Received: 10 May 2021 | Accepted: 8 September 2021

DOI: 10.1002/jgt.22751

ARTICLE

WILEY

4-Separations in Hajós graphs

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Funding information
Office of China Postdoc Council Project, Grant/Award Number: YJ20190133; National Science Foundation

Abstract

As a natural extension of the Four Color Theorem, Hajós conjectured that graphs containing no K_5-subdivision are 4-colorable. Any possible counterexample to this conjecture with minimum number of vertices is called a *Hajós* graph. Previous results show that Hajós graphs are 4-connected but not 5-connected. A k-separation in a graph G is a pair (G_1, G_2) of edge-disjoint subgraphs of G such that $|V(G_1 \cap G_2)| = k$, $G = G_1 \cup G_2$, and $G_i \nsubseteq G_{3-i}$ for $i = 1, 2$. In this paper, we show that Hajós graphs do not admit a 4-separation (G_1, G_2) such that $|V(G_1)| \geq 6$ and G_1 can be drawn in the plane with no edge crossings and all vertices in $V(G_1 \cap G_2)$ incident with a common face. This is a step in our attempt to reduce Hajós' conjecture to the Four Color Theorem.

KEYWORDS

coloring, disjoint paths, graph subdivision, wheels

MATHEMATICAL SUBJECT CLASSIFICATION

05C10, 05C40, 05C83

1 | INTRODUCTION

Using Kuratowski's characterization of planar graphs [13], the Four Color Theorem [1-3,17] can be stated as follows: Graphs containing no K_5-subdivision or $K_{3,3}$-subdivision are 4-colorable. Since $K_{3,3}$ has chromatic number 2, it is natural to expect that graphs containing no K_5-subdivision are also 4-colorable. Indeed, this is part of a more general conjecture made by Hajós in the 1950s (see [23], although reference [6] is often cited): For any positive integer k, every graph not containing K_{k+1}-subdivision is k-colorable. It is not hard to prove this conjecture for $k \leq 3$. However, Catlin [4] disproved Hajós' conjecture for $k \geq 6$. Erdős and Fajtlowicz [5] then showed that Hajós' conjecture fails for almost all graphs. On the other hand, Kühn and Osthus [12] proved that Hajós' conjecture holds for graphs with large girth, and Thomassen [23] pointed out interesting connections between Hajós' conjecture and several important problems, including Ramsey numbers, Max-Cut, and perfect graphs. Hajós' conjecture remains open for $k = 4$ and $k = 5$.

In this paper, we are concerned with Hajós' conjecture for $k = 4$. We say that a graph G is a *Hajós* graph if

(1) G contains no K_5-subdivision,
(2) G is not 4-colorable, and
(3) subject to (1) and (2), $|V(G)|$ is minimum.

Thus, if no Hajós graph exists then graphs not containing K_5-subdivisions are 4-colorable.

Recently, He, Wang, and Yu [7–10] proved that every 5-connected nonplanar graph contains a K_5-subdivision, establishing a conjecture of Kelmans [11] and, independently, of Seymour [18] (also see Mader [15]). Therefore, Hajós graphs cannot be 5-connected. On the other hand, Yu and Zickfeld [25] proved that Hajós graphs must be 4-connected, and Sun and Yu [21] proved that for any 4-cut T in a Hajós graph G, $G - T$ has exactly 2 components.

The goal of this paper is to prove a result useful for modifying the recent proof of the Kelmans–Seymour conjecture in [7–10] to make progress on the Hajós conjecture; in particular, for the class of graphs containing K_4^- as a subgraph, where K_4^- is the graph obtained from K_4 by removing an edge.

To state our result precisely, we need some notation. Let G_1, G_2 be two graphs. We use $G_1 \cup G_2$ (respectively, $G_1 \cap G_2$) to denote the graph with vertex set $V(G_1) \cup V(G_2)$ (respectively, $V(G_1) \cap V(G_2)$) and edge set $E(G_1) \cup E(G_2)$ (respectively, $E(G_1) \cap E(G_2)$). Let G be a graph and k a nonnegative integer; then a *k-separation* in G is a pair (G_1, G_2) of edge-disjoint subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $|V(G_1 \cap G_2)| = k$, and $G_i \not\subseteq G_{3-i}$ for $i = 1, 2$.

Let G be a graph and $S \subseteq V(G)$. For convenience, we say that (G, S) is *planar* if G has a drawing in a closed disc in the plane with no edge crossings and with vertices in S on the boundary of the disc. We often assume that we work with such an embedding when we say (G, S) is planar. Two elements of $V(G) \cup E(G)$ are said to be *cofacial* if they are incident with a common face. Our main result can be stated as follows, it will be used in subsequent work to derive further useful structure of Hajós graphs.

Theorem 1.1. *If G is a Hajós graph and G has a 4-separation (G_1, G_2) such that $(G_1, V(G_1 \cap G_2))$ is planar then $|V(G_1)| \leq 5$.*

To prove Theorem 1.1, we first find a special wheel inside G_1, then extend the wheel to $V(G_1 \cap G_2)$ by four disjoint paths inside G_1, and form a K_5-subdivision with two disjoint paths in G_2. By a *wheel* we mean a graph which consists of a cycle C, a vertex v not on C (known as the *center* of the wheel), and at least three edges from v to a subset of $V(C)$. The wheels in this paper are special—they are inside a plane graph consisting of vertices and edges that are cofacial with a given vertex. For any positive integer k, let $[k] := \{1, ..., k\}$.

To effectively describe the process of extending wheel to a K_5-subdivision, we introduce the following. Let H be a plane graph and $T \subseteq V(H)$ such that $|T| \geq 4$ and all vertices in T are incident with a common face of H. Let $w \in V(H) \setminus S$ such that the vertices and edges of H cofacial with w form a wheel, denoted as W. We say that W is *T-good* if $T \cap V(W) \subseteq N_H(w)$.

For any $S \subseteq T$ with $|S| \leq 4$, we say that W is (T, S)-*extendable* if H has four paths P_1, P_2, P_3, P_4 from w to T such that

- $V(P_i \cap P_j) = \{w\}$ for all distinct $i, j \in [4]$,
- $|V(P_i - w) \cap V(W)| = 1$ for $i \in [4]$, and
- for any $s \in S$ there exists $i \in [4]$ such that P_i is from w to s.

Note that each P_i may use more than one vertex from T. When $S = \emptyset$ we simply say that W is T-*extendable*.

Remark: These concepts about wheel will be applied to separations (H, L) of a Hajós graph, where H is a plane graph in which all vertices in $T := V(H \cap L)$ are incident with a common face of H.

For the proof of Theorem 1.1, we suppose G has a 4-separation (G_1, G_2) such that $(G_1, V(G_1 \cap G_2))$ is planar and $|V(G_1)| \geq 6$. A result from [24] shows that G_1 has a $V(G_1 \cap G_2)$-good wheel. However, we need to allow the separation (G_1, G_2) to be a 5-separation to deal with issues when such wheels are not $V(G_1 \cap G_2)$-extendable. Another result from [24] characterizes all such 5-separations (G_1, G_2) with G_1 containing no $V(G_1 \cap G_2)$-good wheel. In Sections 2 and 3, we characterize the situations where good wheels are also extendable. We complete the proof of Theorem 1.1 in Section 4.

It will be convenient to use a sequence of vertices to represent a path or cycle, with consecutive vertices representing an edge in the path. Let G be a graph. For $v \in V(G)$, we use $N_G(v)$ to denote the neighborhood of v in G. Let $T \subseteq V(G)$. We use $G - T$ to denote the subgraph of G induced by $V(G) \backslash T$ and write $G - x$ when $T = \{x\}$. For any set S of 2-element subsets of $V(G)$, we use $G + S$ to denote the graph with $V(G + S) = V(G)$ and $E(G + S) = E(G) \cup S$, and write $G + xy$ if $S = \{\{x, y\}\}$.

Let C be a cycle in a plane graph, and let $u, v \in V(C)$. If $u = v$ let $uCv = u$, and if $u \neq v$ let uCv denote the subpath of C from u to v in clockwise order.

2 | EXTENDING A WHEEL

In [25] it is shown that Hajós graphs are 4-connected, and in [10] it is shown that Hajós graphs are not 5-connected. So we have the following result.

Lemma 2.1. *Hajós graphs are* 4*-connected but not* 5*-connected.*

We also need a result from [24] which characterizes the 4-separations and 5-separations (G_1, G_2) with $(G_1, V(G_1 \cap G_2))$ planar such that G_1 has no $V(G_1 \cap G_2)$-good wheel. See Figure 1 for the graph G_1, where the solid vertices are in $V(G_1) \backslash V(G_2)$.

Lemma 2.2. *Let G be a Hajós graph and (G_1, G_2) be a separation in G such that $4 \leq |V(G_1 \cap G_2)| \leq 5$, $V(G_1 \cap G_2)$ is independent in G_1, $(G_1, V(G_1 \cap G_2))$ is planar, and $V(G_1) \backslash V(G_2) \neq \emptyset$. Then, one of the following holds:*

(i) *G_1 contains a $V(G_1 \cap G_2)$-good wheel.*
(ii) $|V(G_1 \cap G_2)| = 4$ *and* $|V(G_1)| = 5$.

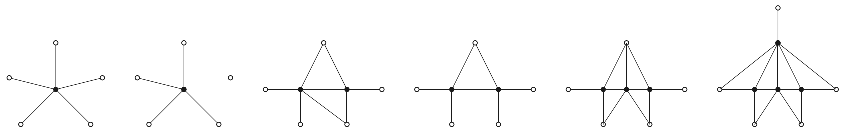

FIGURE 1 Obstructions to good wheels inside 5-separations

(iii) $|V(G_1 \cap G_2)| = 5$, G_1 *is one of the graphs in Figure* 1 *with* $V(G_1)\backslash V(G_2)$ *consisting of the solid vertices, and if* $|V(G_1)| = 8$ *then the degree* 3 *vertex in* G_1 *has degree at least* 5 *in* G.

Let G be a Hajós graph and (G_1, G_2) be a separation in G such that $|V(G_1 \cap G_2)| \in \{4, 5\}$ and $(G_1, V(G_1 \cap G_2))$ is planar, and assume W is a $V(G_1 \cap G_2)$-good wheel in G_1. We wish to extend W to a K_5-subdivision by adding two disjoint paths, which must be routed through the nonplanar part G_2. The following lemma provides four paths extending one such good wheel to $V(G_1 \cap G_2)$.

Lemma 2.3. *Let* G *be a Hajós graph and let* (G_1, G_2) *be a separation in* G *with* $V(G_1 \cap G_2)$ *independent in* G_1 *such that*

(i) $|V(G_1 \cap G_2)| \leq 5$, $(G_1, V(G_1 \cap G_2))$ *is planar, and* G_1 *has a* $V(G_1 \cap G_2)$*-good wheel,*
(ii) *subject to* (i), G_1 *is minimal, and*
(iii) *subject to* (ii), $V(G_1 \cap G_2)$ *is minimal.*

Then any $V(G_1 \cap G_2)$*-good wheel in* G_1 *is* $V(G_1 \cap G_2)$*-extendable.*

Proof. By our convention, G_1 is drawn in a closed disc in the plane with no edge crossing such that $V(G_1 \cap G_2)$ is on the boundary of that disc. Let W be a $V(G_1 \cap G_2)$-good wheel in G_1 with center w, $U = V(W - w)\backslash N_G(w)$, and $G_1' = G_1 - U$. If G_1' has four disjoint paths from $N_G(w)$ to $V(G_1 \cap G_2)$ then extending these paths to w (by adding one edge for each path), we see that W is $V(G_1 \cap G_2)$-extendable. So we may assume that such four paths do not exist. Then G_1' has a separation (H_1, H_2) such that $|V(H_1 \cap H_2)| \leq 3$, $N_G(w) \cup \{w\} \subseteq V(H_1)$, and $V(G_1 \cap G_2) \subseteq V(H_2)$. We choose (H_1, H_2) with $|V(H_1 \cap H_2)|$ minimum.

We see that $V(H_1 \cap H_2) \cup U$ is a cut in G_1 separating H_1 from $V(G_1 \cap G_2)$. Thus, by the planarity of $(G_1, V(G_1 \cap G_2))$, we can draw a simple closed curve γ in the plane such that $\gamma \cap G_1 \subseteq V(H_1 \cap H_2) \cup U$, H_1 is inside γ, and H_2 is outside γ. We choose such γ that $|\gamma \cap G_1|$ is minimum.

Note $V(H_1 \cap H_2) \subseteq \gamma$ by the minimality of $|V(H_1 \cap H_2)|$. Also, $\gamma \cap N_G(w) \subseteq V(H_1 \cap H_2)$. Moreover, $\gamma \cap U \neq \emptyset$ as, otherwise, $V(H_1 \cap H_2)$ would be a cut in G, a contradiction as G is 4-connected.

For convenience, let $N_G(w) = \{w_1, \ldots, w_t\}$, and assume that the notation is chosen so that $w_1, \ldots, w_t$ occur on $W - w$ in clockwise order. Moreover, for $i \in [t]$, let W_i denote the path in $W - w$ from w_i to w_{i+1} in clockwise order, where $w_{t+1} = w_1$. We claim that

(1) any two vertices of $\gamma \cap U$ consecutive on γ must be contained in the same W_i, for some $i \in [t]$.

For, otherwise, let $u, v \in \gamma \cap U$ be consecutive on γ such that $u \in V(W_i)$ and $v \in V(W_j)$, with $i < j$. Then we see that G has a separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{u, v, w\}$, $w_{i+1} \in V(L_1 - L_2)$, and $G_2 \subseteq L_2$. This contradicts the fact that G is 4-connected.

Let $k = |V(H_1 \cap H_2)|$. Then $k \geq 2$. For, otherwise, it follows from (1) that $\gamma \cap U \subseteq W_i$ for some $i \in [k]$. Choose $u, v \in \gamma \cap U$ with uW_iv maximal. Then $\{u, v\} \cup V(H_1 \cap H_2)$ is a cut in G, a contradiction.

Let $V(H_1 \cap H_2) = \{v_1, \ldots, v_k\}$, where $2 \leq k \leq 3$, and for $i \in [k]$, let γ_i be the open curve in γ from v_i to v_{i+1} in clockwise order, where $v_{k+1} = v_1$. We further claim that

(2) there exist unique $i \in [k]$ and unique $j \in [t]$, for which $\gamma_i \cap W_j \neq \emptyset$.

For, suppose otherwise. First, assume that there exist γ_i and γ_l with $i \neq l$ such that for some W_j, $\gamma_i \cap W_j \neq \emptyset$ and $\gamma_l \cap W_j \neq \emptyset$. Without loss of generality, we may assume $i = 1$ and $l = 2$. Then, by planarity and by (1), $U \cup (V(H_1 \cap H_2) \backslash \{v_2\})$ is a cut in G_1 separating $\{w\} \cup N_G(w)$ from $V(G_1 \cap G_2)$; so G_1' has a separation (H_1', H_2') such that $V(H_1' \cap H_2') = V(H_1 \cap H_2) \backslash \{v_2\}$, $N_G(w) \cup \{w\} \subseteq V(H_1')$, and $V(G_1 \cap G_2) \subseteq V(H_2')$. This contradicts the choice of (H_1, H_2) that $|V(H_1 \cap H_2)|$ is minimum.

Hence, by (1), there exist $p \neq q$ and $i \neq j$ such that $\gamma_p \cap W_i \neq \emptyset$ and $\gamma_q \cap W_j \neq \emptyset$. Without loss of generality, we may further assume that $p = 1, q = 2$, and $i < j$. Let $v_2' \in V(W_i)$ such that v_2, v_2' are consecutive on γ, and $v_2'' \in V(W_j)$ such that v_2, v_2'' are consecutive on γ. Then, by (1), G has a 4-separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{v_2, v_2', v_2'', w\}$ is independent in L_1, $\{w_{i+1}, \ldots, w_j\} \subseteq V(L_1)$, and $G_2 \subseteq L_2$. If $|V(L_1)| \geq 6$ then, by Lemma 2.2, L_1 has a $V(L_1 \cap L_2)$-good wheel; so (L_1, L_2) contradicts the choice of (G_1, G_2). Hence, $|V(L_1)| \leq 5$ and $j = i + 1$.

We may assume $k = 3$. For, suppose $k = 2$. Let $v' \in V(W_i)$ such that v_1, v' are consecutive on γ, and let $v'' \in V(W_{i+1})$ such that v_1, v'' are consecutive on γ. By (1), G has a 4-separation (L_1', L_2') such that $V(L_1' \cap L_2') = \{v_1, v', v'', w_{i+1}\}$ is independent in L_1', $\{w\} \cup (N_G(w) \backslash \{w_{i+1}\}) \subseteq V(L_1')$, and $G_2 \subseteq L_2'$. Since $|V(L_1')| \geq 6$, it follows from Lemma 2.2 that L_1' contains a $V(L_1' \cap L_2')$-good wheel. So (L_1', L_2') contradicts the choice of (G_1, G_2).

Now let $v_1' \in V(W_i)$ such that v_1, v_1' are consecutive on γ, and let $v_3' \in V(W_{i+1})$ such that v_3, v_3' are consecutive on γ.

Suppose $\gamma_3 \cap U = \emptyset$. Then $v_1 \neq w_1$ or $v_3 \neq w_3$; otherwise, $\{v_1, v_3, w_{i+1}\}$ would be a 3-cut in G. If $v_1 = w_1$ then by (1), G has a separation (L_1', L_2') such that $V(L_1' \cap L_2') = \{v_1, v_3, v_3', w_{i+1}\}$ is independent in L_1', $\{w\} \cup (N_G(w) \backslash \{w_{i+1}\}) \subseteq V(L_1')$, and $G_2 \subseteq L_2'$; so by Lemma 2.2, L_1' has a $V(L_1' \cap L_2')$-good wheel and, hence, (L_1', L_2') contradicts the choice of (G_1, G_2). So $v_1 \neq w_1$. Similarly, $v_3 \neq w_3$. Then by (1), G has a separation (L_1', L_2') such that $V(L_1' \cap L_2') = \{v_1, v_1', v_3, v_3', w_{i+1}\}$ is independent in L_1', $\{w\} \cup (N_G(w) \backslash \{w_{i+1}\}) \subseteq V(L_1')$, $G_2 \subseteq L_2'$, and $|V(L_1') \backslash V(L_2')| \geq 4$. By the choice of (G_1, G_2), L_1' does not admit a $V(L_1' \cap L_2')$-good wheel. So by Lemma 2.2, $|V(L_1' - L_2')| = 4$ and $(L_1', V(L_1' \cap L_2'))$ is the 9-vertex graph in Figure 1, which means that the only neighbor of w_{i+1} in L_1', namely w, should have degree 6 in L_1' and must be adjacent to v_1' and v_3'. But this is a contradiction as $v_1', v_3' \notin N_G(w)$.

So $\gamma_3 \cap U \neq \emptyset$. But then by (1) and 4-connectedness of G, there exist $l \in \{1, 3\}$ and vertex v_l'' such that v_l', v_l, v_l'' are consecutive on γ in order listed and G has a 4-separation (L_1', L_2') with $V(L_1' \cap L_2') = \{w, v_l', v_l, v_l''\}$ independent in L_1', $G_2 \subseteq L_2'$, and $|V(L_1')| \geq 6$. Then by Lemma 2.2, L_1' contains a $V(L_1' \cap L_2')$-good wheel. Hence (L_1', L_2') contradicts the choice of (G_1, G_2).

Thus, by (1) and (2), we may assume that $\gamma_1 \cap W_1 \neq \emptyset$ and, for $i \in [k]\backslash\{1\}$ and $j \in [t]\backslash\{1\}$, $\gamma_i \cap W_j = \emptyset$. Let $v_1', v_2' \in V(W_1)$ such that, for $i = 1, 2$, v_i and v_i' are consecutive on γ. Then G has a separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{v_i : i \in [k]\} \cup \{v_1', v_2'\}$ is independent in L_1, $N_G(w) \cup \{w\} \subseteq V(L_1)$, and $G_2 \subseteq L_2$. Note that $w \in V(L_1)\backslash V(L_2)$. Also note that $v_1 \neq w_1$ or $v_2 \neq w_2$; otherwise, $V(L_1 \cap L_2)$ would be a 3-cut in G. If $v_1 = w_1$ or $v_2 = w_2$ then $|V(L_1 \cap L_2)| = 4$ and $|V(L_1)\backslash V(L_2)| \geq 6$; hence, by Lemma 2.2, L_1 has a $V(L_1 \cap L_2)$-good wheel and, hence, (L_1, L_2) contradicts the choice of (G_1, G_2). So $v_1 \neq w_1$ and $v_2 \neq w_2$. Hence $|V(L_1)| \geq 9$ and w is not adjacent to $\{v_1, v_2, v_1', v_2'\}$. It follows from Lemma 2.2 that L_1 contains a $V(L_1 \cap L_2)$-good wheel. Hence (L_1, L_2) contradicts the choice of (G_1, G_2). □

To extend a wheel to a K_5-subdivision, we need the following weaker version of a result of Seymour [19], with equivalent forms proved in [16, 20, 22]. For a graph G and vertices $v_1, v_2, \ldots, v_n$ of G, we say that $(G, v_1, v_2, \ldots, v_n)$ is planar if G can be drawn in a closed disc in the plane with no edge crossings such that $v_1, v_2, \ldots, v_n$ occur on the boundary of the disc in clockwise order.

Lemma 2.4. *Let G be a graph and $s_1, s_2, t_1, t_2 \in V(G)$ be distinct such that, for any $S \subseteq V(G)$ with $|S| \leq 3$, every component of $G - S$ must contain a vertex from $\{s_1, s_2, t_1, t_2\}$. Then either G contains disjoint paths from s_1, s_2 to t_1, t_2, respectively, or (G, s_1, s_2, t_1, t_2) is planar.*

The next result shows that in a Hajós graph, we cannot extend a wheel in certain way.

Lemma 2.5. *Let G be a Hajós graph. Suppose there exists a 4-separation (G_1, G_2) in G such that $(G_1, V(G_1 \cap G_2))$ is planar. If W is a $V(G_1 \cap G_2)$-good wheel in G_1 then W is not $V(G_1 \cap G_2)$-extendable.*

Proof. For, suppose W is $V(G_1 \cap G_2)$-extendable. Let $V(G_1 \cap G_2) = \{t_1, t_2, t_3, t_4\}$, and assume that the notation is chosen so that $(G_1, t_1, t_2, t_3, t_4)$ is planar. Then there exist four paths P_1, P_2, P_3, P_4 in G_1 from w to t_1, t_2, t_3, t_4, respectively, such that $V(P_i \cap P_j) = \{w\}$ for any distinct $i, j \in [4]$ and $|V(P_i \cap W)| = 2$ for $i \in [4]$.

If $(G_2, t_1, t_2, t_3, t_4)$ is planar then G is planar and, hence, 4-colorable, a contradiction. So $(G_2, t_1, t_2, t_3, t_4)$ is not planar. Then, by Lemma 2.4, G_2 has disjoint paths Q_1, Q_2 from t_1, t_2 to t_3, t_4, respectively. But then $W \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup Q_1 \cup Q_2$ is a K_5-subdivision in G, a contradiction. □

3 | EXTENDING PATHS FROM 5-CUTS TO 4-CUTS

The goal of this section is to describe the situations where a good wheel cannot be extended from a 5-cut to a 4-cut in the desired way. We achieve this goal in three steps (formulated as lemmas), by gradually reducing the number of possibilities. The first lemma has four possibilities.

Lemma 3.1. *Suppose G is a Hajós graph and (G_1, G_2) is a 4-separation in G such that $(G_1, V(G_1 \cap G_2))$ is planar and $|V(G_1)| \geq 6$, and, subject to this, G_1 is minimal. Moreover,* *suppose that* G_1 *has a* 5*-separation* (H, L) *with* $V(G_1 \cap G_2) \subseteq V(L)$ *and* $V(H \cap L)$ *independent in* H*, such that*

(a) $V(G_1 \cap G_2) \nsubseteq V(H \cap L)$ *and* H *has a* $V(H \cap L)$*-good wheel,*
(b) *subject to* (a), $|S|$ *is minimum, where* $S = V(H \cap L) \cap V(G_1 \cap G_2)$*, and*
(c) *subject* (b), H *is minimal.*

Then H *has a* $(V(H \cap L), S)$*-extendable wheel, or* G_1 *has a* $V(G_1 \cap G_2)$*-extendable wheel, or, for each* $V(H \cap L)$*-good wheel* W *with center* w *in* H*, one of the following holds:*

(i) *There exist* $s \in S \backslash V(W)$ *and* $a, b \in V(W - w) \backslash N_G(w)$ *such that* $N_H(s) = \{a, b\}$ *and either* $a = b$ *or* $ab \in E(W)$*.*
(ii) *There exist* $s_1, s_2 \in S \backslash V(W)$*,* $a, b \in V(W - w) \backslash N_G(w)$*, and separation* (H_1, H_2) *in* H *such that* $V(H_1 \cap H_2) = \{a, b, w\}$*,* $\{s_1, s_2\} \subseteq V(H_1)$*,* $|N_G(w) \cap V(H_1)| = 1$*, and* $V(H \cap L) \backslash \{s_1, s_2\} \subseteq V(H_2)$*.*
(iii) $|S| = 3$*, and there exist* $s_1, s_2 \in S$*,* $a, b \in V(W - w) \backslash N_G(w)$*, and separation* (H_1, H_2) *in* H *such that* $V(H_1 \cap H_2) = \{a, b, s_1, s_2\}$*,* $S \subseteq V(H_1)$*, and* $\{w\} \cup (V(H \cap L) \backslash S) \subseteq V(H_2)$*.*
(iv) *There exist* $a, b \in V(W - w) \backslash N_G(w)$*,* $c \in V(H) \backslash V(W)$*, and a separation* (H_1, H_2) *in* H *such that* $V(H_1 \cap H_2) = \{a, b, c\}$*,* $|V(H_1) \cap V(H \cap L)| = 2$*,* $V(H_1) \cap V(H \cap L) \subseteq S$*, and* $N_G(w) \cup \{w\} \subseteq V(H_2) \backslash V(H_1)$*.*

Proof. Note that $|S| \leq 3$ as $V(G_1 \cap G_2) \nsubseteq V(H \cap L)$. We may assume that G_1 is drawn in a closed disc in the plane with no edge crossing such that $V(G_1 \cap G_2)$ is on the boundary of that disc. For convenience, let $V(H \cap L) = \{t_i : i \in [5]\}$ such that $(H, t_1, t_2, t_3, t_4, t_5)$ is planar. Let D denote the outer walk of H. Let W be a $V(H \cap L)$-good wheel in H with center w, and let $F = W - w$ (which is a cycle).

By Lemma 2.3, W is $V(H \cap L)$-extendable in H. Without loss of generality, assume that H has four paths P_1, P_2, P_3, P_4 from w to t_1, t_2, t_3, t_4, respectively, such that $|V(P_i \cap F)| = 1$ and $t_5 \notin V(P_i)$ for $i \in [4]$. Moreover, we may assume $t_5 \in S$ as, otherwise, these paths show that W is $(V(H \cap L), S)$-extendable. Then $t_5 \notin V(W)$; for, if $t_5 \in V(W)$ then $t_5 w \in E(H)$ (as W is $V(H \cap L)$-good) which, combined with three of $\{P_1, P_2, P_3, P_4\}$, shows that W is $(V(H \cap L), S)$-extendable. Let $V(P_i \cap F) = \{w_i\}$ for $i \in [4]$. Since $(H, t_1, t_2, t_3, t_4, t_5)$ is planar, w_1, w_2, w_3, w_4 occur on F in clockwise order.

We choose P_1, P_4 so that $w_4 F w_1$ is minimal. Then

$$N_G(w) \cap V(w_4 F w_1 - \{w_1, w_4\}) = \emptyset.$$

For, suppose not and let $w' \in N_G(w) \cap V(w_4 F w_1 - \{w_1, w_4\})$. Since G is 4-connected and $(H, t_1, t_2, t_3, t_4, t_5)$ is planar, H must contain a path P from w' to $(P_4 - \{w, w_4\}) \cup (P_1 - \{w, w_1\}) \cup \{t_5\}$ and internally disjoint from $P_4 \cup P_1 \cup F$. (For, if such P does not exist then there exist $a \in V(w_4 F w' - w')$ and $b \in V(w' F w_1 - w')$ such that $\{a, b, w\}$ is a 3-cut in H separating w' from $P_4 \cup P_1 \cup \{t_5\}$. Thus, $\{a, b, w\}$ is a 3-cut in G, a contradiction). If P ends at t_5 then P and three of $\{P_1, P_2, P_3, P_4\}$ show that W is $(V(H \cap L), S)$-extendable. So by symmetry we may assume P ends at $P_4 - \{w, w_4\}$. Then replacing P_4 with the path in $P \cup (P_4 - w_4) \cup ww'$ from w to t_4, we obtain a contradiction to the minimality of $w_4 F w_1$.

Note that H has a path R from t_5 to $(P_4 \cup w_4Fw_1 \cup P_1)\backslash\{t_1, t_4\}$ and internally disjoint from $P_1 \cup P_4 \cup w_4Fw_1$. For otherwise, G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t_1, t_2, t_3, t_4\}$, $W \subseteq G_1'$, and $G_2 + t_5 \subseteq G_2'$. By the choice of (G_1, G_2), $|V(G_1')| = 5$. This implies that $w_i = t_i$ for $i \in [4]$ and, hence, $t_1t_2 \in E(H)$, a contradiction.

We may assume that H has a path from t_5 to $P_1 \cup P_4$ and internally disjoint from $P_1 \cup P_4 \cup F$. For, suppose not. Then, by planarity, there exist $a, b \in V(w_4Fw_1 - \{w_1, w_4\})$ (not necessarily distinct) such that w_4, a, b, w_1 occur on F in clockwise order and all paths in H from t_5 to $P_1 \cup P_4 \cup (w_4Fw_1 - aFb)$ must intersect aFb first. We choose a, b so that aFb is minimal. Since $N_G(w) \cap V(w_4Fw_1 - \{w_1, w_4\}) = \emptyset$, H has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{a, b, t_5\}$, $aFb + t_5 \subseteq H_1$, and $bFa + \{t_1, t_2, t_3, t_4\} \subseteq H_2$. Thus $V(H_1) = \{a, b, t_5\}$ as G is 4-connected. Now, by the existence of R, (i) holds with $s := t_5$.

Case 1. H has paths from t_5 to both P_1 and P_4 and internally disjoint from $P_1 \cup P_4 \cup F$. Then $t_1, t_4 \in S$ as otherwise we may reroute P_1 or P_4 to t_5; and the new path, P_2 and P_3, and P_4 or P_1 show that W is $(V(H \cap L), S)$-extendable. So $S = \{t_1, t_4, t_5\}$. Let $v \in V(G_1 \cap G_2)\backslash S$.

We further choose P_1, P_4 so that, subject to the minimality of w_4Fw_1, the subgraph K of H contained in the closed region bounded by $(P_1 - w) \cup (P_4 - w) \cup t_4Dt_1 \cup w_4Fw_1$ is maximal. Then, every vertex in $V(P_1)\backslash\{w, w_1, t_1\}$ is cofacial with some vertex in $V(w_1Fw_2 - w_1)$ $\cup V(P_2 - w)$; and every vertex in $V(P_4)\backslash\{w, w_4, t_4\}$ is cofacial with some vertex in $V(w_3Fw_4 - w_4) \cup V(P_3 - w)$. Let $T_1 := \{x \in V(P_1 \cup w_4Fw_1)\backslash\{t_1, w, w_4\} : x \text{ is cofacial with } t_4\}$ and $T_4 := \{x \in V(P_4 \cup w_4Fw_1)\backslash\{t_4, w, w_1\} : x \text{ is cofacial with } t_1\}$. Note that $t_1 \notin T_4$ and $t_4 \notin T_1$ by the existence of the path R.

We may assume that $T_1 = \emptyset$ or $T_4 = \emptyset$. For otherwise, suppose $T_1 \neq \emptyset$ and $T_4 \neq \emptyset$. Then let $a \in T_1$ and $b \in T_4$. Now, by the existence of the path R, the vertices w_4, a, b, w_1 occur on F in clockwise order. Since $N_G(w) \cap V(w_4Fw_1 - \{w_1, w_4\}) = \emptyset$, H has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{t_1, t_4, a, b\}$, $S \subseteq V(H_1)$, and $\{w\} \cup (V(H \cap L)\backslash S) \subseteq V(H_2)$; hence, we have (iii) with $s_1 = t_1$ and $s_2 = t_4$.

We may also assume that if $L - t_1$ has disjoint paths R_3, R_2 from t_3, t_2 to t_4, v, respectively, then $T_1 \neq \emptyset$; for, otherwise, $K - t_4$ contains a path P from w_4 to t_5 and internally disjoint from $w_4Fw_1 \cup P_1$, and the paths $P_1, P_2 \cup R_2, P_3 \cup R_3, P \cup ww_4$ show that W is $V(G_1 \cap G_2)$-extendable in G_1. Similarly, we may assume that if $L - t_4$ has disjoint paths Q_3, Q_2 from t_3, t_2 to v, t_1, respectively, then $T_4 \neq \emptyset$.

Hence, since $T_1 = \emptyset$ or $T_4 = \emptyset$, R_2, R_3 do not exist or Q_2, Q_3 do not exist.

Subcase 1.1. Q_2, Q_3 or R_2, R_3 exist.

Without loss of generality, assume that R_2, R_3 exist, and Q_2, Q_3 do not exist. Then $T_1 \neq \emptyset$ and $T_4 = \emptyset$. Since Q_2, Q_3 do not exist, v and t_1Dt_2 are cofacial in G_1. (For, otherwise, L has a 2-separation (L_1, L_2) with $t_4 \in V(L_1 \cap L_2)$, $\{t_2, t_3\} \subseteq V(L_1)$, and $\{t_1, v\} \subseteq V(L_2)$. Now $(H \cup L_1, L_2 \cup L \cup G_2)$ is a 4-separation in G, contradicting the choice of (G_1, G_2)). Since $T_4 = \emptyset$, there exists a path P in $K - t_1$ from w_1 to t_5 and internally disjoint from $w_4Fw_1 \cup P_4$. We choose P so that the subgraph K' of K in the closed region bounded by $P_1 \cup P \cup t_5Dt_1$ is maximal.

We may assume that there exists a vertex $t \in V(t_1Dt_2 - t_1) \cap V(P \cup (w_1Fw_2 - w_2))$. For, suppose not. Then let P_2' be a path in $P_2 \cup t_1Dt_2$ from w_2 to t_1; now $P_3, P_4, P \cup ww_1$, and $P_2' \cup ww_2$ show that W is $(V(H \cap L), S)$-extendable.

Suppose $t \in V(P)$. Choose t so that t_1Dt is maximal. Then note that $t_1Dt = t_1P_1t$ (by the maximality of K) and, for any vertex $t^* \in V(t_1P_1t)$, $t^* \notin T_1$ to avoid the 3-cut $\{t^*, t_4, v\}$ in G. Thus, since $T_1 \neq \emptyset$, it follows from the maximality of K' that t is cofacial with some vertex $t' \in V(w_4Fw_1 - w_1)$. Choose t' so that $t'Fw_1$ is maximal. Now $t_4 \neq w_4$; otherwise, G has a

4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t, t', t_4, v\}$, $G_1' \subseteq G_1 - t_5$, $G_2 + t_5 \subseteq G_2'$, and $|V(G_1')| \geq 6$, contradicting the choice of (G_1, G_2). Now, since t, t' are cofacial and $t^* \notin T_1$ for any vertex $t^* \in V(t_1Pt)$, it follows from planarity that $T_1 \subseteq V(w_4Ft' - w_4)$, and we choose $u_1 \in T_1$ with w_4Fu_1 maximal. Thus G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{t', t, v, t_4, u_1\}$ is independent in H', $\{w, w_2, w_3, w_4\} \subseteq V(H')\backslash V(L')$, and $L + \{t_1, t_5\} \subseteq L'$. Note that $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$ (since $t \neq t_1$). So by the choice of (H, L), H' contains no $V(H' \cap L')$-good wheel. Hence by Lemma 2.2, $(H', V(H' \cap L'))$ must be the 9-vertex graph in Figure 1. However, this is impossible as $w_4w_2 \notin E(H')$ but w_4 is the unique neighbor of u_1 in $H' - L'$.

Thus, $t \in V(w_1Fw_2)\backslash\{w_1, w_2\}$ for all choices of t. Choose t so that w_1Ft is minimal. Now, $V(P_1 \cap t_1Dt) = \{t_1\}$ and, by the maximality of K, each vertex of P_1 is cofacial with some vertex in $V(w_1Ft - w_1)$.

Suppose there exists $u_1 \in V(P_1 - w_1) \cap T_1$. Then there exists $u_1' \in V(w_1Ft - w_1)$ such that u_1' and u_1 are cofacial. Choose such u_1, u_1' that $u_1P_1t_1$ and $u_1'Ft$ are minimal. Suppose there exists $w' \in N_G(w) \cap V(u_1'Ft - \{u_1', t\})$. Then since G is 4-connected, it follows from the choice of u_1, u_1' that H has a path P_1' from w' to t_1 and internally disjoint from $F \cup P$. Now $P_1' \cup ww'$, $P_2 \cup R_2$, $P_3 \cup R_3$, $P \cup ww_1$ show that W is $V(G_1 \cap G_2)$-extendable. So we may assume $N_G(w) \cap V(u_1'Ft - \{u_1', t\}) = \emptyset$. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{t, v, t_4, u_1, u_1'\}$ is independent in H', $\{w, w_1, w_2, w_3\} \subseteq V(H')\backslash V(L')$, and $L + \{t_1, t_5\} \subseteq L'$. Note that $w_1w_2, w_1w_3 \notin E(H')$; so $(H', V(H' \cap L'))$ cannot be any graph in Figure 1. Thus, by Lemma 2.2, H' has a $V(H' \cap L')$-good wheel. Hence, (H', L') contradicts the choice of (H, L) as $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$.

Thus, we may assume $V(P_1 - w_1) \cap T_1 = \emptyset$. So there exists $u_1 \in V(w_4Fw_1 - w_4) \cap T_1$. Choose u_1 so that u_1Fw_1 is minimal.

If $t_4 = w_4$ then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t, v, w_4, w\}$, $G_1' \subseteq G_1 - t_5$, $G_2 \subseteq G_2'$, and $|V(G_1')| \geq 6$; which contradicts the choice of (G_1, G_2). So $t_4 \neq w_4$. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{t, v, t_4, u_1, w\}$ is independent in H', $\{w_2, w_3, w_4\} \subseteq V(H')\backslash V(L')$, and $L + \{t_1, t_5\} \subseteq L'$. Now H' has no $V(H' \cap L')$-good wheel; otherwise, (H', L') contradicts the choice of (H, L) as $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$. Hence, by Lemma 2.2, $(H', V(H' \cap L')$ is the 8-vertex or 9-vertex graph in Figure 1. Note that w is adjacent to all of w_2, w_3, w_4. Thus, $(H', V(H' \cap L')$ must be the 8-vertex graph in Figure 1. However, this forces $t_2 = w_2$ and $t_3 = w_3$; so $t_2t_3 \in E(W) \subseteq E(H)$, a contradiction as $V(H \cap L)$ is independent in H.

Subcase 1.2. Neither Q_2, Q_3 nor R_2, R_3 exist.

Then, by the choice of (G_1, G_2), we see that t_1Dt_2 and v are cofacial, and that t_3Dt_4 and v are cofacial. Moreover, since G is 4-connected, $\{t_2, t_3, v\}$ is not a cut in G. Hence, $V(L) = \{t_1, t_2, t_3, t_4, t_5, v\}$ and, by the choice of (G_1, G_2), we have $vt_2, vt_3 \in E(G)$.

Suppose there exist $a \in V(t_1Dt_2) \cap V(w_1Fw_2 - w_2)$ and $b \in V(t_3Dt_4) \cap V(w_3Fw_4 - w_3)$. Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, b, v, w\}$, $\{w_2, w_3\} \subseteq V(G_1')\backslash V(G_2')$, and $G_2 + \{t_1, t_4, t_5\} \subseteq G_2'$. Now (G_1', G_2') contradicts the choice of (G_1, G_2).

So by symmetry, we may assume that $t_3Dt_4 \cap (w_3Fw_4 - w_3) = \emptyset$. Thus, $t_3Dt_4 \cup P_3$ contains a path P from w_3 to t_4 and internally disjoint from F. If H has a path Q from w_4 to t_5 and internally disjoint from $P \cup w_3Fw_1 \cup P_1$ then $P_1, P_2, P \cup ww_3, Q \cup ww_4$ show that W is $(V(H \cap L), S)$-extendable. So assume that Q does not exist. Then there exist $x \in V(P) \cup V(w_3Fw_4 - w_3)$ and $y \in V(P_1) \cup V(w_4Fw_1 - w_4)$ such that x and y are cofacial. By the choice of P and planarity of H, $x \in V(t_3Dt_4) \cap V(P_4 - w_4)$. Choose x to minimize xDt_4.

First, suppose $t_1Dt_2 \cap (w_1Fw_2 - w_2) \neq \emptyset$ and $y \in V(w_4Fw_1 - w_4)$ for all choices of y. Choose $a \in V(t_1Dt_2) \cap V(w_1Fw_2 - w_2)$ such that w_1Fa is minimal, and choose x, y so that yFw_1 is minimal. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, v, x, y, w\}$ is independent in H', $\{w_2, w_3, w_4\} \subseteq V(H') \backslash V(L')$, and $L + \{t_1, t_4, t_5\} \subseteq L'$. Since $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$, we see that H' has no $V(H' \cap L')$-good wheel. Hence, by Lemma 2.2, $(H', V(H' \cap L'))$ is the 8-vertex or 9-vertex graph in Figure 1. Since w is adjacent to all of w_2, w_3, w_4, we see that $|V(H')| = 8$ which forces $t_2 = w_2$ and $t_3 = w_3$. Hence, $t_2t_3 \in E(W) \subseteq E(H)$, a contradiction as $V(H \cap L)$ is independent in H.

Now suppose $t_1Dt_2 \cap (w_1Fw_2 - w_2) \neq \emptyset$ and $y \in V(P_1 - w_1)$ for some choice of y. Choose such y so that yP_1t_1 is minimal; so K has a path P_5 from y to t_5 and internally disjoint from $P_1 \cup P_4$. Let $a \in V(t_1Dt_2) \cap V(w_1Fw_2 - w_2)$ such that w_1Fa is minimal. Note that $y \notin V(t_1Dt_2)$ (to avoid the 3-cut $\{v, x, y\}$ in G). So y is not cofacial with $P_2 - w_2$ and, by the maximality of K, there exists $y' \in V(w_1Fa - w_1)$ such that y and y' are cofacial. We choose y' so that $y'Fa$ is minimal. If there exists $w' \in N_G(w) \cap V(y'Fa - \{y', a\})$ then by the minimality of yP_1t_1 and $y'Fa$ and by the 4-connectedness of G, H has a path Z from w' to t_1 and internally disjoint from $F \cup P_5$; now $Z \cup ww'$, P_2, P_4, $wP_1y \cup P_5$ show that W is $(V(H \cap L), S)$-extendable. Hence, we may assume $N_G(w) \cap V(y'Fa) = \emptyset$. So G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, v, x, y, y'\}$ is independent in H', $\{w, w_1, w_2, w_3, w_4\} \subseteq V(H') \backslash V(L')$, and $L + \{t_1, t_4, t_5\} \subseteq L'$. By Lemma 2.2, H' contains a $V(H' \cap L')$-good wheel. Now (H', L') contradicts the choice of (H, L), as $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$.

Hence, we may assume that $t_1Dt_2 \cap (w_1Fw_2 - w_2) = \emptyset$. Then $t_1Dt_2 \cup P_2$ has a path Q from w_2 to t_1 and internally disjoint from F. Similar to the argument for showing the existence of x and y above, we may assume that there exist $p \in V(t_1Dt_2) \cap V(P_1 - w_1)$ and $q \in V(P_4) \cup V(w_4Fw_1 - w_1)$ such that p and q are cofacial.

Note that x and p are not cofacial in G_1; as otherwise $\{x, p, v\}$ would be a 3-cut in G. Thus, w_4, y, q, w_1 occur on F in clockwise order, or $q \in V(w_4P_4x)$ and $p \in V(w_1P_1y)$, or $x \in V(w_4P_4q)$ and $y \in V(w_1P_1p)$. In the later two cases, we see that $\{x, y, v\}$ or $\{p, q, v\}$ is a 3-cut in G, a contradiction. Thus, w_4, y, q, w_1 occur on F in clockwise order.

So G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{p, v, x, y, q\}$ is independent in H', $\{w, w_1, w_2, w_3, w_4\} \subseteq V(H') \backslash V(L')$, and $L + \{t_1, t_4, t_5\} \subseteq L'$. By Lemma 2.2, H' has a $V(H' \cap L')$-good wheel. If $\{x, p\} \neq \{t_1, t_4\}$ then $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$; and hence (H', L') contradicts the choice of (H, L). So $x = t_4$ and $p = t_1$; hence (iii) holds.

Case 2. Case 1 does not occur.

We choose P_1, P_4, subject to the minimality of w_4Fw_1, to maximize the subgraph K of H contained in the closed region bounded by $(P_1 - w) \cup (P_4 - w) \cup t_4Dt_1 \cup w_4Fw_1$.

Without loss of generality, we may assume that G_1 has no path from t_5 to P_4 and internally disjoint from $P_4 \cup P_1 \cup w_4Fw_1$, but G_1 has a path P_5' from t_5 to P_1 and internally disjoint from $P_4 \cup P_1 \cup w_4Fw_1$. Then $t_4Dt_5 \cap ((w_4Fw_1 - w_4) \cup P_1) \neq \emptyset$. Moreover, we may assume $t_1 \in S$; as otherwise we could reroute P_1 to end at t_5 which, along with P_2, P_3, P_4, shows that W is $(V(H \cap L), S)$-extendable.

Subcase 2.1. $t_4Dt_5 \cap (w_4Fw_1 - w_4) = \emptyset$ and $t_1Dt_2 \cap (w_1Fw_2 - w_2) = \emptyset$.

Then there exists $a \in V(t_4Dt_5) \cap V(P_1 - w_1)$, and we choose such a with t_1P_1a minimal. Note $t_1 \neq a$ by the existence of R. Let $b \in V(t_1Dt_2) \cap V(P_1 - w_1)$ with t_1P_1b maximal. By the maximality of K, $t_1Db = t_1P_1b$. If $a \in V(t_1P_1b)$ then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, t_2, t_3, t_4\}$, $\{w, w_1\} \subseteq V(G_1')\backslash V(G_2')$, and $G_2 + \{t_1, t_5\} \subseteq G_2'$, which contradicts the choice of (G_1, G_2).

Hence, $a \in V(bP_1w_1)\backslash\{b, w_1\}$. Let $P_5 := wP_1a \cup aDt_5$. We consider the paths P_2, P_3, P_4, P_5. By the minimality of t_1P_1a, we see that t_1P_1a is a path from t_1 to P_5 and internally disjoint from $P_5 \cup P_2 \cup F$. Since $t_1Dt_2 \cap (w_1Fw_2 - w_2) = \emptyset$, t_1Dt_2 contains a path from t_1 to P_2 and internally disjoint from $P_5 \cup P_2 \cup F$. Hence, we are back to Case 1 (with t_5, t_1, t_2, t_3, t_4 as t_4, t_5, t_1, t_2, t_3, respectively).

Subcase 2.2. Either $t_4Dt_5 \cap (w_4Fw_1 - w_4) \neq \emptyset$ or $t_1Dt_2 \cap (w_1Fw_2 - w_2) \neq \emptyset$.

First, we may assume that $t_4Dt_5 \cap (w_4Fw_1 - w_4) \neq \emptyset$. For, if not, then there exists $a \in V(P_1 - w_1) \cap V(t_4Dt_5)$ and choose a so that t_1P_1a is minimal. Moreover, $t_1Dt_2 \cap (w_1Fw_2 - w_2) \neq \emptyset$; so H has no path from t_1 to P_2 and internally disjoint from F. Note that $t_1 \neq a$ by the existence of R. Let $P_5 := wP_1a \cup aDt_5$. Now consider the paths P_2, P_3, P_4, P_5. We see that t_1P_1a is a path from t_1 to P_5 and internally disjoint from $P_5 \cup P_2 \cup F$. Hence, since $t_1Dt_2 \cap (w_1Fw_2 - w_2) \neq \emptyset$, we could take the mirror image of G_1 and view t_2, t_1, t_5, t_4, t_3 as t_4, t_5, t_1, t_2, t_3, respectively; and, thus, may assume $t_4Dt_5 \cap (w_4Fw_1 - w_4) \neq \emptyset$.

Then $t_1Dt_2 \cap (w_1Fw_2 - w_2) = \emptyset$, and we let $a \in V(t_4Dt_5) \cap V(w_4Fw_1 - w_4)$ with w_4Fa minimal. Let $t \in V(t_1Dt_2 \cap P_1)$ with t_1Dt maximal. Then $t \neq w_1$ and, by the maximality of K, $t_1Dt = t_1P_1t$.

Note that $t_4Dt_5 \cap t_1Dt_2 = \emptyset$. For, otherwise, let $p \in V(t_4Dt_5) \cap V(t_1Dt_2)$. Then G has a 4-separation (G_1', G_2') with $V(G_1' \cap G_2') = \{p, t_2, t_3, t_4\}$, $w, w_1 \in V(G_1' - G_2')$, and $G_2 + \{t_1, t_5\} \subseteq G_2'$. Clearly, (G_1', G_2') contradicts the choice of (G_1, G_2).

If there exist $c \in V(t_1Dt)$ and $b \in V(aFw_1 - w_1)$ such that b and c are cofacial, then (iv) holds. So assume such b, c do not exist. Then K contains a path P from w_1 to t_5 and internally disjoint from $F \cup t_1Dt_2$. By the existence of a path in P_1 from t_1 to P and the path t_1Dt_2, we are back to Case 1 (with t_5, t_1, t_2, t_3, t_4 playing the roles of t_4, t_5, t_1, t_2, t_3, respectively).

Subcase 2.3. $t_4Dt_5 \cap (w_4Fw_1 - w_4) \neq \emptyset$ and $t_1Dt_2 \cap (w_1Fw_2 - w_2) \neq \emptyset$.

Let $a \in V(t_4Dt_5) \cap V(w_4Fw_1 - w_4)$ and $b \in V(t_1Dt_2) \cap V(w_1Fw_2 - w_2)$, and we choose a, b to minimize aFb. Consider the separation (H_1, H_2) in G_1 such that $V(H_1 \cap H_2) = \{a, b, w\}$, $V(H_1) \cap \{t_i : i \in [5]\} = \{t_1, t_5\} \subseteq S$, and $bFa + \{t_2, t_3, t_4\} \subseteq H_2$.

(1) $|N_G(w) \cap V(aFb)| \geq 2$.

For, suppose $|N_G(w) \cap V(aFb)| = 1$. If $w_1 \notin \{a, b\}$ then we have (ii). Since in this proof of (1) we do not make use of the minimality of w_4Fw_1, we may use the symmetry between t_1 and t_5 and assume $w_1 = b$. Consider the 5-separation (H', L') in G_1 such that $V(H' \cap L') = \{a, b, t_2, t_3, t_4\}$ is independent in H', $bFa + w \subseteq H'$, and $L \cup H_1 \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is one of the graphs in Figure 1.

Suppose $w_3 \neq t_3$. Then $w_4 \neq t_4$ to avoid the 4-separation (G_1', G_2') with $V(G_1' \cap G_2') = \{b, t_2, t_3, w_4\}$, $\{w, w_3\} \subseteq V(G_1' - G_2')$, and $G_2 + \{t_1, t_5\} \subseteq G_2'$. So $ww_3w_4w \subseteq H' - L'$, and $(H', V(H' \cap L'))$ must be the 9-vertex graph in Figure 1. However, this is impossible, as $w_4b \notin E(H')$ and one of the following holds: w_4 is the unique neighbor of a in $H' - L'$, or w is the unique neighbor of b in $H' - L'$.

Therefore, $w_3 = t_3$. Then $t_2 \neq w_2$; for, otherwise, $w_2Fw_3 = t_2t_3 \in E(H)$ as G is 4-connected, a contradiction. Similarly, $t_4 \neq w_4$. Thus, $w_2ww_4 \subseteq H' - L'$. So by Lemma 2.2, $(H', V(H' \cap L'))$ must be the 8-vertex or 9-vertex graph in Figure 1. But this is not possible, as $w_4b \notin E(H')$ and one of the following holds: w_4 is the unique neighbor of a in $H' - L'$, or w is the unique neighbor of b in $H' - L'$.

We may assume $t_1 \neq w_1$; since otherwise $b = t_1$ by the minimality of aFb, and we would have $|N_G(w) \cap V(aFb)| = 1$, contradicting (1). We may also assume

(2) $N_G(w) \cap V(aFb) \neq \{a, b\}$.

For, otherwise, $a, b \in N_G(w)$ *(so* $w_1 = a$*) and* G *has a 4-separation* (G_1', G_2') *such that* $V(G_1' \cap G_2') = \{a, b, t_1, t_5\}$, $aFb \subseteq G_1'$, *and* $G_2 \cup bFa + w \subseteq G_2'$. *Hence, by the choice of* (G_1, G_2), $|V(G_1')| \leq 5$.

If $|V(G_1')| = 4$ then $P_1 = wat_1$; so $bt_1 \in E(G)$ (by the minimality of aFb) and $at_5 \in E(G)$ (by the path R), and, hence, wat_5, wbt_1 and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable.

Hence, we may assume $|V(G_1')| = 5$ and let $u \in V(G_1') \backslash V(G_2')$. Then $N_G(u) = \{a, b, t_1, t_5\}$. Since $a = w_1$, $u \notin V(W)$ and $P_1 = waut_1$. If $t_5a \in E(G)$ then $wat_5, wbut_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. If $t_1b \in E(G)$ then $waut_5, wbt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So assume $t_5a, t_1b \notin E(G)$. Then G has a 4-separation (G_1'', G_2'') such that $V(G_1'' \cap G_2'') = \{t_2, t_3, t_4, u\}$, $\{a, b, w\} \subseteq V(G_1'') \backslash V(G_2'')$, and $G_2 + \{t_1, t_5\} \subseteq G_2'$. Hence, (G_1'', G_2'') contradicts the choice of (G_1, G_2).

Now consider the 5-separation (H', L') in G_1 with $V(H' \cap L') = \{a, b, w, t_1, t_5\}$, $bFa \subseteq L'$, and $aFb + \{t_1, t_5\} \subseteq H'$. Note that $|V(H' \cap L') \cap V(G_1 \cap G_2)| \leq |S|$ and $H' \subseteq H$ but $H' \neq H$; so by the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. Thus, since $N_G(w) \cap V(aFb) \neq \{a, b\}$, $(H', V(H' \cap L'))$ is one of the graphs in Figure 1. Recall that $t_5t_1 \notin E(H)$ as $V(H \cap L)$ is independent in H.

First, suppose $|V(H')| = 6$ and let $u \in V(H') \backslash V(L')$. Then by (1) and (2), $aFb = aub$ and $u \in N_G(w)$. By the minimality of aFb, $t_1b \in E(G)$. If $u = w_1$ then $t_5u \in E(G)$ (because of R) and $wb \in E(G)$; so wut_5, wbt_1 and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. Hence, we may assume $a = w_1$. Then $at_5, at_1 \in E(G)$ (because of P_1 and R) and, hence, $ut_1 \in E(G)$ (because the degree of u_1 is at least 4 as G is 4-connected); so wat_5, wut_1 and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable.

Now assume $|V(H')| = 7$. First, suppose $|V(aFb)| \geq 4$ and let $aFb = auvb$. If $P_1 = wat_1$ then G has a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, t_1, b, w\}, \{u, v\} \subseteq V(G_1' - G_2')$, and $G_2 + t_5 \subseteq G_1$; and (G_1', G_2') contradicts the choice of (G_1, G_2). If $P_1 = wvt_1$ then $wa, wu \notin E(H)$; so $ut_1, ut_5 \in E(H)$, contradicting the existence of the path P_5'. So $P_1 = wut_1$ then $t_5u \in E(H)$ (by P_5') and $vw, vt_1 \in E(H)$ (by 4-connectedness of G); so wut_5, wvt_1 and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So we may assume $|V(aFb)| = 3$ and let $aFb = aub$ and $v \in V(H') \backslash (V(L') \cup \{u\})$. Then $wu \in E(G)$ by (1) and (2). If $t_5u \in E(G)$ then $N_G(v) = \{b, t_1, t_5, u\}$ and $t_5a \in E(G)$ (by the minimality of aFb); now $wat_5, wuvt_1$ (when $wa \in E(G)$) or $wut_5, wbvt_1$ (when $wb \in E(G)$), and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So assume $t_5u \notin E(G)$. By the same argument, we may assume $t_1u \notin E(G)$. Then $t_1v, t_5v \in E(G)$. Note that $t_1b \in E(G)$ or $t_5a \in E(G)$; otherwise, $(H - \{t_1, t_5\}, G_2 \cup L \cup t_1vt_5)$ is a 4-separation in G contradicting the choice of (G_1, G_2). So by symmetry, we may assume $t_5a \in E(G)$. If $wa \in E(G)$ then $wat_5, wuvt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So assume $wa \notin W(G)$; hence, $wb \in E(G)$ by (1). If $t_1b \in E(G)$ then $wbt_1, wuvt_5$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So assume $t_1b \notin E(G)$. Now G_1 has a 5-separation (H^*, L^*) such that $H^* = (H - t_1) - t_5v$ and $L^* = L \cup t_1vt_5$. Note that $V(H^* \cap L^*)$ is independent in H^* and W is $V(H^* \cap L^*)$-good. So (H^*, L^*) contradicts the choice of (H, L) as $|V(H^* \cap L^*) \cap V(G_1 \cap G_2)| < |S|$.

Suppose $|V(H')| = 8$ and let $H' - L' = xyz$. Recall that w_4, a, w_1, b, w_2 occur on F in clockwise order. Note that exactly one vertex in $V(H' \cap L')$ is adjacent to all of $\{x, y, z\}$, and call that vertex t. If $t = w$ then we may let $aFb = axyzb$; we see that wxt_5, wzt_1 and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. If $t = t_5$ then we may let $aFb = axyb$; we see that $wxt_5, wyzt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. Similarly, if $t = t_1$ then W is $(V(H \cap L), S)$-extendable. Now assume $t = a$; the argument for $t = b$ is symmetric. Then we may let $aFb = axb$. If $wa \in E(H)$ then $wazt_5, wxyt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So assume $wa \notin E(H)$. Then $wb \in E(H)$ by (1). If $t_1b \in E(H)$ then $wxyt_5, wbt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. So $t_1b \notin E(H)$. Let $H^* = (H - t_5) - t_1z$ and $L^* = L \cup t_1zt_5$. Note that $V(H^* \cap L^*)$ is independent in H^* and W is $V(H^* \cap L^*)$-good. So (H^*, L^*) contradicts the choice of (H, L) as $|V(H^* \cap L^*) \cap V(G_1 \cap G_2)| < |S|$.

Finally, assume $|V(H')| = 9$. Let $V(H' - L') = \{u, x, y, z\}$ such that $xz \notin E(H)$, and u is the unique neighbor of some vertex $t \in V(H' \cap L')$. If $t = w$ then we see that $aFb = aub$ and let $ax, zb \in E(H)$; now $waxt_5, wuyt_1$ (when $wa \in E(H)$) or $wuxt_5, wbzt_1$ (when $wb \in E(H)$), and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. If $t = a$ then we may let $aFb = auxb$; then $wut_5, wxyzt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. If $t = b$ then may let $aFb = axub$; then $wxyt_5, wut_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. If $t = t_5$ then we may let $aFb = axyb$; now $wxut_5, wyzt_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. If $t = t_1$ then we may let $aFb = ayxb$; now $wyzt_5, wxut_1$, and two of P_2, P_3, P_4 show that W is $(V(H \cap L), S)$-extendable. □

Next, we eliminate the possibility (iv) of Lemma 3.2 by working with more than one wheel.

Lemma 3.2. *With the same assumptions of Lemma 3.1, H has a $(V(H \cap L), S)$-extendable wheel, or G_1 has a $V(G_1 \cap G_2)$-extendable wheel, or* (i) *or* (ii) *or* (iii) *of Lemma 3.1 holds for any $w \in V(H - L)$ and for any $V(H \cap L)$-good wheel W in H with center w.*

Proof. Suppose (iv) of Lemma 3.1 holds for some $V(H \cap L)$-good wheel W with center w. Then there exist $a, b \in V(W - w) \backslash N_G(w)$, $c \in V(H) \backslash V(W)$, and separation (H_1, H_2) in H such that $V(H_1 \cap H_2) = \{a, b, c\}$, $|V(H_1) \cap V(H \cap L)| = 2$, $V(H_1) \cap V(H \cap L) \subseteq S$, and $(N_G(w) \cup \{w\}) \cap V(H_1) = \varnothing$. Let G_1 be drawn in a closed disc in the plane with no edge crossings such that $V(G_1 \cap G_2)$ is contained in the boundary of that disc. Let $V(H \cap L) = \{t_i : i \in [5]\}$ and we may assume that $(H, t_1, t_2, t_3, t_4, t_5)$ is planar. Recall from the assumptions in Lemma 3.1 that (H, L) is chosen to minimize $|S|$, where $S \coloneqq V(G_1 \cap G_2) \cap V(H \cap L)$. Without loss of generality, we may assume that $V(H_1) \cap V(H \cap L) = \{t_1, t_5\}$. So $t_1, t_5 \in S$.

By Lemma 2.3, W is $V(H \cap L)$-extendable in H. So there are four paths P_i, $i \in [4]$, in H from w to $\{t_i : i \in [5]\}$, such that $V(P_i \cap P_j) = \{w\}$ for $i \neq j$, $|V(P_i) \cap W| = 2$ for $i \in [4]$, $|V(P_i) \cap \{t_j : j \in [5]\}| = 1$ for $i \in [4]$. Without loss of generality, we may assume that $t_i \in V(P_i)$ for $i \in [4]$. Note that P_2, P_3, P_4 are disjoint from H_1, and we may assume by planarity that $P_1 \cap H_1 = cP_1t_1$. We further choose a, b, c so that aFb and cP_1t_1 are minimal.

Now $t_3 \notin S$. For, suppose $t_3 \in S$. Then, t_3 is cofacial with t_1 or t_5. If t_3 is cofacial with t_5 then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t_1, t_2, t_3, t_5\}$, $W \subseteq G_1'$, and $G_2 + t_2 \subseteq G_2'$; which contradicts the choice of (G_1, G_2). We derive a similar contradiction if t_3 is cofacial with t_1, using the cut $\{t_1, t_3, t_4, t_5\}$.

Let $F = W - w$ and let D denote the outer walk of H. We choose P_1, P_2, P_3, P_4 so that the following are satisfied in the order listed: w_4Fw_1 is minimal, w_3Fw_2 is minimal, and the subgraph K of H contained inside the region bounded by $P_4 \cup t_4Dt_1 \cup P_1$ is minimal. Then every vertex of P_4 is cofacial with a vertex in $w_4Fa - w_4$, every vertex of P_1 is cofacial with a vertex in $bFw_1 - w_1$, and

(1) $N_G(w) \cap V(w_3Fw_2) = \{w_1, w_2, w_3, w_4\}$.

For, suppose (1) fails and let $w' \in N_G(w) \cap V(w_3Fw_2)\setminus\{w_1, w_2, w_3, w_4\}$. First, assume $w' \in V(w_4Fw_1)\setminus\{w_1, w_4\}$. If $w' \in V(w_4Fa - w_4)$ then since G is 4-connected, K has a path P from w' to P_4 and internally disjoint from $P_4 \cup F$. Hence, we can replace P_4 by a path in $P \cup (P_4 - \{w, w_4\})$ from w to t_4, contradicting the minimality of K. We get the same contradiction if $w' \in V(bFw_1 - w_1)$.

Now assume $w' \in V(w_3Fw_4)\setminus\{w_3, w_4\}$. Consider the subgraph J of H contained in the closed region bounded by $P_3 \cup t_3Dt_4 \cup P_4$. By the minimality of w_3Fw_2, J has no path from w' to t_3 and internally disjoint from $F \cup P_4$. Thus, there exist $x \in V(w_3Fw' - w')$ and $y \in V(w'Fw_4 - w') \cup V(P_4)$ such that x, y are cofacial. Since G is 4-connected, $y \in V(P_4 - w_4)$. Note that y is cofacial with some vertex on $w_4Fa - w_4$, say z. Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{w, x, y, z\}$, $w'Fw_4 \subseteq G_1' - G_2'$, and $G_2 + \{t_1, t_2, t_3, t_5\} \subseteq G_2'$. However, (G_1', G_2') contradicts the choice of (G_1, G_2).

Similarly, if $w' \in V(w_1Fw_2)\setminus\{w_1, w_2\}$ then we derive a contradiction.

(2) $w_i \neq t_i$ for $i \in \{2, 3, 4\}$.

First, $w_3 \neq t_3$. For, suppose $w_3 = t_3$. Then $w_2 \neq t_2$ as, otherwise, since G is 4-connected, $w_2Fw_3 = t_2t_3 \in E(W) \subseteq E(H)$, a contradiction. Now by (1), G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{b, c, t_2, w_3, w_4\}$ is independent in H', $ww_1w_2w \subseteq H' - L'$, and $L + \{t_1, t_4, t_5\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. This is not possible, as w is the unique neighbor of w_4 in $H' - L'$ and $wb \notin E(H')$.

Next, $w_4 \neq t_4$. For, suppose $w_4 = t_4$. Then by (1), G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{b, c, t_2, t_3, w_4\}$ is independent in H', $w_1ww_3 \subseteq H' - L'$, and $L + \{t_1, t_5\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 8-vertex or 9-vertex graph in Figure 1. Now $|V(H')| = 9$; as otherwise $w_2 = t_2$ is adjacent to all vertices in $H' - L'$, which implies $bw \in E(H')$, a contradiction. Let $v \in V(H' - L')\setminus\{w, w_1, w_3\}$. Since $w_1w_3 \notin E(H)$, w and v both have degree 3 in $H' - L'$. Therefore, $v = w_2$ is the unique neighbor of t_2 in $H' - L'$, which implies $wb \in E(H)$, a contradiction.

Now $w_2 \neq t_2$. For, suppose $w_2 = t_2$. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, w_2, t_3, t_4\}$ is independent in H', $ww_3w_4w \subseteq H' - L'$, and $L + \{t_1, t_5\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. Since w is the unique neighbor of w_1 in $H' - L'$, $wa \in E(H')$, a contradiction.

(3) $a \neq b$.

For, if $a = b$ then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, c, t_2, t_3, t_4\}$ is independent in H', $|V(H' - L')| \geq 5$ (by (2)), and $L + \{t_1, t_5\} \subseteq L'$. So by Lemma 2.2, H' has a $V(H' \cap L')$-good wheel. Now (H', L') contradicts the choice of (H, L), as $|V(H' \cap L') \cap V(G_1 \cap G_2)| < |S|$.

We may assume $(w_1Fw_2 - w_2) \cap t_1Dt_2 = \emptyset$. For, suppose not. Let $b' \in V(w_1Fw_2 - w_2) \cap V(t_1Dt_2)$ with $b'Fw_2$ minimal. If $b' \neq w_1$ then, by (1), (ii) of Lemma 3.1 holds, with $\{t_1, t_5\}$ and $\{a, b'\}$ as $\{s_1, s_2\}$ and $\{a, b\}$, respectively, in (ii) of Lemma 3.1. So $b' = w_1$. Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, t_1, t_5, w_1\}$, $b \in V(G_1' - G_2')$, and $G_2 + \{t_i : i \in [5]\} \subseteq G_2'$. Now $V(G_1') \backslash V(G_2') = \{b\}$ as otherwise (G_1', G_2') contradicts the choice of (G_1, G_2). But then we see that $N_H(t_5) \subseteq \{a, b\}$; so (i) of Lemma 3.1 holds.

We wish to consider the wheel W_2 consisting of those vertices and edges of H cofacial with w_2.

(4) w_2 and t_1 are not cofacial in H, and w_2, t_3 are not cofacial in H.

First, suppose w_2 and t_3 are cofacial. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, w_2, t_3, t_4\}$ is independent in H', $ww_3w_4w \subseteq H' - L'$, and $L + \{t_1, t_2, t_5\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. This is impossible, as w is the unique neighbor of w_1 in $H' - L'$ and $wa \notin E(H')$.

Now assume that w_2, t_1 are cofacial. Then c, w_2 are cofacial as $c \in V(t_1Dt_2)$. So G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{b, c, w_2, w\}$, $w_1 \in V(G_1' - G_2')$, and $G_2 + \{t_i : i \in [5]\} \subseteq G_2'$. By the choice of (G_1, G_2), $|V(G_1')| = 5$.

Suppose $c = t_1$. Then G has a 4-separation (G_1'', G_2'') such that $V(G_1'' \cap G_2'') = \{a, w_1, t_1, t_5\}$, $b \in V(G_1'' - G_2'')$, and $G_2 + \{t_2, t_3, t_4\} \subseteq G_2''$. By the choice of (G_1, G_2), $|V(G_1'')| = 5$; so $N_G(b) = \{a, w_1, t_1, t_5\}$ and, hence, $N_H(t_5) = \{a, b\}$ and (i) of Lemma 3.1 holds.

Therefore, we may assume $c \neq t_1$. Now consider the 5-separation (H', L') in G_1 such that $V(H' \cap L') = \{a, t_5, t_1, w_2, w\}$ is independent in H', $bw_1c \subseteq H' - L'$ (by (3)), and $L + \{t_2, t_3, t_4\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 8-vertex or 9-vertex graph in Figure 1. This is impossible as w_1 is the unique neighbor of w in $H' - L'$ and $w_1a \notin E(H')$.

Suppose $w_2t_2 \in E(H)$ or w_2 and t_2 are not cofacial. Then W_2 is $V(H \cap L)$-good. By Lemma 3.1, we may assume that (i) or (ii) or (iii) or (iv) of Lemma 3.1 holds for W_2 (with t_2 as s). By the separation (H_1, H_2) we see that only (i) of Lemma 3.1 can hold for W_2. Hence, there exists $t_2', t_2'' \in V(W_2)$ such that $N_H(t_2) = \{t_2', t_2''\}$ and $N_G(w_2) \cap V(t_2'F_2t_2'') = \emptyset$, where $F_2 = W_2 - w_2$ and t_2', t_2, t_2'' occur on F_2 in clockwise order.

We define $t_2' = t_2'' = t_2$ when $w_2t_2 \notin E(H)$ and w_2 and t_2 are cofacial. Then

(5) w_1 and t_2' are not cofacial in H.

For, suppose they are. Then, since $w_2t_2' \notin E(H)$, it follows from (1) that, to avoid the cut $\{w_1, w_2, t_2''\}$ in G, w_1 and t_2'' must be cofacial in G_1 and $w_1Fw_2 = w_1w_2$. Thus, G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, t_2'', t_3, t_4\}$ is independent in H', $\{w, w_2, w_3, w_4\} \subseteq V(H' - L')$, and $L + \{t_1, t_5\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. But this is impossible as w is the unique neighbor of w_1 in $H' - L'$ and $wa \notin E(H')$.

Consider the 5-separation (H', L') in G_1 such that $V(H' \cap L') = \{b, c, t_2', w_2, w\}$ is independent in H', $w_1 \in V(H' - L')$, and $L + \{t_i : i \in [5]\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is one of the graphs in Figure 1. By (5), $|V(H')| \geq 7$.

(6) If $|V(H')| \geq 8$ then $bFw_2 = bw_1w_2$; H_1 has a path Q from b to t_5 and internally disjoint from $aFb \cup cP_1t_1$; and $P_3 \cup t_3Dt_4 \cup (w_3Fw_4 - w_4)$ has a path R from w_3 to t_4.

Note that w_1 is the unique neighbor of w in $H' - L'$; so if $|V(H')| \geq 8$ then $bFw_2 = bw_1w_2$. Also note that, by the choice of $\{b, c\}$, H_1 has a path Q from b to t_5 and internally disjoint from $aFb \cup cP_1t_1$.

Moreover, $(P_3 - t_3) \cup t_3Dt_4 \cup (w_3Fw_4 - w_4)$ has a path R from w_3 to t_4. For, otherwise, $w_4 \in V(t_3Dt_4)$. Hence H has a separation (H'', L'') such that $V(H'' \cap L'') = \{b, c, t_2, t_3, w_4\}$ is independent in H'', $\{w, w_1, w_2, w_3\} \subseteq V(H'' - L'')$, and $L' + \{t_1, t_5\} \subseteq L''$. Since $|V(H'' \cap L'') \cap V(G_1 \cap G_2)| < |S|$, we see from the choice of (H, L) that H'' has no $V(H'' \cap L'')$-good wheel. Then $(H'', V(H'' \cap L''))$ is the 9-vertex graph in Figure 1. However, this is not possible, as w_1 is the unique neighbor of b in $H'' - L''$ and $w_1w_4 \notin E(G)$.

(7) We may assume $|V(H')| = 7$.

First, suppose $|V(H')| = 9$. Note that w has a unique neighbor in $H' - L'$, namely w_1. Hence, $H' - \{w, w_1, c, t_2'\}$ has a path $bv_1v_2v_3w_2$ such that $v_i \in N_G(w_1)$ for $i \in [3]$, $v_1, v_2 \in N_G(c)$, and $v_2, v_3 \in N_G(t_2')$. Since $t_3 \notin S$, we see that W_1, the wheel consisting of vertices and edges of H cofacial with w_1, is $(V(H \cap L), S)$-extendable, using the paths $w_1b \cup Q$, $w_1vc \cup cP_1t_1$, $w_1v_2t_2't_2$, and $w_1ww_3 \cup R$ (where Q and R are from (6)).

Now suppose $|V(H')| = 8$. Then $H' - \{w, w_1, c, t_2'\}$ has a path bv_1w_2 such that $v_1 \in N_G(w_1)$, $H' - L'$ is a path $w_1v_1v_2$, and either $v_2 \in N_G(b) \cap N_G(c)$ and $v_1, v_2 \in N_G(t_2')$, or $v_2 \in N_G(w_2) \cap N_G(t_2')$ and $v_1, v_2 \in N_G(c)$. Again, since $t_3 \notin S$ and because of Q and R, we see that W_1 is $(V(H \cap L), S)$-extendable.

Thus, let $V(H' - L') = \{w_1, v\}$. Suppose $v \notin V(w_1P_1c)$. Then $N_G(v) = \{c, t_2', w_2, w_1\}$. Since G is 4-connected, it follows from the choice of $\{b, c\}$ that H_1 has a path Q' from b to t_5 internally disjoint from $aFb \cup W_1 \cup cP_1t_1$. Now, since $t_3 \notin S$ and because of Q' and R, we see that W_1 is $(V(H \cap L), S)$-extendable.

Hence, we may assume $v \in V(w_1P_1c)$. Then $vw_2 \in E(H)$, since w_1, t_2' are not cofacial. Note that $t_2'v \in E(H)$. If $bv \in E(H)$ then, since $t_2 \notin S$ and because of Q, R, we see that W_1 is $(V(H \cap L), S)$-extendable. So $bv \notin E(H)$. If $ct_2' \in E(H)$ then let W_v denote the wheel consisting of vertices and edges of H cofacial with v; then by the choice of $\{b, c\}$, H_1 has a path Q' from b to t_5 internally disjoint from $aFb \cup W_v \cup cP_1t_1$, and, hence, since $t_3 \notin S$ and because of R, W_v is $(V(H \cap L), S)$-extendable. So assume $ct_2' \notin E(H)$.

We may assume $c \neq t_1$. For, if $c = t_1$ then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, w_1, c, t_5\}$, $b \in V(G_1' - G_2')$, and $G_2 + \{t_2, t_3, t_4\} \subseteq G_2'$. By the choice of (G_1, G_2), we see that $|V(G_1')| = 5$ and $N_G(b) = \{a, t_5, t_1, w_1\}$, which implies that (i) of Lemma 3.1 holds for W.

Then G_1 has a 5-separation (H'', L'') such that $V(H'' \cap L'') = \{a, t_5, t_1, v, w_1\}$ is independent in H'', $\{b, c\} \subseteq V(H'' - L'')$, and $L + \{t_2, t_3, t_4\} \subseteq L''$. By the choice of (H, L), H'' has no $V(H'' \cap L'')$-good wheel. So by Lemma 2.2, $(H'', V(H'' \cap L''))$ is one of the graphs in Figure 1. Since b is the unique neighbor of w_1 in $H'' - L''$ and c is the unique neighbor of v in $H'' - L''$, $|V(H'')| = 7$. If $t_1b \in E(H)$ then $N_H(t_5) \subseteq \{a, b\}$ and (i) of Lemma 3.1 holds. So assume $t_1b \notin E(H)$. Then $N_G(b) = \{a, t_5, c, w_1\}$ and $N_G(c) = \{b, t_1, t_5, v\}$. Thus, $((H - t_1) - t_5c, L \cup t_5ct_1)$ is a 5-separation in G_1 that contradicts the choice of (H, L). □

We further eliminate possibilities (i) and (iii) of Lemma 3.1.

Lemma 3.3. *With the same assumptions of Lemma 3.1, H has a $(V(H \cap L), S)$-extendable wheel, or G_1 has a $V(G_1 \cap G_2)$-extendable wheel, or* (ii) *of Lemma 3.1 holds for any $w \in V(H - L)$ and for any $V(H \cap L)$-good wheel W in H with center w.*

Proof. By Lemma 3.2, we may assume that (i) or (iii) of Lemma 3.1 holds for some $V(H \cap L)$-good wheel W. Let w be the center of W, and let $F = W - w$. Let $V(H \cap L) = \{t_1, t_2, t_3, t_4, t_5\}$. We may assume that G_1 is drawn in a closed disc in the plane with no edge crossings such that the vertices in $V(G_1 \cap G_2)$ occur on the boundary of that disc. Further, we may assume that $(H, t_1, t_2, t_3, t_4, t_5)$ is planar.

By Lemma 2.3, W is $V(H \cap L)$-extendable. So let P_1, P_2, P_3, P_4 be paths in H from w to t_1, t_2, t_3, t_4, respectively, such that $V(P_i \cap P_j) = \{w\}$ for distinct $i, j \in [4]$, and $|V(P_i) \cap V(W)| = 2$ for $i \in [4]$. Let $V(P_i) \cap V(F) = \{w_i\}$ for $i \in [4]$.

Since (i) or (iii) of Lemma 3.1 holds for W, we may assume that there exist $a, b \in V(w_4Fw_1)$ and separation (H_1, H_2) in H, such that w_4, a, b, w_1 occur on F in clockwise order, $N_G(w) \cap V(aFb) = \emptyset$, $V(H_1 \cap H_2) = \{a, b, t_1, t_4\}$, $aFb + t_5 \subseteq H_1$, and $bFa + \{w, t_2, t_3\} \subseteq H_2$. Moreover, $t_5 \in S$; and $t_1, t_4 \in S$, or H_1 consists of the triangle abt_5a (or the edge $t_5a = t_5b$) and two isolated vertices t_1 and t_4.

We choose a, b so that aFb is minimal. We further choose P_1, P_2, P_3, P_4 to minimize w_4Fw_1 and then w_3Fw_2. By the same argument in the proof of Lemma 3.2, we have

(1) $N_G(w) \cap V(w_3Fw_2) = \{w_1, w_2, w_3, w_4\}$.

Note that $w_2 \neq t_2$ or $w_3 \neq t_3$. Since, otherwise, $w_2Fw_3 = t_2t_3 \in E(H)$ (as G is 4-connected), contradicting the fact that $V(H \cap L)$ is independent in H. We claim that

(2) w_1, t_2 are not cofacial in H and that w_4, t_3 are not cofacial in H.

For, suppose otherwise and assume by symmetry that w_1 and t_2 are cofacial in H. Then $w_4 \neq t_4$, to avoid the 4-separation (G_1', G_2') in G such that $V(G_1' \cap G_2') = \{t_2, t_3, w_4, w_1\}$, $\{w, w_2\} \subseteq V(G_1' - G_2')$ or $\{w, w_3\} \subseteq V(G_1' - G_2')$, and $G_2 + t_5 \subseteq G_2'$.

Suppose $w_3 = t_3$. Then $w_2 \neq t_2$ and G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{w_1, t_2, w_3, w_4\}$, $\{w, w_2\} \subseteq V(G_1' - G_2')$, and $G_2 + t_5 \subseteq G_2'$. Now (G_1', G_2') contradicts the choice of (G_1, G_2).

So $w_3 \neq t_3$. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, t_2, t_3, t_4\}$ is independent in H', $ww_3w_4w \subseteq H' - L'$, and $L + \{b, t_1, t_5\} \subseteq L'$. By the choice of (H, L), H' does not contain any $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. This is impossible because one of the following holds: w_4 is the unique neighbor of a in $H' - L'$ but $w_1w_4 \notin E(H')$, or w is the unique neighbor of w_1 in $H' - L'$ and $aw \notin E(H')$.

Thus, $w_1 \neq t_1$ (as t_1, t_2 are cofacial in H), $w_2 \neq t_2$ (as w_1, w_2 are cofacial in H), $w_3 \neq t_3$ (as w_3, w_4 are cofacial in H), and $w_4 \neq t_4$ (as t_4, t_3 are cofacial in H). Moreover,

(3) $a \neq b$.

For, suppose $a = b$. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{b, t_1, t_2, t_3, t_4\}$ is independent in H', $\{w, w_1, w_2, w_3, w_4\} \subseteq V(H' - L')$, $L + t_5 \subseteq L'$. Hence, by Lemma 2.2, H' has a $V(H' \cap L')$-good wheel. Now (H', L') contradicts the choice of (H, L).

(4) t_5 is not cofacial in H with w_1 or w_4.

For, otherwise, assume by symmetry that w_1 and t_5 are cofacial. Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, t_4, t_5, w_1\}$, $b \in V(G_1' - G_2')$, and $w_1Fa \cup G_2 \subseteq G_2'$. Hence, by the choice of (G_1, G_2), $|V(G_1')| = 5$ and $N_G(b) = \{a, t_4, t_5, w_1\}$. Therefore, we could have chosen $a = b$, contradicting (3) and the minimality of aFb.

(5) w_2, t_3 are not cofacial in H and that w_3, t_2 are not cofacial in H.

For, suppose this is false and assume by symmetry that w_2 and t_3 are cofacial in H. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, w_2, t_3, t_4\}$ is independent in H', $ww_3w_4w \subseteq H' - L'$, and $L + \{t_1, t_2, t_5\} \subseteq L'$. By the choice of (H, L), H' does not contain any $V(H' \cap L')$-good wheel. So $(H', V(H' \cap L'))$ must be the 9-vertex graph in Figure 1. However, this is not possible, because w is the unique neighbor of w_1 in $H' - L'$ and $aw \notin E(H')$.

Suppose $\{t_2, t_3\} \subseteq S$. Then G has a separation (G_1', G_2') such that $V(G_1' \cap G_2') = S \cup \{t_4\}$ or $V(G_1' \cap G_2') = S \cup \{t_1\}$, $H \subseteq G_1'$, and $G_2 + t_1 \subseteq G_2'$ or $G_2 + t_4 \subseteq G_2'$. However, (G_1', G_2') contradicts the choice of (G_1, G_2). Thus, we may assume by symmetry that

(6) $t_2 \notin S$.

We will consider wheels W_i (for $i \in [2]$) consisting of the vertices and edges of H that are cofacial with w_i.

(7) H_2 has disjoint paths P, Q from w_2, w_3 to t_3, t_4, respectively, and internally disjoint from $w_4Fa \cup bFw_2 \cup P_1$; and H_1 has a path R from b to t_5 and internally disjoint from $W_1 + t_4$.

First, suppose P, Q do not exist. Then there exist $v_3 \in V(P_3)\backslash\{t_3, w_3\}$ and separation (G_1', G_2') in G such that $V(G_1' \cap G_2') = \{w_1, w_2, v_3, w_4\}$, $\{w, w_3\} \subseteq V(G_1' - G_2')$, and $G_2 \cup L \subseteq G_2'$. Now (G_1', G_2') contradicts the choice of (G_1, G_2).

Now assume that the path R does not exist. Then H_1 has a 2-cut $\{p, q\}$ separating b from t_5 such that $p \in V(W_1 - b) \cup \{t_1\}$ and $q \in V(aFb - b) \cup \{t_4\}$.

If $p = t_1$ then $q = t_4$ by the minimality of aFb. So G has a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t_1, t_2, t_3, t_4\}$, $G_1' \subseteq G_1 - t_5$, and $G_2 + t_5 \subseteq G_2'$. Now (G_1', G_2') contradicts the choice of (G_1, G_2).

Hence, $p \in V(W_1 - b)$. Then $q \notin V(aFb - b)$ to avoid the 3-cut $\{p, q, w_1\}$ in G. So $q = t_4$. Now G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{p, t_1, t_2, t_3, t_4\}$ is independent in H', $\{w, w_1, w_2, w_3, w_4\} \subseteq V(H' - L')$, and $L + t_5 \subseteq L'$. By Lemma 2.2, H' has a $V(H' \cap L')$-good wheel. So (H', L') contradicts the choice of (H, L).

(8) We may assume that w_2 and t_1 are not cofacial in H.

For, otherwise, G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{b, t_1, w_2, w\}$, $w_1 \in V(G_1' - G_2')$, and $G_2 + \{t_2, t_3, t_4, t_5\} \subseteq G_2'$. Hence, $V(G_1' - G_2') = \{w_1\}$ by the choice of (G_1, G_2). Since w_1 and t_2 are not cofacial in H, $w_2t_1 \in E(H)$. Now the paths $w_1t_1, R \cup w_1b, P \cup w_1w_2, Q \cup w_1ww_3$ show that W_1 is $(V(H \cap L), S)$-extendable.

Then $w_2t_2 \notin E(H)$ and w_2 and t_2 are cofacial. For, otherwise, W_2 is a $V(H \cap L)$-good wheel in H. So by Lemma 3.2, (i) or (ii) or (iii) of Lemma 3.1 occurs for W_2. Note that $w \in W_2$ and W_2 is disjoint from $w_3Fw_1 - \{w_1, w_3\}$. Thus, there do not exist vertices $a, b \in V(W_2 - w_2)$ such that in H, $\{a, b, w\}$ separates two vertices in $\{t_1, t_2, t_3, t_4, t_5\}$ from the other three. So (ii) and (iii) of Lemma 3.1 do not occur for W_2. Moreover, if (i) of Lemma 3.1 occurs for W_2 then $t_2 \in S$, contradicting (6).

Hence, G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{b, t_1, t_2, w_2, w\}$ is independent in H', $w_1 \in V(H' - L')$, and $L + \{t_3, t_4, t_5\} \subseteq L'$. By the choice of (H, L), H' does not contain any $V(H' \cap L')$-good wheel. Hence by Lemma 2.2, $(H', V(H' \cap L'))$ is one of the graphs in Figure 1. Note that $|V(H')| \geq 7$ by (2). We may assume that

(9) $|V(H')| = 7$ and $N_G(t) = \{t_1, t_2, w_2, w_1\}$ with $t \in V(H' - L')\backslash\{w\}$.

First, we may assume $|V(H')| = 7$. For, suppose $|V(H')| \geq 8$. Then, since w_1 is the only neighbor of w in $H' - L'$, we see, by checking the 8-vertex and 9-vertex graph in Figure 1, that $bFw_2 = bw_1w_2$, W_1 is defined, and P_1 can be chosen so that $W_1 - w_1$ intersects $P_1 - w$ just once. So the paths $w_1P_1t_1, R \cup w_1b, P \cup w_1w_2, Q \cup w_1ww_3$ show that W_1 is $(V(H \cap L), S)$-extendable.

Now let $t \in V(H' - L')\backslash\{w\}$. We may assume $N_G(t) = \{t_1, t_2, w_2, w_1\}$. This is clear if $P_1 = ww_1t_1$. So assume $P_1 = ww_1tt_1$. Then $tw_2 \in E(H')$ by (2). If $tb \in E(H')$ then W_1 is a $V(H \cap L)$-good wheel, and $P_1 - w, R \cup w_1b, P \cup w_1w_2, Q \cup w_1ww_3$ show that W_1 is $(V(H \cap L), S)$-extendable. So assume $tb \notin E(H')$. Hence, $N_G(t) = \{t_1, t_2, w_2, w_1\}$.

(10) $t_1b \in E(H)$.

For, suppose $t_1b \notin E(H)$. Consider the 5-separation (H'', L'') in G_1 such that $V(H'' \cap L'') = \{a, t_4, t_5, t_1, w_1\}$ is independent in H'', $b \in V(H'' - L'')$, $|V(H'' - L'')| \geq 2$ (because of R and W_1), and $L + \{t_2, t_3\} \subseteq L''$. By the choice of (H, L) and by Lemma 2.2,

$(H'', V(H'' \cap L''))$ is one of the graphs in Figure 1. Note that b is the only neighbor of w_1 in $V(H'' - L'')$. Since $t_1b \notin E(H)$, $|V(H'')| = 7$. Because of R and W_1, we see that $R = bt_5$ and, hence, $\{b, t_1, t_5\}$ is a 3-cut in G, a contradiction.

Suppose $P_1 = ww_1t_1$. If $w_1Fw_2 = w_1w_2$ then $w_1t_1, R \cup w_1b, P \cup w_1w_2, Q \cup w_1ww_3$ show that W_1 is $(V(H \cap L), S)$-extendable. So assume $w_1Fw_2 = w_1tw_2$. If there are disjoint paths P', Q' in H from t_2, w_3 to t_3, t_4, respectively, and internally disjoint from $w_4Fa \cup bFw_2 \cup P_1$, then $w_1t_1, R \cup w_1b, P' \cup w_1tt_2, Q' \cup w_1ww_3$ show that W_1 is $(V(H \cap L), S)$-extendable. Hence, we may assume that P', Q' do not exist. Then there exist $v_3 \in V(P_3)\backslash\{t_3, w_3\}$ and separation (H'', L'') in H such that $V(H'' \cap L'') = \{w_4, w_1, t, t_2, v_3\}$ is independent in H'', $ww_2w_3w \subseteq H'' - L''$, and $L' + \{t_1, t_5\} \subseteq L''$. By the choice of (H, L), H'' does not contain any $V(H'' \cap L'')$-good wheel. Hence by Lemma 2.2, $(H'', V(H'' \cap L''))$ is one of the graphs in Figure 1. But this is not possible as w is the unique neighbor of w_1 in $H'' - L''$ and $wt \notin E(H'')$.

Therefore, $P_1 = ww_1tt_1$. Let $G' := G - \{t, w_1\} + t_1w$, which does not contain a K_5-subdivision as t_1w can be replaced by t_1tw_1w. So G' admits a 4-coloring, say σ. We now have a contradiction by extending σ to a 4-coloring of G as follows: If $\sigma(t_1) = \sigma(w_2)$ then greedily color w_1, t in order; if $\sigma(t_1) \neq \sigma(w_2)$ then assign $\sigma(t_1)$ to w_1 and greedily color t. □

4 | PROOF OF THEOREM 1.1

Suppose that G is a Hajós graph and that G has a 4-separation (G_1, G_2) such that $(G_1, V(G_1 \cap G_2))$ is planar and $|V(G_1)| \geq 6$, and choose such (G_1, G_2) that G_1 is minimal. Further, we assume that G_1 is drawn in a closed disc in the plane with no edge crossings such that $V(G_1 \cap G_2)$ is contained in the boundary of that disc.

By Lemma 2.2, G_1 has a $V(G_1 \cap G_2)$-good wheel. Moreover, by Lemma 2.5, any $V(G_1 \cap G_2)$-good wheel in G_1 is not $V(G_1 \cap G_2)$-extendable. Hence, by Lemma 2.3, there exists a 5-separation (H, L) in G_1 such that $V(H \cap L)$ is independent in H, $V(G_1 \cap G_2) \subseteq V(L)$, $V(G_1 \cap G_2) \not\subseteq V(H \cap L)$, and H has a $V(H \cap L)$-good wheel. Let $S = V(H \cap L) \cap V(G_1 \cap G_2)$. We further choose (H, L) such that

(1) $|S|$ is minimum and, subject to this, H is minimal.

Then by Lemma 2.3,

(2) any $V(H \cap L)$-good wheel in H is $V(H \cap L)$-extendable.

Let $V(H \cap L) = \{t_1, t_2, t_3, t_4, t_5\}$ such that $(H, t_1, t_2, t_3, t_4, t_5)$ is planar. Note that

(3) the vertices in S must occur consecutively in the cyclic ordering t_1, t_2, t_3, t_4, t_5.

For, suppose not. Then, without loss of generality, assume that $t_1, t_3 \in S$ but $t_2, t_5 \notin S$. Let $V(G_1 \cap G_2) = \{t_1, t_3, x, y\}$.

If (G_1, t_1, x, t_3, y) is planar then there exists a 4-separation (G_1', G_2') in G such that $V(G_1' \cap G_2') = \{t_1, t_2, t_3, y\}$, $H \subseteq G_1'$, $x \notin V(G_1')$, and $G_2 \subseteq G_2'$; which contradicts the choice of (G_1, G_2). Similarly, if (G_1, t_1, y, t_3, x) is planar we obtain a contradiction.

If (G_1, t_1, x, y, t_3) or (G_1, t_1, y, x, t_3) is planar then $\{t_1, t_2, t_3\}$ would be a 3-cut in G.

So assume (G, t_1, t_3, x, y) is planar (by renaming x, y if necessary). Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t_1, t_3, t_4, t_5\}$, $H \subseteq G_1'$, $\{x, y\} \nsubseteq V(G_1')$, and $G_2 \subseteq G_2'$, which contradicts the choice of (G_1, G_2).

We claim that

(4) no $V(H \cap L)$-good wheel in H is $(V(H \cap L), S)$-extendable.

For, suppose W is a $V(H \cap L)$-good wheel in H that is also $(V(H \cap L), S)$-extendable. Let w be the center of W and assume that H has four paths P_1, P_2, P_3, P_4 from w to t_1, t_2, t_3, t_4, respectively, such that $V(P_i \cap P_j) = \{w\}$ for distinct $i, j \in [4]$, $|V(P_i) \cap V(W)| = 2$ for $i \in [4]$, and $S \subseteq \{t_1, t_2, t_3, t_4\}$.

Let $k = 4 - |S|$. Since W is not $V(G_1 \cap G_2)$-extendable, $L - (S \cup \{t_5\})$ does not contain k disjoint paths from $\{t_i : i \in [4]\} \backslash S$ to $V(G_1 \cap G_2) \backslash S$. Thus, $L - (S \cup \{t_5\})$ has a cut T of size at most $k - 1$ separating $\{t_i : i \in [4]\} \backslash S$ from $V(G_1 \cap G_2) \backslash S$. Hence $T \cup S \cup \{t_5\}$ is a cut in G, and $|T \cup S \cup \{t_5\}| = 4$ since G is 4-connected. Thus, G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = T \cup S \cup \{t_5\}$, $H \subseteq G_1'$, G_1' is a proper subgraph of G_1, and $G_2 \subseteq G_2'$. Note that $|V(G_1')| \geq 6$ because $W \subseteq H \subseteq G_1'$ and $V(H \cap L)$ is independent in H; so (G_1', G_2') contradicts the choice of (G_1, G_2).

Thus, by (4) and Lemma 3.3,

(5) for any $V(H \cap L)$-good wheel W in H (with center w, say), (ii) of Lemma 3.1 holds.

By (2) (and without loss of generality), let P_1, P_2, P_3, P_4 be paths in H from w to t_1, t_2, t_3, t_4, respectively, such that $V(P_i \cap P_j) = \{w\}$ for distinct $i, j \in [4]$, and $|V(P_i) \cap V(W)| = 2$ for $i \in [4]$. Let $F = W - w$ (which is a cycle) and let $V(P_i) \cap V(F) = \{w_i\}$ for $i \in [4]$. By (4), $t_5 \in S$.

By (5), there exist $s_1, s_2 \in S \backslash V(W)$, $a, b \in V(W - w) \backslash N_G(w)$, and a separation (H_1, H_2) in H such that $|V(aFb) \cap N_G(w)| = 1$, $V(H_1 \cap H_2) = \{a, b, w\}$, $V(aFb) \cup \{s_1, s_2\} \subseteq V(H_1)$, and $V(H \cap L) \backslash \{s_1, s_2\} \subseteq V(H_2)$. Without loss of generality, we may assume that $s_1 = t_1, s_2 = t_5$, $aFb \subseteq w_4Fw_2$, and $w_1 \in V(aFb)$. We choose P_i, $i \in [4]$, to minimize w_4Fw_2. Then it is easy to see that

(6) $N_G(w) \cap V(w_4Fw_2) = \{w_1, w_2, w_4\}$.

We claim that

(7) $w_i \neq t_i$ for $i = 2, 3, 4$.

First, we show $w_2 \neq t_2$ and $w_4 \neq t_4$. For, suppose the contrary and, by symmetry, assume $w_2 = t_2$. Then $w_3 \neq t_3$ as otherwise $w_2Fw_3 = w_2w_3$ (since G is 4-connected); so $t_2t_3 \in E(H)$, contradicting the fact that $W \subseteq H$ and $V(H \cap L)$ is independent in H. So $w_4 \neq t_4$ to avoid the 4-separation (G_1', G_2') with $V(G_1' \cap G_2') = \{w_1, w_2, t_3, w_4\}$, $\{w, w_3\} \subseteq V(G_1' - G_2')$, and $G_2 \subseteq G_2'$. Now G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, w_2, t_3, t_4\}$ is independent in H', $ww_3w_4w \subseteq H' - L'$, and $L \cup H_1 \subseteq L'$. By the choice of (H, L), H' does not have any $V(H' \cap L')$-good wheel. Hence, by Lemma 2.2, $(H', V(H' \cap L'))$ must be the 9-vertex graph in Figure 1. However, this is not possible, as w is the only neighbor of w_1 in $H' - L'$ and $wa \notin E(H)$.

Now suppose $w_3 = t_3$. Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{b, t_2, w_3, w\}$, $w_2 \in V(G_1' - G_2')$, and $G_2 \cup w_3Fw_1 \subseteq G_2'$. Hence, by the choice of (G_1, G_2), we have $|V(G_1')| = 5$. So $N_G(w_2) = \{b, t_2, w_3, w\}$ and b has degree at most 2 in H_2. Similarly, by considering the 4-cut $\{a, t_4, w, w_3\}$, we have $N_G(w_4) = \{a, t_4, w_3, w\}$ and a has degree at most 2 in H_2. Thus, since G is 4-connected, a, b each have degree at least 2 in H_1.

Now consider the 5-separation (H', L') in G_1 such that $V(H' \cap L') = \{a, w, b, t_1, t_5\}$ is independent in H', $w_1 \in V(H' - L')$, and $H_2 \cup L \subseteq L'$. By the choice of (H, L), H' does not contain any $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is one of the graphs in Figure 1. Note that w_1 is the only neighbor of w in $H' - L'$. So $aw_1, bw_1 \in E(H)$ as a and b each have degree at least 2 in H'. Note that $H' \subseteq H_1 - \{at_5, bt_1\}$.

We may assume that a or b has degree exactly 2 in H_1. For, otherwise, by checking the graphs in Figure 1, we see that $|V(H')| = 8$ or $|V(H')| = 9$, and H_1 contains a wheel W' with center w' such that $N_G(w') = V(W' - w')$ and $|V(W')| \in \{4, 5\}$. If $|V(W')| = 4$ then, since G is 4-connected, $G_1 - w'$ has four disjoint paths from $V(W')$ to $V(G_1 \cap G_2)$; which shows that W' is $(V(H \cap L), S)$-extendable, contradicting (4). So $|V(W')| = 5$. Then, by the choice of (H, L), $H - w$ has 5 disjoint paths from $V(W')$ to $V(H \cap L)$; which, again, shows that W' is $(V(H \cap L), S)$-extendable, contradicting (4).

Thus, we may assume by symmetry that a has degree exactly 2 in H_1. Then a has degree 4 in G and $at_4 \in E(G)$. Let σ be a 4-coloring of $G - \{a, w, w_2, w_4\}$. If $\sigma(w_1) = \sigma(t_4)$ then by greedily coloring w_2, w, w_4, a in order we obtain a 4-coloring of G, a contradiction. If $\sigma(w_1) = \sigma(t_3)$ then by greedily coloring a, w_4, w_2, w in order we obtain a 4-coloring of G, a contradiction. So $\sigma(w_1) \notin \{\sigma(t_3), \sigma(t_4)\}$. Then assigning $\sigma(w_1)$ to w_4 and greedily coloring w_2, w, a in order, we obtain a 4-coloring of G, a contradiction.

(8) $w_2, w_4 \notin V(D)$, where D denotes the outer walk of H.

First, $w_2 \notin V(t_2Dt_3)$ and $w_4 \notin V(t_3Dt_4)$. For, suppose not and assume by symmetry that $w_4 \in V(t_3Dt_4)$. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{b, t_2, t_3, w_4, w_1\}$ is independent in H', $ww_2w_3w \subseteq H' - L'$, $bFw_4 + w \subseteq H'$, and $L + \{t_1, t_5\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. Hence, by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. However, this is impossible since w is the only neighbor of w_1 in H' and $wb \notin E(H')$.

Now suppose (8) fails. Then we may assume by symmetry that $w_4 \in V(D)$. So $w_4 \in V(t_4Dt_5)$. Then $w_4Fa = w_4a$ (by (6) and 4-connectedness of G) and G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{b, t_1, t_5, w_4, w\}$ is independent in H', $aFw_1 \subseteq V(H')\backslash V(L')$, and $bFw_4 + \{t_2, t_3, t_4\} \subseteq L'$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. Hence, since w_4 and w each have exactly one neighbor in $V(H')\backslash V(L')$, it follows from Lemma 2.2 that $V(H')\backslash V(L') = \{a, w_1\}$. Since G is 4-connected, $N_G(a) = \{t_1, t_5, w_1, w_4\}$ and $N_G(w_1) = \{a, b, t_1, w\}$. However, G_1 now has a 5-separation (H'', L'') such that $H'' = (H - t_5) - at_1$ and $L'' = L \cup t_1at_5$. Note that $\{w, w_1, w_2, w_3, w_4\} \subseteq V(H'')\backslash V(L'')$; so by Lemma 2.2, (H'', L'') has a $V(H'' \cap L'')$-good wheel, contradicting the choice of (H, L).

(9) w_2, t_3 are not cofacial, and w_4, t_3 are not cofacial.

Otherwise, suppose by symmetry that w_2, t_3 are cofacial. Then G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, w_1, w_2, t_3, t_4\}$ is independent in H', $ww_3w_4w \subseteq H' - L'$, and $L + \{b, t_1, t_5\} \subseteq L'$. By the choice of (H, L), H' does not contain any $V(H' \cap L')$-good wheel. So by Lemma 2.2, $(H', V(H' \cap L'))$ is the 9-vertex graph in Figure 1. But this is not possible as w is the only neighbor of w_1 in $H' - L'$ and $wa \notin E(H')$.

For $i = 2, 4$, let W_i denote the wheel consisting of all vertices and edges of H that are cofacial with w_i. Then

(10) for each $i \in \{2, 4\}$, W_i is not a $V(H \cap L)$-good wheel in H.

For, suppose W_2 is a $V(H \cap L)$-good wheel in H. Since $W_2 \cap (w_3Fw_1 - \{w_1, w_3\}) = \emptyset$ and $t_1, t_5 \in S$, it follows from (3) that (ii) of Lemma 3.1 does not hold for W_2, contradicting (5).

Thus, by (7)–(10), $w_2t_2, w_4t_4 \notin E(H)$, w_2 and t_2 are cofacial in H, and w_4 and t_4 are cofacial in H. Since G is 4-connected, $\{b, t_2, w_2\}$ and $\{a, t_4, w_4\}$ are not cuts in G. So by (8), $N_{H_2}(a) = \{t_4, w_4\}$ and $N_{H_2}(b) = \{t_2, w_2\}$.

We claim that there exists some $i \in \{2, 3, 4\}$ such that w_3, t_i are cofacial and $w_3t_i \notin E(G)$. For, otherwise, W_3 is a $V(H \cap L)$-good in H. Since W_3 is disjoint from $w_4Fw_2 - \{w_2, w_4\}$, it follows from (3) that (ii) of Lemma 3.1 does not hold for W_3, contradicting (5).

First, suppose $i \in \{2, 4\}$ and, by symmetry, assume w_3, t_4 are cofacial and $w_3t_4 \notin E(G)$. Then G has a 4-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, t_4, w_3, w\}$, $w_4 \in V(G_1' - G_2')$, and $G_2 \cup aFw_3 \subseteq G_2'$. Since $w_4t_4 \notin E(G)$, $|V(G_1' - G_2')| \geq 2$. Hence (G_1', G_2') contradicts the choice of (G_1, G_2).

Thus, w_3, t_3 are cofacial and $w_3t_3 \notin E(G)$. By symmetry, we may assume that $P_3 - w$ and w_4 are on the same side of the face which is incident with both t_3 and w_3. Now G_1 has a 5-separation (H', L') such that $V(H' \cap L') = \{a, t_4, t_3, w_3, w\}$ is independent in H', $(P_3 - w) \cup (P_4 - w) \subseteq H'$, and $aFw_3 + \{t_1, t_2, t_5\} \subseteq L'$. Moreover, $|V(H' - L')| \geq 3$ since $(P_3 - w) \cap (P_4 - w) = \emptyset$, $w_3P_3t_3 \neq w_3t_3$, and $w_4P_4t_4 \neq w_4t_4$. By the choice of (H, L), H' has no $V(H' \cap L')$-good wheel. Therefore, by Lemma 2.2, $(H', V(H' \cap L'))$ is the 8-vertex graph or 9-vertex graph in Figure 1. This is impossible, as w_4 is the only neighbor of a in $H' - L'$ and $w_4t_4 \notin E(H)$, a contradiction [14].

ACKNOWLEDGMENTS

We thank the anonymous referees for carefully reading the manuscript and providing helpful suggestions. Partially supported by NSF grants DMS-1600738 and DMS-1954134. Qiqin Xie is supported by Office of China Postdoc Council Project YJ20190133 (the Talent Introduction Program of Postdoctoral International Exchange Program).

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How to cite this article: Q. Xie, S. Xie, X. Yu, and X. Yuan, *4-Separations in Hajós graphs*, J Graph Theory. 2022;99:485–508. https://doi.org/10.1002/jgt.22751