s} \operatorname{Jac}(D\Phi(x)) \operatorname{Jac}(D\Phi(x + rh))$$

we have

$$\mathcal{L}_{K,s}(\tilde{u}, \psi) = \mathcal{L}_{\mathcal{M},s}(u, \psi \circ \Phi^{-1}).$$

Observe that

$$K(x, 0, h) \approx |D\Phi(x)h|^{-n-2s}$$

which is uniformly bounded from below because $D\Phi$ has maximal rank as a diffeomorphism. In particular, we have the following corollary.

Corollary 1.3 *The conclusion of Theorem 1.1 is true for the equation*

$$\int_\Omega \int_\Omega \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|\phi(x) - \phi(y)|^{n+2s}} \, dx \, dy = g[\varphi] \quad \forall \varphi \in C_c^\infty(\Omega)$$

where $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ *is bi-Lipschitz and* $C^{1,\alpha}$-*differentiable.*

Lastly, let us illustrate a consequence of our arguments for the higher regularity theory of the fractional p-Laplacian in the regime where $|\nabla u(x_0)| > 0$ for $s \approx 1$.

Corollary 1.4 *Let* $s \in (0, 1)$, $p \geq 2$ *such that* $s > \frac{p}{p+1}$. *For a given* $\Omega \subseteq \mathbb{R}^n$ *open set, let* $u \in W^{s,p} \cap L^\infty(\Omega)$ *be a solution to*

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \, (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy = \int_\Omega f\varphi dx \quad \forall \varphi \in C_c^\infty(\Omega). \quad (1.5)$$

If $f \in C_c^\infty(\mathbb{R}^n)$, *u is continuously differentiable around some point* $x_0 \in \Omega$, *and* $|\nabla u(x_0)| > 0$, *then there exists a small neighborhood* $B(x_0, r)$ *such that* $u \in C^2(B(x_0, r))$.

Let us stress a subtlety of Corollary 1.4. The idea here is to take the solution u as part of the kernel $K(x, r, h) = \frac{|u(x) - u(x+rh)|^{p-2}}{|h|^{s(p-2)}}$. Even if we assume that $|\nabla u(x_0)| \neq 0$, we still might have

$$\frac{|u(x_0) - u(x_0 + rh)|}{r} \approx |\nabla u(x_0) \cdot h| \approx 0,$$

that for sufficiently small $r > 0$, for example, whenever h is perpendicular to $\nabla u(x_0)$. However, we know that for a substantial subset $\Sigma \subset \mathbb{S}^{n-1}$, that depends on x_0, we have for all $h \in \Sigma$ that $|\nabla u(x_0)h| \gtrsim |\nabla u(x_0)|$, namely this is true for all $h \in S^{n-1}$ with $|\angle(h, \nabla u(x_0))| > \lambda > 0$. Condition (1.2) on the kernel in Theorem 1.1 will allow us to apply the theorem to handle such cases. The details of Corollary 1.4 are explained in Sect. 6.

Let us stress that Corollary 1.4 is an expected result but likely not optimal. The a priori L^∞-condition on u can be waived, indeed in view of [6] we can even assume that u is Hölder continuous. Also, it may be possible to remove the conditions on p and s in Corollary 1.4 and as well obtain differentiability for the solution above C^2. The crucial condition $|\nabla u(x_0)| > 0$ makes the equation "somewhat uniformly elliptic" with coefficients which regularize with u. All these are the crucial ingredients to bootstrap towards C^∞-regularity – however, there is some subtlety here that we do not address in this paper: differentiating the equation even for $p > 2$ the fractional p-Laplacian becomes singular elliptic (even around points where $|\nabla u(x_0)| > 0$).

Another point to keep in mind with respect to Corollary 1.4 is that the more challenging case is around points x_0 with vanishing derivative, $|\nabla u(x_0)| = 0$ and the (here a priori assumed) C^1-regularity cannot be obtained by Theorem 1.1. Perhaps other techniques such as those developed by Brasco–Lindgren [2,3] may be needed.

The statement of Corollary 1.4 is also somewhat motivated by the recent extention formula for the (sublinear, $p < 2$) p-Laplacian obtained in [5] – where they assumed $u \in C^2$ and $|\nabla u(x_0)| > 0$. While our arguments do not immediately apply for $p < 2$ we hope that our techniques have some use for this equation to remove the (very strong) C^2-assumption.

We conclude the introduction by describing the organization of the paper as well as the approach we follow to prove the main result. We use a perturbation argument to obtain

the interior regularity estimate stated in Theorem 1.1. We compare the nonlocal equation of interest locally with a nonlocal equation on the n-torus whose solutions are periodic functions with better regularity. Results pertaining to regularity theory to solutions to nonlocal equation on the n-torus are presented in Sect. 2.

2 The torus and periodic maps—Dong-Kim's regularity result

To facilitate the perturbation argument that we use to obtain the main result, we develop regularity theory for nonlocal PDE with convolution-type kernel on the periodic domain the n-torus which is the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$ with opposite sides identified. The regularity results that we obtain in this section are a reformulation of the results obtained in [8,12] where a regularity theory is developed for nonlocal equations over $\mathbb{R}^n$.

First we fix some notations. The n-torus $\mathbb{T}^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ is given as the set of equivalence class $\mathbb{R}^n / \sim$ where the equivalence relation $\sim$ on $\mathbb{R}^n$ defined as follows: we say that, for $x, y \in \mathbb{R}^n$, $x \sim y$ if $x - y \in \mathbb{Z}^n$. With this identification, periodic functions on $\mathbb{T}^n$ are characterized as f defined on $\mathbb{R}^n$ and satisfy $f(x + m) = f(x)$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}^n$. The space of infinitely differentiable functions on $\mathbb{T}^n$ denoted by $C^\infty(\mathbb{T}^n)$. For $1 \le p \le \infty$, f is in $L^p(\mathbb{T}^n)$ if $\|f\|_{L^p(\mathbb{T}^n)} < \infty$. For a complex-valued function f in $L^1(\mathbb{T}^n)$, we define the Fourier transform of f, $\mathcal{F}(f)$, and for $m \in \mathbb{Z}^n$ as

$$\mathcal{F}(f)(m) = \int_{\mathbb{T}^n} f(x)e^{-2\pi \iota \langle m, x \rangle}\, dx,$$

where $\langle m, x \rangle = \sum_{i=1}^n m_i x_i$ is the dot product of m and x. The inverse Fourier transform (also known as Fourier series of f) at $x \in \mathbb{T}^n$ is the series

$$\sum_{m \in \mathbb{Z}^n} \mathcal{F}(f)(m)e^{2\pi \iota \langle m, x \rangle}.$$

We can define other operators on function spaces of $\mathbb{T}^n$-periodic functions. For example, for $s > 0$, we define the fractional Laplacian on $\mathbb{T}^n$, $\Delta_{\mathbb{T}^n}^{s/2}$, via the Fourier series as

$$\Delta_{\mathbb{T}^n}^{\frac{s}{2}} f(x) = \sum_{m \in \mathbb{Z}^n} |m|^s \mathcal{F}(f)(m)e^{2\pi \iota \langle m, x \rangle}.$$

See [19] for more on the fractional Laplacian on $\mathbb{T}^n$.

For $1 < p < \infty$ and $s > 0$ we define the inhomogeneous Sobolev and Bessel potential spaces on the torus are defined, respectively, as

$$W^{s,p}(\mathbb{T}^n) = \left\{ u \in L^p(\mathbb{T}^n) : [u]_{W^{s,p}(\mathbb{T}^n)}^p = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|u(x) - u(y)|}{|x - y|^{d+ps}} dx dy < \infty \right\}$$

and

$$H^{s,p}(\mathbb{T}^n) = \left\{ u \in L^p(\mathbb{T}^n) : |u|_{\dot{H}^{s,p}(\mathbb{T}^n)} = \|\Delta_{\mathbb{T}^n}^{\frac{s}{2}} u\|_{L^p(\mathbb{T}^n)} < \infty \right\}.$$

For $p = 2$, $W^{s,2}(\mathbb{T}^n) = H^{s,2}(\mathbb{T}^n)$, which can be shown via Plancherel Theorem. We denote $H^{s,2}(\mathbb{T}^n)$ just simply by $H^s(\mathbb{T}^n)$. The homogeneous Bessel potential space $\dot{H}^{s,p}(\mathbb{T}^n)$ on the torus collects all periodic distributions u such that $|u|_{\dot{H}^{s,p}(\mathbb{T}^n)} = \|\Delta_{\mathbb{T}^n}^{\frac{s}{2}} u\|_{L^p(\mathbb{T}^n)} < \infty$. We

should mention the Sobolev inequality for potential spaces: for $s > 0$ and $1 < p < q < \infty$ such that $\frac{s}{n} \geq \frac{1}{p} - \frac{1}{q}$, we have

$$\|u - (u)_{\mathbb{T}^n}\|_{L^q(\mathbb{T}^n)} \leq C|u|_{\dot{H}^{s,p}(\mathbb{T}^n)},$$

where $C = C(s, n, p, q) > 0$ and $(\cdot)_{\mathbb{T}^n}$ represents the average operator, see [1].

The main result of this section is stated in the following theorem which obtains a Calderón-Zygmond type result for nonlocal equations on the torus with kernels that are homogeneous of degree zero. We denote by $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ the $n - 1$ sphere.

Theorem 2.1 *Let $\eta > 0$, $s_1 + s_2 = 2s$, $s_2 < s$, and $s_1, s_2, s \in (0, 1)$. Suppose that $p \in (1, \infty)$ and $\Sigma \subset \mathbb{S}^{n-1}$ such that $|\Sigma| > 0$. Let $K : \mathbb{S}^{n-1} \to [0, \Lambda]$ be a symmetric nonnegative, bounded and measurable function, such that*

$$0 < \eta := \inf_{\Sigma} K. \tag{2.1}$$

Then there exists a constant $C = C(\eta, s_1, s_2, s, \Sigma) > 0$ such that the following holds: for any g with $\|g\|_{W^{-s_2,p}(\mathbb{R}^n)} \equiv \|g\|_{\left(W^{s_2,p'}(\mathbb{R}^n)\right)^*} \lesssim \Lambda$, i.e. $|g[\varphi]| \leq \Lambda \, [\varphi]_{W^{s_2,p'}(\mathbb{R}^n)}$, and a solution $u \in W^{s,2}(\mathbb{T}^n)$ to the equation

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \mu \, (x - y) \, (u(x) - u(y))(\varphi(x) - \varphi(y)) \, dx \, dy = g[\varphi] \quad \forall \varphi \in C^\infty(\mathbb{T}^n)$$

with

$$\mu(h) := \sum_{m \in \mathbb{Z}^n} \frac{K\left(\frac{h+m}{|h+m|}\right)}{|h + m|^{n+2s}},$$

we have $u \in W^{s_1,p}(\mathbb{T}^n)$. Moreover, we have the estimate

$$[u]_{W^{s_1,p}(\mathbb{T}^n)} \leq C \, \Lambda + \|u\|_{L^2(\mathbb{T}^n)}.$$

Our proof of the theorem follows that of a similar result proved by H. Dong and D. Kim in [8] for nonlocal equations posed on $\mathbb{R}^n$. There are some distinctions however. First, the nonlocal equation we study is posed over the torus. Second, we have a relaxed ellipticity condition, namely, ellipticity of K is assumed over subset of $\mathbb{S}^d$ with positive measure as opposed to the whole of $\mathbb{S}^d$. As the following lemma demonstrates, this is enough to establish the ellipticity of the associated operator.

Lemma 2.2 *Let $K : \mathbb{S}^{n-1} \to [0, 1]$ be measurable and satisfy (2.1) with $\Sigma \subset \mathbb{S}^{n-1}$ and $|\Sigma| > 0$. Then, for any $\bar{r}, R, R' > 0$ with $RR' \geq \bar{r} > 0$, there exists $c = c(n, s, \Sigma, \bar{r}) > 0$ such that for all $\xi \in \mathbb{R}^N \setminus B(0, R')$,*

$$\int_{B(0,R)} \frac{1 - \cos(2\xi \cdot h)}{|h|^{n+2s}} K(h/|h|) \, dh \geq c \, \eta |\xi|^{2s}. \tag{2.2}$$

If $R = +\infty$ and $R' = 0$, then (2.2) holds with $\bar{r} = 1$. In particular, for any $k \in \mathbb{Z}^n$, we have

$$\int_{\mathbb{T}^n} (1 - \cos(2\pi \langle k, h \rangle)) \sum_{m \in \mathbb{Z}^n} \frac{K\left(\frac{h+m}{|h+m|}\right)}{|h + m|^{n+2s}} \, dh \geq c(n, s, \Sigma)\eta |k|^{2s}.$$

Proof By (2.1), for $\xi \in \mathbb{R}^N \setminus B(0, R')$, we have

$$
\begin{aligned}
\int_{B(0,R)} \frac{1 - \cos(2\xi \cdot h)}{|h|^{n+2s}} K(h/|h|)\, dh &= \int_{\mathbb{S}^{N-1}} \int_0^R \frac{1 - \cos(2\xi \cdot t\theta)}{t^{1+2s}} K(\theta)\, d\theta dt \\
&= |\xi|^{2s} \int_{\mathbb{S}^{N-1}} \int_0^{|\xi|R} \frac{1 - \cos(2\xi/|\xi| \cdot r\theta)}{r^{1+2s}} K(\theta)\, d\theta dr \\
&\geq \eta |\xi|^{2s} \int_\Sigma \int_0^{R'R} \frac{1 - \cos(2\xi/|\xi| \cdot r\theta)}{r^{1+2s}}\, d\theta dr \\
&\geq \eta |\xi|^{2s} \int_\Sigma \int_0^{\bar{r}} \frac{1 - \cos(2\xi/|\xi| \cdot r\theta)}{r^{1+2s}}\, d\theta dr. \qquad (2.3)
\end{aligned}
$$

From the Taylor expansion of cosine, we can find a constant $r_0 \in (0, \bar{r})$ such that

$$
1 - \cos(2\sigma \cdot r\theta) \geq r^2 \frac{|\sigma \cdot \theta|^2}{4} \qquad \text{for all } \sigma, \theta \in \mathbb{S}^{n-1} \text{ and } r \in (0, r_0). \qquad (2.4)
$$

Moreover, it is clear that the map

$$
\mathbb{S}^{n-1} \to \mathbb{R}, \qquad \sigma \mapsto f(\sigma) := \int_\Sigma |\sigma \cdot \theta|^2\, d\theta
$$

attains its minimum at some $\sigma_0 \in \mathbb{S}^{n-1}$. It is plain that $f(\sigma_0) > 0$, because otherwise $\sigma \cdot \theta = 0$ for all $\theta \in \Sigma$, which contradicts that $|\Sigma| > 0$. Using this and (2.4) in (2.3), we obtain

$$
\int_{B(0,R)} \frac{1 - \cos(2\xi \cdot h)}{|h|^{n+2s}} K(h/|h|)\, dh \geq \eta \frac{|\xi|^{2s}}{4} \int_\Sigma \int_0^{r_0} \frac{|\xi/|\xi| \cdot \theta|^2}{r^{-1+2s}}\, d\theta dr \geq \frac{\eta r_0^{2-2s} f(\sigma_0)}{8(1-s)} |\xi|^{2s}.
$$

This gives (2.2). $\qquad \square$

Lemma 2.3 *With K and μ as in Theorem 2.1, let*

$$
Lu(x) := \int_{\mathbb{T}^n} (u(x) - u(y))\, \mu(x - y)\, dy.
$$

Let $u \in W^{s,2}(\mathbb{T}^n)$ satisfying $Lu = g$ in the distributional sense in $\mathbb{T}^n$, where $g \in L^2(\mathbb{T}^n)$. Then

$$
|u|_{H^{2s}(\mathbb{T}^n)} \lesssim \|g\|_{L^2(\mathbb{T}^n)}.
$$

Proof First, we consider the Fourier transform of Lu on torus. Observe that for $u \in W^{s,2}(\mathbb{T}^n)$ the expression Lu is a distibution on $\mathbb{T}^n$ - in particular $\mathcal{F} Lu(k)$ is well-defined since $e^{-2\pi \iota \langle k, x \rangle} \in C^\infty(\mathbb{T}^n)$. Then

$$
\mathcal{F}(Lu)(k) = \left\langle Lu(x), e^{-2\pi \iota \langle k, x \rangle} \right\rangle = \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (u(x) - u(y))\, \mu(x - y) \left(e^{-2\pi \iota \langle k, x \rangle} - e^{-2\pi \iota \langle k, y \rangle} \right) dy\, dx.
$$

Changing variable $x - y = h$, we obtain

$$
\begin{aligned}
\mathcal{F}(Lu)(k) &= \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (u(x) - u(x - h))\, \mu(h) \left(e^{-2\pi \iota \langle k, x \rangle} - e^{-2\pi \iota \langle k, x - h \rangle} \right) dx\, dh \\
&= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (u(x) - u(x - h))\, \mu(h)\, e^{-2\pi \iota \langle k, x \rangle}\, dx\, dh \\
&= \mathcal{F}(u)(k) \int_{\mathbb{T}^n} (1 - e^{2\pi \iota \langle k, h \rangle})\, \mu(h)\, dh.
\end{aligned}
$$

By Plancherel's identity, we get

$$
\begin{aligned}
\|Lu\|_{L^2(\mathbb{R}^n)}^2 &= \sum_{k \in \mathbb{Z}^n} |\mathcal{F}(Lu)(k)|^2 \\
&= \sum_{k \in \mathbb{Z}^n} |\mathcal{F}(u)(k)|^2 \left| \int_{\mathbb{T}^n} (1 - e^{2\pi \iota \langle k, h \rangle}) \, \mu(h) \, dh \right|^2 \\
&\geq \sum_{k \in \mathbb{Z}^n} |\mathcal{F}(u)(k)|^2 \left| \Re \int_{\mathbb{T}^n} (1 - e^{2\pi \iota \langle k, h \rangle}) \, \mu(h) \, dh \right|^2 \\
&= \sum_{k \in \mathbb{Z}^n} |\mathcal{F}(u)(k)|^2 \left(\int_{\mathbb{T}^n} (1 - \cos(2\pi \langle k, h \rangle)) \, \mu(h) \, dh \right)^2 \\
&= \sum_{k \in \mathbb{Z}^n} |\mathcal{F}(u)(k)|^2 \left(\int_{\mathbb{T}^n} (1 - \cos(2\pi \langle k, h \rangle)) \sum_{m \in \mathbb{Z}^n} \frac{K(\frac{h+m}{|h+m|})}{|h+m|^{n+2s}} \, dh \right)^2 .
\end{aligned}
$$

In view of Equation 2.2, we find that

$$
\|g\|_{L^2(\mathbb{T}^n)}^2 = \|Lu\|_{L^2(\mathbb{T}^n)}^2 \gtrsim \sum_{k \in \mathbb{Z}^n} ||k|^{2s} \mathcal{F}(u)(k)|^2 = |u|_{H^{2s}(\mathbb{T}^n)}^2.
$$

$\square$

The above lemmata and related estimates not only say that for any $\lambda > 0$, the operator $\lambda I - L : H^{2s}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is bounded, but also that if for any $g \in L^2(\mathbb{T}^n)$, there is $u \in H^{2s}(\mathbb{T}^n)$ such that $\lambda u - Lu = g$. See [8, Remark 3.6.]. Moreover, if $u \in W^{s,2}(\mathbb{T}^n)$ solves $Lu = g$, then $\|(-\Delta)_{\mathbb{T}^n}^{\frac{2s}{2}} u\|_{L^2(\mathbb{T}^n)} \lesssim \|g\|_{L^2(\mathbb{T}^n)}$. In the event, $g \in L^p(\mathbb{T}^n)$, we follow almost the same arguments as in [8] to obtain the corresponding L^p estimates. The argument relies on mean oscillation estimates on u and $(-\Delta)_{\mathbb{T}^n}^{\frac{2s}{2}} u$ which in turn uses a Hölder-regularity estimates on smooth solution of $Lu = g$. A careful examination of the proof of [8, Theorem 2.1] reveals that in fact a Hölder estimate of smooth solutions for some small exponent is sufficient in order to carry out the argument. An alternative argument to a proof of such small exponent Hölder regularity estimate is sketched in Appendix 1. We summarize the L^p-regularity result in the following proposition.

Proposition 2.4 *Let u, μ, g be as in Theorem 2.1. Then,*

$$
\|(-\Delta)_{\mathbb{T}^n}^{\frac{2s}{2}} u\|_{L^p(\mathbb{T}^n)} \lesssim \|g\|_{L^p(\mathbb{T}^n)}.
$$

We also observe that we can differentiate the convolution type equation. Namely, we have

Lemma 2.5 *Let u, μ, g be as in Theorem 2.1. Assume that u satisfies*

$$
\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) \, \mu(x - y) \, dx \, dy = g[\varphi], \quad \forall \varphi \in C_{per}^\infty(\mathbb{T}^n).
$$

Then $v := (-\Delta)^{\frac{\sigma}{2}} u$, for $\sigma \in \mathbb{R}$, satisfies

$$
\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (v(x) - v(y))(\varphi(x) - \varphi(y)) \, \mu(x - y) \, dx \, dy = g[(-\Delta)^{\frac{\sigma}{2}} \varphi], \quad \forall \varphi \in C_{per}^\infty(\mathbb{T}^n).
$$

Proof For every fixed $y \in \mathbb{T}^n$, by changing of variable choosing $h = x - y$, we get

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (v(x) - v(y))(\varphi(x) - \varphi(y)) \, \mu(x - y) \, dx \, dy$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} ((-\Delta)^{\frac{\sigma}{2}} u(x) - (-\Delta)^{\frac{\sigma}{2}} u(y))(\varphi(x) - \varphi(y)) \, \mu(x - y) \, dx \, dy$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} ((-\Delta)^{\frac{\sigma}{2}} u(y + h) - (-\Delta)^{\frac{\sigma}{2}} u(y))(\varphi(y + h) - \varphi(y)) \, \mu(h) \, dh \, dy.$$

Then, using Parseval's relation, we get

$$= \int_{\mathbb{T}^n} \sum_{m \in \mathbb{Z}^n} |e^{2\pi \imath m \cdot h} - 1|^2 |m|^\sigma \hat{u}(m) \hat{\varphi}(m) \, \mu(h) \, dh$$

$$= \int_{\mathbb{T}^n} \sum_{m \in \mathbb{Z}^n} |e^{2\pi \imath m \cdot h} - 1|^2 \hat{u}(m) \widehat{(-\Delta)^{\frac{\sigma}{2}} \varphi}(m) \, \mu(h) \, dh$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (u(y + h) - u(y))((-\Delta)^{\frac{\sigma}{2}} \varphi(y + h) - (-\Delta)^{\frac{\sigma}{2}} \varphi(y)) \, \mu(h) \, dh \, dy$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (u(x) - u(y))((-\Delta)^{\frac{\sigma}{2}} \varphi(x) - (-\Delta)^{\frac{\sigma}{2}} \varphi(y)) \, \mu(x - y) \, dx \, dy$$

$$= g[(-\Delta)^{\frac{\sigma}{2}} \varphi].$$

$\square$

Now we obtain the full result of Theorem 2.1 from interpolation.

Proof of Theorem 2.1 Let T be the solution operator, i.e., for given g, Tg is the solution of

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (Tg(x) - Tg(y))(\varphi(x) - \varphi(y)) \, \mu(x - y) \, dx \, dy = g[\varphi] \quad \forall \varphi \in C^\infty(\mathbb{T}^n).$$

Observe that $Tg = u$, where such a solution can be shown to exist in $W^{s,2}(\mathbb{T}^n)$ via variational methods. From Proposition 2.4, we obtain

$$\|(-\Delta)^{\frac{2s}{2}} Tg\|_{L^p(\mathbb{T}^n)} \lesssim \|g\|_{L^p(\mathbb{T}^n)}.$$

In view of Lemma 2.5 the above estimate implies that for any $\sigma \in \mathbb{R}$

$$\|(-\Delta)^{\frac{2s+\sigma}{2}} Tg\|_{L^p(\mathbb{T}^n)} \lesssim \|(-\Delta)^{\frac{\sigma}{2}} g\|_{L^p(\mathbb{T}^n)}.$$

That is for any $\sigma \in \mathbb{R}$,

$$T : \dot{H}^{\sigma,p}(\mathbb{T}^n) \mapsto \dot{H}^{2s+\sigma,p}(\mathbb{T}^n)$$

is a linear, bounded operator. Take $\sigma_0 := -s$ and $\sigma_1 := 0$. Observe that $H^{\sigma,p} = F^\sigma_{p,2}$. Then we have

$$T : \dot{H}^{-s,p}(\mathbb{T}^n) \mapsto \dot{H}^{s,p}(\mathbb{T}^n),$$
$$T : \dot{H}^{s,p}(\mathbb{T}^n) \mapsto \dot{H}^{2s,p}(\mathbb{T}^n)$$

are both bounded operators.

By [20, 2.5.1, p.86, Proposition and Remark] we obtain by interpolation

$$T : B^\sigma_{p,p}(\mathbb{T}^n) \mapsto B^{2s+\sigma}_{p,p}(\mathbb{T}^n)$$

is continuous and bounded for any $\sigma \in (-s, 0)$, that is

$$\|Tg\|_{B_{p,p}^{2s+\sigma}(\mathbb{T}^n)} \lesssim \|g\|_{B_{p,p}^{\sigma}(\mathbb{T}^n)}.$$

Since $B_{p,p}^{\sigma} = W^{\sigma,p}$ we conclude for $\sigma := -s_2$,

$$\|Tg\|_{W^{s_1,p}(\mathbb{T}^n)} = \|Tg\|_{B_{p,p}^{s_1}(\mathbb{T}^n)} \lesssim \|g\|_{B_{p,p}^{-s_2}(\mathbb{T}^n)} \equiv \|g\|_{\left(W^{s_2,p'}(\mathbb{T}^n)\right)^*}.$$

Since $u = Tg$ we conclude.   □

3 Local estimates

Our proof of Theorem 1.1 uses a perturbative argument where we view the nonlocal equation of interest as a small perturbation of a convolution-type equation on the torus. This is achieved by a freezing of coefficient method facilitated by the uniform continuity assumption on the kernel K given by (1.3). This leads to a convolution-type equation as treated in [8], see Theorem 2.1. Applying Theorem 2.1 on the convolution-type equation leads to local priori estimates (because of the term (3.3)). We convert these a priori estimates to regularity estimates with the help of a fixed point argument in Sect. 3.2.

3.1 Local *a priori* estimates via freezing coefficients argument

Proposition 3.1 *(Freezing the coefficient) Let* $s \in (0, 1)$, $t \in [s, 1)$ *and* $p \in [2, \infty)$. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\Omega' \subset\subset \Omega$. *Let* $u \in W^{s,2}(\mathbb{R}^n) \cap W^{t,p}(\Omega')$ *be a solution to*

$$\int_{\Omega} \int_{\Omega} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dx\, dy = g[\varphi] \quad \forall \varphi \in C_c^\infty(\Omega'). \quad (3.1)$$

where K is bounded, nondegenerate, and satisfies Equation 1.2. Let $x_0 \in \Omega'$ and $R > 0$ such that $B(x_0, 60\sqrt{n}R) \subset \Omega'$, and $\eta \in C_c^\infty(B(x_0, 6R))$ with $\eta \equiv 1$ in $B(x_0, 5R)$. Set $\tilde{\mathbb{T}} := x_0 + 30R\left[\frac{-1}{2}, \frac{1}{2}\right]^n \subset B(x_0, 60\sqrt{n}R)$ and define the $\tilde{\mathbb{T}}$-periodic function

$$\mu(h) := \sum_{k \in 30R\mathbb{Z}^n} \frac{K\left(x_0, 0, \frac{h+k}{|h+k|}\right)}{|h+k|^{n+2s}}, \quad for\ h \in \tilde{\mathbb{T}}.$$

Denote by $v := \eta u$. Then

$$\int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x-y)\,(v(x)-v(y))\,(\psi(x)-\psi(y))\,dx\,dy$$
$$= g[\eta\psi] + \mathcal{H}(v, \psi) + \mathcal{G}(u, \psi), \quad (3.2)$$

for all $\psi \in C_{per}^\infty\left(\overline{\tilde{\mathbb{T}}}\right)$, where $\mathcal{H}$ and $\mathcal{G}$ are bilinear forms with the property that for any $s_1, s_2 \in (0, 1)$ with $s_1 + s_2 = 2s$ and any $p_1, p_2 \in (1, \infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ we have

$$|\mathcal{H}(a, b)| \leq \sup_{x \in B(x_0, 10R), |h|=1, r \in [0, 10R]}$$
$$|K(x, r, h) - K(x_0, 0, h)|\ [a]_{W_{per}^{s_1, p_1}(\tilde{\mathbb{T}})}\ [b]_{W_{per}^{s_2, p_2}(\tilde{\mathbb{T}})}, \quad (3.3)$$

Moreover for any $\max\{0, 2s - 1\} < \tilde{t} < 2s - t$ *and* $q \le \frac{p}{p-1}$ *such that*

$$\min\left\{t - \frac{n}{q}, \tilde{t} - \frac{n}{q}\right\} \ge -\frac{n}{p'} \tag{3.4}$$

we have $|\mathcal{G}(a, b)| \le C(R, \eta, \|K\|_{L^\infty}) \left(\|b\|_{L^q(\mathbb{T}^n)} + [b]_{W^{\tilde{t}, q}(\mathbb{T})}\right) \left(\|a\|_{L^2(\tilde{\mathbb{T}})} + [a]_{W^{s, 2}(\tilde{\mathbb{T}})} + [a]_{W^{t, p}(\tilde{\mathbb{T}})}\right).$

Proof Without loss of generality (otherwise we rescale and translate) we may assume that $x_0 = 0$ and $R = \frac{1}{30}$ so that $0 \in \Omega' \subset\subset \Omega$, $\tilde{\mathbb{T}}$ becomes the unit torus $\mathbb{T}^n = \left[\frac{-1}{2}, \frac{1}{2}\right]^n$, and $B(0, 2\sqrt{n})$ contains a number of copies of $\mathbb{T}^n$. The function μ now becomes

$$\mu(h) = \sum_{k \in \mathbb{Z}^n} \frac{K(0, 0, \frac{h+k}{|h+k|})}{|h + k|^{n+2s}}, \quad \text{for } h \in \mathbb{T}^n,$$

and $\eta \in C_c^\infty(B(0, \frac{1}{5}))$ with $\eta \equiv 1$ in $B(0, \frac{1}{6})$.

For $\psi \in C_{per}^\infty(\overline{\mathbb{T}^n})$, by a slight abuse of notation, we can assume that ψ is equal to its periodic extension $\psi \in C^\infty(\mathbb{R}^n)$.

After writing μ as

$$\mu(x - y) = \frac{K(0, 0, \frac{x-y}{|x-y|})}{|x - y|^{n+2s}} + \frac{\tilde{K}(x, y)}{|x - y|^{n+2s}}$$

with $\tilde{K}(x, y) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} K\left(0, 0, \frac{x-y+k}{|x-y+k|}\right) \frac{|x-y|^{n+2s}}{|x-y+k|^{n+2s}}$, we view it as a perturbation of $\frac{K(0, 0, \frac{x-y}{|x-y|})}{|x-y|^{n+2s}}$. This is motivated by the fact that

$$\sup_{x \in B(0, \frac{1}{5}), y \in \mathbb{T}^n} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{K\left(0, 0, \frac{x-y+k}{|x-y+k|}\right)}{|x - y + k|^{n+2s}} + \sup_{y \in B(0, \frac{1}{5}), x \in \mathbb{T}^n} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{K\left(0, 0, \frac{x-y+k}{|x-y+k|}\right)}{|x - y + k|^{n+2s}} \lesssim 1. \tag{3.5}$$

and thus for $(x, y) \in \mathbb{T}^n \times \mathbb{T}^n$ and either $x \in \operatorname{supp} v$ or $y \in \operatorname{supp} v$ (or both) then

$$\left|\mu(x - y) - \frac{K(0, 0, \frac{x-y}{|x-y|})}{|x - y|^{n+2s}}\right| \lesssim 1.$$

Consequently we obtain the decomposition

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \mu(x - y) (v(x) - v(y)) (\psi(x) - \psi(y)) \, dx \, dy$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K\left(0, 0, \frac{x - y}{|x - y|}\right) \frac{(v(x) - v(y)) (\psi(x) - \psi(y))}{|x - y|^{n+2s}} dx \, dy + \mathcal{G}_1(u, \psi) \tag{3.6}$$

where

$$\mathcal{G}_1(u, \psi) = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \tilde{K}(x, y) \frac{(v(x) - v(y)) (\psi(x) - \psi(y))}{|x - y|^{n+2s}} dx \, dy.$$

We are going to view the first term in the right hand side of (3.6) as a 'frozen coefficient' of the original nonlocal equation. To make proper comparison, we set $\tilde{\psi} := \tilde{\eta}(\psi - (\psi)_{\mathbb{T}^n})$ for

a reasonable cutoff function $\tilde{\eta} \in C_c^\infty(2\mathbb{T}^n)$ and $\tilde{\eta} \equiv 1$ in $\mathbb{T}^n$. Notice that $2\mathbb{T}^n \subset \Omega$. It then follows that

$$
\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K\left(0, 0, \frac{x-y}{|x-y|}\right) \frac{(v(x) - v(y))\,(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\,dx\,dy
$$
$$
= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K\left(0, 0, \frac{x-y}{|x-y|}\right) \frac{(v(x) - v(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy
$$
$$
= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(v(x) - v(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy
$$
$$
+ \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} E_K(x, y) \frac{(v(x) - v(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy,
$$

where $E_K(x, y) = K\left(0, 0, \frac{x-y}{|x-y|}\right) - K\left(x, |x-y|, \frac{x-y}{|x-y|}\right)$ is the measure of the pointwise deviation of the main kernel from $K\left(0, 0, \frac{x-y}{|x-y|}\right)$. We may thus write

$$
\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K\left(0, 0, \frac{x-y}{|x-y|}\right) \frac{(v(x) - v(y))\,(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\,dx\,dy
$$
$$
= \mathcal{H}_1(u, \tilde{\psi}) + \mathcal{H}_2(v, \psi) + \mathcal{G}_2(u, \psi) + \mathcal{G}_3(u, \tilde{\psi}) + \mathcal{G}_4(u, \tilde{\psi}),
$$

where $\mathcal{H}_1(u, \tilde{\psi})$

$$
\mathcal{H}_1(u, \tilde{\psi}) := \int_\Omega \int_\Omega K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(v(x) - v(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy
$$

and the remaining terms come from decomposing the deviation from the frozen term and are given by

$$
\mathcal{H}_2(v, \psi) := \int_{B(x_0, \frac{1}{3})} \int_{B(x_0, \frac{1}{3})} E_K(x, y) \frac{(v(x) - v(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy,
$$

$$
\mathcal{G}_2(u, \psi) := \int_{B(x_0, \frac{1}{3})} \int_{\Omega \setminus \mathbb{T}^n} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(\eta(x)u(x) - \eta(y)u(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy,
$$

$$
\mathcal{G}_3(u, \psi) := \int_{\Omega \setminus \mathbb{T}^n} \int_{B(x_0, \frac{1}{3})} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(\eta(x)u(x) - \eta(y)u(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy,
$$

and after recalling that the support of η is in $B(0, \frac{1}{5}))$

$$
\mathcal{G}_4(u, \psi) := \int_{B(0, \frac{1}{5})} \int_{\mathbb{T}^n \setminus B(0, \frac{1}{3})} E_K(x, y) \frac{(\eta(x)u(x) - \eta(y)u(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy
$$
$$
+ \int_{\mathbb{T}^n \setminus B(0, \frac{1}{3})} \int_{B(x_0, \frac{1}{3})} E_K(x, y) \frac{(\eta(x)u(x) - \eta(y)u(y))\left(\tilde{\psi}(x) - \tilde{\psi}(y)\right)}{|x-y|^{n+2s}}\,dx\,dy
$$

Using the identity $(a - b)(cd - ef) + (c - e)(bd - af) = (ac - be)(d - f)$ we may rewrite the expression in the integrand of $\mathcal{H}_1$ as

$$(v(x) - v(y))(\tilde{\psi}(x) - \tilde{\psi}(y))$$
$$= (u(x) - u(y))(\eta(x)\tilde{\psi}(x) - \eta(y)\tilde{\psi}(y)) + (\eta(x) - \eta(y))\left(u(y)\tilde{\psi}(x) - u(x)\tilde{\psi}(y)\right).$$

Consequently, using the fact that u solves the nonlocal equation (3.1) we have

$$\mathcal{H}_1(u, \tilde{\psi}) = g[\eta\tilde{\psi}] + \mathcal{G}_5(u, \psi),$$

where

$$\mathcal{G}_5(u, \psi) := \int_\Omega \int_\Omega K\left(x, |x - y|, \frac{x - y}{|x - y|}\right) \frac{(\eta(x) - \eta(y))\left(u(y)\tilde{\psi}(x) - u(x)\tilde{\psi}(y)\right)}{|x - y|^{n+2s}}$$
$$dx\,dy.$$

We may now summarize the above discussion to rewrite (3.6) as

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \mu(x - y)\,(v(x) - v(y))\,(\psi(x) - \psi(y))\,dx\,dy = g[\eta\tilde{\psi}] + \mathcal{H}(v, \psi) + \mathcal{G}(u, \psi),$$

where $\mathcal{H}(v, \psi) = \mathcal{H}_2(v, \psi)$, and $\mathcal{G}(u, \psi) = \sum_{i=1}^5 \mathcal{G}_i(u, \psi)$, which is precisely the expression in (3.2). What remains is to estimate to each of the terms to conclude the proof of the proposition. To that end, the estimate (3.3) follows from the expression $\mathcal{H}_2(v, \psi)$ and from Hölder's inequality. For the estimate for $\mathcal{G}$ will estimate each of the terms of $\mathcal{G}$. We begin estimating $\mathcal{G}_1$. First observe that using (3.5) and since $v = \eta u \in W_0^{t,p}(\mathbb{T}^n)$ applying Poincaré inequality to obtain

$$\mathcal{G}_1(u, \psi) \lesssim \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |v(x) - v(y)||\psi(x) - \psi(y)|\,dx\,dy$$
$$\lesssim \|u\|_{W^{t,p}(\mathbb{T}^n)} \left(\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\psi(x) - \psi(y)|^{p'}\right)^{\frac{1}{p'}}.$$

By Sobolev-Poincarè embedding the last factor can be estimates as

$$\left(\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\psi(x) - \psi(y)|^{p'}\,dx\,dy\right)^{\frac{1}{p'}} \lesssim [\psi]_{W^{\tilde{t},q}(\mathbb{T}^n)},$$

so that we have

$$\mathcal{G}_1(u, \psi) \lesssim \|u\|_{W^{t,p}(\mathbb{T}^n)} [\psi]_{W^{\tilde{t},q}(\mathbb{T}^n)}.$$

To estimate of $\mathcal{G}_2$, we notice that for $x \in \Omega\backslash\mathbb{T}^n$ and $y \in B(0, \frac{1}{3})$ we have $|x - y| \approx |x - y| + 1$. Thus,

$$|\mathcal{G}_2(u, \psi)| \lesssim \int_{B(0,\frac{1}{3})} \int_{\Omega\backslash\mathbb{T}^n} \frac{1}{|x - y|^{n+2s} + 1} |u(y)| \left|\tilde{\psi}(x) - \tilde{\psi}(y)\right| dx\,dy$$
$$\lesssim \|\tilde{\psi}\|_{L^1(\Omega)} \|u\|_{L^1(B(0,\frac{1}{3}))} + \|u\tilde{\psi}\|_{L^1(B(0,\frac{1}{3}))}.$$

Observe that from periodicity and Sobolev embedding we obtain that

$$\|\tilde{\psi}\|_{L^1(\mathbb{R}^n)} + \|\tilde{\psi}\|_{L^{p'}(\mathbb{R}^n)} \lesssim [\psi]_{W^{\tilde{t},q}(\mathbb{T}^n)},$$

and

$$\|u\|_{L^1(B(0,\frac{1}{3}))} + \|u\|_{L^p(B(0,\frac{1}{3}))} \lesssim \left(\|u\|_{L^2(\mathbb{T})} + [u]_{W^{s,2}(\mathbb{T})} + [u]_{W^{t,p}(\mathbb{T})} \right).$$

Thus we obtain the estimate

$$\mathcal{G}_2(u, \psi) \lesssim [\psi]_{W^{\tilde{t},\tilde{q}}(\mathbb{T}^n)} \left(\|u\|_{L^2(\mathbb{T})} + [u]_{W^{s,2}(\mathbb{T})} + [u]_{W^{t,p}(\mathbb{T})} \right).$$

The procedure to estimate $\mathcal{G}_3$ is exactly the same as that of $\mathcal{G}_2$ and so we have

$$\mathcal{G}_3(u, \psi) \lesssim \int_{\Omega \setminus \mathbb{T}^n} \int_{B(0,\frac{1}{3})} \frac{1}{|x-y|^{n+2s}+1} |u(x)| \left| \tilde{\psi}(x) - \tilde{\psi}(y) \right| dx \, dy$$

$$\lesssim [\psi]_{W^{\tilde{t},\tilde{q}}(\mathbb{T}^n)} \left(\|u\|_{L^2(\mathbb{T})} + [u]_{W^{s,2}(\mathbb{T})} + [u]_{W^{t,p}(\mathbb{T})} \right).$$

To estimate of $\mathcal{G}_4$ first notice that since the support of η is contained in $B(0, \frac{1}{5})$,

$$|\mathcal{G}_4(u, \psi)| = \left| \int_{B(0,\frac{1}{5})} \int_{\mathbb{T}^n \setminus B(0,\frac{1}{3})} \frac{\left| -\eta(y)u(y) \left(\tilde{\psi}(x) - \tilde{\psi}(y) \right) \right|}{|x-y|^{n+2s}} dx \, dy \right|$$

Now arguing in exactly similar way as the estimate for $\mathcal{G}_2$ we obtain

$$|\mathcal{G}_4(u, \psi)| \lesssim \int_{B(0,\frac{1}{5}))} \int_{\mathbb{T}^n \setminus B(0,\frac{1}{3})} \frac{1}{|x-y|^{n+2s}+1} |u(y)| \left| \tilde{\psi}(x) - \tilde{\psi}(y) \right| dx \, dy$$

$$\lesssim [\psi]_{W^{\tilde{t},q}(\mathbb{T}^n)} \left(\|u\|_{L^2(\mathbb{T})} + [u]_{W^{s,2}(\mathbb{T})} + [u]_{W^{t,p}(\mathbb{T})} \right).$$

Finally, we estimate $\mathcal{G}_5$ as follows: using $|\eta(x) - \eta(y)| \lesssim |x-y|$ for $x, y \in \Omega'$,

$$|\mathcal{G}_5(u, \tilde{\psi})| \lesssim \int_{\Omega'} \int_{\Omega'} \frac{|u(x)||\tilde{\psi}(x) - \tilde{\psi}(y)| + |\tilde{\psi}(x)||u(x) - u(y)|}{|x-y|^{n+2s-1}} dx \, dy$$

$$+ \int_{\Omega \setminus \Omega'} \int_{\Omega'} \chi_{|x-y| \gtrsim R} \frac{|u(x)||\tilde{\psi}(y)| + |\tilde{\psi}(x)||u(y)|}{|x-y|^{n+2s}} dx \, dy.$$

For the first term, we observe that

$$\int_{\Omega'} \int_{\Omega'} \frac{|u(x)||\tilde{\psi}(x) - \tilde{\psi}(y)|}{|x-y|^{n+2s-1}} dx \, dy$$

$$= \int_{\Omega'} \int_{\Omega'} |x-y|^{\tilde{t}-(2s-1)} |u(x)| \frac{|\tilde{\psi}(x) - \tilde{\psi}(y)|}{|x-y|^{n+\tilde{t}}} dx \, dy$$

$$\lesssim [\tilde{\psi}]_{W^{\tilde{t},q}} \left(\int_{\Omega'} \int_{\Omega'} |x-y|^{\tilde{t}q'-(2s-1)q'-n} |u(x)|^{q'} dx \, dy \right)^{\frac{1}{q'}}$$

$$\lesssim [\tilde{\psi}]_{W^{\tilde{t},q}} \|u\|_{L^{q'}(\Omega')}$$

$$\lesssim [\tilde{\psi}]_{W^{\tilde{t},q}} \left(\|u\|_{L^2(\Omega')} + [u]_{W^{t,p}(\Omega')} \right).$$

The the second inequality follows from the fact that $\tilde{t} > 2s - 1$ and

$$\sup_{x \in \Omega'} \int_{\Omega'} |x-y|^{\tilde{t}q'-(2s-1)q'-n} dy \leq C(\text{diam}(\Omega'), q', n, s, \tilde{t}).$$

Also since $t > s$ and so $t > 2s - 1$ and $s \in (0, 1)$, similarly as above we have

$$\int_{\Omega'} \int_{\Omega'} \frac{|\tilde{\psi}(x)||u(x) - u(y)|}{|x-y|^{n+2s-1}} dx \, dy \lesssim [u]_{W^{t,p}(\Omega')} \|\tilde{\psi}\|_{L^{p'}(\Omega')} \lesssim [u]_{W^{t,p}(\Omega')} [\psi]_{W^{\tilde{t},p'}(\mathbb{T}^n)}.$$

As for the second term: as in [11, Lemma 5.2.] for any $r \in [1, \infty]$,

$$\sup_{x \in \Omega'} \int_{\Omega \setminus \Omega'} \chi_{|x-y| \gtrsim R} \frac{|g(y)|}{|x-y|^{n+2s}} \, dy \lesssim \|g\|_{L^r(\Omega)}$$

and thus

$$\left| \int_{\Omega \setminus \Omega'} \int_{\Omega'} \chi_{|x-y| \gtrsim R} \frac{|u(x)||\tilde{\psi}(y)| + |\tilde{\psi}(x)||u(y)|}{|x-y|^{n+2s}} \, dx \, dy \right| \lesssim \|\tilde{\psi}\|_{L^1(\Omega')} \|u\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega')} \|\tilde{\psi}\|_{L^q(\Omega)}$$

$$\lesssim \|\tilde{\psi}\|_{L^q(\Omega')} \|u\|_{L^2(\Omega)}$$

$$\lesssim [\psi]_{W^{\tilde{t},q}(\mathbb{T}^n)} \|u\|_{L^2(\Omega)}.$$

Collecting the estimates for $\mathcal{G}_i$ for $i = 1, 2 \cdots, 5$ completes the proof of the proposition. □

Corollary 3.2 *Fix $u \in W^{t,p} \cap W^{s,2}(\Omega')$ under the assumptions for K, t, $\tilde{t}$, s, p, and q as well as R and $\tilde{\mathbb{T}}$ as in Proposition 3.1.*

For a given distribution g such that there is some $\Lambda > 0$ with the

$$|g[\varphi]| \le \Lambda \left(\|\varphi\|_{L^q(\mathbb{R}^n)} + [\varphi]_{W^{\tilde{t},q}(\mathbb{R}^n)} \right),$$

for all $\varphi \in C_c^\infty(B(x_0, 30R))$, let $v \in W^{s,2}(\tilde{\mathbb{T}})$ be a solution to (3.2). If in addition $v \in W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$, then there exists a small enough $\varepsilon > 0$ (depending on x_0, t, s, p) such that for any K satisfying (1.3) we have

$$[v]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} \lesssim \|u\|_{L^2(\tilde{\mathbb{T}})} + [u]_{W^{s,2}(\tilde{\mathbb{T}})} + [u]_{W^{t,p}(\tilde{\mathbb{T}})} + \Lambda. @ \tag{3.7}$$

Proof We notice that (3.2) can be rewritten as

$$\int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x-y)(v(x) - v(y))(\psi(x) - \psi(y)) \, dx \, dy = \Upsilon[\psi],$$

where $\Upsilon[\psi] := g[\eta\psi] + \mathcal{H}(v, \psi) + \mathcal{G}(u, \psi)$ for any $\psi \in C_{per}^\infty(\tilde{\mathbb{T}})$. Next, under all the assumptions of Proposition 3.1, the estimates for the bilinear forms $\mathcal{H}$ and $\mathcal{G}$ combined with (1.3) we have the estimate for

$$|\Upsilon[\psi]| \lesssim [\psi]_{W^{\tilde{t},q}(\tilde{\mathbb{T}})} \left(\varepsilon [v]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} + \|u\|_{L^2(\tilde{\mathbb{T}})} + [u]_{W^{s,2}(\tilde{\mathbb{T}})} + [u]_{W^{t,p}(\tilde{\mathbb{T}})} + \Lambda \right).$$

We can now apply Theorem 2.1 and obtain

$$[v]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} \lesssim \varepsilon [v]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} + \|u\|_{L^2(\tilde{\mathbb{T}})} + [u]_{W^{s,2}(\tilde{\mathbb{T}})} + [u]_{W^{t,p}(\tilde{\mathbb{T}})} + \Lambda.$$

Since by assumption $[v]_{W^{2s-\tilde{t},\tilde{q}'}(\tilde{\mathbb{T}})} < \infty$ we obtain (3.7) by absorption of the $\varepsilon[v]_{W^{2s-\tilde{t},\tilde{q}'}(\mathbb{T}^n)}$ term after choosing ε small enough. □

3.2 Local improved regularity via a fixed point theorem

We remark that the estimate given in Corollary 3.2 is an *a priori* estimate because it relies on the regularity assumption $v \in W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$. In this subsection we prove this local regularity result under the assumption of Proposition 3.1

Theorem 3.3 *Fix $u \in W^{t,p} \cap W^{s,2}(\Omega')$ under the assumptions for t, $\tilde{t}$, s, p, and q as in Proposition 3.1. Assume that g is a distribution such that there is some $\Lambda > 0$ with the*

$$|g[\varphi]| \le \Lambda \left(\|\varphi\|_{L^q(\mathbb{R}^n)} + [\varphi]_{W^{\tilde{t},q}(\mathbb{R}^n)} \right),$$

for all $\varphi \in C_c^\infty(B(x_0, 30R))$. Then there exists a small enough $\varepsilon > $ (depending on x_0, t, s, p such that for any K satisfying (1.3), any solution $v \in W^{s,2}(\tilde{\mathbb{T}})$ to (3.2) belongs to $W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$.

Proof We prove the theorem in two steps. In the first step we show that any $W^{s,2}(\tilde{\mathbb{T}})$ solution to (3.2) is unique up a constant. Next, we demonstrate that there is a solution $v \in W^{2s-\tilde{t},q'} \cap W^{s,2}(\tilde{\mathbb{T}})$ to (3.2). As a consequence, we conclude that any solution $v \in W^{s,2}(\tilde{\mathbb{T}})$ to (3.2) belongs to $W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$.

For the uniqueness, assume there are two solutions v and $\tilde{v}$ in $W^{s,2}(\tilde{\mathbb{T}})$. Set $w := v - \tilde{v}$, then (3.2) implies (recall that u and g are fixed)

$$\int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x - y)\,(w(x) - w(y))\,(\psi(x) - \psi(y))\,dx\,dy = \mathcal{H}(w, \psi).$$

Testing (by a density argument) with $\psi = w$ we find in view of Equation 3.3

$$[w]^2_{W^{s,2}(\tilde{\mathbb{T}})} \lesssim \sup_{x \in B(x_0, 10R), |h|=1, r \le 20R} |K(x, r, h) - K(x_0, 0, h)|\; [w]^2_{W^{s,2}(\tilde{\mathbb{T}})}.$$

Then choosing ε sufficiently small, for any K satisfying (1.3), it follows that $[w]^2_{W^{s,2}(\tilde{\mathbb{T}})} = 0$ and thus $w \equiv const$.

Let us now demonstrate that there is a solution v that belongs to $W^{2s-\tilde{t},q'} \cap W^{s,2}(\tilde{\mathbb{T}})$. To that end, let

$$X := \left\{ v \in W^{s,2}(\tilde{\mathbb{T}}) :\ (v)_{\tilde{\mathbb{T}}} = 0 \right\}.$$

By Poincaré inequality X endowed with the $[\cdot]_{W^{s,2}}$-seminorm is a Banach space.

Starting from $v_0 := 0$ define $v_{k+1} \in X$ inductively as the minimizer of $\inf_X \mathcal{E}_k(\cdot)$ where

$$\mathcal{E}_k(v) := \frac{1}{2} \int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x - y)\,|v(x) - v(y)|^2\,dx\,dy - g[\eta v] - \mathcal{H}(v_k, v) - \mathcal{G}(u, v).$$

The minimizer $v_{k+1} \in X$ exists by the direct method in the calculus of variations. The minimizer v_{k+1} satisfies the Euler-Lagrange equation

$$\int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x - y)(v_{k+1}(x) - v_{k+1}(y))\,(\psi(x) - \psi(y))dx\,dy = g[\eta\psi] + \mathcal{H}(v_k, \psi) + \mathcal{G}(u, \psi),$$

for all $\psi \in C_{per}^\infty\left(\overline{\tilde{\mathbb{T}}}\right)$.

Denote by $(v_{k+1})_\tau = v_{k+1} * \phi_\tau$ the convolution with respect to $\tilde{\mathbb{T}}$ where ϕ_τ is a approximate identity in $C^\infty(\overline{\tilde{\mathbb{T}}})$. By the convolution type of the equation for v_{k+1} we have

$$\int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x - y)((v_{k+1})_\tau(x) - (v_{k+1})_\tau(y))\,(\psi(x) - \psi(y))dx\,dy$$
$$= \int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x - y)\,(v_{k+1}(x) - v_{k+1}(y))\,(\psi_\tau(x) - \psi_\tau(y))\,dx\,dy$$
$$= g[\eta\psi_\tau] + \mathcal{H}(v_k, \psi_\tau) + \mathcal{G}(u, \psi_\tau) \quad \forall \psi \in C_{per}^\infty\left(\overline{\tilde{\mathbb{T}}}\right).$$

We observe that the map

$$\psi \in C_{per}^\infty\left(\overline{\tilde{\mathbb{T}}}\right) \mapsto \Upsilon_{(k,\tau)}(\psi) := g[\eta\psi_\tau] + \mathcal{H}(v_k, \psi_\tau) + \mathcal{G}(u, \psi_\tau)$$

satisfies the estimate that

$$|\Upsilon_{(k,\tau)}(\psi)| \le C\|\psi\|_{W^{\tilde{t},q}} \left(\varepsilon[v_k]_{W^{2s-\tilde{t},\tilde{q}'}(\tilde{\mathbb{T}})} + \left(\|u\|_{L^2(\tilde{\mathbb{T}})} + [u]_{W^{s,2}(\tilde{\mathbb{T}})} + [u]_{W^{t,p}(\tilde{\mathbb{T}})} \right) + \Lambda \right)$$

for a given $\epsilon > 0$ and K satisfying (1.3) where the constant C is uniform in τ, ϵ, k for any ϵ. Since $(v_{k+1})_\tau \in C^\infty_{per}(\tilde{\mathbb{T}}) \subset W^{2s-\tilde{t},q'}_{per}(\tilde{\mathbb{T}})$ we can apply Corollary 3.2. and obtain uniform $W^{2s-\tilde{t},q'}_{per}(\tilde{\mathbb{T}})$-estimates for $(v_{k+1})_\tau$, and as $\tau \to 0$ we thus obtain $v_{k+1} \in W^{2s-\tilde{t},q'}_{per}(\tilde{\mathbb{T}})$ with the estimate

$$[v_{k+1}]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} \lesssim \varepsilon[v_k]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} + \left(\|u\|_{L^2(\tilde{\mathbb{T}})} + [u]_{W^{s,2}(\tilde{\mathbb{T}})} + [u]_{W^{t,p}(\tilde{\mathbb{T}})} \right) + \Lambda.$$

Repeating this argument for $w_{k+1} := v_{k+1} - v_k$, which satisfies the equation

$$\int_{\tilde{\mathbb{T}}} \int_{\tilde{\mathbb{T}}} \mu(x-y)(w_{k+1}(x) - w_{k+1}(y))(\psi(x) - \psi(y))dx\,dy = \mathcal{H}(w_k, \psi) \quad \forall \psi \in C^\infty_{per}\left(\tilde{\mathbb{T}}\right).$$

we obtain the estimate (recall (1.3))

$$[w_k]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})} \leq C\varepsilon[w_{k-1}]_{W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})}.$$

Since $(w_k)_{\tilde{\mathbb{T}}} = 0$, this implies that v_k is convergent in $W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$, and passing to the limit we find a solution $v \in W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$ of (3.2).

Since $2s - \tilde{t} > s$ and $q' \geq 2$ we have $W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}}) \subset W^{s,2}(\tilde{\mathbb{T}})$ and thus by uniqueness (up to a constant) of v for $W^{s,2}$-maps we have that any solution belongs to $W^{2s-\tilde{t},q'}(\tilde{\mathbb{T}})$. That concludes the proof. □

4 $W^{t,p}$-estimates, $t > 1$: differentiating the equation

Up to now we obtained $W^{t,p}$-estimates for $t < 1$ – under suitable assumptions on the right-hand side, without assuming anything but continuity of the kernel involved. For $W^{t,p}$-estimates for $t \geq 1$, we need $t - 1$-Hölder continuity of the kernel in order to differentiate the equation.

Theorem 4.1 *Let $s \in (0, 1)$ and $\alpha \in (0, 1)$. Assume K is as in Theorem 1.1, and additionally we have Hölder continuity in the first variable,*

$$|K(x, r, h) - K(y, r, h)| \leq \Lambda|x-y|^\alpha \quad \forall x, y \in \mathbb{R}^n, |h| = 1, r \geq 0.$$

Suppose also that, for $t \in (s, 1)$, $p \geq 2$, $u \in W^{t,p} \cap W^{s,2}(\mathbb{R}^n)$ is a solution of

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dx\,dy = g[\varphi] \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n).$$

Then for any $r \in (0, \min\{t, \alpha\})$, $(-\Delta)^{\frac{r}{2}}u \in W^{t-r,2}(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{((-\Delta)^{\frac{r}{2}}u(x) - (-\Delta)^{\frac{r}{2}}u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dx\,dh \tag{4.1}$$
$$= g[(-\Delta)^{\frac{r}{2}}\varphi] + \mathcal{H}(u, \varphi) \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n),$$

where $\mathcal{H}(u, \varphi)$ is a bilinear form. Moreover, we have the following estimate for any for any $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$, and $R > 0$

$$\mathcal{H}(u, \varphi) \lesssim C(\varepsilon, R)\,[u]_{W^{t,p}(\mathbb{R}^n)}(\|\varphi\|_{L^{p'}(\mathbb{R}^n)} + \|\varphi\|_{W^{2s-t+\varepsilon,p'}(\mathbb{R}^n)}) \tag{4.2}$$

for any $\varphi \in C^\infty_c(B(x_0, R))$.

Proof Let $r \in (0, t - s)$. Then $(-\Delta)^{\frac{r}{2}} u \in W^{s,2}(\mathbb{R}^n)$, and we have for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{((-\Delta)^{\frac{r}{2}} u(x) - (-\Delta)^{\frac{r}{2}} u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K\left(x, |h|, \frac{h}{|h|}\right) \frac{((-\Delta)_x^{\frac{r}{2}} u(x) - (-\Delta)_x^{\frac{r}{2}} u(x+h))(\varphi(x) - \varphi(x+h))}{|h|^{n+2s}} \, dx \, dh$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(x+h))((-\Delta)_x^{\frac{r}{2}} \left(K\left(x, |h|, \frac{h}{|h|}\right) \varphi(x)\right) - (-\Delta)_x^{\frac{r}{2}} \left(K\left(x, |h|, \frac{h}{|h|}\right) \varphi(x+h)\right))}{|h|^{n+2s}} \, dx \, dh$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K\left(x, |x-y|, \frac{x-y}{|x-y|}\right) \frac{(u(x) - u(y))((-\Delta)^{\frac{r}{2}} \varphi(x) - (-\Delta)^{\frac{r}{2}} \varphi(y))}{|x-y|^{n+2s}} \, dx \, dh$$

$$+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(x+h)) \kappa(x, |h|, \frac{h}{|h|})}{|h|^{n+2s}} \, dx \, dh,$$

where

$$\kappa(x, \rho, \frac{h}{|h|}) = c \int_{\mathbb{R}^n} \frac{\left(K\left(x, \rho, \frac{h}{|h|}\right) - K\left(z, \rho, \frac{h}{|h|}\right)\right)(\varphi(z) - \varphi(z+h))}{|x-z|^{n+r}} \, dz.$$

Here we have applied the elementary formula

$$(-\Delta)^{\frac{r}{2}}(ab)(x) - a(x)(-\Delta)^{\frac{r}{2}} b(x) = c \int_{\mathbb{R}^n} \frac{(a(x) - a(z))b(z)}{|x-z|^{n+r}} \, dz.$$

So we have shown (4.1) for

$$\mathcal{H}(u, \varphi) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y)) \kappa(x, |x-y|, \frac{x-y}{|x-y|})}{|x-y|^{n+2s}} \, dx \, dy.$$

For a given $\epsilon > 0$ and $R > 0$, it remains to prove the estimate (4.2) for $\mathcal{H}(u, \varphi)$ for any $\varphi \in C_c^\infty(B(x_0, R))$. To that end, first by Hölder's inequality,

$$|\mathcal{H}(u, \varphi)| \leq [u]_{W^{t,p}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\kappa\left(x, |x-y|, \frac{x-y}{|x-y|}\right)|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy \right)^{\frac{1}{p'}}.$$

Recall (cf. [21, Proposition 6.6]) that for any $\sigma \in (0, 1)$,

$$|f(z) - f(z+h)| \lesssim |h|^\sigma \left(\mathcal{M}(-\Delta)^{\frac{\sigma}{2}} f(z) + \mathcal{M}(-\Delta)^{\frac{\sigma}{2}} f(z+h) \right),$$

where $\mathcal{M}$ is a finite power of the maximal function. Then, by boundedness and the α-Hölder continuity of K, (1.4), for any $\beta \in (r, \alpha)$ and $\sigma \in (0, 1)$ we have

$$|\kappa(x, |x-y|, \frac{x-y}{|x-y|})|$$

$$\lesssim |x-y|^\sigma \int_{\mathbb{R}^n} |x-z|^{\beta-r-n} \left(\mathcal{M}(-\Delta)^{\frac{\sigma}{2}} \varphi(z) + \mathcal{M}(-\Delta)^{\frac{\sigma}{2}} \varphi(z+y-x) \right) dz \quad (4.3)$$

$$\approx |x-y|^\sigma \left(I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma}{2}} \varphi(x) + I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma}{2}} \varphi(y) \right).$$

Now writing

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| \kappa \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \right|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy$$

$$\leq \int_{\mathbb{R}^n} \int_{|x-y| \leq 1} \frac{\left| \kappa \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \right|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy + \int_{\mathbb{R}^n} \int_{|x-y| \geq 1} \frac{\left| \kappa \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \right|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy$$

$$:= I_1 + I_2.$$

We estimate I_1 and I_2 separately.

We estimate I_1 first. Choose $\epsilon > 0$ such that $2s - t + \varepsilon < 1$. This choice is possible because $t > s$. Now for any $\sigma_1 \in (2s - t, 2s - t + \varepsilon)$ we apply (4.3) to obtain that

$$I_1 \lesssim \int_{\mathbb{R}^n} \int_{|x-y| \leq 1} \frac{|I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma_1}{2}} \varphi(x)|^{p'}}{|x-y|^{n+(2s-t-\sigma_1)p'}} dxdy + \int_{\mathbb{R}^n} \int_{|x-y| \leq 1} \frac{|I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma_1}{2}} \varphi(y)|^{p'}}{|x-y|^{n+(2s-t-\sigma_1)p'}} dxdy$$

$$\lesssim \int_{\mathbb{R}^n} |I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma_1}{2}} \varphi(x)|^{p'} dx,$$

where use the fact that $\int_{|h| \leq 1} \frac{1}{|h|^{n+(2s-t-\sigma_1)p'}} dh < \infty$, since $2s - t - \sigma_1 < 0$. Similarly, to estimate I_2 we apply (4.3) for $\sigma_2 \in (0, 2s - t)$ and use the fact that $\int_{|h| \geq 1} \frac{1}{|h|^{n+(2s-t-\sigma_2)p'}} dh < \infty$ to obtain that

$$I_2 \lesssim \int_{\mathbb{R}^n} |I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma_2}{2}} \varphi(x)|^{p'} dx.$$

Thus we have shown that

$$\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| \kappa \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \right|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy \right)^{1/p'}$$

$$\lesssim \| I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma_1}{2}} \varphi \|_{L^{p'}(\mathbb{R}^n)} + \| I^{\beta-r} \mathcal{M}(-\Delta)^{\frac{\sigma_2}{2}} \varphi \|_{L^{p'}(\mathbb{R}^n)}.$$

Now we use the the continuous embedding $H^{t,p}(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n)$ for $0 < s < t$ and $1 < p < \infty$, see [20], with the inequality $[f]_{H^{s,p}} \lesssim \|f\|_{L^p} + [f]_{H^{t,p}}$, for any given $\delta > 0$ small, we may choose $\beta > r$ sufficiently close to r so that

$$\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| \kappa \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \right|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy \right)^{1/p'} \lesssim \| \varphi \|_{L^{p'-\delta}(\mathbb{R}^n)} + \| (-\Delta)^{\frac{\sigma_1}{2}} \varphi \|_{L^{p'-\delta}(\mathbb{R}^n)}.$$

After noting that φ is supported on $B(x_0, R)$, we have that $\| \varphi \|_{L^{p'-\delta}(\mathbb{R}^n)} \lesssim \| \varphi \|_{L^{p'}(\mathbb{R}^n)}$. Also using [11, Lemma 2.3] and the usual disjoint support argument we have

$$\| (-\Delta)^{\frac{\sigma_1}{2}} \varphi \|_{L^{p'-\delta}(\mathbb{R}^n)} \lesssim \| \varphi \|_{L^{p'}(\mathbb{R}^n)} + \| (-\Delta)^{\frac{\sigma_1}{2}} \varphi \|_{L^{p'}(\mathbb{R}^n)}$$

Since $\sigma_1 < 2s - t + \varepsilon$ we can use yet again Sobolev embedding to obtain (here the constant depends on $\varepsilon > 0$)

$$\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| \kappa \left(x, |x-y|, \frac{x-y}{|x-y|} \right) \right|^{p'}}{|x-y|^{n+(2s-t)p'}} \, dx \, dy \right)^{1/p'} \lesssim \| \varphi \|_{L^{p'}(\mathbb{R}^n)} + [\varphi]_{W^{2s-t+\varepsilon, p'}(\mathbb{R}^n)}.$$

This proves estimate (4.2). $\qquad \square$

5 Proof of main results

Proof of Theorem 1.1 The proof uses an iterative argument similar to the one used in [22]. Here we present a sketch of the argument. Let $k \in \mathbb{N}$, and choose bounded sets $\Omega' \subset \Omega_k \subset \Omega_{k-1} \subset \dots \Omega_1 \subset \Omega = B(x_0, \mathfrak{R})$. Cover Ω_1 by finitely many balls $B(x_i, 5R)$ where R is chosen such that $B(x_i, 60\sqrt{n}R) \subset\subset \Omega$, $i = 1, 2, \cdots, l$. Applying Proposition 3.1 and then Theorem 3.3 we obtain that $u \in W^{s_1, p_1}(B(x_i, 5R))$ (since $u = v$ in $B(x_i, 5R)$), and thus $u \in W^{s_1, p_1}(\Omega_1)$, for some $s_1 \geq s$ and $p_1 \geq p$. Repeat this procedure on Ω_2 and we find for some $s_2 \geq s_1$ and $p_2 \geq p_1$ that $u \in W^{s_2, p_2}(\Omega_2)$. After k steps we have obtained $u \in W^{s_k, p_k}(\Omega')$. We can choose s_k and p_k so that Proposition 3.1 and Theorem 3.3 are applicable in each step and after k steps we have $p_k = p$ and $s_k = s$, and thus the theorem is proven. $\square$

Proof of Theorem 1.2 We may assume that $2s > 1$, otherwise there is nothing to show. By the Theorem 1.1 we have already obtained a $W_{loc}^{t,p}$-estimate for any $t < 1$. Let $r := t - s$ for $t < 1$, $t \approx 1$ then applying Theorem 4.1, we obtain that $(-\Delta)^{\frac{r}{2}} u$ satisfies an equation to which we can apply (locally) Theorem 1.1. In this way we can keep bootstrapping to the claim. $\square$

Proof of Corollary 1.3 Set $K(x, r, h) := \left(\frac{r}{|\phi(x) - \phi(x+rh)|} \right)^{n+2s}$ and recall that ϕ bi-Lipschitz, that is

$$0 < \inf_{x \in \Omega, r \geq 0, |h|=1} \frac{|\phi(x) - \phi(x+rh)|}{|x-y|} < \sup_{x \in \Omega, r \geq 0, |h|=1} \frac{|\phi(x) - \phi(x+rh)|}{|x-y|} < \infty. \quad (5.1)$$

We conclude that K is bounded from below (by a positive number) and above. Also K is differentiable in C^α with respect to x. Indeed

$$\left| \frac{r}{|\phi(x) - \phi(x+rh)|} - \frac{r}{|\phi(y) - \phi(y+rh)|} \right|$$
$$\leq \frac{r}{|\phi(x) - \phi(x+rh)| \, |\phi(y) - \phi(y+rh)|} \, ||\phi(y) - \phi(y+rh)| - |\phi(x) - \phi(x+rh)||$$
$$\leq \frac{r}{|\phi(x) - \phi(x+rh)| \, |\phi(y) - \phi(y+rh)|} \, |\phi(y) - \phi(y+rh) - (\phi(x) - \phi(x+rh))|.$$

By the fundamental theorem of calculus,

$$|\phi(y) - \phi(y+rh) - \phi(x) - \phi(x+rh)|$$
$$\leq |r| \int_0^1 |D\phi(y+trh) - D\phi(x+trh)| \, dt$$
$$\lesssim |r| |x-y|^\alpha$$

Thus we have shown

$$\left| \frac{r}{|\phi(x) - \phi(x+rh)|} - \frac{r}{|\phi(y) - \phi(y+rh)|} \right|$$
$$\lesssim \frac{r^2}{|\phi(x) - \phi(x+rh)| \, |\phi(y) - \phi(y+rh)|} |x-y|^\alpha$$
$$\lesssim |x-y|^\alpha,$$

where the last inequality follows from (5.1). Since $K(x, r, h)$ is a positive power of this expression and $K(x, r, h)$ is bounded, we also get the C^α-Hölder continuity of K.

Now the claim follows from Theorem 1.1 and Theorem 1.2 if necessary. □

6 Applications to the *p*-Laplacian: Proof of Corollary 1.4

The main idea of Corollary 1.4 is that locally around a point x_0,

$$|u(x) - u(y)|^{p-2} \approx |x - y|^{p-2} \left| \nabla u(x_0) \frac{x - y}{|x - y|} \right|^{p-2}.$$

The kernel $K(h) := |\nabla u(x_0) \frac{h}{|h|}|^{p-2}$ satisfies the ellipticity assumptions as in (1.2).

So one expects that we can rewrite (1.5) into

$$\int_\Omega \int_\Omega \frac{K(\frac{x-y}{|x-y|})(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp-p+2}} \, dx \, dy = \mathcal{F}(u, \varphi) + \int_\Omega f\varphi$$

where $\mathcal{F}$ is a small distortion. Since by assumption $sp - p + 2 > sp - p + s > 0$ this is still a nonlocal differential equation of the type treated in our paper. Differentiating both sides, we expect an equation of the form

$$\int_\Omega \int_\Omega \frac{K(\frac{x-y}{|x-y|})(\partial_\alpha u(x) - \partial_\alpha u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp-p+2}} \, dx \, dy = \mathcal{F}(\partial_\alpha u, \varphi) + \int_\Omega \partial_\alpha f\varphi$$

To make this more precise we apply discrete differentiation. Set

$$\delta_\tau f(x) := f(x + \tau) - f(x).$$

Then we have the following result which obtains an equation for $\delta_\tau u$ to which Theorem 1.1 is applicable.

Proposition 6.1 *Let* $s > 1 - \frac{1}{p+1}$, $p \geq 2$, $s \in (0, 1)$, $\Omega \subseteq \mathbb{R}^n$ *open. Let* $u \in L^\infty(\mathbb{R}^n) \cap W^{s,p}(\mathbb{R}^n)$ *be a solution to* (1.5).

Assume that moreover $u \in W^{\frac{sp-p+2}{2}, 2}(B(x_0, 60\sqrt{n}R)) \cap C^1(B(x_0, 60\sqrt{n}R))$ *where the ball* $B(x_0, 60\sqrt{n}R) \subset \Omega$.

For each $\tau \in B(0, R)$ *there exist* $K_\tau(x, r, h)$ *that satisfies the conditions of Theorem* 1.1, *and for all* $\psi \in C_c^\infty\left(B(x_0, \frac{1}{2}R)\right)$,

$$\int_\Omega \int_\Omega K_\tau(x, y)(\delta_\tau u(x) - \delta_\tau u(y))(\psi(x) - \psi(y)) \, dx \, dy = \mathcal{G}_\tau(u, \psi) + \mathcal{H}_\tau(f, \psi) \tag{6.1}$$

where $\mathcal{F}$ *and* $\mathcal{G}$ *and* $\mathcal{H}$ *are bilinear and have the following properties: for any* $\beta > 1$

$$|\mathcal{H}_\tau(f, \psi)| \leq C(\beta, \eta, \tilde{\eta})|\tau| \|f\|_{C^\beta} \|\psi\|_{L^1} \tag{6.2}$$

and for any $q \geq p$,

$$|\mathcal{G}_\tau(u, \psi)| \lesssim \|u\|_{L^\infty(\Omega)}^{p-2} \left(|\tau| + \|\delta_\tau u\|_{L^q(B(x_0, R))} + \|\delta_\tau u\|_{L^2(\mathbb{R}^n)}\right) \|\psi\|_{L^{q'}(B(x_0, R))}$$

Moreover we have that if $u \in C^{1,\gamma}(B(x_0, R))$ *then* K_τ *can be chosen in* $C^\gamma(\Omega)$.

Proof In the following we center all balls at x_0, i.e. we write $B(R)$ instead of $B(x_0, R)$.

Let $\psi \in C_c^\infty(B(\lambda R))$ and set

$$\mathcal{H}_\tau(f, \psi) := \int_{\mathbb{R}^n} \psi \delta_\tau f.$$

Observe that ψ is localizing this so that the integration domain lies well within Ω, and thus we have (6.2)

$$|\mathcal{H}_\tau(f, \psi)| \leq C(\beta, \eta, \tilde{\eta})|\tau|\|f\|_{C^\beta}\|\psi\|_{L^1(\Omega)}.$$

which readily implies (6.2).

We begin now our computations by observing that from (1.5)

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\delta_{-\tau}\psi(x) - \delta_{-\tau}\psi(y))}{|x - y|^{n+sp}}\,dx\,dy$$
$$= \mathcal{H}_\tau(f, \psi).$$

Next we have by the discrete integration by parts of δ_τ,

$$\int_\Omega \int_\Omega \frac{\delta_\tau\left(|u(x) - u(y)|^{p-2}(u(x) - u(y))\right)(\psi(x) - \psi(y))}{|x - y|^{n+sp}}\,dx\,dy$$
$$= \mathcal{G}_{1,\tau}(u, \psi) + \mathcal{H}_\tau(f, \psi).$$

where

$$\mathcal{G}_{1,\tau}(u, \psi) = \iint_D \frac{\left(|u(x + \tau) - u(y + \tau)|^{p-2}(u(x + \tau) - u(y + \tau))\right)(\psi(x) - \psi(y))}{|x - y|^{n+sp}}\,dx\,dy$$

with $D = [\Omega \times \Omega]\Delta[(\Omega - \tau) \times (\Omega - \tau)]$. We can now define K_τ. Set

$$\tilde{K}_\tau(x, r, h) := r^{2-p}(p - 2)\int_0^1 |t(u(x) - u(x + rh)) + (1 - t)(u(x + \tau) - u(x + \tau + rh))|^{p-2}\,dt.$$

We then have by the fundamental theorem of calculus (and since for $g(a) = |a|^{p-2}a$ we have $g'(a) = (p - 1)|a|^{p-2}$)

$$\delta_\tau\left(|u(x) - u(y)|^{p-2}(u(x) - u(y))\right) = |x - y|^{p-2}\tilde{K}_\tau\left(x, |x - y|, \frac{x - y}{|x - y|}\right)(\delta_\tau u(x) - \delta_\tau u(y)).$$

Thus we have shown

$$\int_\Omega \int_\Omega \frac{\tilde{K}_\tau\left(x, |x - y|, \frac{x-y}{|x-y|}\right)(\delta_\tau u(x) - \delta_\tau u(y))(\psi(x) - \psi(y))}{|x - y|^{n+sp+2-p}}\,dx\,dy$$
$$= \mathcal{G}_{1,\tau}(u, \psi) + \mathcal{H}_\tau(f, \psi).$$

Observe that for $x \in B(R)$ and $|r| \leq R$ and $|h| = 1$ we have by the Lipschitz continuity of u in $B(0, 10R)$ that

$$|\tilde{K}_\tau(x, r, h)| \leq C.$$

Moreover, since u is continuously differentiable in $B(R)$, for each $x \in B(x_0, \lambda R)$, λ, $|\tau|$ r and h small,

$$\tilde{K}_\tau(x, r, h) = (p - 2)\int_0^1 |t(Du(x) + o_r(1))h + (1 - t)(Du(x + \tau) + o_r(1))h|^{p-2}$$

$$= (p - 2)\int_0^1 |t(Du(x_0) + o_\lambda(1) + o_r(1))h + (1 - t)(Du(x_0) + o_{\lambda+|\tau|}(1) + o_r(1))h|^{p-2}$$

Since $p - 2 > 0$ (if $p = 2$ there is nothing to show),

$$|\tilde{K}_\tau(x, r, h) - \tilde{K}_\tau(x_0, 0, h)| \lesssim o_r(1) + o_{\lambda+|\tau|}(1),$$

where

$$\tilde{K}_\tau(x_0, 0, h) = (p - 2) \int_0^1 |Du(x_0) \cdot h + (1 - t)o_{|\tau|}(1)h|^{p-2}.$$

Thus we have the required continuity from Theorem 1.1.

Next, we show the positivity of $\tilde{K}_\tau(x_0, 0, h)$. Let $v := |\nabla u(x_0)|^{-1}\nabla u(x_0) \in \mathbb{S}^{n-1}$ (by assumption $|\nabla u(x_0)| > 0$). Then, we can take Σ a small neighbourhood of v, so that

$$\inf_{h \in \Sigma} |\langle h, v \rangle| \geq \sigma > 0$$

for a given small σ given. For each $h \in \Sigma$ let $t_0 > 0$ such that $|\nabla u(x_0)|^{-1}|(1-t_0)o_{|\tau|}(1)| \leq \frac{\sigma}{2}$. Then

$$|\nabla u(x_0)|^{-1} |Du(x_0) \cdot h + (1 - t)o_{|\tau|}(1)h| \geq \frac{\sigma}{2} \quad \forall t \in [t_0, 1].$$

Thus for each $h \in \Sigma$,

$$\tilde{K}_\tau(x_0, 0, h) \geq (p - 2) \int_{t_0}^1 |Du(x_0) \cdot h + (1 - t)o_{|\tau|}(1)h|^{p-2} \geq C > 0.$$

That is (1.2) is satisfied. It now easy to extend $\tilde{K}_\tau$ to a kernel that satisfies the conditions of Theorem 1.1 in all of Ω, and we call this kernel K_τ.

$$\int_\Omega \int_\Omega \frac{K_\tau\left(x, |x - y|, \frac{x-y}{|x-y|}\right)(\delta_\tau u(x) - \delta_\tau u(y))(\psi(x) - \psi(y))}{|x - y|^{n+sp+2-p}} \, dx \, dy$$
$$= \mathcal{G}_{1,\tau}(u, \psi) + \mathcal{G}_{2,\tau}(u, \psi) + \mathcal{H}_\tau(f, \psi)$$

where

$$\mathcal{G}_{2,\tau}(u, \psi) = \int_\Omega \int_\Omega \frac{(\tilde{K}_\tau - K_\tau)\left(x, |x - y|, \frac{x-y}{|x-y|}\right)(\delta_\tau u(x) - \delta_\tau u(y))(\psi(x) - \psi(y))}{|x - y|^{n+sp+2-p}} \, dx \, dy.$$

Next we estimate each of the terms in the right hand side.

Estimate of $\mathcal{G}_{1,\tau}$ Observe that $|\Omega \Delta [\Omega - \tau]| \lesssim |\tau|$, and so also $|[\Omega \times \Omega] \Delta [(\Omega - \tau) \times (\Omega - \tau)]| \lesssim |\tau|$. Also if $x, y \in [\Omega \times \Omega] \Delta [(\Omega - \tau) \times (\Omega - \tau)]$ and either x or $y \in \text{supp}\,\psi = B(x_0, \frac{1}{2}R)$ then $|x - y| \approx |x - y| + 1$ (with constant depending on R). So

$$\mathcal{G}_{1,\tau}(u, \psi) \lesssim |\tau| \|u\|_{L^\infty}^{p-1} \|\psi\|_{L^1}.$$

Estimate of $\mathcal{G}_2$ Observe that since $\psi \in C_c^\infty(B(x_0, \frac{1}{2}R))$ we have that

$$x, y \in \Omega : (\tilde{K}_\tau - K_\tau)\left(x, |x - y|, \frac{x-y}{|x-y|}\right)(\psi(x) - \psi(y)) \neq 0$$

necessarily implies that $|x - y| \gtrsim R$. So we have

$$\mathcal{G}_{2,\tau}(u, \psi) \lesssim \int_\Omega \int_\Omega \left(1 + |u(x) - u(y)|^{p-2} + |u(x + \tau) - u(y + \tau)|^{p-2}\right)$$
$$\times |\delta_\tau u(x) - \delta_\tau u(y)| |\psi(x) - \psi(y)| \frac{dx \, dy}{1 + |x - y|^{n+sp-p+2}}.$$

Arguing again with the support of ψ we should get for any $q \geq p$,

$$\mathcal{G}_{2,\tau}(u, \psi) \lesssim \|u\|_{L^\infty(B(R))}^{p-2} \left(\|\delta_\tau u\|_{L^q(B(R))} + \|\delta_\tau u\|_{L^2(\mathbb{R}^n)} \right) \|\psi\|_{L^{q'}(B(R))}.$$

$\square$

Proof of Corollary 1.4 Set $\alpha := \min\{sp - p + 2, 1\}$. Iteratively applying Theorem 1.1 to the equation in Proposition 6.1 on balls $B(R_i)$ with $R_{i+1} < R_i$, for any q_i such that the right-hand side is small, for arbitrarily small $\varepsilon > 0$.

$$[\delta_h u]_{W^{\alpha-\varepsilon, q_i-\varepsilon}(B(R_{i+1}))} \lesssim \left(|\tau| + \|\delta_\tau u\|_{L^{q_i}(B(R))} + \|\delta_\tau u\|_{L^2(\mathbb{R}^n)} \right).$$

We choose $q_1 = p$ and $q_{i+1} > q_i$ such that $q_{i+1} < \frac{nq_i}{n-\alpha q_i}$ ($q_{i+1} = \infty$ if $n - \alpha q_i < 0$) then by Sobolev inequality (choosing ε appropriately small)

$$\|\delta_\tau u\|_{L^{q_{i+1}}(B(R_{i+1}))} \lesssim \left(|\tau| + \|\delta_\tau u\|_{L^{q_i}(B(R_i))} + \|\delta_\tau u\|_{L^2(\mathbb{R}^n)} \right).$$

After finitely many steps N we reach $q_N = \infty$. Thus we have obtained

$$[\delta_\tau u]_{C^{\alpha-\varepsilon}(B(R_{N+1}))} \lesssim \left(|\tau| + \|\delta_\tau u\|_{L^p(B(R_i))} + \|\delta_\tau u\|_{L^2(\mathbb{R}^n)} \right).$$

Dividing by $|\tau|^s$ we find that

$$[u]_{C^{s+\alpha-\varepsilon}(B(R_{N+1}))} < \infty.$$

By assumptions we have $s + \alpha > 1$, so we can apply Theorem 1.2, since $\left(\frac{|u(x)-u(y)|}{|x-y|} \right)^{p-2}$ is Hölder continuous. Thus we continue to bootstrap regularity with the help of Theorem 1.2, and finally obtain $u \in C^{s+\min\{sp-p+2, 1\}-\varepsilon}$.

$\square$

Appendix A: A remark on Hölder regularity for equations with coercive kernels

We consider symmetric kernels $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ measurable and satisfying

$$0 \leq K(x, y) \leq \kappa \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{A.1}$$

Moreover, we assume that for all $\varphi \in \dot{H}^s(\mathbb{R}^n)$,

$$\begin{aligned}
\frac{1}{\kappa}[\varphi]_{H^s(\mathbb{R}^n)}^2 &:= \frac{1}{\kappa} \iint_{\mathbb{R}^{2n}} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2s}} \, dx dy \\
&\leq \iint_{\mathbb{R}^{2n}} K(x, y) \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2s}} dx dy := [\varphi]_{H^s_K(\mathbb{R}^n)}^2.
\end{aligned} \tag{A.2}$$

This includes translation anisotropic kernels K supported on cones as in (1.2), see (2.2).

We recall that

$$\mathcal{L}_K u[\varphi] := \iint_{\mathbb{R}^{2n}} K(x, y) \frac{(u(y) - u(x))(\varphi(y) - \varphi(x))}{|x - y|^{n+2s}} \, dx \, dy.$$

We denote by $\mathcal{L}_s^1$ the space of L^1_{loc} functions u such that $x \mapsto \frac{u(x)}{1+|x|^{n+2s}}$ is $L^1(\mathbb{R}^n)$. Our aim is to re-prove the following theorem.

Theorem A.1 *Let $f \in L^\infty(\mathbb{R}^n)$ and $u \in H^s_{loc}(\mathbb{R}^n) \cap \mathcal{L}^1_s$ such that $\mathcal{L}_K u = f$ in B_2. Then, there exist $\alpha, C > 0$ such*

$$\|u\|_{C^\alpha(B_1)} \leq C(\|u\|_{L^2(B_2)} + \|u\|_{\mathcal{L}^1_s} + \|f\|_{L^\infty(\mathbb{R}^n)}).$$

To prove this result, we need some classical fundamental results: Caccioppoli inequality, a nonlinear Sobolev inequality, and a Logarithmic lemma.

Lemma A.2 (Caccioppoli inequality)
  Let $v \in H^s(B(R)) \cap \mathcal{L}^1_s$ be nonnegative and $f \in L^\infty(B(R))$ satisfy

$$\mathcal{L}_K v \leq f \quad \text{in } B(R). \tag{A.3}$$

Then there exists $c = c(n, s, \kappa) > 0$ such that for every $\varphi \in C^\infty_c(B(R))$, we have

$$\frac{1}{2}[\varphi v]^2_{H^s(\mathbb{R}^n)} \leq c\|\nabla\varphi\|^2_{L^\infty(B(R))} R^{2-2s} \|v\|^2_{L^2(B(R))}$$

$$+ c\|\varphi\|_{L^\infty(B(R))} \left[\int_{\mathbb{R}^n \setminus B(R)} v(y)|x-y|^{-2s-n} \, dy + \|f\|_{L^\infty(B(R))} \right] \|v\varphi\|_{L^1(B(R))}.$$

Proof By direct computations, we get

$$(v(x) - v(y))[v(x)\varphi^2(x) - v(y)\varphi^2(y)] = (\varphi(x)v(x) - \varphi(y)v(y))^2 - v(x)v(y)(\varphi(x) - \varphi(y))^2.$$

Testing the equation (A.3) with $v\varphi^2$, and using the identity above together with the symmetry of K, we get

$$\frac{1}{2}[\varphi v]^2_{H^s_K(\mathbb{R}^n)} \leq \frac{1}{2} \iint_{\mathbb{R}^{2n}} v(x)v(y)(\varphi(x) - \varphi(y))^2 \frac{K(x,y)}{|x-y|^{n+2s}} \, dydx + \int_{\mathbb{R}^n} f(x)v(x)\varphi^2(x) \, dx.$$
$$\tag{A.4}$$

By Hölder and Young's inequalities, we get

$$\int_{\mathbb{R}^n} |f(x)|v(x)\varphi^2(x) \, dx \leq \|f\|_{L^\infty(B(R))} \|\varphi\|_{L^\infty(B(R))} \|v\varphi\|_{L^1(\mathbb{R}^n)}. \tag{A.5}$$

Next, we write

$$\iint_{\mathbb{R}^{2n}} v(x)v(y)(\varphi(x) - \varphi(y))^2 \frac{K(x,y)}{|x-y|^{n+2s}} \, dydx \tag{A.6}$$

$$= \iint_{B(R) \times B(R)} v(x)v(y)(\varphi(x) - \varphi(y))^2 \frac{K(x,y)}{|x-y|^{n+2s}} \, dydx$$

$$+ 2 \iint_{B(R) \times \mathbb{R}^n \setminus B(R)} v(x)v(y)(\varphi(x) - \varphi(y))^2 \frac{K(x,y)}{|x-y|^{n+2s}} \, dydx. \tag{A.7}$$

We put $\mu_1(x, y) = |x - y|^{-2s-n}$. By (A.1), Hölder and Young's inequalities, we get

$$\iint_{B(R) \times B(R)} v(x)v(y)(\varphi(x) - \varphi(y))^2 \frac{K(x,y)}{|x-y|^{n+2s}} \, dydx \tag{A.8}$$

$$\leq 4\kappa \int_{B(R)} v^2(y) \int_{B(R)} (\varphi(x) - \varphi(y))^2 \mu_1(x, y) \, dydx$$

$$\leq 4\|\nabla\varphi\|^2_{L^\infty(B(R))} \kappa \int_{B(R)} |v|^2(y) \int_{B(R)} |x - y|^2 \mu_1(x, y) \, dydx$$

$$\leq c\|\nabla\varphi\|^2_{L^\infty(B(R))} R^{2-2s} \int_{B(R)} v^2(x) \, dx. \tag{A.9}$$

Here we used that $\int_{B(R)} |x-y|^{2-2s-n} dy \approx R^{2-2s}$ whenever $x \in B(R)$.

Moreover, we estimate

$$\iint_{B(R) \times \mathbb{R}^n \setminus B(R)} v(x)v(y)(\varphi(x) - \varphi(y))^2 K(x,y)\,dy dx$$

$$\leq \kappa \iint_{B(R) \times \mathbb{R}^n \setminus B(R)} v(x)v(y)\varphi(x)^2 \mu_1(x,y)\,dy dx$$

$$\leq 2\kappa \int_{B(R)} v(x)|\varphi(x)|\,dx \sup_{x \in B(R)} |\varphi(x)| \int_{\mathbb{R}^n \setminus B_R} v(y)\mu_1(x,y)\,dy. \quad \text{(A.10)}$$

Using (A.2), (A.10), (A.9) and (A.7) we get the result.   □

An immediate consequence of the above Caccioppoli inequality and (A.2), which implies the (nonlinear) Sobolev inequality, we can carry over the De Giorgi iteration, [10], to get the following local L^∞ estimate.

Theorem A.3 *Let $u \in H^s(B_{2r}) \cap \mathcal{L}_s^1$ and $f \in L^\infty(\mathbb{R}^n)$ satisfy*

$$\mathcal{L}_K u = f \quad \text{in } B_{2r}.$$

Then there exists $C = C(n, s, \kappa) > 0$ such that

$$\sup_{B_r} |u| \leq C \left(r^{-n/2} \|u\|_{L^2(B_{2r})} + r^{2s} \int_{|y| \geq r/2} |u(y)||y|^{-n-2s}\,dy + r^{2s}\|f\|_{L^\infty(B_{2r})} \right).$$

To obtain Hölder estimates, we need the following result.

Lemma A.4 (Logarithmic lemma) *Let $v \in H^s(B(R)) \cap \mathcal{L}_s^1$ and $f \in L^\infty(B(R))$ such that (A.3) holds and $u \geq 0$ in $B(R)$. Then there exists $c = c(n, s, \kappa) > 0$ such that for every $r \in (0, R/2), d > 0$, we have*

$$\int_{B_r \times B_r} K(x,y) \left| \log \frac{u(x)+d}{u(y)+d} \right|^2 dx dy$$

$$\leq c r^{n-2s} \left(\frac{r^{2s}}{R^{2s}} d^{-1} \int_{|y| \geq R/2} |v_-(y)||y|^{-n-2s}\,dy + r^{2s}d^{-1}\|f\|_{L^\infty(B_r)} + 1 \right).$$

Proof The proof uses precisely the argument in [7, proof of Lemma 3.1], where only the upper bound of K was used. Note that in [7] a right hand side f was not considered, however since the proof uses $\frac{\varphi^2}{u+d}$, with $\varphi \in C_c^\infty(B_{2r})$, as a test function, we can easily obtain the estimate of the term involving f.   □

Note that in view of (A.2), we have the following Poincaré inequality,

$$c \int_{B_r} |u - (u)_{B_r}|^2 \, dx$$

$$\leq r^{2s-n} \int_{B_r \times B_r} (u(x) - u(y))^2 K(x,y)\,dx dy \quad \text{for all } u \in H_{loc}^s(\mathbb{R}^n). \quad \text{(A.11)}$$

In view of this and Lemma A.4, we have

Corollary A.5 *Under the assumptions of Lemma A.4, we have*

$$c \int_{B_r} |w - (w)_{B_r}|^2 \, dx$$

$$\leq \left(\frac{r^{2s}}{R^{2s}} d^{-1} \int_{|y| \geq R/2} |v_-(y)||y|^{-n-2s}\,dy + r^{2s}d^{-1}\|f\|_{L^\infty(B_r)} + 1 \right), \quad \text{(A.12)}$$

where $w = \min((\log(a+d) - \log(u+d))_+, \log b)$ for all $a, d > 0$ and $b > 1$.

The Hölder continuity is a consequence of Theorem A.1, Corollary A.5, Lemma A.2 and the following growth estimates of the oscillation of u.

Lemma A.6 *Let $f \in L^\infty(\mathbb{R}^n)$ and $u \in H^s_{loc}(\mathbb{R}^n) \cap \mathcal{L}^1_s$ such that $\mathcal{L}_K u = f$ in B_2. Then, there exists $\sigma, \alpha \in (0, 1)$ depending only on n, s, κ such that, for $i \in \mathbb{N}$,*

$$\sup_{B_{\sigma^i}} u - \inf_{B_{\sigma^i}} u \leq \sigma^{\alpha i} \left(\int_{|y| \geq 1/2} |u(y)||y|^{-n-2s} \, dy + \|f\|_{L^\infty(B_2)} + \|u\|_{L^2(B_2)} \right).$$

The proof is exactly the same as the [7, proof of Lemma 5.1]. We skip the details.

References

1. Benyi, A., Oh, T.: The sobolev inequality on the torus revisited. Publ. Math. Debrecen, **83**(3) (2013)
2. Brasco, L., Lindgren, E.: Higher Sobolev regularity for the fractional p-Laplace equation in the superquadratic case. Adv. Math. **304**, 300–354 (2017)
3. Brasco, L., Lindgren, E., Schikorra, A.: Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case. Adv. Math. **338**, 782–846 (2018)
4. Cozzi, M.: Interior regularity of solutions of non-local equations in Sobolev and Nikol'skii spaces. Ann. Mat . Pura Appl. (4) **196**(2), 555–578 (2017)
5. del Teso, F., Gómez-Castro, D., Vázquez, J. L.: Three representations of the fractional p-laplacian: semigroup, extension and balakrishnan formulas (2020)
6. Di Castro, A., Kuusi, T., Palatucci, G.: Nonlocal harnack inequalities. J. Funct. Anal. **267**, 1807–1836 (2014)
7. Di Castro, A., Kuusi, T., Palatucci, G.: Local behavior of fractional p-minimizers. Ann. Inst . H. Poincaré Anal. Non Linéaire **33**(5), 1279–1299 (2016)
8. Dong, H., Kim, D.: On Lp-estimates for a class of non-local elliptic equations. J. Funct. Anal. **262**(3), 1166–1199 (2012)
9. Fall, M. M.: Constant nonlocal mean curvatures surfaces and related problems. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 1613–1637. World Sci. Publ., Hackensack, NJ (2018)
10. Kassmann, M.: The theory of De Giorgi for non-local operators. C. R. Math. Acad. Sci. Paris **345**(11), 621–624 (2007)
11. Mengesha, T., Schikorra, A., Yeepo, S.: Calderon-Zygmund type estimates for nonlocal PDE with Hölder continuous kernel. Adv. Math. **383**, 107692, 64 (2021)
12. Mikulevičius, R., Pragarauskas, H.: On the Cauchy problem for integro-differential operators in Sobolev classes and the martingale problem. J. Differ. Equ. **256**(4), 1581–1626 (2014)
13. Moustapha Fall, M.: Regularity results for nonlocal equations and applications. *Calc. Var. and PDE (accepted)*, June (2018)
14. Nowak, S.: $H^{s,p}$ regularity theory for a class of nonlocal elliptic equations. Nonlinear Anal. **195**, 111730, 28 (2020)
15. Nowak, S.: Higher Hölder regularity for nonlocal equations with irregular kernel. Calc. Var. Partial Differ. Equ. **60**(1), 24, 37 (2021)
16. Nowak, S.: Improved Sobolev regularity for linear nonlocal equations with VMO coefficients. Math. Ann. (2022)
17. Nowak, S.: Regularity theory for nonlocal equations with VMO coefficients. *Ann.H.Poinc. (accepted)* (2021)
18. Nowak, S.: Higher integrability for nonlinear nonlocal equations with irregular kernel. Adv. Anal. Geom. **3**, 459–492 (2021)
19. Roncal, L., Stinga, P.R.: Fractional Laplacian on the torus. Commun. Contemp. Math. **18**(3), 1550033, 26 (2016)
20. Runst, T., Sickel, W.: Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. De Gruyter Series in Nonlinear Analysis and Applications, vol. 3. Walter de Gruyter & Co., Berlin (1996)
21. Schikorra, A.: Boundary equations and regularity theory for geometric variational systems with Neumann data. Arch. Ration. Mech. Anal. **229**(2), 709–788 (2018)
22. Yeepo, S., Lewkeeratiyutkul, W., Khomrutai, S., Schikorra, A.: On the Calderon-Zygmund property of Riesz-transform type operators arising in nonlocal equations. Commun. Pure Appl. Anal. **20**(9), 2915–2939 (2021)