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Partitioning digraphs with outdegree at least 4

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Abstract

Scott asked the question of determining c_d such that if D is a digraph with m arcs and minimum outdegree $d \geqslant 2$ then $V(D)$ has a partition V_1, V_2 such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant c_d m$, where $e(V_1, V_2)$ (respectively, $e(V_2, V_1)$) is the number of arcs from V_1 to V_2 (respectively, from V_2 to V_1). Lee, Loh, and Sudakov showed that $c_2 = 1/6 + o(1)$ and $c_3 = 1/5 + o(1)$, and conjectured that $c_d = \frac{d-1}{2(2d-1)} + o(1)$ for $d \geqslant 4$. In this paper, we show $c_4 = 3/14 + o(1)$ and prove some partial results for $d \geqslant 5$.

KEYWORDS
digraph, judicious partition, outdegree

MATHEMATICAL SUBJECT CLASSIFICATION
05C70, 05C20, 05C35, 05D40

1 | INTRODUCTION

Judicious partitioning problems concern partitions of graphs and hypergraphs that provide bounds for several parameters simultaneously, while classical partitioning problems seek for partitions that optimize a single parameter. For a graph G and $A, B \subseteq V(G)$, we use $e(A, B)$ to denote the number of edges in G between A and B, and we write $e(A) := e(A, A)$. An example of a classical partitioning result is Edwards' theorem [4,5] that if G is a graph with m edges then $V(G)$ has a partition V_1, V_2 such that $e(V_1, V_2) \geqslant m/2 + (\sqrt{2m + 1/4} - 1/2)/4$, and the inequality is tight for complete graphs of odd order. Bollobás and Scott [2] proved the following judicious version of Edwards' result: The vertex set of any m-edge graph has a bipartition V_1, V_2 such that $e(V_1, V_2) \geqslant m/2 + (\sqrt{2m + 1/4} - 1/2)/4$ and $\max\{e(V_1), e(V_2)\} \leqslant m/4 + (\sqrt{2m + 1/4} - 1/2)/8$, and both bounds are tight for complete graphs of odd order.

Bollobás and Scott [3,16] initiated a systematic study of judicious partitioning problems, which has lead to a large amount of research in this area, see, for instance [7,10,11,13,13-15,17,17-20].

Partitioning problems concerning digraphs (i.e., directed graphs) may be more difficult. For a digraph D and $A, B \subseteq V(D)$, we use $e(A, B)$ to denote the number of arcs in D directed from A to B and write $e(A) := e(A, A)$. Edwards' result above implies that every digraph D with m arcs has a vertex partition V_1, V_2 such that $e(V_1, V_2) \geqslant m/4 + (\sqrt{2m + 1/4} - 1/2)/8$, and the bound is tight for complete graphs of odd order with an Eulerian orientation. On the other hand, Alon et al. [1] constructed digraphs whose maximum directed cut is $m/4 + O(m^{4/5})$.

A natural judicious version of Edwards' result is to bound both $e(V_1, V_2)$ and $e(V_2, V_1)$. Indeed, Scott [16] asked the following question for digraphs without loops or parallel arcs in the same direction. (Throughout this paper, all digraphs have no loops or parallel arcs in the same direction.) Note that the *outdegree* of a vertex in a digraph is the number of arcs directed away from that vertex.

Problem 1.1 (Scott [16]). What is the maximum constant c_d such that every digraph D with m arcs and minimum outdegree $d \geqslant 2$ admits a bipartition $V(D) = V_1 \cup V_2$ such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant c_d m?$$

The reason for the requirement $d \geqslant 2$ in Problem 1.1 is the following: Take the star $K_{1,n-1}$ with $n \geqslant 4$, and add a single edge between two vertices of degree 1. Orient the unique triangle so that it becomes a directed cycle, and orient all other edges so that they are directed towards the unique vertex of degree $n - 1$. This digraph has minimum outdegree 1, and $e(V_1, V_2) \leqslant 1$ for any bipartition V_1, V_2 of its vertex set with V_1 containing the unique vertex of degree $n - 1$. Thus, $c_1 = 0$.

Lee et al. [11] proved that $c_2 = 1/6 + o(1)$ and $c_3 = 1/5 + o(1)$, and they made the following conjecture for $d \geqslant 4$.

Conjecture 1.2 (Lee et al. [11]). *Let d be an integer satisfying $d \geqslant 4$. Every digraph D with m arcs and minimum outdegree at least d admits a bipartition $V(D) = V_1 \cup V_2$ with*

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} + o(1)\right)m.$$

The main term $\frac{d-1}{2(2d-1)}$ in Conjecture 1.2 is best possible, because of examples constructed in [11] using copies of K_{2d-1} and one copy of K_{2d+1}. Lee et al. [11] also noted that their tools for $d = 2, 3$ appear to be insufficient for $d \geqslant 4$. Hence, much effort has been devoted to studying variations of this problem, for instance, by considering minimum total degree conditions, see [6-9]. In this paper, we show that Conjecture 1.2 holds under certain natural conditions. In particular, we prove Conjecture 1.2 for $d = 4$.

Theorem 1.3. *Every digraph D with m arcs and minimum outdegree at least 4 admits a bipartition $V(D) = V_1 \cup V_2$ with*

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant (3/14 + o(1))m.$$

In Section 2, we set up notation and list previous results needed in our proof of Theorem 1.3. In Section 3, we describe and discuss our approach for all d and obtain information in terms of "huge" vertices, vertices whose indegree and outdegree have a large gap. In Section 4, we show that Conjecture 1.2 holds under some additional conditions on the number of huge vertices. We complete the proof of Theorem 1.3 in Section 5 and offer some concluding remarks in Section 6.

2 | NOTATION AND LEMMAS

We start with notation and terminology that will be used in this paper. Let D be a digraph. For $x \in V(D)$, let $N_D^+(x) = \{y : xy \in E(D)\}$ and $N_D^-(x) = \{y : yx \in E(D)\}$. Then $d_D^+(x) := |N_D^+(x)|$ and $d_D^-(x) := |N_D^-(x)|$ are the the *outdegree* and *indegree* of x, respectively. The *degree* of $x \in V(D)$ is defined as $d_D(x) = d_D^+(x) + d_D^-(x)$. We use $\Delta(D) = \max\{d_D(x) : x \in V(D)\}$ to denote the *maximum degree* of D. For any $X \subseteq V(D)$, the subgraph of D induced by X is denoted by $D[X]$. We will often omit the subscript D in the above notation when there is no danger of confusion. It will be convenient to write $[k]$ for $\{1, ..., k\}$, where k is any positive integer.

Lee et al. [11] proved that certain partial partitions of a digraph may be extended to a good partition of the entire digraph.

Lemma 2.1 (Lee et al. [11]). *Let D be a digraph with m arcs. Let p be a real satisfying $p \in [0, 1]$, and let $\varepsilon > 0$. Suppose that a subset $X \subseteq V$ and its partition $X = X_1 \cup X_2$ are given, and let $Y = V \backslash X$. Further suppose that $\max_{y \in Y} d(y) \leqslant \varepsilon^2 m/4$. Then there exists a bipartition $V(D) = V_1 \cup V_2$ with $X_i \subseteq V_i$ for $i \in [2]$ such that*

$$e(V_1, V_2) \geqslant e(X_1, X_2) + (1-p) \cdot e(X_1, Y) + p \cdot e(Y, X_2) + p(1-p) \cdot e(Y) - \varepsilon m,$$

$$e(V_2, V_1) \geqslant e(X_2, X_1) + p \cdot e(X_2, Y) + (1-p) \cdot e(Y, X_1) + p(1-p) \cdot e(Y) - \varepsilon m.$$

By applying Lemma 2.1 with $p = 1/2$ and $X_1 = X_2 = \emptyset$ and by noting that $d_D(v) \leqslant 2|V(D)|$, we obtain the following.

Corollary 2.2 (Lee et al. [11]). *Let D be a digraph with n vertices and m arcs. For any $\varepsilon > 0$, if $m \geqslant 8n/\varepsilon^2$ or $\Delta(D) \leqslant \varepsilon^2 m/4$, then D admits a bipartition $V(D) = V_1 \cup V_2$ with $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant m/4 - \varepsilon m$.*

From Corollary 2.2 we see that if the maximum degree of a digraph D is not too large, then $V(D)$ admits a partition V_1, V_2 such that both $e(V_1, V_2)$ and $e(V_2, V_1)$ are close to $m/4$. We will see that the vertices causing problems for obtaining the desired partition in Conjecture 1.2 are those whose outdegree and indegree differ significantly. Hence, for $x \in V(D)$, let

$$s^+(x) := d^+(x) - d^-(x), s^-(x) := d^-(x) - d^+(x), \quad \text{and} \quad s(x) := \max\{s^+(x), s^-(x)\}.$$

Note that $d(x) - s(x)$ is an even integer, and we often write

$$2b = \sum_{x \in X} (d(x) - s(x)).$$

To study those vertices x with large $s(x)$, we need the concept of the gap of a partition. Let D be a digraph and let X, Y be a partition of $V(D)$. For each partition X_1, X_2 of X, the *gap* of X_1, X_2 is defined as

$$\theta(X_1, X_2) = (e(X_1, Y) + e(Y, X_2)) - (e(X_2, Y) + e(Y, X_1)).$$

The *huge* vertices of D with respect to the partition X, Y are the vertices x such that

$$s(x) \geqslant \min\{|\theta(X_1, X_2)| : X_1, X_2 \quad \text{is a partition of} \quad X\}.$$

Let D be a digraph, X, Y a partition of $V(D)$, and X_1, X_2 a partition of X. For convenience, let $m_f(X_1, X_2) = e(X_1, Y) + e(Y, X_2)$ and $m_b(X_1, X_2) = e(X_2, Y) + e(Y, X_1)$; so

$$\theta(X_1, X_2) = m_f(X_1, X_2) - m_b(X_1, X_2).$$

Note that

$$\theta(X_1, X_2) = (e(X_1, Y) - e(Y, X_1)) - (e(X_2, Y) - e(Y, X_2)).$$

Thus, if $e(X) = 0$ then

$$\theta(X_1, X_2) = \left(\sum_{x \in X_1} s^+(x)\right) - \left(\sum_{x \in X_2} s^+(x)\right) \tag{1}$$

For any $x \in X$, we say that x is

(X_1, X_2)-*forward* if $x \in X_1$ and $s^+(x) > 0$, or $x \in X_2$ and $s^-(x) > 0$, and
(X_1, X_2)-*backward* if $x \in X_1$ and $s^-(x) > 0$, or $x \in X_2$ and $s^+(x) > 0$.

Let $X_f := \{x \in X : x \text{ is } (X_1, X_2)\text{-forward}\}$ and $X_b := \{x \in X : x \text{ is } (X_1, X_2)\text{-backward}\}$. By (1), if $e(X) = 0$ then

$$\theta(X_1, X_2) = \sum_{x \in X_f} s(x) - \sum_{x \in X_b} s(x). \tag{2}$$

We will need the following result from [9].

Lemma 2.3 (Hou et al. [9]). *Let D be a digraph and $V(D) = X \cup Y$ be a partition of D with $e(X) = 0$. Let $X = X_1 \cup X_2$ be a partition of X that minimizes $|\theta(X_1, X_2)|$ among all partitions of X. Then*

(1) $|\theta(X_1, X_2)| \leqslant |Y|$, *and*
(2) $g := \sum_{\{v \in X : s(v) < |\theta(X_1, X_2)|\}} s(v) \leqslant |Y| - |\theta(X_1, X_2)|$.

3 | PROPERTIES OF PARTITIONS WITH MINIMUM GAP

In this section, we explore the probabilistic approach used by Lee et al. [10,11]. In particular, we investigate digraph partitions whose gaps have minimum absolute value. We will prove several properties about gaps and huge vertices, by considering various ways to partition the set of huge vertices. Those properties may be useful for the eventual resolution of Conjecture 1.2.

Lemma 3.1. *Let D be a digraph with m arcs and minimum outdegree $d \geqslant 4$, and let X, Y be a partition of $V(D)$ with $e(X) = 0$. Let $\theta = \min\{|\theta(X_1, X_2)| : X_1, X_2 \text{ is a partition of } X\}$, and let $X' = \{x \in X : s(x) \geqslant \theta\}$. Let $\varepsilon > 0$ such that $\max_{y \in Y} d(y) \leqslant \varepsilon^2 m/4$. Then there exists a partition V_1, V_2 of $V(D)$ such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$, or the following statements hold:*

(1) $\theta > m/(2d - 1)$;
(2) $|X'|$ *is an odd integer;*

(3) *letting* $g = \sum_{x \in X \setminus X'} s(x)$ *and* $X' = \{v_1, \ldots, v_{2k+1}\}$ *such that* $s(v_1) \geqslant s(v_2) \geqslant \cdots \geqslant s(v_{2k+1})$, *we have* $\sum_{j=k+1}^{2k+1} s(v_j) - \sum_{j=1}^{k} s(v_j) \geqslant g + \theta$.

Proof. Suppose, for any partition V_1, V_2 of $V(D)$, $\min\{e(V_1, V_2), e(V_2, V_1)\} < \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$. We show that (1), (2), and (3) hold. First, we prove (1). Let $m_1 = e(X, Y) + e(Y, X)$ and $m_2 = e(Y)$. Thus, $m = m_1 + m_2$, as $e(X) = 0$. Let (X_1, X_2) be a partition of X such that $\theta(X_1, X_2) = \theta$. Applying Lemma 2.1 with $p = 1/2$, there is a bipartition V_1, V_2 of $V(D)$ such that $X_i \subseteq V_i$ for $i \in [2]$, and

$$\begin{aligned}
&\min\{e(V_1, V_2), e(V_2, V_1)\} \\
&\geqslant \frac{1}{2}\min\{e(X_1, Y) + e(Y, X_2), e(X_2, Y) + e(Y, X_1)\} + \frac{e(Y)}{4} - \varepsilon m \\
&= \frac{m_1 - \theta}{4} + \frac{m_2}{4} - \varepsilon m \\
&= \frac{m - \theta}{4} - \varepsilon m.
\end{aligned}$$

If $\theta \leqslant m/(2d-1)$ then $(m - \theta)/4 \geqslant (d-1)m/(2(2d-1))$; so $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$, a contradiction. Thus, $\theta > m/(2d-1)$, and (1) holds.

Let $X = \{v_1, \ldots, v_{|X|}\}$ such that $s(v_1) \geqslant s(v_2) \ldots \geqslant s(v_{|X|})$. To prove (2), let us assume $|X'| = 2k$ for some nonnegative integer k. Then $X' = \{v_1, \ldots, v_{2k}\}$.

First, suppose $k = 0$. Then $s(v_1) < \theta$ by the definition of X'. Let X_1^*, X_2^* be the partition of X such that $v_1, v_3, \ldots, v_{2p-1}$ are (X_1^*, X_2^*)-forward, where $p = \lceil |X|/2 \rceil$, and all other vertices are (X_1^*, X_2^*)-backward. If $|X|$ is even then, by (2),

$$\left|\theta\left(X_1^*, X_2^*\right)\right| = s(v_1) - \sum_{i=1}^{p-1}(s(v_{2i}) - s(v_{2i+1})) - s(v_{|X|}) \leqslant s(v_1) < \theta,$$

a contradiction. If $|X|$ is odd then, by (2),

$$\left|\theta\left(X_1^*, X_2^*\right)\right| = s(v_1) - \sum_{i=1}^{p-1}(s(v_{2i}) - s(v_{2i+1})) \leqslant s(v_1) < \theta,$$

a contradiction.

Now suppose $k > 0$. Then $s(v_{2k}) \geqslant \theta$. Let X_1^*, X_2^* be the partition of X such that $v_1, v_3, \ldots, v_{2k-1}$ are (X_1^*, X_2^*)-forward, and all other vertices in X are (X_1^*, X_2^*)-backward. Then by (2), $|\theta(X_1^*, X_2^*)| = |\sum_{i=1}^{k} s(v_{2i-1}) - (\sum_{i=1}^{k} s(v_{2i}) + g)|$. Note that

$$\begin{aligned}
\sum_{i=1}^{k} s(v_{2i-1}) - \left(\sum_{i=1}^{k} s(v_{2i}) + g\right) &= s(v_1) - \sum_{i=1}^{k-1}(s(v_{2i}) - s(v_{2i+1})) - s(v_{2k}) - g \\
&\leqslant s(v_1) - s(v_{2k}) \\
&\leqslant |Y| - \theta,
\end{aligned}$$

and, since $g \leqslant |Y| - \theta$ (by Lemma 2.3),

$$\left(\sum_{i=1}^{k} s(v_{2i}) + g\right) - \sum_{i=1}^{k} s(v_{2i-1}) = \sum_{i=1}^{k} (s(v_{2i}) - s(v_{2i-1})) + g \leqslant g \leqslant |Y| - \theta.$$

Hence, $|\theta(X_1^*, X_2^*)| \leqslant |Y| - \theta$. Because $\theta > m/(2d - 1)$ (by 1) and $m \geqslant d|V(D)|$, we see that $\theta > |V(D)|/2 \geqslant |Y|/2$. Thus, $|\theta(X_1^*, X_2^*)| \leqslant |Y| - \theta < \theta$, a contradiction. Thus, $|X'|$ must be odd, and we have (2).

By (2), let $X' := \{v_1, ..., v_{2k+1}\}$ for some $k \geqslant 0$. Recall that $d(x) - s(x)$ is an even integer for all $x \in X$, and we write $2b = \sum_{x \in X}(d(x) - s(x))$. To prove (3), we consider the partition X_1^1, X_2^1 of X such that $\{v_1, v_3, ..., v_{2k-1}\} \cup (X \backslash X')$ is the set of (X_1^1, X_2^1)-forward vertices, and $\{v_2, v_4, ..., v_{2k}, v_{2k+1}\}$ is the set of (X_1^1, X_2^2)-backward vertices. Then $m_f(X_1^1, X_2^1) = \sum_{j=1}^{k} s(v_{2j-1}) + g + b$ and $m_b(X_1^1, X_2^1) = \sum_{j=1}^{k} s(v_{2j}) + s(v_{2k+1}) + b$. Note that

$$\begin{aligned}
\theta\left(X_1^1, X_2^1\right) &= m_f\left(X_1^1, X_2^1\right) - m_b\left(X_1^1, X_2^1\right) \\
&= \sum_{j=1}^{k} s(v_{2j-1}) + g - \sum_{j=1}^{k} s(v_{2j}) - s(v_{2k+1}) \\
&= (s(v_1) - s(v_{2k}) - s(v_{2k+1})) + \sum_{j=1}^{k-1} (s(v_{2j+1}) - s(v_{2j})) + g \\
&\leqslant s(v_1) - s(v_{2k}) - s(v_{2k+1}) + |Y| - \theta \quad (\text{since} \quad g \leqslant |Y| - \theta \quad \text{by Lemma} \quad 2.3) \\
&\leqslant s(v_1) + |Y| - 3\theta \\
&\leqslant 2|V(D)| - 3\theta \\
&< \theta \quad (\text{as} \quad \theta > |V(D)|/2 \quad \text{because} \quad m \geqslant d|V(D)| \quad \text{and by} \quad (1)).
\end{aligned}$$

Thus, since $|m_f(X_1^1, X_2^1) - m_b(X_1^1, X_2^1)| = |\theta(X_1^1, X_2^1)| \geqslant \theta$, we have

$$m_b\left(X_1^1, X_2^1\right) - m_f\left(X_1^1, X_2^1\right) = \sum_{j=1}^{k} s(v_{2j}) + s(v_{2k+1}) - \sum_{j=1}^{k} s(v_{2j-1}) - g \geqslant \theta.$$

Now exchange the sides for v_2 and v_{2k-1}, and consider the partition X_1^2, X_2^2 of X such that $X_1^2 = (X_1^1 \backslash \{v_{2k-1}\}) \cup \{v_2\}$ and $X_2^2 = (X_2^1 \backslash \{v_2\}) \cup \{v_{2k-1}\}$. Then $m_f(X_1^2, X_2^2) = m_f(X_1^1, X_2^1) - s(v_{2k-1}) + s(v_2)$ and $m_b(X_1^2, X_2^2) = m_b(X_1^1, X_2^1) - s(v_2) + s(v_{2k-1})$. Hence,

$$m_b\left(X_1^2, X_2^2\right) - m_f\left(X_1^2, X_2^2\right) = \left(m_b\left(X_1^1, X_2^1\right) - m_f\left(X_1^1, X_2^1\right)\right) - 2(s(v_2) - s(v_{2k-1})),$$

which implies that

$$m_b\left(X_1^2, X_2^2\right) - m_f\left(X_1^2, X_2^2\right) \geqslant \theta - 2(|Y| - \theta) > \theta - 2\theta = -\theta.$$

Therefore, since $|m_b(X_1^2, X_2^2) - m_f(X_1^2, X_2^2)| = |\theta(X_1^2, X_2^2)| \geqslant \theta$, we see that

$$m_b\left(X_1^2, X_2^2\right) - m_f\left(X_1^2, X_2^2\right) \geqslant \theta.$$

Repeating the same argument by exchanging the sides for $v_{2(k-i)+1}$ and v_{2i}, one step at a time in the order $i = 2, \ldots, \lfloor k/2 \rfloor$, we arrive at the partition $X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor}$ of X, such that $\{v_1, v_2, \ldots, v_k\} \cup (X \backslash X')$ is the set of $(X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor})$-forward vertices, $\{v_{k+1}, v_{k+2}, \ldots, v_{2k+1}\}$ is the set of $(X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor})$-backward vertices, and

$$m_b\left(X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor}\right) - m_f\left(X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor}\right) \geqslant \theta.$$

On the other hand, we have, by (2), that

$$m_b\left(X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor}\right) - m_f\left(X_1^{\lfloor k/2 \rfloor}, X_2^{\lfloor k/2 \rfloor}\right) = \sum_{j=k+1}^{2k+1} s(v_j) - \sum_{j=1}^{k} s(v_j) - g.$$

Hence, (3) holds. □

Lemma 3.2. *Let D be a digraph with m arcs and minimum outdegree $d \geqslant 4$, and let X, Y be a partition of $V(D)$ with $e(X) = 0$. Let $\theta = \min\{|\theta(X_1, X_2)| : X_1, X_2$ is a partition of $X\}$, $X' = \{x \in X : s(x) \geqslant \theta\}$, $g = \sum_{x \in X \backslash X'} s(x)$, and $2b = \sum_{x \in X}(d(x) - s(x))$. Let $\varepsilon > 0$ and assume that $\max_{y \in Y} d(y) \leqslant \varepsilon^2 m/4$. Then there exists a partition V_1, V_2 of $V(D)$ such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$; or $|X'|$ is odd and if we let $X' = \{v_1, \ldots, v_{2k+1}\}$ such that $s(v_1) \geqslant s(v_2) \geqslant \cdots \geqslant s(v_{2k+1})$ and write $\Delta_j = s(v_j)$ for $j \in [2k+1]$ then*

(1) $d(\sum_{j=1}^{k}\Delta_j + g) - (d-1)\sum_{j=k+1}^{2k+1}\Delta_j + b + e(Y)/2 < 0$,

(2) $b > \frac{d^2+2d-1}{d-1}\sum_{j=k}^{2k-1}\Delta_j - d(\sum_{j=1}^{k-1}\Delta_j + \Delta_{2k} + \Delta_{2k+1}) + (d-1)g + \frac{d-1}{2d}e(Y)$,

(3) $b < \frac{2(2d-1)(k+1)}{3d-1}|V(D)| + \frac{d^2-5d+2}{3d-1}(\sum_{j=1}^{k-1}\Delta_j + \Delta_{2k} + \Delta_{2k+1}) - \frac{d^2+2d-1}{3d-1}\sum_{j=k}^{2k-1}\Delta_j$ $+ \frac{d(d-1)}{3d-1}g - \frac{(d-1)^2}{2d(3d-1)}e(Y)$, *and*

(4) *when $d = 4$ and $k = 1$, $2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3e(Y)/14 < 0$, or both $6\Delta_1 - 3\Delta_2 - 3\Delta_3 + 2g - b + 3e(Y)/14 < 0$ and $6\Delta_3 - 3\Delta_1 - 3\Delta_2 - 3g + b/3 + 3e(Y)/14 < 0$.*

Proof. For convenience, we introduce two functions which we will use to compare $\min\{e(V_1, V_2), e(V_2, V_1)\}$ with $\left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$ for any partition V_1, V_2 of $V(D)$. For any partition X_1, X_2 of X, let $z(X_1, X_2) = e(X_1, Y)$ and $z'(X_1, X_2) = e(Y, X_2)$; so $m_f(X_1, X_2) = z(X_1, X_2) + z'(X_1, X_2)$. Let $m_1 := e(X, Y) + e(Y, X)$, $m_2 := e(Y)$, and

$$\ell(p, X_1, X_2) := (d-1)\sum_{j=1}^{2k+1}\Delta_j + (d-1)g + (2d-2)b$$
$$- (2(2d-1)p(1-p) - (d-1))m_2.$$

Define

$f(p, X_1, X_2) = 2(1-p)(2d-1)z(X_1, X_2) + 2p(2d-1)z'(X_1, X_2) - \ell(p, X_1, X_2)$, and
$h(p, X_1, X_2) = 2p(2d-1)(m_1 - z(X_1, X_2) - z'(X_1, X_2)) - \ell(p, X_1, X_2)$.

By Lemma 2.1, for any $0 \leqslant p \leqslant 1$, there is a partition $V(D) = V_1 \cup V_2$ such that $X_i \subseteq V_i$ for $i \in [2]$, and

$$\begin{cases} e(V_1, V_2) \geqslant (1-p)\cdot e(X_1, Y) + p\cdot e(Y, X_2) + p(1-p)\cdot e(Y) - \varepsilon m, \\ e(V_2, V_1) \geqslant p\cdot e(X_2, Y) + (1-p)\cdot e(Y, X_1) + p(1-p)\cdot e(Y) - \varepsilon m. \end{cases} \tag{3}$$

Without loss of generality, we may assume $p \leqslant 1 - p$; so $p \leqslant 1/2$. Then, from (3), we have

$$\begin{cases} e(V_1, V_2) \geqslant (1-p)z(X_1, X_2) + pz'(X_1, X_2) + p(1-p)m_2 - \varepsilon m, \\ e(V_2, V_1) \geqslant p(m_1 - z(X_1, X_2) - z'(X_1, X_2)) + p(1-p)m_2 - \varepsilon m. \end{cases} \tag{4}$$

Note that

$$m = m_1 + m_2 = \sum_{j=1}^{2k+1} \Delta_j + g + 2b + m_2. \tag{5}$$

By (4) and (5), we have

$$\begin{aligned} & e(V_1, V_2) - \left(\frac{d-1}{2(2d-1)}m - \varepsilon m\right) \\ & \geqslant (1-p)z(X_1, X_2) + pz'(X_1, X_2) + p(1-p)m_2 - \frac{d-1}{2(2d-1)}\left(\sum_{j=1}^{2k+1} \Delta_j + g + 2b + m_2\right) \\ & = \frac{1}{2(2d-1)} f(p, X_1, X_2), \end{aligned}$$

and

$$\begin{aligned} & e(V_2, V_1) - \left(\frac{d-1}{2(2d-1)}m - \varepsilon m\right) \\ & \geqslant p(m_1 - z(X_1, X_2) - z'(X_1, X_2)) + p(1-p)m_2 - \frac{d-1}{2(2d-1)}\left(\sum_{j=1}^{2k+1} \Delta_j + g + 2b + m_2\right) \\ & = \frac{1}{2(2d-1)} h(p, X_1, X_2). \end{aligned}$$

If $f(p, X_1, X_2) \geqslant 0$ and $h(p, X_1, X_2) \geqslant 0$ for some choice of p, X_1, X_2, we see that the there is a partition V_1, V_2 of $V(D)$ such that for $i \in [2]$, $X_i \subseteq V_i$ and $e(V_i, V_{3-i}) \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$. Hence, we may assume that

$$f(p, X_1, X_2) < 0 \quad \text{or} \quad h(p, X_1, X_2) < 0 \quad \text{for any choice of} \quad p, X_1, X_2. \tag{6}$$

To see (1), we consider the partition X_1^1, X_2^1 of X such that $\{v_1, v_2, ..., v_k\} \cup (X \backslash X')$ is the set of (X_1^1, X_2^1)-forward vertices, and $\{v_{k+1}, v_{k+2}, ..., v_{2k+1}\}$ is the set of (X_1^1, X_2^1)-backward vertices. Then $z(X_1^1, X_2^1) + z'(X_1^1, X_2^1) = m_f(X_1^1, X_2^1) = \sum_{j=1}^{k} \Delta_j + g + b$ and $m_b(X_1^1, X_2^1) = \sum_{j=k+1}^{2k+1} \Delta_j + b$. Setting $p = 1/2$, it follows from a simple calculation that

$$\begin{cases} f\left(1/2, X_1^1, X_2^1\right) = d\left(\sum_{j=1}^{k} \Delta_j + g\right) - (d-1)\sum_{j=k+1}^{2k+1} \Delta_j + b + m_2/2, \\ h\left(1/2, X_1^1, X_2^1\right) = d\sum_{j=k+1}^{2k+1} \Delta_j - (d-1)\left(\sum_{j=1}^{k} \Delta_j + g\right) + b + m_2/2. \end{cases}$$

By (3) of Lemma 3.1, we may assume $h(1/2, X_1^1, X_2^1) > 0$; so by (6), we have $f(1/2, X_1^1, X_2^1) < 0$. Thus, (1) holds.

For (2) and (3), we note that at least k members of $\{s^+(v_1), s^+(v_2), \ldots, s^+(v_{2k-1})\}$ have the same sign. We may assume that $s^+(v_{j_1}), s^+(v_{j_2}), \ldots, s^+(v_{j_k})$ are positive, where $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant 2k-1$; otherwise, we may consider the digraph D' obtained from D by reversing orientations of all arcs in D.

To prove (2), let X_1^2, X_2^2 be the partition of X such that $\{v_{j_1}, v_{j_2}, \cdots, v_{j_k}\} \cup (X \backslash X')$ is the set of (X_1^2, X_2^2)-forward vertices, and all other vertices in X are (X_1^2, X_2^2)-backward. Then

$$m_f\left(X_1^2, X_2^2\right) = \sum_{i=1}^{k} \Delta_{j_i} + g + b$$

and

$$m_b\left(X_1^2, X_2^2\right) = \sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i} + b \geqslant \sum_{j=k+1}^{2k+1} \Delta_j + b.$$

Note that $z(X_1^2, X_2^2) = e(X_1^2, Y) \geqslant \sum_{i=1}^{k} \Delta_{j_i}$ and $z'(X_1^2, X_2^2) = e(Y, X_2^2) = m_f(X_1^2, X_2^2) - e(X_1^2, Y)$. Setting $p = (d-1)/(2d)$, we see that

$$\begin{aligned} & f\left(\frac{d-1}{2d}, X_1^2, X_2^2\right) \\ & \geqslant \frac{(d+1)(2d-1)}{d} \sum_{i=1}^{k} \Delta_{j_i} + \frac{(d-1)(2d-1)}{d}(g+b) - (d-1)\sum_{j=1}^{2k+1} \Delta_j \\ & - (d-1)g - (2d-2)b + \frac{(d-1)^2}{2d^2} m_2 \\ & = \frac{d-1}{d}\left(\frac{d^2+2d-1}{d-1} \sum_{i=1}^{k} \Delta_{j_i} - d\left(\sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i}\right) + (d-1)g - b + \frac{d-1}{2d} m_2\right) \\ & \geqslant \frac{d-1}{d}\left(\frac{d^2+2d-1}{d-1} \sum_{j=k}^{2k-1} \Delta_j - d\left(\sum_{j=1}^{k-1} \Delta_j + \Delta_{2k} + \Delta_{2k+1}\right) + (d-1)g - b + \frac{d-1}{2d} m_2\right), \end{aligned}$$

and

$$\begin{aligned} & h\left(\frac{d-1}{2d}, X_1^2, X_2^2\right) \\ & \geqslant \frac{(d-1)(2d-1)}{d}\left(\sum_{j=k+1}^{2k+1} \Delta_j + b\right) - (d-1)\sum_{j=1}^{2k+1} \Delta_j - (d-1)g - (2d-2)b + \frac{(d-1)^2}{2d^2} m_2 \\ & = \frac{d-1}{d}\left((d-1)\sum_{j=k+1}^{2k+1} \Delta_j - d\left(\sum_{j=1}^{k} \Delta_j + g\right) - b + \frac{d-1}{2d} m_2\right). \end{aligned}$$

By (1), $h\left(\frac{d-1}{2d}, X_1^2, X_2^2\right) > 0$. So $f\left(\frac{d-1}{2d}, X_1^2, X_2^2\right) < 0$ by (6). Hence, (2) holds.

To prove (3), consider the partition X_1^3, X_2^3 of X such that $\{v_{j_1}, v_{j_2}, ..., v_{j_k}\} \cup (X \backslash X') \subseteq X_1^3$, and the vertices in $X' \backslash \{v_{j_1}, ..., v_{j_k}\}$ are (X_1^3, X_2^3)-backward. Since $s^+(v_{j_i}) > 0$ for $i \in [k]$, the vertices $v_{j_1}, v_{j_2}, ..., v_{j_k}$ are (X_1^3, X_2^3)-forward. Hence,

$$m_f\left(X_1^3, X_2^3\right) \geqslant \sum_{i=1}^{k} \Delta_{j_i} + b$$

and

$$m_b\left(X_1^3, X_2^3\right) \geqslant \sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i} + b \geqslant \sum_{j=k+1}^{2k+1} \Delta_j + b.$$

Let $n := |V(D)|$. Note that $X_2^3 \subseteq X' \backslash \{v_{j_1}, ..., v_{j_k}\}$; so

$$z'\left(X_1^3, X_2^3\right) = e\left(Y, X_2^3\right) \leqslant (k+1)n - \left(\sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i}\right).$$

Hence,

$$z\left(X_1^3, X_2^3\right) = e\left(X_1^3, Y\right) \geqslant \sum_{i=1}^{k} \Delta_{j_i} + b - \left((k+1)n - \left(\sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i}\right)\right).$$

Setting $p = (d-1)/(2d)$, we have

$$\begin{aligned}
& f\left(\tfrac{d-1}{2d}, X_1^3, X_2^3\right) \\
& \geqslant \tfrac{(d+1)(2d-1)}{d}\left[\sum_{i=1}^{k} \Delta_{j_i} + b - \left((k+1)n - \left(\sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i}\right)\right)\right] \\
& \quad + \tfrac{(d-1)(2d-1)}{d}\left((k+1)n - \left(\sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i}\right)\right) - (d-1)\sum_{j=1}^{2k+1} \Delta_j \\
& \quad - (d-1)g - (2d-2)b + \tfrac{(d-1)^2}{2d^2} m_2 \\
& = \tfrac{3d-1}{d}\left(\tfrac{d^2+2d-1}{3d-1}\sum_{i=1}^{k} \Delta_{j_i} - \tfrac{d^2-5d+2}{3d-1}\left(\sum_{j=1}^{2k+1} \Delta_j - \sum_{i=1}^{k} \Delta_{j_i}\right) - \tfrac{2(2d-1)(k+1)}{3d-1} n\right. \\
& \quad \left. - \tfrac{d(d-1)}{3d-1} g + b + \tfrac{(d-1)^2}{2d(3d-1)} m_2\right) \\
& \geqslant \tfrac{3d-1}{d}\left(\tfrac{d^2+2d-1}{3d-1}\sum_{j=k}^{2k-1} \Delta_j - \tfrac{d^2-5d+2}{3d-1}\left(\sum_{j=1}^{k-1} \Delta_j + \Delta_{2k} + \Delta_{2k+1}\right) - \tfrac{2(2d-1)(k+1)}{3d-1} n\right. \\
& \quad \left. - \tfrac{d(d-1)}{3d-1} g + b + \tfrac{(d-1)^2}{2d(3d-1)} m_2\right),
\end{aligned}$$

and

$$\begin{aligned}
& h\left(\frac{d-1}{2d}, X_1^3, X_2^3\right) \\
& \geqslant \frac{(d-1)(2d-1)}{d}\left(\sum_{j=k+1}^{2k+1} \Delta_j + b\right) - (d-1)\sum_{j=1}^{2k+1} \Delta_j - (d-1)g - (2d-2)b + \frac{(d-1)^2}{2d^2} m_2 \\
& = \frac{d-1}{d}\left((d-1)\sum_{j=k+1}^{2k+1} \Delta_j - d\left(\sum_{j=1}^{k} \Delta_j + g\right) - b + \frac{d-1}{2d} m_2\right).
\end{aligned}$$

By (1), $h\left(\frac{d-1}{2d}, X_1^3, X_2^3\right) > 0$. So $f\left(\frac{d-1}{2d}, X_1^3, X_2^3\right) < 0$ by (6). Hence, (3) holds.

Now we prove (4); so assume $d = 4$ and $k = 1$. First, let X_1^4, X_2^4 be the partition of X such that $\{v_1\} \cup (X \backslash X')$ is the set of (X_1^4, X_2^4)-forward vertices, and v_2, v_3 are (X_1^4, X_2^4)-backward. Then $m_f(X_1^4, X_2^4) = \Delta_1 + g + b$ and $m_b(X_1^4, X_2^4) = \Delta_2 + \Delta_3 + b$. Also, we have $e(X_1^4, Y) \geqslant \Delta_1$. Setting $p = 5/14$, we see that

$$\begin{aligned} h\left(5/14, X_1^4, X_2^4\right) &\geqslant 5(\Delta_2 + \Delta_3 + b) - 3(\Delta_1 + \Delta_2 + \Delta_3) - 3g - 6b + 3m_2/14 \\ &= 2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3m_2/14 \end{aligned}$$

and

$$\begin{aligned} f\left(5/14, X_1^4, X_2^4\right) &\geqslant 9\Delta_1 + 5(g + b) - 3(\Delta_1 + \Delta_2 + \Delta_3) - 3g - 6b + 3m_2/14 \\ &= 6\Delta_1 - 3\Delta_2 - 3\Delta_3 + 2g - b + 3m_2/14. \end{aligned}$$

Thus, we have $2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3m_2/14 < 0$ or $6\Delta_1 - 3\Delta_2 - 3\Delta_3 + 2g - b + 3m_2/14 < 0$.

Next, we choose some $i \in [3]$ such that the number of arcs from v_i to Y counted in b is maximum. Consider the partition X_1^5, X_2^5 of X such that $\{v_i\} \cup (X \backslash X') \subseteq X_1^5$, and the vertices in $X' \backslash \{v_i\}$ are (X_1^5, X_2^5)-backward. Then, clearly, $m_f(X_1^5, X_2^5) \geqslant \Delta_i + b \geqslant \Delta_3 + b$ and $m_b(X_1^5, X_2^5) \geqslant \Delta_2 + \Delta_3 + b$. Also, we have $e(X_1^5, Y) \geqslant (\Delta_i + b) - 2b/3 \geqslant \Delta_3 + b/3$. Setting $p = 5/14$, we see that

$$\begin{aligned} h\left(5/14, X_1^5, X_2^5\right) &\geqslant 5(\Delta_2 + \Delta_3 + b) - 3(\Delta_1 + \Delta_2 + \Delta_3) - 3g - 6b + 3m_2/14 \\ &\geqslant 2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3m_2/14, \end{aligned}$$

and

$$\begin{aligned} f\left(5/14, X_1^5, X_2^5\right) &\geqslant 5(\Delta_3 + b) + 4(\Delta_3 + b/3) - 3(\Delta_1 + \Delta_2 + \Delta_3) - 3g - 6b + 3m_2/14 \\ &= 6\Delta_3 - 3\Delta_1 - 3\Delta_2 - 3g + b/3 + 3m_2/14. \end{aligned}$$

Thus, $2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3m_2/14 < 0$ or $6\Delta_3 - 3\Delta_1 - 3\Delta_2 - 3g + b/3 + 3m_2/14 < 0$. This completes the proof of (4). □

4 | HUGE VERTICES

In this section, we show that if $V(D)$ has a partition X, Y such that $e(X) = 0$, $\max_{y \in Y} d(y) \leqslant \varepsilon^2 m/4$, and X has at least d huge vertices or a unique huge vertex then Conjecture 1.2 holds.

Proposition 4.1. *Let $d \geqslant 4$ be an integer and $\varepsilon > 0$ be a real. Let D be a digraph with m arcs and minimum outdegree at least d. Let X, Y be a partition of $V(D)$ with $e(X) = 0$ and $\max_{y \in Y} d(y) \leqslant \varepsilon^2 m/4$. Let $\theta = \min\{|\theta(X_1, X_2)| : X_1, X_2$ is a partition of $X\}$ and $X' = \{x \in X : s(x) \geqslant \theta\}$. Suppose $|X'| \geqslant d$. Then $V(D)$ admits a partition V_1, V_2 such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$.*

Proof. Suppose the desired partition V_1, V_2 does not exist. By (2) of Lemma 3.1, let $X' = \{v_1, ..., v_{2k+1}\}$. Then $2k + 1 \geqslant d \geqslant 4$ by assumption. Let $\Delta_i = s(v_i)$ for $i \in [2k + 1]$ and assume, without loss of generality, $\Delta_1 \geqslant \Delta_2 \geqslant \cdots \geqslant \Delta_{2k+1}$. Let X_1, X_2 be a partition of X such that $\theta(X_1, X_2) = \theta$. Then, by (1) and (2) of Lemma 3.2, we have

$$\begin{aligned}
0 &> d\left(\sum_{j=1}^{k}\Delta_j + g\right) - (d-1)\sum_{j=k+1}^{2k+1}\Delta_j + b + m_2/2 \\
&> d\left(\sum_{j=1}^{k-1}\Delta_j + \Delta_{2k}\right) - (d-1)\left(\sum_{j=k}^{2k-1}\Delta_j + \Delta_{2k+1}\right) + \frac{d^2+2d-1}{d-1}\sum_{j=k}^{2k-1}\Delta_j \\
&\quad - d\left(\sum_{j=1}^{k-1}\Delta_j + \Delta_{2k} + \Delta_{2k+1}\right) \\
&= \frac{4d-2}{d-1}\sum_{j=k}^{2k-1}\Delta_j - (2d-1)\Delta_{2k+1} \\
&\geqslant \frac{k(4d-2)}{d-1}\Delta_{2k+1} - (2d-1)\Delta_{2k+1} \\
&= \frac{(2d-1)(2k+1-d)}{d-1}\Delta_{2k+1},
\end{aligned}$$

This is a contradiction, as $2k + 1 \geqslant d$. □

Remark 4.2. The requirement $e(X) = 0$ in Proposition can be replaced by $e(X) = o(m)$.

Next, we show that if $V(D)$ admits a partition X, Y such that $|X| = o(|V(D)|)$, $\max_{y \in Y} d(y) \leqslant \varepsilon^2 m/4$, and D has a unique huge vertex in X then the conclusion of Conjecture 1.2 holds. For this, we need another concept introduced by Lee et al. [10], and we use the result of Lu et al. in [12] to give its definition. We say that a connected graph is *tight* if all its blocks are odd cliques. If a disconnected graph G is the underlying graph of a digraph D, the *tight components* of D are the components of G that are tight. (The *underlying graph* of D is obtained from D by ignoring arc orientations and removing redundant parallel edges.) For a tight component T of D, we say T is *essential* if $D[V(T)]$, the subgraph of D induced by $V(T)$, does not contain any parallel arcs in opposite directions. Recently, Hou et al. [7] proved the following.

Lemma 4.3 (Hou et al. [7]). *For any positive constants C and ε, there exist γ, $n_0 > 0$ for which the following holds. Let D be a digraph with $n \geqslant n_0$ vertices and at most Cn arcs. Suppose $X \subseteq V(D)$ is a set of at most γn vertices and X_1, X_2 is a partition of X. Let $Y = V(D) \backslash X$ and let τ be the number of essential tight components in $D[Y]$. If every vertex in Y has degree at most γn in D, then there is a bipartition $V(D) = V_1 \cup V_2$ with $X_i \subseteq V_i$ for $i = 1, 2$ such that*

$$\begin{aligned}
e(V_1, V_2) &\geqslant e(X_1, X_2) + \frac{e(X_1, Y) + e(Y, X_2)}{2} + \frac{e(Y)}{4} + \frac{n-\tau}{8} - \varepsilon n, \\
e(V_2, V_1) &\geqslant e(X_2, X_1) + \frac{e(X_2, Y) + e(Y, X_1)}{2} + \frac{e(Y)}{4} + \frac{n-\tau}{8} - \varepsilon n.
\end{aligned}$$

Proposition 4.4. *Let $d \geqslant 4$ be an integer and let C, ε be positive reals. Let D be a digraph with n vertices, $m \leqslant Cn$ arcs, and minimum outdegree at least d. Then there exists γ with*

$0 < \gamma < \varepsilon$ such that the following holds: Let X, Y be a partition of $V(D)$ with $|X| \leqslant \gamma n$, $e(X) = 0$, and $\max_{y \in Y} d(y) \leqslant \gamma n$. Let $\theta = \min\{|\theta(X_1, X_2)| : X_1, X_2 \text{ is a partition of } X\}$, $X' = \{x \in X : s(x) \geqslant \theta\}$, $g = \sum_{x \in X \setminus X'} s(x)$, and $2b = \sum_{x \in X}(d(x) - s(x))$. Then the following statements hold.

(1) *$\tau \leqslant (n + 2g + 2b)/(2d - 2|X'| + 1)$, where τ is the number of essential tight components in $D[Y]$.*
(2) *Suppose $|X'| = 1$. Then $V(D)$ admits a partition V_1, V_2 such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m$.*

Proof. First, we prove (1). Let $\alpha := |X'|$. For $i = 1, 3, \ldots, 2d - 2\alpha - 1$, let τ_i be the number of essential tight components of order i; and let τ' be the number of essential tight components of order at least $2d - 2\alpha + 1$. Then $\tau = \sum_{i=1}^{d-\alpha} \tau_{2i-1} + \tau'$ and

$$\tau_1 + 3\tau_3 + \cdots + (2d - 2\alpha - 1)\tau_{2d-2\alpha-1} + (2d - 2\alpha + 1)\tau' \leqslant n. \tag{7}$$

For each essential tight component D_i of order i, we see that $e(D_i) \leqslant i(i-1)/2$ and $e(D_i, X') \leqslant \alpha i$. Thus, since the outdegree of D is at least d, we see that $e(D_i, X \backslash X') \geqslant di - \alpha i - i(i-1)/2$. Viewing $di - \alpha i - i(i-1)/2$ as a function of i over the interval $[1, 2d - 2\alpha]$, we see that it achieves its minimum at $i = 1$ (as well as at $i = 2d - 2\alpha$). Hence, $e(D_i, X \backslash X') \geqslant d - \alpha$ for $i \in [2d - 2\alpha]$. Thus, $e(Y, X \backslash X') \geqslant \sum_{i=1}^{d-\alpha} (d - \alpha)\tau_{2i-1}$. On the other hand, we have $e(Y, X \backslash X') \leqslant g + b$. Hence,

$$\sum_{i=1}^{d-\alpha} (d - \alpha)\tau_{2i-1} \leqslant g + b. \tag{8}$$

Multiplying (8) by 2 and adding the resulting inequality to (7), we derive that $(2d - 2\alpha + 1)\tau \leqslant n + 2g + 2b$, completing the proof of (1).

To prove (2), let $X' = \{v_0\}$ and let $\Delta = s(v_0)$. Let X_1, X_2 be the partition of X such that v_0 is the only (X_1, X_2)-forward vertex. Then $m_f(X_1, X_2) = \Delta + b$ and $m_b(X_1, X_2) = g + b$. By Lemma 4.3 (with $p = 1/2$), there is a bipartition V_1, V_2 of $V(D)$ such that $X_i \subseteq V_i$ for $i \in [2]$ and

$$\begin{aligned}
&\min\{e(V_1, V_2), e(V_2, V_1)\} \\
&\geqslant \frac{1}{2}\min\left\{e(X_1, Y) + e(Y, X_2), e(X_2, Y) + e(Y, X_1)\right\} + e(Y)/4 + (n - \tau)/8 - \varepsilon n \\
&= (m - \theta)/4 + (n - \tau)/8 - \varepsilon n.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\min\{e(V_1, V_2), e(V_2, V_2)\} - \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m \\
&\geqslant (m - \theta)/4 + (n - \tau)/8 - \varepsilon n - \left(\frac{d-1}{2(2d-1)} - \varepsilon\right)m \\
&\geqslant \frac{1}{4}\left(\frac{m}{2d-1} + \frac{n}{2} + 4(d-1)\varepsilon n - \theta - \frac{\tau}{2}\right) \quad (\text{since} \quad m \geqslant dn).
\end{aligned}$$

Since the minimum outdegree of D at least d and $|Y| = n - |X| > n - \varepsilon n$, we have

$$m \geqslant b + d|Y| \geqslant b + dn - d\varepsilon n.$$

Therefore,

$$
\begin{aligned}
&\frac{m}{2d-1}+\frac{n}{2}+4(d-1)\varepsilon n-\theta-\frac{\tau}{2}\\
&\geqslant \frac{b+dn-d\varepsilon n}{2d-1}+\frac{n}{2}+4(d-1)\varepsilon n-\theta-\frac{n+2g+2b}{2(2d-1)}\quad (\text{by }(1))\\
&\geqslant \frac{1}{2(2d-1)}(2b+2dn+(2d-1)n-(4d-2)\theta-n-2g-2b)\\
&=\frac{1}{2(2d-1)}((4d-2)n-(4d-2)\theta-2g)\\
&\geqslant 0\quad (\text{since}\quad n\geqslant |Y|\geqslant \theta+g\quad \text{by Lemma 2.3}).
\end{aligned}
$$

So V_1, V_2 gives the desired partition for (2). □

We now use Lemma 4.3 to define (1) of Lemma 3.2 for the case when there are only three huge vertices.

Lemma 4.5. *Let D be a digraph with m arcs and minimum outdegree $d \geqslant 4$, and let X, Y be a partition of $V(D)$ with $e(X)=0$. Let $\theta=\min\{|\theta(X_1,X_2)| : X_1, X_2 \text{ is a partition of } X\}$, $X'=\{x\in X : s(x)\geqslant\theta\}$, $g=\sum_{x\in X\setminus X'} s(x)$, $2b=\sum_{x\in X}(d(x)-s(x))$, and τ the number of essential tight components in $D[Y]$. Let $\varepsilon>0$ and assume that $\max_{y\in Y} d(y)\leqslant \varepsilon^2 m/4$. Suppose $|X'|=3$. Then there exists a partition V_1, V_2 of $V(D)$ such that $\min\{e(V_1,V_2), e(V_2,V_1)\}\geqslant (3/14-\varepsilon)m$; or if we write $X'=\{v_1,v_2,v_3\}$ with $s(v_1)\geqslant s(v_2)\geqslant s(v_3)$ and $\Delta_j=s(v_j)$ for $j\in[3]$ then*

$$b<3\Delta_2+3\Delta_3-4\Delta_1-4g-m_2/2-7(|V(D)|-\tau)/4.$$

Proof. We consider the partition X_1, X_2 of X such that $\{v_1\}\cup(X\setminus X')$ is the set of (X_1,X_2)-forward vertices, and $\{v_2,v_3\}$ is the set of (X_1,X_2)-backward vertices. Then $m_f(X_1,X_2)=\Delta_1+g+b$ and $m_b(X_1,X_2)=\Delta_2+\Delta_3+b$. Note that

$$m=m_1+m_2=\sum_{j=1}^{3}\Delta_j+g+2b+m_2. \tag{9}$$

Let $n:=|V(D)|$. By applying Lemma 4.3 with $p=1/2$, there is a partition $V(D)=V_1\cup V_2$ such that $X_i\subseteq V_i$ for $i\in[2]$, and

$$\begin{cases} e(V_1,V_2)\geqslant(\Delta_1+g+b)/2+m_2/4+(n-\tau)/8-\varepsilon n,\\ e(V_2,V_1)\geqslant(\Delta_2+\Delta_3+b)/2+m_2/4+(n-\tau)/8-\varepsilon n.\end{cases} \tag{10}$$

Define

$$
\begin{aligned}
&f(1/2,X_1,X_2)=4\Delta_1-3\Delta_2-3\Delta_3+4g+b+m_2/2+7(n-\tau)/4, \text{ and}\\
&h(1/2,X_1,X_2)=4\Delta_2+4\Delta_3-3\Delta_1-3g+b+m_2/2+7(n-\tau)/4.
\end{aligned}
$$

By (3) of Lemma 3.1, we may assume $\Delta_2+\Delta_3-\Delta_1\geqslant g+\theta$; for otherwise the desired partition of $V(D)$ exists. Hence, $h(1/2,X_1,X_2)>0$.

By (9) and (10), we have

$$
\begin{aligned}
& e(V_1, V_2) - (3m/14 - \varepsilon m) \\
\geqslant\ & (\Delta_1 + g + b)/2 + m_2/4 + (n - \tau)/8 - (3/14)\left(\sum_{j=1}^{3} \Delta_j + g + 2b + m_2\right) \\
=\ & (1/14)f(1/2, X_1, X_2),
\end{aligned}
$$

and

$$
\begin{aligned}
& e(V_2, V_1) - (3m/14 - \varepsilon m) \\
\geqslant\ & (\Delta_2 + \Delta_3 + b)/2 + m_2/4 + (n - \tau)/8 - (3/14)\left(\sum_{j=1}^{3} \Delta_j + g + 2b + m_2\right) \\
=\ & (1/14)h(1/2, X_1, X_2).
\end{aligned}
$$

Thus, if $f(1/2, X_1, X_2) \geqslant 0$ then, since $h(1/2, X_1, X_2) > 0$, the desired partition of $V(D)$ exists. So, we may assume $f(1/2, X_1, X_2) < 0$ which implies the desired inequality. □

5 | PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3, by using Propositions 4.1 and 4.4 and Lemma 4.5 and by choosing X to consist of vertices of degree at most $n^{3/4}$. Our proof is much simpler when applied to the cases $d = 2$ and $d = 3$, and gives the results of Lee et al. [11] that Conjecture 1.2 is true for these cases.

Proof of Theorem 1.3. Let D be a digraph with n vertices and m arcs, and assume that the minimum outdegree of D is at least 4. We wish to find a partition $V(D) = V_1 \cup V_2$, such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant (3/14 + o(1))m$. We may assume that n is sufficiently large so that all lemmas in the previous sections can be applied. We claim that

(1) $m \geqslant 4n$ and we may assume $m < 128 \cdot 7^2 n$.

Since D has minimum outdegree at least 4, we have $m \geqslant 4n$. Now suppose $m \geqslant 128 \cdot 7^2 n$. Then applying Corollary 2.2 with $\varepsilon = 1/28$ we obtain a partition $V(D) = V_1 \cup V_2$ such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant (1/4 - 1/28)m = 3m/14.$$

So we may assume $m < 128 \cdot 7^2 n$. □

Consider the partition X, Y of $V(D)$ such that $X = \{v \in V(D) : d(v) \geqslant n^{3/4}\}$ and $Y = V(D) \backslash X$. Then, by (1),

$$|X| \cdot n^{3/4} \leqslant \sum_{v \in X} d(v) \leqslant \sum_{v \in V(D)} d(v) = 2m < 256 \cdot 7^2 n.$$

Hence, $|X| = O(n^{1/4})$ and, thus, $e(X) \leqslant |X|^2 = O(n^{1/2}) = o(m)$. Therefore, we may assume

(2) $e(X) = 0$.

Let $\theta = \min\{|\theta(S, T)| : S, T \text{ is a partition of} X\}$. Let X_1, X_2 be a partition of X such that $\theta(X_1, X_2) = \theta$, and let $X' = \{x \in X : s(x) \geqslant \theta\}$. If $|X'| = 1$ or $|X'| \geqslant 5$ then the desired partition exists by Propositions 4.1 and 4.4. So let $X' = \{v_1, v_2, v_3\}$ and $\Delta_i = s(v_i)$ for $i \in [3]$ such that $\Delta_1 \geqslant \Delta_2 \geqslant \Delta_3$. Since $|X| = o(n)$,

$$\Delta_1 + \Delta_2 + \Delta_3 + g + b + m_2 \geqslant \sum_{y \in Y} d^{+(y)} \geqslant 4|Y| = 4n - o(n).$$

Hence, writing $t = 2\Delta_1 - (\Delta_2 + \Delta_3)$, we have

$$b \geqslant 4n - \Delta_1 - \Delta_2 - \Delta_3 - g - m_2 - o(n) = 4n - 3\Delta_1 - g - m_2 + t - o(n), \tag{11}$$

Next, we will derive bounds on m_2, Δ_1, t and b in terms of n and g, so that we can use (4) of Lemma 3.2.

By Lemma 4.5,

$$b < 3\Delta_2 + 3\Delta_3 - 4\Delta_1 - 4g - m_2/2 - 7(n - \tau)/4 = 2\Delta_1 - 3t - 4g - m_2/2 - 7(n - \tau)/4,$$

which, combined with $\tau \leqslant (n + 2g + 2b)/3$ (by (1) of Proposition 4.4), imlpies

$$b > 7n + 3m_2 + 17g - 12\Delta_1 + 18t. \tag{12}$$

We may assume that (2) and (3) of Lemma 3.2 hold; for, otherwise, by Lemma 3.2, the desired partition of $V(D)$ exists. Thus,

$$b > 3g + 3m_2/8 - \Delta_1/3 + 4t, \tag{13}$$

and

$$b < \frac{28}{11}n - \frac{27}{11}\Delta_1 + \frac{12}{11}g - \frac{9}{88}m_2 + \frac{2}{11}t. \tag{14}$$

Combining (14) with (11), (12), (13), respectively, we obtain (by eliminating b)

$$\frac{79}{8}m_2 + 23g > 16n - 6\Delta_1 + 9t - o(n), \tag{15}$$

$$\frac{273}{8}m_2 + 175g < 105\Delta_1 - 49n - 196t. \tag{16}$$

$$\frac{63}{4}m_2 + 63g < 84n - 70\Delta_1 - 126t. \tag{17}$$

Noting that $\Delta_1 \leqslant n - |X|$ (by (2)) and n is large, we see from (15) that $\frac{79}{8}m_2 + 23g > 9.99n$; so

$$m_2 + 2.31g > n \tag{18}$$

We now combine (15) with (16) and (17), respectively, and we get (by eliminating m_2)

$$1.806t + 0.759g < \Delta_1 - 0.829n, \tag{19}$$

$$2.322t + 0.435g < 0.968n - \Delta_1. \tag{20}$$

From (19), we have

$$\Delta_1 > 0.829n. \tag{21}$$

Combining (19) and (20) to eliminate Δ_1, we have

$$3t + g < 0.117n. \tag{22}$$

From (14), we have

$$b < \frac{28}{11}n - \frac{27}{11}\Delta_1 - \left(\frac{9}{88}m_2 + \frac{21}{88}g\right) + \left(\frac{2}{11}t + \frac{117}{88}g\right).$$

Hence, by (18), (21), and (22), we have

$$b < \frac{28}{11}n - \frac{27}{11} \times 0.829n - \frac{9}{88}n + \frac{117}{88} \times 0.117n < 0.564n. \tag{23}$$

We wish to use (4) of Lemma 3.2. Note that

$$2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3m_2/14 = \Delta_1 + (3m_2/14 + g/2) - b - (2t + 7g/2).$$

So by (18), (21), (22), and (23), we see that

$$\begin{aligned}
&2\Delta_2 + 2\Delta_3 - 3\Delta_1 - 3g - b + 3m_2/14 \\
&> 0.829n + 3n/14 - 0.564n - (7/2) \times 0.117n \\
&> 0.069n \\
&> 0.
\end{aligned}$$

Hence, by (4) of Lemma 3.2, we have $6\Delta_1 - 3\Delta_2 - 3\Delta_3 + 2g - b + 3m_2/14 < 0$. Thus,

$$b > 6\Delta_1 - 3\Delta_2 - 3\Delta_3 + 3m_2/14 + 2g = 3m_2/14 + 2g + 3t. \tag{24}$$

Combining (14) and (24) (by eliminating b), we have

$$\frac{195}{56}m_2 + 10g < 28n - 27\Delta_1 - 31t,$$

which, combined with (15) (by eliminating m_2), gives

$$1.373t + 0.075g < 0.899n - \Delta_1, \tag{25}$$

which implies $\Delta_1 < 0.899n$; so by (11), we have

$$g + b + m_2 > 4n - 3\Delta_1 + t - o(n) > 4n - 3 \times 0.9n = 1.3n. \tag{26}$$

Combining (19) and (25) (to eliminate Δ_1), we derive

$$3t + g < 0.084n. \tag{27}$$

Again by (4) of Lemma 3.2, we have

$$\begin{aligned}
0 > 6\Delta_3 - 3\Delta_1 - 3\Delta_2 - 3g + b/3 + 3m_2/14 &= 3(m_2 + b + g)/14 + 5b/42 \\
&\quad - (6t + 2g + 17g/14) + 9(\Delta_1 - \Delta_2).
\end{aligned}$$

Hence, noting that $b > 3n/14$ (by 18 and 24) and by (26) and (27),

$$0 > (3/14) \times 1.3n + (5/42) \times (3n/14) - (2 + 17/14)) \times 0.084n > 0.034n.$$

This is a contradiction, completing the proof of Theorem 1.3.

6 | CONCLUDING REMARKS

We studied partitions of digraphs with minimum outdegree $d \geqslant 4$ and proved Conjecture 1.2 in the case when $d = 4$. We used a typical approach for finding a partition V_1, V_2 in a digraph D that bounds $e(V_1, V_2)$ and $e(V_2, V_1)$ simultaneously: Start with a partition X, Y of $V(D)$ such that X consists of large degree vertices; partition X by considering the "huge" vertices in X, those vertices with large gap between their outdegree and their indegree; and randomly partition the vertices in Y. Huge vertices play an important role in the process for obtaining the desired partition. For instance, we showed that Conjecture 1.2 holds when there exists a partition of $V(D)$ for which the number of huge vertices is at least d or exactly 1. We hope that our work would shed light on how the set $V(D)$ should be partitioned into X, Y and how the set X should be partitioned.

In [11], Lee, Loh, and Sudakov point out that one needs to combine both Lemma 2.1 and Lemma 4.3 to prove Conjecture 1.2. They also remarked that a naive combination is not adequate for $d \geqslant 4$ because of the following example. Let D' be the digraph obtained from $K_{5,n-5}$ (with $n > 9$) by orienting the edges so that one vertex, say v_1, has outdegree $n - 5$ and four vertices, say v_2, v_3, v_4, v_5, each have indegree $n - 5$. Let D be obtained from D' by adding an arc directed from v_i to v_j for each ordered pair (i, j) with $1 \leqslant i \neq j \leqslant 5$. Then the minimum outdegree of D is 4 and the number of arcs in D is $m = 5n - 5$. Let $X = \{v_1, ..., v_5\}$. If we partition $V(D)$ to X (consisting of large degree vertices) and $Y = V(D)\backslash X$ (consisting of small degree vertices), then X is the set of huge vertices. One can check that $X_1 = \{v_2\}$ and $X_2 = \{v_1, v_3, v_4, v_5\}$ form a partition of X with minimum gap. However, for any partition Y_1, Y_2 of Y, we see that $e(X_2 \cup Y_2, X_1 \cup Y_1) = n - 5 + 4 = n - 1 = m/5$, which is smaller than $3m/14$. What this means is that one need to consider different partitions of the set of huge vertices. In this paper, we have managed to prove Conjecture 1.2 in the case when $d = 4$ by carefully partitioning the huge vertices.

For digraphs with minimum outdegree $d \geqslant 5$, new ideas seem needed (in addition to better partitioning the huge vertices), as shown by the following example. Let D' be the digraph obtained from $K_{3,n-3}$ (with $n > 900$) by orienting all edges from the part Y of size $n - 3$ to the part X of size 3. Let D be obtained from D' by adding six arcs directed from each vertex in X to 6 vertices in Y (so that no two arcs get directed towards the same vertex in Y), and adding a 3-out-regular graph on Y. Hence, the minimum outdegree of D is 6, and X is the set of huge vertices with respect to the partition X, Y. It is not difficult to verify that for any value p and any partition X_1, X_2 of X, we have $f(p, X_1, X_2) < 0$ or $h(p, X_1, X_2) < 0$ (see Section 3). Therefore, one needs to better partition $V(D)\backslash X$ to achieve the bound in Conjecture 1.2.

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