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NEARLY PERFECT MATCHINGS IN UNIFORM HYPERGRAPHS*

HONGLIANG LU†, XINGXING YU‡, AND XIAOFAN YUAN†

Abstract. We prove that, for any integers k, l with $k \geq 3$ and $k/2 < l \leq k-1$, there exists a positive real μ such that, for all sufficiently large integers m, n satisfying $\frac{n}{k} - \mu n \leq m \leq \frac{n}{k} - 1 - (1 - \frac{l}{k}) \left\lceil \frac{k-l}{2l-k} \right\rceil$, if H is a k-uniform hypergraph on n vertices and $\delta_l(H) > \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l}$, then H has a matching of size $m+1$. This improves upon an earlier result of Hàn, Person, and Schacht for the range $k/2 < l \leq k-1$. In many cases, our result gives a tight bound on $\delta_l(H)$ for near perfect matchings (e.g., when $l \geq 2k/3$, $n \equiv r \pmod{k}$, $0 \leq r < k$, and $r + l \geq k$, we can take $m = \lceil n/k \rceil - 2$). When $k = 3$, using an absorbing lemma of Hàn, Person, and Schacht, our proof also implies a result of Kühn, Osthus, and Treglown (and, independently, of Khan) on perfect matchings in 3-uniform hypergraphs.

Key words. perfect matching, near perfect matching, hypergraph

AMS subject classifications. 05C65, 05C70, 05C35

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1. Introduction.

A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. For a positive integer k, a hypergraph H is *k-uniform* if $E(H) \subseteq \binom{V(H)}{k}$, and a k-uniform hypergraph is also called a *k-graph*. Let H be a hypergraph. For $S \subseteq V(H)$, we use $H - S$ to denote the hypergraph obtained from H by deleting S and all edges of H with a vertex in S, and we use $H[S]$ to denote the hypergraph with vertex set S and edge set $\{e \in E(H) : e \subseteq S\}$. For $S \subseteq R \subseteq V(H)$, let $N_{H-R}(S) = \{T \subseteq V(H) \setminus R : S \cup T \in E(H)\}$, and let $N_H(S) := N_{H-S}(S)$. For any positive integer n, let $[n] := \{1, \ldots, n\}$.

Let H be a hypergraph. A *matching* in H is a set of pairwise disjoint edges of H. (If M is a matching in H, we write $V(M) := \bigcup_{e \in M} e$.) The size of a largest matching in H is denoted by $\nu(H)$, known as the *matching number* of H. A matching in H is *perfect* if it covers all vertices of H. A matching is *nearly perfect* in H if it covers all but a constant number of vertices. Moreover, a matching in a k-graph is *near perfect* if it covers all but at most k vertices.

We are interested in degree conditions for the existence of a nearly perfect matching in a hypergraph. Let H be a hypergraph. For any $T \subseteq V(H)$, we use $d_H(T)$ to denote the *degree* of T in H, i.e., the number of edges of H containing T. Let l be a nonnegative integer. Then $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}\}$ is the minimum *l-degree* of H. Note that $\delta_0(H)$ is the number of edges in H, and $\delta_1(H)$ is often called the minimum *vertex degree* of H. When H is a k-graph for some positive integer k, $\delta_{k-1}(H)$ is known as the minimum *codegree* of H.

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†School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049 China (luhongliang215@sina.com).
‡School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 USA (yu@math.gatech.edu, xyuan@gatech.edu).

Bollobás, Daykin, and Erdős [5] considered minimum vertex degree conditions for matchings in k-graphs. They proved that if H is a k-graph of order $n \geq 2k^2(m+2)$ and $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$ then $\nu(H) \geq m$. For 3-graphs, Kühn, Osthus, and Treglown [17] and, independently, Khan [14] proved the following stronger result: There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geq n_0$, $m \leq n/3$, and $\delta_1(H) > \binom{n-1}{2} - \binom{n-m}{2}$ then $\nu(H) \geq m$.

In [15], Kuhn and Osthus proved that there exists $n_0 \in \mathbb{N}$ such that if H is a k-graph of order $n \geq n_0$ and $\delta_{k-1}(H) \geq n/2 + 3K^2\sqrt{n \log n}$, then H has a perfect matching. Rödl, Ruciński, and Szemerédi [21] determined the minimum codegree threshold for the existence of a perfect matching in a k-graph. Treglown and Zhao [23, 24] extended this result to include l-degrees for $k/2 \leq l \leq k-2$. Hàn, Person, and Schacht [12] considered the minimum l-degree condition for perfect matchings in the range $1 \leq l \leq k/2$. In particular, they showed that if H is a 3-graph and $\delta_1(H) > (1+o(1))\frac{5}{9}\binom{|V(H)|}{2}$ then H has a perfect matching. Two surveys of these and other related results appear in [19, 25].

For near perfect matchings, Han [11] proved a conjecture of Rödl, Ruciński, and Szemerédi [21] that, for $n \not\equiv 0 \pmod{k}$, the codegree threshold for the existence of a near perfect matching in a k-graph H is $\lfloor n/k \rfloor$. This is much smaller than the codegree threshold (roughly $n/2$) obtained by Rödl, Ruciński, and Szemerédi [21] for perfect matchings.

For nearly perfect matchings, Hàn, Person, and Schacht [12] proved the following result: For any integers $k > l > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ with $n \in k\mathbb{Z}$ and for every n-vertex k-graph H with

$$\delta_l(H) \geq \frac{k-l}{k}\binom{n}{k-l} + k^{k+1}(\ln n)^{1/2} n^{k-l-1/2},$$

H contains a matching covering in all but $(l-1)k$ vertices. Our main result improves this bound for the range $k/2 < l \leq k-1$, by providing an exact l-degree threshold for the existence of a matching covering in all but at most $(k-l)\lceil (k-l)/(2l-k) \rceil + k - 1$ vertices.

Theorem 1.1. *For any integers k, l satisfying $k \geq 3$ and $k/2 < l \leq k-1$, there exists a positive real μ such that, for all sufficiently large integers m, n satisfying*

$$\frac{n}{k} - \mu n \leq m \leq \frac{n}{k} - 1 - \left(1 - \frac{l}{k}\right)\left\lceil \frac{k-l}{2l-k} \right\rceil, \tag{1.1}$$

if H is a k-graph on n vertices and $\delta_l(H) > \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l}$ then $\nu(H) \geq m+1$.

When $l \geq 2k/3$, we have $(k-l)/(2l-k) \leq 1$. Moreover, if $n \equiv r \pmod{k}$, $0 \leq r < k$, and $r + l \geq k$ then Theorem 1.1 with $m = \lceil n/k \rceil - 2$ implies that H has a matching covering in all but at most k vertices. In general, if the interval $[n/k - 2, n/k - 1 - (1 - l/k)\lceil (k-l)/(2l-k) \rceil]$ contains an integer, then by letting m be that integer, Theorem 1.1 implies that H has a near perfect matching.

The bound on $\delta_l(H)$ in Theorem 1.1 is best possible. To see this, consider the k-graph $H_k^k(U,W)$, where U, W is a partition of $V(H_k^k(U,W))$ and the edges of $H_k^k(U,W)$ are precisely those k-subsets of $V(H_k^k(U,W))$ intersecting W at least once. For integers k, l, n with $k \geq 2$ and $0 < l < k$ and for large n, $\delta_l(H_k^k(U,W)) = \binom{n-l}{k-l} - \binom{(n-l)-|W|}{k-l}$ and the matching number of $H_k^k(U,W)$ is $|W|$. Thus, the bound on $\delta_l(H)$ in Theorem 1.1 is best possible (by letting $|W| = m$).

We need to refine the definition of $H_k^k(U,W)$ to $H_k^s(U,W)$ for all $s \in [k]$. Again, U,W is a partition of $V(H_k^s(U,W))$ and the edges of $H_k^s(U,W)$ are precisely those k-subsets of $V(H_k^s(U,W))$ intersecting W at least once and at most s times.

Given two hypergraphs H_1, H_2 and a real number $\varepsilon > 0$, we say that H_2 is *ε-close* to H_1 if $V(H_1) = V(H_2)$ and $|E(H_1)\backslash E(H_2)| \le \varepsilon |V(H_1)|^k$. Our proof of Theorem 1.1 consists of two parts by considering whether or not H is "close" to $H_k^s(U,W)$, which is similar to arguments in [21]. In the next two paragraphs, we give an outline for each case.

We first consider the case when $V(H)$ has a partition U,W with $|W| = m$ such that H is close to $H_k^{k-l}(U,W)$. If every vertex of H is "good" (to be made precise later) with respect to $H_k^{k-l}(U,W)$ then we find the desired matching by a greedy argument. Otherwise, we find the desired matching in two steps by first finding a matching M' such that every vertex in $H - V(M')$ is good, thereby reducing the problem to the previous case.

The other case is when H is not close to $H_k^{k-l}(U,W)$ for any partition $V(H)$ into U,W with $|W| = m$. We will see that such H does not have any sparse subset of very large size. To deal with this case, we will use the following approach of Alon et al. [1]:

- Find a small absorbing matching M_a in H;
- find random subgraphs of $H - V(M_a)$ with perfect fractional matchings (see section 4 for a definition);
- use those random subgraphs and a theorem of Frankl and Rödl to find an almost perfect matching M' in $H - V(M_a)$ (see Lemma 5.7); and
- use the matching M_a to absorb the remaining vertices in $V(H) \setminus (V(M_a) \cup V(M'))$.

To find a perfect fractional matching in certain random subgraphs of $H - V(M_a)$ we need to prove a stability version of a result of Frankl [8] on the Erdős matching conjecture [6], which might be of independent interest. We also need to use the hypergraph container result of Balogh, Morris, and Samotij [3] to bound the independence number of random subgraphs of H.

Our paper is organized as follows. In section 2, we prove Theorem 1.1 for k-graphs H such that $V(H)$ has a partition U,W with $|W| = m$ and H is ε-close to $H_k^{k-l}(U,W)$ (for any $\varepsilon < (8^{k-1}k^{5(k-1)}k!)^{-3}$). In fact, in this case, the degree threshold works for all $m < n/k$. In section 3, we prove an absorbing lemma that ensures the existence of a small matching M_a in H with the following property: For any small set S, the subgraph of H induced by $V(M_a) \cup S$ has a nearly perfect matching. This is done by a standard second moment method. In section 4, we show that if a k-graph does not have a very large independence number but has a large minimum l-degree then it has a perfect fractional matching. This is done by proving a stability version of a result of Frankl. In section 5, we first prove Lemma 5.3, which is used to control the independence number of H when it is not close to $H_k^{k-l}(U,W)$ for any partition of $V(H)$ into U,W with $|W| = m$. This in turn allows us to apply the hypergraph container result to control the independence number of random subgraphs of $H - V(M_a)$. We then use the approach in [1] to find random subgraphs of $H - V(M_a)$ with perfect fractional matchings. Those random subgraphs enable us to use a result of Frankl and Rödl [10] (see Lemma 5.1) to find an almost perfect matching in $H - V(M_a)$. In section 6, we complete the proof of Theorem 1.1 by applying the absorbing lemma from section 3. We also show how our proof implies a result on perfect matchings in 3-graphs proved by Kühn, Osthus, and Treglown [17] and, independently, by Khan [14].

2. Hypergraphs close to $H_k^{k-l}(U,W)$. In this section, we prove Theorem 1.1 for the case when $V(H)$ has a partition U, W with $|W| = m$ such that H is close to $H_k^{k-l}(U,W)$. Actually, in this case, the assertion of Theorem 1.1 holds for all $m \le n/k - 1$. Moreover, in the case when $m \le n/(2k^4)$, we do not require H to be close to $H_k^{k-l}(U,W)$ or $l > k/2$.

LEMMA 2.1. *Let n, m, k, l be positive integers such that $k \ge 3$, $m \le n/(2k^4)$, and $l \in [k-1]$. Let H be a k-graph on n vertices and $\delta_l(H) > \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l}$. Then $\nu(H) \ge m+1$.*

Proof. We apply induction on m. When $m = 0$, we have $\delta_l(H) > 0$; so $\nu(H) \ge 1$. Now assume $m \ge 1$ and that the assertion holds when m is replaced with $m-1$. Let M be a maximum matching in H, and assume $|M| \le m$.

Since M is a maximum matching in H, every edge of H intersects M. So there exists a vertex $v \in V(M)$ such that

$$d_H(v) > \frac{e(H)}{km}.$$

Note that $e(H) \ge \delta_l(H)\binom{n}{l}/\binom{k}{l}$, and

$$\begin{aligned}
\delta_l(H) &> \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l} \quad \text{(by assumption)}\\
&= \binom{n-l}{k-l}\left(1 - \prod_{i=0}^{m-1}\frac{n-k-i}{n-l-i}\right)\\
&> \binom{n-l}{k-l}\left(1 - \left(1-\frac{k-l}{n-l}\right)^m\right)\\
&> \binom{n-l}{k-l}\left(1 - \left(1 - m\frac{k-l}{n-l} + \binom{m}{2}\left(\frac{k-l}{n-l}\right)^2\right)\right)\\
&> \frac{m(k-l)}{2(n-l)}\binom{n-l}{k-l} \quad \text{(since } m \le n/(2k^4)).
\end{aligned}$$

Thus we have

$$d_H(v) > \frac{e(H)}{km} \ge \frac{\delta_l(H)\binom{n}{l}}{km\binom{k}{l}} > \frac{(k-l)}{2nk}\binom{n-l}{k-l}\frac{\binom{n}{l}}{\binom{k}{l}} = \frac{(k-l)}{2k^2}\binom{n-1}{k-1} \ge \frac{1}{2k^2}\binom{n-1}{k-1},$$

where the last inequality holds because $l \le k-1$.

Note that

$$\begin{aligned}
\delta_l(H-v) &\ge \delta_l(H) - \binom{n-(l+1)}{k-(l+1)}\\
&> \binom{n-l}{k-l} - \binom{(n-l)-m}{k-l} - \binom{n-(l+1)}{k-(l+1)}\\
&= \binom{(n-1)-l}{k-l} - \binom{((n-1)-l)-(m-1)}{k-l}.
\end{aligned}$$

Recall that $m \le n/(2k^4)$, so $m-1 \le (n-1)/(2k^4)$. Hence, by the induction hypothesis, $H-v$ has a matching of size m, say M'.

The number of edges of H containing v and intersecting $V(M')$ is at most $km\binom{n-2}{k-2}$. Since $m \le n/(2k^4)$,

$$km\binom{n-2}{k-2} < \frac{1}{2k^2}\binom{n-1}{k-1} < d_H(v).$$

Thus $H - V(M')$ contains an edge e such that $v \in e$. Now $M' \cup \{e\}$ is a matching in H of size $m+1$. ∎

For the case when $m > n/(2k^4)$, we use the structure of $H_k^{k-l}(U,W)$ to help us construct the desired matching in H. First, we prove a lemma for the case where, for each vertex $v \in V(H)$, only a small number of edges of $H_k^{k-l}(U,W)$ containing v do not belong to H.

Let H be a k-graph and let U, W be a partition of $V(H)$ and let $n = |U| + |W|$. Given real number α with $0 < \alpha < 1$, a vertex $v \in V(H)$ is called *α-good* with respect to $H_k^{k-l}(U,W)$ if

$$\left|N_{H_k^{k-l}(U,W)}(v) \setminus N_H(v)\right| \le \alpha n^{k-1},$$

and, otherwise, v is called *α-bad*. This notion quantifies the closeness of H to $H_k^{k-l}(U,W)$ at a vertex. Clearly, if H is ε-close to $H_k^{k-l}(U,W)$, then the number of α-bad vertices in H is at most $k\varepsilon n/\alpha$, otherwise,

$$\begin{aligned}|E(H_k^{k-l}(U,W)) \setminus E(H)| &\ge \frac{1}{k}\sum_{v\in V(H)}\left|N_{H_k^{k-l}(U,W)}(v) \setminus N_H(v)\right| \\ &\ge \frac{1}{k}(k\varepsilon n/\alpha)(\alpha n^{k-1}) = \varepsilon n^k,\end{aligned}$$

a contradiction. Note that in the statement of the lemma below we use $m \ge n/(2k^5)$ rather than $m \ge n/(2k^4)$ as opposed to Lemma 2.1. The reason is for its application in the proof of Lemma 2.3.

LEMMA 2.2. *Let k, l, m, n be integers and α be a positive real, such that $k \ge 3$, $l \in [k-1]$, $\alpha < (8^{k-1}k^{5(k-1)}k!)^{-1}$, $n \ge 8k^6$, and $n/(2k^5) \le m \le n/k$. Suppose that H is a k-graph on n vertices and U, W is a partition of $V(H)$ with $|W| = m$ such that every vertex of H is α-good with respect to $H_k^{k-l}(U,W)$. Then $\nu(H) \ge m$.*

Proof. We find a matching of size m in H using those edges that intersect W just once. Let M be a maximum matching in H such that $|e \cap W| = 1$ for each $e \in M$, and let $t = |M|$. We may assume $t < m$, or else the desired matching exists. So $W \setminus V(M) \ne \emptyset$. By the maximality of M, $N_H(x) \cap \binom{U\setminus V(M)}{k-1} = \emptyset$ for all $x \in W \setminus V(M)$.

We claim that $t \ge m/2$. For, suppose $t < m/2$. Since $m \le n/k$, $t < n/(2k)$, so $|V(H) \setminus V(M)| = n - tk > n - n/2 = n/2$. Hence,

$$|U \setminus V(M)| > |V(H) \setminus V(M)| - |W| \ge n/2 - n/k \ge n/6.$$

Thus, for any $x \in W \setminus V(M)$,

$$\left|N_{H_k^{k-l}(U,W)}(x) \setminus N_H(x)\right| \ge \left|\binom{|U \setminus V(M)|}{k-1}\right| > \binom{n/6}{k-1} > \alpha n^{k-1},$$

contradicting the assumption that every vertex in H is α-good.

Since $t < m \le n/k$ and $|e \cap W| = 1$ for each $e \in M$, there exists a k-set $S = \{u_1, \dots, u_k\} \subseteq V(H) \setminus V(M)$ such that $u_k \in W$, and $S \setminus \{u_k\} \subseteq U$. Since $m \ge n/(2k^5) > 2k$, we have $t \ge m/2 > k$.

Arbitrarily choose $k-1$ pairwise distinct edges $e_1, \dots, e_{k-1}$ from M and write $e_i := \{v_{i,1}, v_{i,2}, \dots, v_{i,k}\}$ such that $v_{i,k} \in W$ and $v_{i,j} \in U$ for $j \in [k-1]$. For convenience, let $v_{k,j} := u_j$ for $j \in [k]$. For $i \in [k]$, define $f_i := \{v_{1,1+i}, v_{2,2+i}, \dots, v_{k-1,(k-1)+i}, v_{k,k+i}\}$, where the addition in the subscripts is modulo k (except that we write k for 0). Then $f_i \notin E(H)$ for some $i \in [k]$ as, otherwise, $(M \setminus \{e_i \ : \ i \in [k-1]\}) \cup \{f_i \ : \ i \in [k]\}$ is a matching in H that contradicts the maximality of M.

Note that for different choices of $e_1, \dots, e_{k-1} \in M$ and $e'_1, \dots, e'_{k-1} \in M$, the corresponding sets $\{f_1, \dots, f_k\}$ and $\{f'_1, \dots, f'_k\}$ constructed in the above paragraph are disjoint. Since there are $\binom{t}{k-1}$ choices of $e_1, \dots, e_{k-1}$ from M, we have

$$\begin{aligned}
&\sum_{i=1}^{k} \left| N_{H_k^{k-l}(U,W)}(u_i) \setminus N_H(u_i) \right| \\
&\ge \binom{t}{k-1} \\
&> \frac{(t-(k-1)+1)^{k-1}}{(k-1)!} \\
&> \frac{(n/(4k^5)-(k-1))^{k-1}}{(k-1)!} \qquad (\text{since } t \ge m/2 > n/(4k^5)) \\
&> \frac{(n/(8k^5))^{k-1}}{(k-1)!} \qquad (\text{since } n \ge 8k^6) \\
&= (8^{k-1}k^{5(k-1)}k!)^{-1} k n^{k-1} \\
&> \alpha k n^{k-1} \qquad (\text{since } \alpha < (8^{k-1}k^{5(k-1)}k!)^{-1}).
\end{aligned}$$

Thus there exists $u_j \in S$ such that

$$\left| N_{H_k^{k-l}(U,W)}(u_j) \setminus N_H(u_j) \right| > \alpha n^{k-1},$$

contradicting the assumption that every vertex in H is α-good. ∎

The next lemma takes care of Theorem 1.1 for the case when $m > n/(2k^4)$ and H is ε-close to $H_k^{k-l}(U,W)$. We first find two matchings (in two steps and using Lemma 2.1) that cover all $\sqrt{\varepsilon}$-bad vertices. We then apply Lemma 2.2 to the hypergraph obtained from H by deleting these two matchings.

LEMMA 2.3. *Let k, l, m, n be integers and let $0 < \varepsilon < (8^{k-1}k^{5(k-1)}k!)^{-3}$, such that $k \ge 3$, $l \in [k-1]$, $n \ge 8k^6/(1-5k^2\sqrt{\varepsilon})$, and $n/(2k^4) < m \le n/k - 1$. Suppose H is a k-graph on n vertices and U, W is a partition of $V(H)$ with $|W| = m$, such that $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l}$ and H is ε-close to $H_k^{k-l}(U,W)$. Then $\nu(H) \ge m+1$ when $m < n/k - 1$ or $l \le k-2$, and $\nu(H) \ge m$ when $l = k-1$ and $m = n/k - 1$.*

Proof. Since H is ε-close to $H_k^{k-l}(U,W)$, all but at most $k\sqrt{\varepsilon}n$ vertices of H are $\sqrt{\varepsilon}$-good with respect to $H_k^{k-l}(U,W)$. Let U^{bad} and W^{bad} denote the set of $\sqrt{\varepsilon}$-bad vertices in U and W, respectively. So $|U^{bad}| + |W^{bad}| \le k\sqrt{\varepsilon}n$. Let $c := |W^{bad}|$, $V_1 := U \cup W^{bad}$, and $W_1 := W \setminus W^{bad}$. Note that possibly $c = 0$. We deal with vertices in W_1 later since at those vertices H and $H_k^{k-l}(U,W)$ are close. We claim that

(1) $H[V_1]$ has a matching M_1 of size $c+1$.

To see this, let s be the maximum number of edges in H intersecting W_1 and containing a fixed l-set in V_1. Then $s \le \binom{n-l}{k-l} - \binom{n-l-(m-c)}{k-l}$ and $\delta_l(H[V_1]) \ge \delta_l(H) - s$. Hence,

$$\delta_l(H[V_1]) \ge \delta_l(H) - s > \binom{(n-m+c)-l}{k-l} - \binom{(n-m+c)-l-c}{k-l}.$$

Since $n/(2k^4) < m < n/k \le n/3$, we have $n-m+c > 2m+c > n/k^4+c$. Thus, since $c \le k\sqrt{\varepsilon}n$, $n-m+c > 2k^4c$ by the choice of ε. So by Lemma 2.1, $H[V_1]$ contains a matching of size $c+1$. This completes the proof of (1).

Let $H_1 := H - V(M_1)$. Next, we cover $U^{bad} \cup W^{bad}$ with two matchings in H_1, using edges intersecting W_1 at most once. First note that, for each l-set $S \subseteq V_1 \setminus V(M_1)$, H_1 has lots of edges containing S and intersecting W_1 just once, or H_1 has lots of edges of containing S and contained in $V_1 \setminus V(M_1)$. More precisely, we show that

(2) for any real number β with $2k^2\sqrt{\varepsilon} < \beta < (2k)^{-(k-l+3)}/2 - k^2\sqrt{\varepsilon}$ (which exists as $\varepsilon < (2k)^{-2k-11}$ and $k \ge 3$) and for any $S \in \binom{V_1 \setminus V(M_1)}{l}$, we have

$$|\{T \in N_{H_1}(S) \ : \ |T \cap W_1| = 1\}| \ge \beta n^{k-l}, \quad \text{or}$$

$$|\{T \in N_{H_1}(S) \ : \ T \subseteq V_1 \setminus V(M_1)\}| \ge \beta n^{k-l}.$$

To prove (2), let $S \in \binom{V_1 \setminus V(M_1)}{l}$ and $|\{T \in N_{H_1}(S) \ : \ |T \cap W_1| = 1\}| < \beta n^{k-l}$. Since

$$|\{T \in N_{H_1}(S) \ : \ |T \cap W_1| \ge 2\}| \le \sum_{i=2}^{k-l} \binom{m}{i}\binom{n-l-m}{k-l-i}$$

and

$$|\{T \in N_H(S) \ : \ |T \cap V(M_1)| \ge 1\}| \le k(c+1)n^{k-l-1} < 2k^2\sqrt{\varepsilon}n^{k-l},$$

we have

$$\begin{aligned}
&|\{T \in N_{H_1}(S) \ : \ T \subseteq V_1 \setminus V(M_1)\}| \\
&> \delta_l(H) - |\{T \in N_{H_1}(S) \ : \ |T \cap W_1| \ge 2\}| \\
&\qquad - |\{T \in N_{H_1}(S) \ : \ |T \cap W_1| = 1\}| - 2k^2\sqrt{\varepsilon}n^{k-l} \\
&> \left(\binom{n-l}{k-l} - \binom{n-l-m}{k-l}\right) - \sum_{i=2}^{k-l}\binom{m}{i}\binom{n-l-m}{k-l-i} - \beta n^{k-l} - 2k^2\sqrt{\varepsilon}n^{k-l} \\
&= m\binom{n-l-m}{k-l-1} - 2k^2\sqrt{\varepsilon}n^{k-l} - \beta n^{k-l} \\
&> n^{k-l}/(2k)^{k-l+3} - 2k^2\sqrt{\varepsilon}n^{k-l} - \beta n^{k-l} \quad (\text{since } n/(2k^4) \le m < n/k \text{ and } n \ge 8k^6) \\
&\ge \beta n^{k-l} \quad (\text{by the choice of } \beta).
\end{aligned}$$

This completes the proof of (2).

To find matchings in H_1 covering $(U^{bad} \cup W^{bad}) \setminus V(M_1)$, we fix a set $B \subseteq V_1 \setminus V(M_1)$ such that $|B| \equiv 0 \pmod l$, $(U^{bad} \cup W^{bad}) \setminus V(M_1) \subseteq B$, and $|B \setminus (U^{bad} \cup W^{bad})| < l$. For convenience, let $q = |B|/l$. Then

$$q \le k\sqrt{\varepsilon}n.$$

We partition B into q disjoint l-sets $B_1, \ldots, B_q$. By (2), we may assume that, for some $q_1 \in [q] \cup \{0\}$, $|\{T \in N_{H_1}(B_i) \ : \ |T \cap W_1| = 1\}| \ge \beta n^{k-l}$ for $1 \le i \le q_1$ and $|\{T \in N_{H_1}(B_j) \ : \ T \subseteq V_1 \setminus V(M_1)\}| \ge \beta n^{k-l}$ for $q_1 < j \le q$. We claim that

(3) there exist disjoint matchings M_{21} and M_{22} in H_1 such that
- $|M_{21}| + |M_{22}| \leq k\sqrt{\varepsilon}n$,
- M_{21} covers $\bigcup_{i=1}^{q_1} B_i$ and each edge in M_{21} intersects W_1 just once, and
- M_{22} covers $\bigcup_{i=q_1+1}^{q} B_i$ and each edge in M_{22} is disjoint from W_1.

First, we find the matching M_{21} covering $\bigcup_{i=1}^{q_1} B_i$ (which is empty if $q_1 = 0$). Suppose for some $0 \leq h < q_1$ we have chosen pairwise disjoint edges $e_1, \ldots, e_h$ of $H_1 = H - V(M_1)$ (which is empty when $h = 0$), such that, for $i \in [h]$, we have $|e_i \cap W| = 1$ and $B_i \subseteq e_i$. Since $|\{T \in N_{H_1}(B_{h+1}) : |T \cap W_1| = 1\}| \geq \beta n^{k-l}$ and $h \leq q_1 - 1 \leq k\sqrt{\varepsilon}n - 1$, the number of edges of H disjoint from $V(M_1) \cup (\bigcup_{i=1}^{h} e_i)$ but containing B_{h+1} and exactly one vertex from W_1 is at least

$$\beta n^{k-l} - k|M_1|n^{k-l-1} - (hk)n^{k-l-1} \geq \beta n^{k-l} - 2k^2\sqrt{\varepsilon}n^{k-l} > 0.$$

Thus, there is an edge e_{h+1} of H_1 such that $|e_{h+1} \cap W_1| = 1$, $B_{h+1} \subseteq e_{h+1}$, and $e_{h+1} \cap (\bigcup_{j=1}^{h} e_j) = \emptyset$. Since $q_1 \leq q \leq k\sqrt{\varepsilon}n$, we may continue this process till $h = q_1 - 1$. Now $M_{21} = \{e_1, \ldots, e_{q_1}\}$ is the desired matching that covers $\bigcup_{i=1}^{q_1} B_i$.

Next, we find the matching $M_{22} = \{e_j : q_1 < j \leq q\}$, such that for $q_1 < j \leq q$, $B_j \subseteq e_j$ and $e_j \subseteq V_1 \setminus (V(M_1) \cup (\bigcup_{s=1}^{j-1} e_s))$. Suppose that we have chosen $e_1, \ldots, e_{q_1}, \ldots, e_s$ for some s with $q_1 \leq s < q$ (which is empty if $q_1 = q$). Since $|\{T \in N_{H_1}(B_{s+1}) : T \subseteq V_1 \setminus V(M_1)\}| \geq \beta n^{k-l}$ and $s \leq q - 1 \leq k\sqrt{\varepsilon}n - 1$, the number of edges in H disjoint from $V(M_1) \cup (\cup_{i=1}^{s} e_i) \cup W_1$ but containing B_{s+1} is at least

$$\beta n^{k-l} - k|M_1|n^{k-l-1} - (sk)n^{k-l-1} \geq \beta n^{k-l} - 2k^2\sqrt{\varepsilon}n^{k-l} > 0.$$

So there exists an edge e_{s+1} of H_1 such that $B_{s+1} \subseteq e_{s+1}$ and $e_{s+1} \cap (\bigcup_{i=1}^{s} e_i) = \emptyset$. Since $q \leq k\sqrt{\varepsilon}n$, we may continue this process till $s = q - 1$. Now $M_{22} = \{e_{q_1+1}, \ldots, e_q\}$ gives the desired matching that covers $\bigcup_{i=q_1+1}^{q} B_i$. This completes the proof of (3).

Now, every vertex in $V(H) \setminus V(M_1 \cup M_{21} \cup M_{22})$ (as a vertex of H) is $\sqrt{\varepsilon}$-good with respect to $H_k^{k-l}(U, W)$. In order to apply Lemma 2.2, we find a matching M_{23} in $H_1 - V(M_{21} \cup M_{22})$ such that every vertex of $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$ is $\varepsilon^{1/3}$-good with respect to $H_k^{k-l}(U^*, W^*)$, where $U^* = U \cap V(H_2)$ and $W^* = W \cap V(H_2)$, $|U^*| + |W^*| \geq 8k^6$, and $(|U^*| + |W^*|)/(2k^4) < |W^*| \leq (|U^*| + |W^*|)/k$. So we need to prove (4) and (5) below.

(4) There exists a matching M_{23} in $H_1 - V(M_{21} \cup M_{22})$ with $|M_{23}| < k\sqrt{\varepsilon}n$ and satisfying the following property: If we let $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$, $U' = U \cap V(H_2)$, $W' = W \cap V(H_2)$, then, for some $r \in \{0, 1\}$ with $r = 0$ for $l \leq k - 2$, we have
- $|W'| - r = m - c - |M_{21}| - |M_{22}| - |M_{23}|$, $|U'| + |W'| - r \geq 8k^6$, and $(|U'| + |W'| - r)/(2k^5) < |W'| - r$,
- $|W'| - r \leq (|U'| + |W'| - r)/k$ when $l \leq k - 2$ or $m < n/k - 1$, and
- $|W'| - r \leq (|U'| + |W'|)/k$ when $l = k - 1$ and $m = n/k - 1$.

We prove (4) by considering two cases. Note $|M_1 \cup M_{21} \cup M_{22}| = (c+1) + q \leq 3k\sqrt{\varepsilon}n$ as $c, q \leq k\sqrt{\varepsilon}n$.

Case 1. $l \leq k - 2$.

In this case, we construct the matching M_{23} as follows. Suppose for some $1 \leq t \leq q - q_1$, we found vertices $x_1, \ldots, x_{t-1}$ in $U \setminus V(M_1 \cup M_{21} \cup M_{22})$ and edges $f_1, \ldots, f_{t-1}$ in $H_1 - V(M_{21} \cup M_{22})$ such that, for $i \in [t-1]$, we have $x_i \in f_i$, $|f_i \cap W_1| = 2$, and $f_i \cap (\bigcup_{j=1}^{i-1} f_j) = \emptyset$. (When $t = 1$, these sequences are empty.) Let $x_t \in U \setminus V(M_1 \cup M_{21} \cup M_{22}) \setminus (\bigcup_{i=1}^{t-1} f_i)$. Since x_t is $\sqrt{\varepsilon}$-good with respect to

$H_k^{k-l}(U,W)$, the number of edges of $H_1 - V(M_{21} \cup M_{22}) - (\bigcup_{i=1}^{t-1} f_i)$ containing x_t and exactly two vertices in W_1 is at least

$$\binom{m-c-2(t-1)}{2}\binom{n-m-1}{k-3} - \sqrt{\varepsilon}n^{k-1} - (3k\sqrt{\varepsilon}n)n^{k-2} - (kt)n^{k-2} > 0,$$

as $n/(2k^4) < m$, $c < k\sqrt{\varepsilon}n$, $t < k\sqrt{\varepsilon}n$, and $\varepsilon < (8^{k-1}k^{5(k-1)}k!)^{-3}$. So there exists an edge f_t in $H_1 - V(M_{21} \cup M_{22}) - (\bigcup_{i=1}^{t-1} f_i)$ such that $x_t \in f_t$ and $|f_t \cap W_1| = 2$. This process works as long as $t \le q - q_1$. Thus, we have a matching $M_{23} = \{f_j : j \in [q-q_1]\}$ such that, for $j \in [q-q_1]$, $f_j \subseteq V(H_1) \setminus (V(M_{21} \cup M_{22}) \cup (\bigcup_{i=1}^{j-1} f_i))$ and $|f_j \cap W_1| = 2$.

Let $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$ and let $U' = U \cap V(H_2)$ and $W' = W \cap V(H_2)$. Note that $|M_{23}| = |M_{22}|$, and note that

$$\begin{aligned}|W'| &= |W| - c - |M_{21}| - 2|M_{23}| = |W| - c - |M_{21}| - |M_{22}| - |M_{23}|, \text{ and}\\ |U'| &= |U| - (k(c+1) - c) - (k-1)|M_{21}| - k|M_{22}| - (k-2)|M_{23}|\\ &= |U| - (k-1)(c+1+|M_{21}| + |M_{22}| + |M_{23}|) - 1.\end{aligned}$$

Hence, we have

$$|U'| + |W'| = |U| + |W| - k(c+1) - k|M_{21}| - k|M_{22}| - k|M_{23}|.$$

Thus, $|U'| + |W'| \ge n - 5k^2\sqrt{\varepsilon}n \ge 8k^6$ and, since $m \le n/k - 1$,

$$\begin{aligned}(|U'| + |W'|)/k &= (|U| + |W|)/k - (c+1) - |M_{21}| - |M_{22}| - |M_{23}|\\ &\ge (|W| + 1) - (c+1) - |M_{21}| - |M_{22}| - |M_{23}|\\ &= |W'|.\end{aligned}$$

Moreover, since $|W| > n/(2k^4)$ and $|W| \ge |W'| \ge |W| - 3k\sqrt{\varepsilon}n$, we have

$$\begin{aligned}&(|U'| + |W'|) - 2k^5|W'|\\ &= |U| + |W| - k(c+1) - k|M_{21}| - k|M_{22}| - k|M_{23}| - 2k^5|W'|\\ &< |U| + |W| - 2k^5|W'|\\ &< 2k^4|W| - 2k^5|W'|\\ &< 0 \quad \text{(since } n \text{ is large and } \varepsilon \text{ is small).}\end{aligned}$$

Case 2. $l = k-1$.

Arbitrarily choose $q - q_1$ pairwise disjoint $(k-1)$-sets in $V(H) \setminus V(M_1 \cup M_{21} \cup M_{22})$, each containing exactly two vertices in W_1. Note that this can be done, because $|W_1| = m - c \ge n/(2k^4) - k\sqrt{\varepsilon}n > 2q$. Since $\delta_{k-1}(H) > m \ge n/(2k^4) > 5k^2\sqrt{\varepsilon}n \ge k((c+1) + 3q)$, we can extend these $q - q_1$ sets to $q - q_1$ pairwise disjoint edges $f_1, \ldots, f_{q-q_1}$ in $H - V(M_1 \cup M_{21} \cup M_{22})$.

Clearly, each f_i contains either two or three vertices from W_1. Thus, there exists some integer p with $0 \le p \le q - q_1$ such that $q - q_1 + p - 1 \le |W_1 \cap (\bigcup_{i=1}^{p} f_i)| \le q - q_1 + p$. Let $M_{23} = \{f_1, \ldots, f_p\}$, $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$, and $U' = U \cap V(H_2)$ and $W' = W \cap V(H_2)$.

Note that $|W_1 \cap V(M_{23})| = |M_{22}| + |M_{23}| - r$ for some $r \in \{0, 1\}$. Hence,

$$|W'| = |W| - c - |M_{21}| - |W_1 \cap V(M_{23})| = |W| - c - |M_{21}| - |M_{22}| - |M_{23}| + r$$

and

$$|U'| = |U| - (k(c+1) - c) - (k-1)|M_{21}| - k|M_{22}| - (k|M_{23}| - |W_1 \cap V(M_{23})|).$$

Therefore,

$$|U'| + |W'| - r = (|U| + |W| - r) - k(c+1) - k|M_{21}| - k|M_{22}| - k|M_{23}|.$$

It is easy to see that the same calculations in Case 1 also allow us to conclude that $|U'| + |W'| - r \geq 8k^6$ and $(|U'| + |W'| - r) - 2k^5(|W'| - r) < 0$. Moreover, if $r = 0$ then the same argument in Case 1 shows that $|W'| \leq (|U'| + |W'|)/k$. So we may assume $r = 1$.

First, suppose $m < n/k - 1$. Then $(|U| + |W|)/k \geq |W| + 1 + 1/k$, so

$$\begin{aligned}(|U'| + |W'| - 1)/k &= (|U| + |W| - 1)/k - (c+1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &\geq (|W| + 1 + 1/k) - 1/k - (c+1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &= |W'| - 1.\end{aligned}$$

Now suppose $m = n/k - 1$ (so $n \in k\mathbb{Z}$). Then

$$\begin{aligned}(|U'| + |W'|)/k &= (|U| + |W|)/k - (c+1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &\geq (|W| + 1) - (c+1) - |M_{21}| - |M_{22}| - |M_{23}| \\ &\geq |W'| - 1.\end{aligned}$$

So $|W'| - r \leq (|U'| + |W'|)/k$, completing the proof of (4).

We now define $W^* \subseteq W'$ and $U^* = V(H) \backslash W^*$ as follows: If $r = 0$ let $W^* = W'$. If $r = 1$ and $n \notin k\mathbb{Z}$ or $m < n/k - 1$ then choose some $w \in W'$ and let $W^* = W' \setminus \{w\}$. If $r = 1$, $n \in k\mathbb{Z}$, and $m = n/k - 1$ then choose $w_1, w_2 \in W'$ and let $W^* = W' \setminus \{w_1, w_2\}$.

(5) Every vertex of $H_2 := H_1 - V(M_{21} \cup M_{22} \cup M_{23})$ is $\varepsilon^{1/3}$-good with respect to $H_k^{k-l}(U^*, W^*)$.

To prove (5), we note that $k|M_1 \cup M_{21} \cup M_{22} \cup M_{23}| + 2 \leq k((c+1) + 3q) + 2 \leq 5k^2\sqrt{\varepsilon}n$. For each $x \in V(H_2)$, since x is $\sqrt{\varepsilon}$-good with respect to $H_k^{k-l}(U, W)$, we have

$$|N_{H_k^{k-l}(U,W)}(x) \setminus N_H(x)| \leq \sqrt{\varepsilon} n^{k-1}.$$

Thus,

$$\begin{aligned}&\left|N_{H_k^{k-l}(U^*,W^*)}(x) \setminus N_{H_2}(x)\right| \\ &\leq \left|N_{H_k^{k-l}(U,W)}(x) \setminus N_H(x)\right| + (k\,|M_1 \cup M_{21} \cup M_{22} \cup M_{23}| + 2)\, n^{k-2} \\ &\leq \sqrt{\varepsilon} n^{k-1} + 5k^2\sqrt{\varepsilon} n^{k-1} \\ &< \varepsilon^{1/3} n^{k-1}.\end{aligned}$$

This completes the proof of (5).

Hence, by (4) and (5), it follows from Lemma 2.2 that there is a matching M_3 in H_2 of size $|W^*|$. Let $M := M_1 \cup M_{21} \cup M_{22} \cup M_{23} \cup M_3$. Then M is a matching in H. By (4), $|W'| - r = m - (|M_1| - 1) - |M_{21}| - |M_{22}| - |M_{23}|$. If $l = k - 1$ and $m = n/k - 1$, then $|W^*| \geq |W'| - r - 1$, so

$$|M| \geq (|W'| - r - 1) + |M_1| + |M_{21}| + |M_{22}| + |M_{23}| = n/k - 1.$$

Otherwise, $|W^*| = |W'| - r$ and $|M| = (|W'| - r) + |M_1| + |M_{21}| + |M_{22}| + |M_{23}| = m + 1$. ∎

3. An absorbing lemma. A typical approach to finding large matchings in a dense k-graph H is to find a small matching M_a in H such that, for each small subset

$S \subseteq V(H) \setminus V(M_a)$, $H[V(M_a) \cup S]$ has a large matching (e.g., an almost perfect matching). Such a matching M_a is known as an *absorbing* matching, often found by applying the second moment method. This approach was initiated by Rödl, Ruciński, and Szemerédi [20].

Let $Bi(n,p)$ be the binomial distribution with parameters n and p. The following lemma on the Chernoff bound can be found in Alon and Spencer [2, page 313] (also see [18]).

LEMMA 3.1 (Chernoff). *Suppose $X_1, \dots, X_n$ are independent random variables taking values in $\{0,1\}$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then, for any $0 < \delta \le 1$,*

$$\mathbb{P}[X \ge (1+\delta)\mu] \le e^{-\delta^2\mu/3} \text{ and } \mathbb{P}[X \le (1-\delta)\mu] \le e^{-\delta^2\mu/2}.$$

In particular, when $X \sim Bi(n,p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}(|X - np| \ge \lambda) \le e^{-\Omega(\lambda^2/np)}.$$

We will frequently use the following fact: For integers $0 \le l' < l \le k-1$ and any k-graph H, if $\delta_l(H) \ge c\binom{n-l}{k-l}$ for some $0 \le c \le 1$, then $\delta_{l'}(H) \ge c\binom{n-l'}{l-l'}\binom{n-l}{k-l}/\binom{k-l'}{l-l'} \ge c\binom{n-l'}{k-l'}$.

The main result of this section is the following lemma for absorbing matchings in k-graphs with large l-degree for $k/2 < l \le k-1$. We are able to do this partly due to the existence of positive integers a, h satisfying $h \le l$, $a \le k-l$, and $al \ge a(k-l) + (k-h)$. (One can check that $a = k-l$ and $h = l$ satisfies this requirement.) We use $\alpha \ll \beta$ to mean that α is sufficiently smaller than β.

LEMMA 3.2. *Let k, l be integers with $k \ge 3$ and $k/2 < l \le k-1$, and let $c > 0$ be a constant with $c < 1/k!$. Then there exist $\rho > 0$ and $c' > 0$ with $0 < \rho \ll c' \ll c$, such that the following holds for all sufficiently large integers n:*

Let a, h be positive integers satisfying $h \le l$, $a \le k-l$, and $al \ge a(k-l) + (k-h)$. Let H be a k-graph on n vertices with $\delta_l(H) \ge c\binom{n-l}{k-l}$. Then there exists a matching M in H such that

- *$|M| \le 2k\rho n$ and*
- *for any subset $S \subseteq V(H)$ with $|S| \le c'\rho n$, $H[V(M) \cup S]$ has a matching covering all but at most $al + h - 1$ vertices.*

Proof. For $R \in \binom{V(H)}{al+h}$ and $Q \in \binom{V(H)}{ak}$, we say that Q is *R-absorbing* if $\nu(H[Q \cup R]) \ge a+1$ and Q is the vertex set of a matching in H. (In particular, this requires $al + h \ge k$, which is guaranteed by assumption.) Let $\mathcal{L}(R)$ denote the collection of all R-absorbing sets in H. We claim that

(1) there exists $c' = c'(c,k) > 0$ such that $|\mathcal{L}(R)| \ge c'n^{ak}$ for every $R \in \binom{V(H)}{al+h}$.

To prove (1), let $R \in \binom{V(H)}{al+h}$. We wish to extend R to a matching of size $a+1$ by adding a set of size $(a+1)k - (al+h) = a(k-l) + (k-h)$. Partition R into $a+1$ pairwise disjoint subsets $R_1, \dots, R_{a+1}$ with $|R_{a+1}| = h$ and $|R_i| = l$ for $i \in [a]$. Next we choose $(k-l)$-sets T_s for $s \in [a]$ and a $(k-h)$-set T_{a+1} such that $\{R_s \cup T_s : s \in [a+1]\}$ form a matching in H.

For $j \in [a]$, since $d_H(R_j) \ge \delta_l(H) \ge c\binom{n-l}{k-l}$, we have, for large n,

$$\left|N_{(H-R)-\bigcup_{s=1}^{j-1} T_s}(R_{j+1})\right| \ge c\binom{n-l}{k-l} - ((al+h)+(k-l)j)\binom{(n-l)-1}{(k-l)-1} > \frac{c}{2}\binom{n-l}{k-l};$$

thus, we have more than $\frac{c}{2}\binom{n-l}{k-l}$ choices for each T_j with $j \in [a]$. Similarly, since $d_H(R_{a+1}) \geq c\binom{n-h}{k-h}$ as $h \leq l$, we have

$$\left|N_{(H-R)-\bigcup_{s=1}^{a} T_s}(R_{a+1})\right| \geq c\binom{n-h}{k-h} - ((al+h)+(k-l)a)\binom{(n-h)-1}{(k-h)-1} > \frac{c}{2}\binom{n-h}{k-h};$$

hence, we have more than $\frac{c}{2}\binom{n-h}{k-h}$ choices for T_{a+1}.

Fix an arbitrary choice of $T_i \in N_{(H-R)-\bigcup_{s=1}^{i-1} T_s}(R_i)$, $i \in [a+1]$, such that $\{R_s \cup T_s : s \in [a+1]\}$ form a matching in H. Let $T = \bigcup_{i=1}^{a+1} T_i$. Next, we form an R-absorbing set Q by extending the set T to a matching of size a. We partition T into subsets $T_1', \ldots, T_a'$ such that $1 \leq |T_i'| \leq l$ for $i \in [a]$. Such a partition exists since $|T| = a(k-l) + (k-h) \leq al$. Similarly to the arguments in the previous paragraph, we can show that there exists $P_i \in N_{(H-(R\cup T))-\bigcup_{s=1}^{i-1} P_s}(T_i')$ for $i \in [a]$, such that

$$\left|N_{(H-(R\cup T))-\bigcup_{s=1}^{i-1} P_s}(T_i')\right| > \frac{c}{2}\binom{n-|T_i'|}{k-|T_i'|}.$$

This means that there are more than $\frac{c}{2}\binom{n-|T_i'|}{k-|T_i'|}$ choices for each P_i with $i \in [a]$. Let $Q = T \cup (\bigcup_{i=1}^{a} P_i)$. Then Q is the vertex set of a matching of size a in H. Hence Q is an R-absorbing set.

Note that each such ak-set Q can be produced at most $(ak)!$ times by the above process, and recall that $\sum_{i=1}^{a} |T_i'| = a(k-l) + (k-h)$. Hence, for large n (compared with k), we have

$$\begin{aligned}
|\mathcal{L}(R)| &> ((ak)!)^{-1}\left(\frac{c}{2}\binom{n-l}{k-l}\right)^{a}\left(\frac{c}{2}\binom{n-h}{k-h}\right)\prod_{i=1}^{a}\left(\frac{c}{2}\binom{n-|T_i'|}{k-|T_i'|}\right) \\
&> (2(ak)!)^{-1}\left(\frac{c}{2}\right)^{2a+1}\left(\frac{n^{a(k-l)}}{((k-l)!)^{a}}\right)\left(\frac{n^{k-h}}{(k-h)!}\right)\left(\frac{n^{ak-(a(k-l)+(k-h))}}{(ak-(a(k-l)+(k-h)))!}\right) \\
&> c'n^{ak}
\end{aligned}$$

by choosing $c' < (2(ak)!)^{-1}(c/2)^{2a+1}(((k-l)!)^{a}(k-h)!(al+h-k)!)^{-1}$. This completes the proof of (1).

Choose $\rho < c'/(2a^2k^2)$. We form a family $\mathcal{F} \subseteq \binom{V(H)}{ak}$ by choosing each member of $\binom{V(H)}{ak}$ independently at random with probability

$$p = \frac{\rho n}{\binom{n}{ak}}.$$

Then

(2) with probability $1/2 - o(1)$, all of the following hold:
- (2a) $|\mathcal{F}| \leq 2\rho n$,
- (2b) $|\mathcal{L}(R) \cap \mathcal{F}| \geq 2c'\rho n$ for all $(al+h)$-sets R, and
- (2c) $\mathcal{F}$ contains less than $c'\rho n$ intersecting pairs.

Clearly, $\mathbb{E}(|\mathcal{F}|) = \rho n$ and, by (1), $\mathbb{E}(|\mathcal{L}(R) \cap \mathcal{F}|) > c'n^{ak}p > 4c'\rho n$ (as $a \geq 1$ and $k \geq 3$). So by Lemma 3.1, with probability $1 - o(1)$,

$$|\mathcal{F}| \leq 2\rho n,$$

and, for each fixed $(al+h)$-set R, with probability at least $1 - e^{-\Omega(\rho n)}$, $\mathcal{F}$ satisfies

$$|\mathcal{L}(R) \cap \mathcal{F}| \geq 2c'\rho n.$$

Hence given n sufficiently large, it follows from the union bound that, with probability $1 - o(1)$, (2a) and (2b) hold.

Furthermore, the expected number of intersecting pairs in $\mathcal{F}$ is at most

$$\binom{n}{ak}\binom{ak}{1}\binom{n-1}{ak-1}p^2 = a^2k^2\rho^2 n < c'\rho n/2.$$

Thus, using Markov's inequality, we derive that with probability at least $1/2$, $\mathcal{F}$ contains less than $c'\rho n$ intersecting pairs of ak-sets. Hence, by the union bound, (2a), (2b), (2c) hold with probability $1/2 - o(1)$, completing the proof of (2).

Let $\mathcal{F}'$ denote the family obtained from $\mathcal{F}$ by deleting one ak-set from each intersecting pair of sets in $\mathcal{F}$ and removing all ak-sets that are not the vertex set of a matching in H. (Note that the latter are not in $\mathcal{L}(R)$ for any $(al+h)$-set R.) Then $\mathcal{F}'$ consists of pairwise disjoint vertex sets of matchings of size a in H. Moreover, for all $(al+h)$-sets R,

$$|\mathcal{L}(R) \cap \mathcal{F}'| \geq 2c'\rho n - c'\rho n \geq c'\rho n.$$

For each $F \in \mathcal{F}'$, let M_F be a matching in H with $V(M_F) = F$. Then $M = \bigcup_{F\in\mathcal{F}'} M_F$ is a perfect matching in $H[V(\mathcal{F}')]$, and $|M| \leq a|\mathcal{F}| \leq k|\mathcal{F}| \leq 2k\rho n$. It remains to show that M absorbes small sets.

Let S be an arbitrary subset of $V(H) \setminus V(M)$ with $|S| \leq c'\rho n$. We use M to absorb $(al+h)$-sets iteratively, starting with an arbitrary $(al+h)$-subset of S. Let $S_0 := S$ and let $R_0 \subseteq S_0$ with $|R_0| = al+h$. Since $|\mathcal{L}(R_0) \cap \mathcal{F}'| \geq c'\rho n$, we can find $Q_0 \in \mathcal{F}'$ such that $H[R_0 \cup Q_0]$ has a matching M_0 with $|M_0| = a+1$. Let $t \geq 0$ be the maximal integer such that there exist

- sets $S_0, \dots, S_t$ with $|S_i| \geq al+h$ for $i \in [t] \cup \{0\}$,
- $(al+h)$-sets $R_0, \dots, R_t$ with $R_i \subseteq S_i$ for $i \in [t] \cup \{0\}$,
- pairwise disjoint sets $Q_0, \dots, Q_t \in F'$ with Q_i being R_i-absorbing for $i \in [t] \cup \{0\}$,
- and pairwise disjoint $(a+1)$-matchings $M_0, \dots, M_t$, with M_i in $H[R_i \cup Q_i]$ for $i \in [t] \cup \{0\}$,

satisfying the property that $S_i = (S_{i-1} \cup Q_{i-1}) \setminus V(M_{i-1})$ for $i \in [t]$

Then $|S_i| = |S_{i-1}| - k$ for $i \in [t]$. Let $S_{t+1} = (S_t \cup Q_t) \setminus V(M_t)$. If $|S_{t+1}| < al+h$ then M is the desired matching. So assume $|S_{t+1}| \geq al+h$ and let R_{t+1} be an $(al+h)$-subset of S_{t+1}. Since $|\mathcal{L}(R_{t+1}) \cap \mathcal{F}'| \geq c'\rho n$ and $t+1 \leq |S|/k+1 \leq c'\rho n - 1$, there exists $Q_{t+1} \in \mathcal{F}' \setminus \{Q_0, \dots, Q_t\}$ such that $H[R_{t+1} \cup Q_{t+1}]$ has a matching M_{t+1} with $|M_{t+1}| = a+1$. This contradicts the maximality of t. ◻

4. Perfect fractional matchings. A *fractional matching* in a k-graph H is a function $w : E(H) \to [0,1]$ such that for any $v \in V(H)$, $\sum_{\{e\in E(H): v\in e\}} w(e) \leq 1$. A fractional matching is called perfect if $\sum_{e\in E(H)} w(e) = |V(H)|/k$. Any subset $I \subseteq V(H)$ that contains no edge of H is called an *independent set*. We use $\alpha(H)$ to denote the size of a largest independent set in the hypergraph H.

In this section, we show that for any reals $0 < \rho \ll \varepsilon$, if an n-vertex k-graph H has $\alpha(H) \leq (1 - 1/k - \varepsilon/5)n$ and $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho n^{k-l}$, then H admits a perfect fractional matching. Note the term $-\rho n^{k-l}$, since the result will be applied to a hypergraph after removing an absorbing matching. (In section 5 (see Lemma 5.3) we show that when $\alpha(H) > (1 - 1/k - \varepsilon/5)n$ and $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho n^{k-l}$, H is close to $H_k^{k-l}(U, W)$.)

We need to consider matchings in the "link" graph of an l-set in a k-graph, which is a $(k-l)$-graph. This is related to the following well-known conjecture of Erdős [6] on matchings in uniform hypegraphs: If F is a k-graph on n vertices and $\nu(F) = s$, then $e(F) \leq \max\{\binom{n}{k} - \binom{n-s}{k}, \binom{ks+1}{k}\}$. Frankl [8] proved that if $n \geq (2s+1)k - s$ then $e(F) \leq \binom{n}{k} - \binom{n-s}{k}$ with $H_k^k(U, W)$ (where $|W| = s$ and $|U| = n - s$) as extremal graphs. Very recently, Frankl and Kupavskii [9] further improved the lower bound to $n \geq (5k/3 - 2/3)s$ for large s.

Ellis, Keller, and Lifshitz [7] recently proved the following stability version of Frankl's result, which we state as follows using our notation: For any $s \in \mathbb{N}$, $\eta > 0$, and $\varepsilon > 0$, there exists $\delta = \delta(s, \eta, \varepsilon) > 0$ such that the following holds. Let $n, k \in \mathbb{N}$ with $k \leq (\frac{1}{2s+1} - \eta)n$. Suppose $H \subseteq \binom{[n]}{k}$ with $\nu(H) \leq s$ and $e(H) \geq \binom{n}{k} - \binom{n-s}{k} - \delta\binom{n-s}{k-1}$. Then there exists $W \in \binom{[n]}{s}$ such that $|E(H) \setminus E(H_k^k(U, W))| < \varepsilon\binom{n-s}{k}$.

The lower bound on $e(H)$ in the above result of Ellis, Keller, and Lifshitz is too large for our purpose. Using LP duality we only need to consider "stable" hypergraphs and for such hypergraphs we can improve the bound on $e(H)$ to $\binom{n}{k} - \binom{n-s}{k} - \xi n^k$.

For subsets $e = \{u_1, \ldots, u_k\}, f = \{v_1, \ldots, v_k\} \subseteq [n]$ with $u_i < u_{i+1}$ and $v_i < v_{i+1}$ for $i \in [k-1]$, we write $e \leq f$ if $u_i \leq v_i$ for all $i \in [k]$. A hypergraph $H \subseteq \binom{[n]}{k}$ is said to be *stable* if, for $e, f \in \binom{[n]}{k}$ with $e \leq f$, $f \in E(H)$ implies $e \in E(H)$. Our proof of a stability version of Frankl's theorem for stable hypergraphs uses the same ideas as in [8]. The following result from [8] is an extension of Katona's intersection shadow theorem [13].

LEMMA 4.1. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) = s$. Then $s|\partial\mathcal{F}| \geq |\mathcal{F}|$, where $\partial\mathcal{F}$ is the shadow of $\mathcal{F}$, defined by*

$$\partial\mathcal{F} := \left\{G \in \binom{[n]}{k-1} : G \subseteq F \text{ for some } F \in \mathcal{F}\right\}.$$

We can now state and prove the following stability version of Frankl's result on matchings for stable hypergraphs. Note that we allow $k = 1$.

LEMMA 4.2. *Let k be a positive integer, and let c and ξ be constants such that $0 < c < 1/(2k)$ and $0 < \xi \leq (1 + 18(k-1)!/c)^{-2}$. Let n, m be positive integers such that n is sufficiently large and $cn \leq m \leq n/(2k)$. Let H be a k-graph with vertex set $[n]$ such that H is stable and $\nu(H) \leq m$. If $e(H) > \binom{n}{k} - \binom{n-m}{k} - \xi n^k$, then H is $\sqrt{\xi}$-close to $H_k^k([n] \setminus [m], [m])$.*

Proof. Suppose $e(H) > \binom{n}{k} - \binom{n-m}{k} - \xi n^k$. When $k = 1$, each edge of H consists of a single vertex. In this case, since $e(H) > m - \xi n \geq m - \sqrt{\xi}n$ and because H is stable and $e(H) = \nu(H) \leq m$, we have that H is $\sqrt{\xi}$-close to $H_1^1([n] \setminus [m], [m])$.

Thus, we may assume $k \geq 2$. To show that H is close to $H_k^k([n] \setminus [m], [m])$, we bound $e(H - [m])$ (as edges in $H - [m]$ are not in $H_k^k([n] \setminus [m], [m])$). Since H is stable, the vertex $m+1$ has the maximum degree in $H - [m]$. So

$$e(H - [m]) \leq \frac{(n-m)}{k}|\{e \in E(H - [m]) : m+1 \in e\}|.$$

Hence, our objective is to bound the size of

$$\mathcal{F}(\{m+1\}) := \{e \in E(H - [m]) : m+1 \in e\}.$$

Let

$$\sigma = \frac{2\xi(k-1)!}{c}.$$

First, we may assume that

(1) $|\mathcal{F}(\{m+1\})| \geq 9k\sigma n^{k-1}$.

For, suppose $|\mathcal{F}(\{m+1\})| < 9k\sigma n^{k-1}$, then

$$e(H-[m]) \leq \frac{(n-m)}{k}|\mathcal{F}(\{m+1\})| < 9\sigma n^k.$$

Thus

$$\begin{aligned}
&\left|E\big(H_k^k([n]\setminus[m],[m])\big)\setminus E(H)\right| \\
&= e\big(H_k^k([n]\setminus[m],[m])\big) - \big(e(H) - e(H-[m])\big) \\
&< \left(\binom{n}{k} - \binom{n-m}{k}\right) - \left(\binom{n}{k} - \binom{n-m}{k} - \xi n^k - 9\sigma n^k\right) \\
&= \xi n^k + 9\cdot\frac{2\xi(k-1)!}{c}n^k \\
&\leq \sqrt{\xi}n^k,
\end{aligned}$$

as $\xi \leq (1+18(k-1)!/c)^{-2}$. That is, H is $\sqrt{\xi}$-close to $H_k^k([n]\setminus[m],[m])$, and the assertion of the lemma holds. So we may assume that (1) holds.

To proceed further, we extend the notation $\mathcal{F}(\{m+1\})$ to all $Q \subseteq [m+1]$ by letting

$$\mathcal{F}(Q) = \{e \in E(H) \ : \ e \cap [m+1] = Q\}.$$

Note that $|\mathcal{F}(Q)| \leq \binom{n-(m+1)}{k-|Q|} = \binom{n-m-1}{k-|Q|}$. Also note that, since H is stable, $|\mathcal{F}(\{m+1\})| \geq |\partial\mathcal{F}(\emptyset)|$. So Lemma 4.1 gives

$$m|\mathcal{F}(\{m+1\})| \geq m|\partial\mathcal{F}(\emptyset)| \geq |\mathcal{F}(\emptyset)|.$$

We claim that

(2) $\left(\sum_{i=1}^{m+1}|\mathcal{F}(\{i\})|\right) + m|\mathcal{F}(\{m+1\})| > m\binom{n-m}{k-1}(1-\sigma)$.

To prove (2), it suffices to show $|\mathcal{F}(\emptyset)| + \sum_{i=1}^{m+1}|\mathcal{F}(\{i\})| > m\binom{n-m}{k-1}(1-\sigma)$. Note that

$$\sum_{Q\subseteq[m+1],|Q|\geq 2}|\mathcal{F}(Q)| \leq \sum_{i=2}^{k}\binom{m+1}{i}\binom{n-(m+1)}{k-i}$$

and

$$\begin{aligned}
\binom{n}{k} &= \binom{n-(m+1)}{k} + (m+1)\binom{n-(m+1)}{k-1} + \sum_{i=2}^{k}\binom{m+1}{i}\binom{n-(m+1)}{k-i} \\
&= \binom{n-m}{k} + m\binom{n-(m+1)}{k-1} + \sum_{i=2}^{k}\binom{m+1}{i}\binom{n-(m+1)}{k-i}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|\mathcal{F}(\emptyset)| + \sum_{i=1}^{m+1} |\mathcal{F}(\{i\})| \\
&= e(H) - \sum_{Q\subseteq[m+1],|Q|\geq 2} |\mathcal{F}(Q)| \\
&> \binom{n}{k} - \binom{n-m}{k} - \xi n^k - \sum_{i=2}^{k} \binom{m+1}{i}\binom{n-(m+1)}{k-i} \\
&= m\binom{n-(m+1)}{k-1} - \xi n^k \\
&> m\binom{n-m}{k-1}(1-\sigma) \quad \text{(since } cn \leq m \leq n/(2k) \text{ and } n \text{ large).}
\end{aligned}$$

This proves (2).

Let $t = \lceil (2+1/k)m \rceil$. Since $n \geq 2km$ and $m \geq cn$ (where n is sufficiently large),

$$n-(m+1) \geq 2km-(m+1) = (2+1/(k-1))m(k-1)-1 > t(k-1).$$

Let $\mathcal{M} = \{f_1, \ldots, f_t\}$ be t pairwise disjoint $(k-1)$-subsets of $[n] \setminus [m+1]$ chosen uniformly at random. Let $\mathcal{F}_i := \{e \setminus \{i\} : e \in \mathcal{F}(\{i\})\}$ for $i \in [m+1]$. Then $\mathcal{F}_{m+1} \subseteq \mathcal{F}_m \subseteq \cdots \subseteq \mathcal{F}_1$ (since H is stable) and, for each fixed pair i, j,

$$\mathbb{P}(f_j \in \mathcal{F}_i) = \frac{|\mathcal{F}_i|}{\binom{n-(m+1)}{k-1}}.$$

Let

$$x_i = \begin{cases} 1, & f_i \in \mathcal{F}_{m+1}, \\ 0, & f_i \notin \mathcal{F}_{m+1}, \end{cases}$$

and let $p = \mathbb{P}(x_i = 1)$ (which is the same for all $i \in [t]$). Now $|\mathcal{F}_{m+1}| = p\binom{n-(m+1)}{k-1}$. So by (1), we have

(3) $p > 9k\sigma$.

We claim that

(4) for $1 \leq i < j \leq t$, $\mathbb{P}(x_i x_j = 1) \leq \left(1+\frac{1}{4k}\right)p^2$.

This is because

$$\begin{aligned}
\mathbb{P}(x_i x_j = 1) &= \mathbb{P}(x_j = 1 | x_i = 1)\mathbb{P}(x_i = 1) \\
&\leq \frac{|\mathcal{F}_{m+1}|}{\binom{n-(m+1)-(k-1)}{k-1}} \frac{|\mathcal{F}_{m+1}|}{\binom{n-(m+1)}{k-1}} \\
&= \frac{\binom{n-(m+1)}{k-1}}{\binom{n-(m+1)-(k-1)}{k-1}} \cdot p^2 \\
&\leq \left(1+\frac{1}{4k}\right)p^2,
\end{aligned}$$

as $n-(m+1) \geq (1-1/(2k))n-1$ and n is large. This completes the proof of (4).

Define a bipartite graph G with partition sets $\mathcal{M}$ and $\{\mathcal{F}_1, \dots, \mathcal{F}_{m+1}\}$, where $f_j \in \mathcal{M}$ is adjacent to $\mathcal{F}_i$ if and only if $f_j \in \mathcal{F}_i$. Note that a matching of size $m+1$ in G gives rise to a matching of size $m+1$ in H. Thus, $\nu(G) \leq m$. So by a theorem of König, G has a vertex cover of size m, say T. Let $x = |T \cap \mathcal{M}|$, then $|T \cap \{\mathcal{F}_1, \dots, \mathcal{F}_{m+1}\}| = m - x$. Since $\mathcal{F}_{m+1} \subseteq \mathcal{F}_m \subseteq \cdots \subseteq \mathcal{F}_1$, $d_G(f_j) = m+1$ for $f_j \in \mathcal{F}_{m+1}$; so $f_j \in T$ for all $f_j \in \mathcal{F}_{m+1}$. Hence $0 \leq b \leq x \leq m$, where $b := |\mathcal{M} \cap \mathcal{F}_{m+1}| = \sum_{i=1}^t x_i$. So $pt = \mathbb{E}(b) \leq m \leq t/(2+1/k)$. This implies

(5) $p \leq 1/(2+1/k) < 1/2$.

Moreover,

$$\sum_{i=1}^{m+1} |\mathcal{M} \cap \mathcal{F}_i| = e(G) \leq t(m-x) + x((m+1)-(m-x)) = x^2 - (t-1)x + mt.$$

Thus, letting $h(x,b) := x^2 - (t-1)x + mt + mb$, we have

$$\begin{aligned}
\mathbb{E}(h(x,b)) &\geq \mathbb{E}\left(m|\mathcal{M} \cap \mathcal{F}_{m+1}| + \sum_{i=1}^{m+1} |\mathcal{M} \cap \mathcal{F}_i|\right) \\
&= mt\frac{|\mathcal{F}_{m+1}|}{\binom{n-(m+1)}{k-1}} + \sum_{i=1}^{m+1} t\frac{|\mathcal{F}_i|}{\binom{n-(m+1)}{k-1}} \\
&= \frac{t}{\binom{n-(m+1)}{k-1}}\left(m|\mathcal{F}(\{m+1\})| + \sum_{i=1}^{m+1} |\mathcal{F}(\{i\})|\right) \\
&> mt(1-\sigma) \quad \text{(by (2))}.
\end{aligned}$$

Next we obtain an upper bound on $\mathbb{E}(h(x,b))$. Using the convexity of $h(x,b)$ (as a function of x over the interval $[b,m]$) and the fact that $h(b,b) - h(m,b) = (t-1-m-b)(m-b) \geq 0$, we have

$$h(x,b) \leq \max\{h(b,b), h(m,b)\} = h(b,b) = b^2 - (t-1)b + mt + mb.$$

Thus,

$$\begin{aligned}
\mathbb{E}(h(x,b)) &\leq \mathbb{E}(b^2 - (t-1)b + mt + mb) \\
&= \mathbb{E}\left(\left(\sum_{i=1}^t x_i\right)^2 - (t-1-m)\left(\sum_{i=1}^t x_i\right) + mt\right) \\
&\leq \left(1 + \frac{1}{4k}\right)p^2(t^2-t) + pt - (t-1-m)pt + mt \quad \text{(by (4))}.
\end{aligned}$$

Hence, combining the above bounds on $\mathbb{E}(h(x,b))$, we have

$$\left(1+\frac{1}{4k}\right)p^2(t^2-t) + pt - (t-1-m)pt + mt > mt(1-\sigma).$$

Thus,

$$\begin{aligned}
\sigma mt &> pt\left(t - m - \left(1+\frac{1}{4k}\right)pt - 2 + \left(1+\frac{1}{4k}\right)p\right)\\
&> pt\left(\left(1-\left(1+\frac{1}{4k}\right)p\right)t - m - 2\right)\\
&\ge pt\left(\left(\left(1-\frac{1}{2}\left(1+\frac{1}{4k}\right)\right)\left(2+\frac{1}{k}\right)-1\right)m-2\right) \text{ (by (5) and the definition of } t)\\
&= pt\left(\frac{2k-1}{8k^2}m-2\right)\\
&> ptm/(9k) \quad \text{(since } m \ge cn \text{ and } n \text{ is large)}.
\end{aligned}$$

Thus, $p < 9k\sigma$, contradicting (3). Hence H must be $\sqrt{\xi}$-close to $H_k^k([n]\setminus[m],[m])$. ◻

Remark. In the proof of Lemma 4.2 we require $m \le n/(2k)$ (e.g., when we define t and $\mathcal{M}$ before (3)). We will see in section 6, we can replace it with $n/2-1$ when $k=3$ and $l=1$.

For a hypergraph H, let

$$\nu^*(H) = \max\left\{\sum_{e\in E(H)} w(e) : w \text{ is a fractional matching in } H\right\}.$$

A *fractional vertex cover* of H is a function $w : V(H) \to [0,1]$ such that, for each $e \in E$, $\sum_{v\in e} w(v) \ge 1$. Let

$$\tau^*(H) = \min\left\{\sum_{v\in V(H)} w(v) : w \text{ is a fractional vertex cover of } H\right\}.$$

Then the strong duality theorem of linear programming gives

$$\nu^*(H) = \tau^*(H).$$

We conclude this section by proving the existence of a perfect fractional matching in a uniform hypergraph whose independence number is not too large.

LEMMA 4.3. *Let k,l be integers with $k \ge 3$ and $k/2 \le l < k$, and let ε,ξ be positive reals with $\xi < (\varepsilon/5)^2(3k)^{-4(k-l)}$. Let n be a positive integer such that n is sufficiently large and $n \in k\mathbb{Z}$. Let H be a k-graph of order n such that $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-n/k}{k-l} - \xi n^{k-l}$ and $\alpha(H) \le (1-1/k-\varepsilon/5)n$. Then H contains a perfect fractional matching.*

Proof. For convenience, let $V(H)=[n]$. Let ω be a minimum fractional vertex cover of H and we may assume that $\omega(1) \ge \omega(2) \ge \cdots \ge \omega(n)$. Let $E' = \{e \in \binom{[n]}{k} : e \notin E(H) \text{ and } \sum_{i\in e}\omega(i) \ge 1\}$ and let H' be obtained from H by adding the edges in E'. Then H' is stable and $\tau^*(H') = \tau^*(H)$. Thus $\nu^*(H) = \nu^*(H') \ge \nu(H')$, and it suffices to show that $\nu(H') = n/k$, i.e., H' contains a perfect matching.

Let $S = [n]\setminus[n-l]$, and let G be the hypergraph with $V(G)=[n]$ and $E(G) = N_{H'}(S)$, which is a $(k-l)$-graph on $[n]$. Since H' is stable, G is also stable. We may assume that

(1) $\nu(G) \le n/k - 1$.

For, otherwise, let $f_1, \dots, f_{n/k}$ be a matching in G. Now $[n] \setminus \left(\bigcup_{i=1}^{n/k} f_i\right)$ is a set of size $(n/k)l$ and, hence, can be partitioned into l-sets, say $S_1, \dots, S_{n/k}$. Since H' is stable and $S \cup f_i \in E(H')$ for $i \in [n/k]$, we have $S_i \cup f_i \in E(H')$ for $i \in [n/k]$. Hence, $\{S_i \cup f_i \ : \ i \in [n/k]\}$ is a perfect matching in H'. Hence, we may assume (1).

We may also assume that

(2) $l \le k-2$.

For, suppose $l = k-1$. Then G is a 1-graph. Since H' is stable and $e(G) \ge \delta_{k-1}(H) \ge n/k - \lceil \xi n \rceil$, the first $n/k - \lceil \xi n \rceil$ vertices of G are edges of G.

Note that $H' - [n/k - \lceil \xi n \rceil]$ has $n - n/k + \lceil \xi n \rceil$ vertices. Since $\alpha(H) \le (1 - 1/k - \varepsilon/5)n$, $H' - [n/k - \lceil \xi n \rceil]$ has an edge. In fact, since $\xi < (\varepsilon/5)^2(3k)^{-4(k-l)}$, we can greedily find pairwise disjoint edges $f_1, \dots, f_{\lceil \xi n \rceil}$ in $H' - [n/k - \lceil \xi n \rceil]$. Since

$$n - (n/k - \lceil \xi n \rceil) - \lceil \xi n \rceil k = (k-1)(n/k - \lceil \xi n \rceil),$$

we can partition $[n] \setminus ([n/k - \lceil \xi n \rceil] \cup \bigcup_{i=1}^{\lceil \xi n \rceil} f_i)$ into $(k-1)$-sets $S_1, \dots, S_{n/k - \lceil \xi n \rceil}$. Now $S_i \cup \{i\}$, $i \in [n/k - \lceil \xi n \rceil]$, form a matching in H'. These edges and $\{f_1, \dots, f_{\lceil \xi n \rceil}\}$ form a perfect matching in H'. So we may assume (2).

Let $\eta = \varepsilon/(5k)$ and let $t = n/k - \lfloor \eta n \rfloor$. For $i \in [n]$, we use $d_G(i)$ to denote the degree of i in G. We claim that

(3) $d_G(t) > \binom{n-1}{k-l-1} - \binom{n/(2k)}{k-l-1}$.

For suppose $d_G(t) \le \binom{n-1}{k-l-1} - \binom{n/(2k)}{k-l-1}$. Since H' is stable, $d_G(i) \le \binom{n-1}{k-l-1} - \binom{n/(2k)}{k-l-1}$ for $t \le i \le n/k$. Note that the degree of t in $H_{k-l}^{k-l}([n] \setminus [n/k], [n/k])$ is $\binom{n-1}{k-l-1}$. Thus,

$$\begin{aligned}
&\left| E\left(H_{k-l}^{k-l}\left([n] \setminus [n/k], [n/k]\right)\right) \setminus E(G) \right| \\
&\ge \frac{1}{k-l} \left(\sum_{i=t}^{n/k} \left(d_{H_{k-l}^{k-l}([n]\setminus[n/k],[n/k])}(i) - d_G(i) \right) \right) \\
&\ge \frac{1}{k-l}(n/k - t + 1) \binom{n/(2k)}{k-l-1} \\
&> \frac{1}{k-l} \eta n (3k)^{-(k-l-1)} \binom{n}{k-l-1} \\
&> \sqrt{\xi} n^{k-l},
\end{aligned}$$

as $\xi < (\varepsilon/5)^2(3k)^{-4(k-l)}$.

Hence G is not $\sqrt{\xi}$-close to $H_{k-l}^{k-l}([n] \setminus [n/k], [n/k])$. However, since G is stable and $n/k \le n/(2(k-l))$ (as $l \ge k/2$), we may apply Lemma 4.2 with $n/k, k-l, \xi$ as m, k, ξ, respectively. So $\nu(G) \ge n/k$, contradicting (1) and completing the proof of (3).

Note that $H' - [t]$ has $n - n/k + \lfloor \eta n \rfloor$ vertices. Since $\alpha(H) \le (1 - 1/k - \varepsilon/5)n$, $H' - [t]$ has an edge. In fact, since $\varepsilon n = 5k\eta n$, $H' - [t]$ has $\lfloor \eta n \rfloor$ pairwise disjoint edges, say $f_1, \dots f_{\lfloor \eta n \rfloor}$. Let $T = \bigcup_{i=1}^{\lfloor \eta n \rfloor} f_i$.

Next we find disjoint edges $e_1, \dots, e_t$ of G such that $|e_i \cap [t]| = 1$ and $e_i \cap T = \emptyset$ for all $i \in [t]$. Suppose for some $s \in [t-1]$ we have found pairwise disjoint edges $e_1, \dots, e_s$ of G such that, for $i \in [s]$, $e_i \cap [t] = \{i\}$ and $e_i \cap T = \emptyset$. The number of edges of G containing $s+1$ and intersecting $T \cup ([t] \setminus \{s+1\}) \cup (\bigcup_{i=1}^{s} e_i)$ is at most $\binom{n-1}{k-l-1} - \binom{n-|T|-t-(k-l)s}{k-l-1}$. Note that $n - |T| - t - (k-l)s \ge n/(2k)$, as $l \ge k/2$. Hence, by (3), there exists $e_{s+1} \in E(G)$ such that $e_{s+1} \cap [t] = \{s+1\}$, $e_{s+1} \cap T = \emptyset$, and e_{s+1} is disjoint from $\bigcup_{i=1}^{s} e_i$.

Since $t = n/k - \lfloor \eta n \rfloor$, $(H' - T) - \bigcup_{i=1}^t e_i$ has exactly tl vertices (as $|e_i \cap [t]| = 1$ for $i \in [t]$). Partition the vertices in $(H - T) - \bigcup_{i=1}^t e_i$ into pairwise disjoint l-sets $S_1, \ldots, S_t$. Then, since H' is stable, $S_i \cup e_i \in E(H')$ for $i \in [t]$. Hence, $\{f_i : i \in [\lfloor \eta n \rfloor]\} \cup \{S_j \cup e_j : j \in [t]\}$ is a perfect matching in H'. □

Remark. When we apply Lemma 4.2 in the end of the proof of (3), we require $l \geq k/2$ so that $n/k \leq n(2(k-l))$ (which amounts to $m \leq n/(2k)$ in Lemma 4.2). This is not necessary when $k = 3$ and $l = 1$, as we can use Lemma 6.3 (see section 6) which is the same as Lemma 4.2 except with $m \leq n/(2k) = n/4$ replaced by $m \leq n/2 - 1$.

5. Almost perfect matchings. To complete the proof of Theorem 1.1, we need to consider n-vertex k-graphs H that are not close to $H_k^{k-l}(U, W)$ for any partition of $V(H)$ into U, W with $|W| = m$. We first use the absorbing lemma in section 3 to find a small matching M_a in H such that for any small subset $S \subseteq V(H)$, $H[V(M_a) \cup S]$ has a nearly perfect matching. We then find an almost perfect matching in $H - V(M_a)$ (see Lemma 5.7), and use M_a to absorb the unmatched vertices. To find this almost perfect matching in $H - V(M_a)$, we will find an almost regular subgraph of H with bounded maximum 2-degree, so that the following result of Frankl and Rödl [10] can be applied. For any positive integer l, we use $\Delta_l(H)$ to denote the maximum l-degree of H.

LEMMA 5.1 (Frankl and Rödl, 1985). *For every integer $k \geq 2$ and any real $\varepsilon > 0$, there exist $\tau = \tau(k, \varepsilon)$ and $d_0 = d_0(k, \varepsilon)$ such that, for every $n \geq D \geq d_0$ the following holds: Every k-graph on n vertices with $(1 - \tau)D < d_H(v) < (1 + \tau)D$ and $\Delta_2(H) < \tau D$ contains a matching covering all but at most εn vertices.*

In order to find a subgraph in a k-graph satisfying conditions in Lemma 5.1, we use the two-round randomization technique in [1]. The only difference is that in the first round, we also need to bound the independence number of the subgraph (in order to deal with hypergraphs not close to $H_k^{k-l}(U, W)$). Here we use the hypergraph container result of Balogh, Morris, and Samoti [3]. (A similar result is proved independently by Saxton and Thomason [22].) To state that result, we need additional terminology.

A family $\mathcal{F}$ of subsets of a set V is said to be *increasing* if, for any $A \in \mathcal{F}$ and $B \subseteq V$, $A \subseteq B$ implies $B \in \mathcal{F}$. Let H be a hypergraph. We use $v(H)$ and $e(H)$ to denote the number of vertices and number of edges of H, respectively. Let $\mathcal{I}(H)$ denote the set of all independent sets in H. Let $\varepsilon > 0$, and let $\mathcal{F}$ be a family of subsets of $V(H)$. We say that H is *$(\mathcal{F}, \varepsilon)$-dense* if $e(H[A]) \geq \varepsilon e(H)$ for every $A \in \mathcal{F}$. We use $\overline{\mathcal{F}}$ to denote the family consisting of subsets of $V(H)$ not in $\mathcal{F}$.

LEMMA 5.2 (Balogh, Morris, and Samotij, 2015). *For every $k \in \mathbb{N}$ and all positive c and ε, there exists a positive constant C such that the following holds. Let H be a k-graph and let $\mathcal{F}$ be an increasing family of subsets of $V(H)$ such that $|A| \geq \varepsilon v(H)$ for all $A \in \mathcal{F}$. Suppose that H is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0, 1)$ is such that, for every $l \in [k]$,*

$$\Delta_l(H) \leq cp^{l-1} \frac{e(H)}{v(H)}.$$

Then there exist a family $\mathcal{S} \subseteq \binom{V(H)}{\leq Cpv(H)}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(H) \to \mathcal{S}$ such that, for every $I \in \mathcal{I}(H)$,

$$g(I) \subseteq I \quad \text{and} \quad I \setminus g(I) \subseteq f(g(I)).$$

The next lemma says that, if an n-vertex k-graph H is not ε-close to $H_k^{k-l}(U,W)$ and $\delta_l(H) \geq \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho' n^{k-l}$ then H is $(\mathcal{F}, \varepsilon')$-dense.

LEMMA 5.3. *Let k, l be integers with $k \geq 2$ and $l \in [k-1]$. Let $0 < \varepsilon \ll 1$, $\rho' \leq \varepsilon/8$, and $0 < \mu \leq \varepsilon/40$. Let m, n be sufficiently large integers such that $n/k - \mu n \leq m \leq n/k$. Suppose H is a k-graph with order n such that $\delta_l(H) > \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho' n^{k-l}$, and H is not ε-close to $H_k^{k-l}(U,W)$ for any partition of $V(H)$ into U, W with $|W| = m$. Then H is $(\mathcal{F}, \varepsilon/(2k!))$-dense, where $\mathcal{F} = \{A \subseteq V(H) : |A| \geq (1 - 1/k - \varepsilon/4)n\}$.*

Proof. Suppose to the contrary that there exists $A \subseteq V(H)$ such that $|A| \geq (1 - 1/k - \varepsilon/4)n$ and $e(H[A]) \leq \varepsilon e(H)/(2k!)$. By removing vertices if necessary, we may choose A such that $|V(H) \setminus A| \geq m$ (as $m \leq n/k$). Let $W \subseteq V(H) \setminus A$ such that $|W| = m$. For convenience, let $B = V(H) \setminus (W \cup A)$. Then

$$|B| \leq n - m - (1 - 1/k - \varepsilon/4)n \leq \varepsilon n/4 + n/k - (1/k - \mu)n \leq 11\varepsilon n/40.$$

Let $U = V(H) \setminus W$ and $H_0 = H_k^{k-l}(U,W)$. We derive a contradiction by showing that $|E(H_0) \setminus E(H)| < \varepsilon n^k$.

Note that, for each $f \in E(H_0) \setminus E(H)$, we have $1 \leq |f \cap W| \leq k - l$ (by definition of H_0), so $|f \cap B| > 0$ or $|f \cap A| \geq l$. Thus

$$|E(H_0) \setminus E(H)| \leq |\{f \in E(H_0) : |f \cap B| > 0\}| + |\{f \in E(H_0) \setminus E(H) : |f \cap A| \geq l\}|.$$

It is easy to see that

$$|\{f \in E(H_0) : |f \cap B| > 0\}| \leq |B||W|n^{k-2} \leq (11\varepsilon n/40)(n/k)n^{k-2} = \frac{11\varepsilon}{40k} n^k.$$

Next, we bound $|\{f \in E(H_0) \setminus E(H) : |f \cap A| \geq l\}|$. Fix an arbitrary l-set $S \subseteq A$. Note that

$$|\{f \in E(H) : S \subseteq f \text{ and } f \cap B \neq \emptyset\}| \leq |B| n^{k-l-1} \leq \frac{11\varepsilon}{40} n^{k-l}.$$

For any $f \in E(H)$ and $S \subseteq f$, we have $f \cap B \neq \emptyset$, or $f \subseteq A$, or $f \in E(H_0)$. So

$$\begin{aligned}
&|\{f \in E(H) : S \subseteq f \text{ and } f \in E(H_0)| \\
&\geq d_H(S) - |\{f \in E(H) : S \subseteq f \text{ and } f \cap B \neq \emptyset\}| - |\{f \in E(H) : S \subseteq f \text{ and } f \subseteq A\}| \\
&\geq d_H(S) - \frac{11\varepsilon}{40} n^{k-l} - d_{H[A]}(S).
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\{f \in E(H_0) \setminus E(H) : |f \cap A| \geq l\}| \\
&\leq \sum_{S \in \binom{A}{l}} |\{f \in E(H_0) \setminus E(H) : S \subseteq f\}| \\
&\leq \sum_{S \in \binom{A}{l}} (d_{H_0}(S) - |\{f \in E(H) : f \in E(H_0) \text{ and } S \subseteq f\}|) \\
&\leq \sum_{S \in \binom{A}{l}} \left(d_{H_0}(S) - d_H(S) + \frac{11\varepsilon}{40} n^{k-l} + d_{H[A]}(S) \right).
\end{aligned}$$

Note that for $S \in \binom{A}{l}$, $d_{H_0}(S) = \binom{n-l}{k-l} - \binom{n-l-m}{k-l}$, so $d_{H_0}(S) - d_H(S) < \rho' n^{k-l}$ by the assumption on $\delta_l(H)$. Hence,

$$\begin{aligned} |E(H_0) \setminus E(H)| &< \frac{11\varepsilon}{40k} n^k + \binom{|A|}{l} \left(\rho' + \frac{11\varepsilon}{40}\right) n^{k-l} + \sum_{S \in \binom{A}{l}} d_{H[A]}(S) \\ &\le \left(\frac{11\varepsilon}{40k} + \rho' + \frac{11\varepsilon}{40}\right) n^k + \binom{k}{l} e(H[A]) \\ &\le \left(\frac{11}{120} + \frac{1}{8} + \frac{11}{40}\right) \varepsilon n^k + \binom{k}{l} \frac{\varepsilon n^k}{2k!} \quad \text{(since } k \ge 3 \text{ and } \rho' \le \varepsilon/8) \\ &< \varepsilon n^k, \end{aligned}$$

a contradiction. ∎

We now use Lemma 5.2 to show that one can control, with high probability, the independence number of a subgraph of a k-graph induced by a random subset of vertices.

LEMMA 5.4. *Let c, ε', α be positive reals and let k, n be positive integers. Let H be an n-vertex k-graph such that $e(H) \ge cn^k$ and $e(H[S]) \ge \varepsilon' e(H)$ for all $S \subseteq V(H)$ with $|S| \ge \alpha n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H independently and uniformly at random with probability $n^{-0.9}$. Then, for any positive $\gamma \ll \alpha$, the independence number of $H[R]$ is at most $(\alpha + \gamma + o(1))n^{0.1}$ with probability at least $1 - n^{O(1)} e^{-\Omega(n^{0.1})}$.*

Proof. Define $\mathcal{F} := \{A \subseteq V(H) \ : \ e(H[A]) \ge \varepsilon' e(H) \text{ and } |A| \ge \varepsilon' n\}$. Then $\mathcal{F}$ is an increasing family, and H is $(\mathcal{F}, \varepsilon')$-dense. Let $p = n^{-1}$ and $v(H) = n$. Then

$$\Delta_l(H) \le \binom{n}{k-l} \le n^{k-l} \le c^{-1} n^{-l} e(H) = c^{-1} p^{l-1} \frac{e(H)}{v(H)}.$$

Thus by Lemma 5.2, there exist a constant C (depending only on ε' and c), a family $\mathcal{S} \subseteq \binom{V(H)}{\le C}$, a function $f : \mathcal{S} \to \overline{\mathcal{F}}$, and a family $\mathcal{T} := \{F \cup S : F \in f(\mathcal{S}), S \in \mathcal{S}\}$, such that every independent set in H is contained in some $T \in \mathcal{T}$. Since $\mathcal{S} \subseteq \binom{V(H)}{\le C}$, $|\mathcal{S}| \le Cn^C$, and, hence,

$$|\mathcal{T}| = |\mathcal{S}||f(\mathcal{S})| \le |\mathcal{S}|^2 \le C^2 n^{2C}.$$

We claim that $|T| < \alpha n + C$ for all $T \in \mathcal{T}$. To see this, let $T = F \cup S$ for some $F \in f(\mathcal{S})$ and $S \in \mathcal{S}$. By definition, $F \in \overline{\mathcal{F}}$ and, hence, $e(H[F]) < \varepsilon' e(H)$. Since $e(H[S]) \ge \varepsilon' e(H)$ for any $S \subseteq V(H)$ with $|S| \ge \alpha n$, we have $|F| < \alpha n$. Therefore, $|T| \le |F| + |S| < \alpha n + C$.

We wish to apply Lemma 3.1 and, hence, we need to make sets in $\mathcal{T}$ slightly larger. Take an arbitrary map $h : \mathcal{T} \to \binom{V(H)}{\lfloor \alpha n + C \rfloor}$ such that $T \subseteq h(T)$ for all $T \in \mathcal{T}$, and let $\mathcal{T}' = h(\mathcal{T})$. Then

$$|\mathcal{T}'| \le |\mathcal{T}| \le |\mathcal{S}|^2 \le C^2 n^{2C}.$$

Note that for each fixed $T' \in \mathcal{T}'$, we have $|R \cap T'| \sim Bi\left(|T'|, n^{-0.9}\right)$ and $\mathbb{E}(|R \cap T'|) = n^{-0.9}|T'| = \lfloor \alpha n + C \rfloor n^{-0.9}$. We apply Lemma 3.1 to $|R \cap T'|$ by taking $\lambda = \gamma n^{0.1}$, where γ is fixed and $\gamma \ll \alpha$. Then

$$\mathbb{P}\left(\left||R \cap T'| - n^{-0.9}|T'|\right| \ge \lambda\right) \le e^{-\Omega(\lambda^2/(n^{-0.9}|T'|))} = e^{-\Omega(n^{0.1})}.$$

So with probability at most $e^{-\Omega(n^{0.1})}$, we have $|R \cap T'| \geq n^{-0.9}|T'| + \lambda$. Hence, $|R \cap T'| \geq (\alpha + \gamma + C/n)n^{0.1}$ with probability at most $e^{-\Omega(n^{0.1})}$.

Therefore, with probability at most $C^2 n^{2C} e^{-\Omega(n^{0.1})}$ (from the union bound), there exists some $T' \in \mathcal{T}'$ such that $|R \cap T'| \geq (\alpha + \gamma + C/n)n^{0.1}$. Hence, with probability at least $1 - C^2 n^{2C} e^{-\Omega(n^{0.1})}$, $|R \cap T'| < (\alpha + \gamma + C/n)n^{0.1}$ for all $T' \in \mathcal{T}'$.

It remains to show that, conditioning on $|R \cap T'| < (\alpha + \gamma + C/n)n^{0.1}$ for all $T' \in \mathcal{T}'$, $|J| \leq (\alpha + \gamma + C/n)n^{0.1}$ for every independent set J in $H[R]$. Since such J is also an independent set in H, there exist $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $J \subseteq T \subseteq T'$. Thus $J \subseteq R \cap T'$ and $|J| \leq |R \cap T'| < (\alpha + \gamma + C/n)n^{0.1}$.

Thus $\alpha(H[R]) \leq (\alpha + \gamma + C/n)n^{0.1}$ with probability at least $1 - C^2 n^{2C} e^{-\Omega(n^{0.1})}$. ◻

The following result is the outcome of the first round of the two-round randomization procedure in [1]. We summarize this round as a lemma (see the proof of Claim 4.1 in [1]) and outline a proof, since we need to make some small adjustments. Here we adopt the notation in [1].

LEMMA 5.5. *Let $k > d > 0$ be integers with $k \geq 3$ and let H be a k-graph on n vertices. Let R be chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$ and then arbitrarily deleting less than k vertices so that $|R| \in k\mathbb{Z}$. Take $n^{1.1}$ independent copies of R and denote them by R^i, $1 \leq i \leq n^{1.1}$. For each $S \subseteq V(H)$ with $|S| \leq k$, let $Y_S := |\{i : S \subseteq R^i\}|$ and $\mathrm{DEG}^i_S := |N_H(S) \cap \binom{R^i}{k-|S|}|$. Then with probability at least $1 - o(1)$, all of the following statements hold:*

(i) *For every $v \in V(H)$, $Y_{\{v\}} = (1 + o(1))n^{0.2}$.*
(ii) *$Y_{\{u,v\}} \leq 2$ for every pair $\{u, v\} \subseteq V(H)$.*
(iii) *$Y_e \leq 1$ for every edge $e \in E(H)$.*
(iv) *For all $i = 1, \ldots, n^{1.1}$, we have $|R^i| = (1 + o(1))n^{0.1}$.*
(v) *If μ, ρ' are constants with $0 < \mu \ll \rho'$, $n/k - \mu n \leq m \leq n/k$. $\delta_d(H) \geq \binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-l}$, then for all $i = 1, \ldots, n^{1.1}$ and all $D \in \binom{V(H)}{d}$ and for any positive real $\xi \geq 2\rho'$, we have*

$$\mathrm{DEG}^i_D > \binom{|R^i| - d}{k - d} - \binom{|R^i| - d - |R^i|/k}{k - d} - \xi |R^i|^{k-d}.$$

Proof. Note that the removal of less than k vertices from each R^i does not affect (i)–(iv). Also note that $|Y_S| \sim Bi(n^{1.1}, n^{-0.9|S|})$ for $S \subseteq V(H)$.

Thus, $\mathbb{E}(|Y_{\{v\}}|) = n^{0.2}$ for $v \in V(H)$, and it follows from Lemma 3.1 that

$$\mathbb{P}\left(\left|Y_{\{v\}} - n^{0.2}\right| > n^{0.15}\right) \leq e^{-\Omega(n^{0.1})}.$$

Hence (i) holds with probability at least $1 - e^{-\Omega(n^{0.1})}$.

To prove (ii), let

$$Z_2 = \left| \left\{ \{u, v\} \in \binom{V(H)}{2} : Y_{\{u,v\}} \geq 3 \right\} \right|,$$

and for $k \geq 3$, let

$$Z_k = \left| \left\{ S \in \binom{V(H)}{k} : Y_S \geq 2 \right\} \right|.$$

Then $\mathbb{E}(Z_2) < n^2 (n^{1.1})^3 (n^{-0.9})^6 = n^{-0.1}$ and $\mathbb{E}(Z_k) < n^k (n^{1.1})^2 (n^{-0.9})^{2k} = n^{2.2-0.8k} \leq n^{-0.2}$ (for $k \geq 3$). By Markov's inequality,

$$\mathbb{P}(Z_2 = 0) > 1 - n^{-0.1} \text{ and, for } k \geq 3,\ \mathbb{P}(Z_k = 0) > 1 - n^{-0.2}.$$

Thus (ii) and (iii) hold with probability at least $1 - n^{-0.1}$ and $1 - n^{-0.2}$, respectively.

By Lemma 3.1 (with $\lambda = n^{0.095}$), we have

$$\mathbb{P}\left(\left||R^i| - n^{0.1}\right| \geq n^{0.095}\right) \leq e^{-\Omega(n^{0.09})}$$

for each i. Thus by the union bound, (iv) holds with probability at least $1 - n^{1.1}e^{-\Omega(n^{0.09})}$.

Next, we prove (v). Conditioning on $\left||R^i| - n^{0.1}\right| < n^{0.095}$ for all i and using the assumption that $0 < \mu \ll \rho'$, $n/k - \mu n \leq m \leq n/k$ and n is large, we have

$$\begin{aligned}&\left(\binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-d}\right)(n^{-0.9})^{k-d}\\ &\geq \binom{|R^i|-d}{k-d} - \binom{|R^i|-d-|R^i|/k}{k-d} - 1.5\rho'|R^i|^{k-d}.\end{aligned}$$

So for each $D \in \binom{V(H)}{d}$ and each fixed R^i,

$$\begin{aligned}\mathbb{E}(\mathrm{DEG}_D^i) &= (1-o(1))d_H(D)(n^{-0.9})^{k-d}\\ &\geq (1-o(1))\left(\binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-d}\right)(n^{-0.9})^{k-d}\\ &\geq (1-o(1))\left(\binom{|R^i|-d}{k-d} - \binom{|R^i|-d-|R^i|/k}{k-d} - 1.5\rho'|R^i|^{k-d}\right)\\ &\geq \binom{|R^i|-d}{k-d} - \binom{|R^i|-d-|R^i|/k}{k-d} - 1.8\rho'|R^i|^{k-d}.\end{aligned}$$

In particular,

$$\mathbb{E}(\mathrm{DEG}_D^i) = \Omega(n^{0.1(k-d)}).$$

We apply Janson's inequality (Theorem 8.7.2 in [2]) to bound the deviation of DEG_D^i. Write $\mathrm{DEG}_D^i = \sum_{e\in N_H(D)} X_e$, where $X_e = 1$ if $e \subseteq R^i$ and $X_e = 0$ otherwise. Then

$$\Delta = \sum_{e\cap f\neq\emptyset} \mathbb{P}(X_e = X_f = 1) \leq \sum_{l=1}^{k-d-1} p^{2(k-d)-l}\binom{n-d}{k-d}\binom{k-d}{l}\binom{n-k}{k-d-l}$$

and, thus, $\Delta = O(n^{0.1(2(k-d)-1)})$. By Janson's inequality, for any $\gamma > 0$,

$$\mathbb{P}(\mathrm{DEG}_D^i \leq (1-\gamma)\mathbb{E}(\mathrm{DEG}_D^i)) \leq e^{-\gamma^2\mathbb{E}(DEG_D^i)/(2+\Delta/\mathbb{E}(DEG_D^i))} = e^{-\Omega(n^{0.1})}.$$

Since $\xi \geq 2\rho'$, by taking γ small, the union bound shows that, with probability at least $1 - n^{d+1.1}e^{-\Omega(n^{0.1})}$,

$$DEG_D^i \geq \binom{|R^i|-d}{k-d} - \binom{|R^i|-d-|R^i|/k}{k-d} - \xi|R^i|^{k-d}.$$

Thus, (v) holds with probability at least

$$(1-n^{1.1}e^{-\Omega(n^{0.09})})(1-n^{d+1.1}e^{-\Omega(n^{0.1})}) > 1 - n^{1.1}e^{-\Omega(n^{0.09})} - n^{d+1.1}e^{-\Omega(n^{0.1})}.$$

Hence, it follows from the union bound that, with probability at least

$$1-e^{-\Omega(n^{0.1})}-n^{-0.1}-n^{-0.2}-n^{1.1}e^{-\Omega(n^{0.09})}-n^{1.1}e^{-\Omega(n^{0.09})}-n^{d+1.1}e^{-\Omega(n^{0.1})} = 1-o(1),$$

(i)–(v) hold. ∎

We summarize the second round randomization in [1] as the following lemma (again, see the proof of Claim 4.1 in [1]).

LEMMA 5.6. *Assume* R^i, $i = 1, \ldots, n^{1.1}$, *satisfy* (i)–(v) *in Lemma* 5.5, *and that each* R^i *has a perfect fractional matching* w^i. *Then there exists a spanning subgraph* H'' *of* H *such that* $\mathrm{d}_{H''}(v) = (1 + o(1))n^{0.2}$ *for each* $v \in V$, *and* $\Delta_2(H'') \le n^{0.1}$.

We are now ready to show that for any H satisfying the conditions of Theorem 1.1 and not ε-close to $H_k^{k-l}(U, W)$, $H - V(M_a)$ has an almost perfect matching, where M_a is an absorbing matching from Lemma 3.2.

LEMMA 5.7. *Let* k, l *be integers with* $k \ge 3$ *and* $k/2 \le l \le k - 1$. *Let* $\rho', \varepsilon, \sigma, \mu$ *be positive reals with* $\rho' < \varepsilon^2 (3k)^{-4(k-l)}/100$ *and* $\mu \le \varepsilon/40$. *Let* n, m *be sufficiently large integers such that* $n/k - \mu n \le m \le n/k$. *Suppose* H *is a* k*-graph on* n *vertices such that* $\delta_l(H) \ge \binom{n-l}{k-l} - \binom{n-l-m}{k-l} - \rho' n^{k-l}$, *and* H *is not* ε*-close to* $H_k^{k-l}(U, W)$ *for any partition of* $V(H)$ *into* U, W *with* $|W| = m$. *Then* H *contains a matching covering all but at most* σn *vertices.*

Proof. By Lemma 5.3, $e(H[S]) \ge (\varepsilon/(2k!))e(H)$ for all $S \subseteq V(H)$ with $|S| \ge \alpha n$, where $\alpha = 1 - 1/k - \varepsilon/4$. Note that

$$e(H) = \delta_0(H) \ge \binom{n}{l} \delta_l(H) / \binom{k}{l} \ge cn^k,$$

where $c > 0$ is a constant and $c \ll 1/\binom{k}{l}$.

Let R, R^i be given as in Lemma 5.5. Then it follows from Lemma 5.4 that, with probability $1 - o(1)$, we have $\alpha(H[R^i]) \le (\alpha + \gamma + o(1))n^{0.1}$ for all i, where $\gamma \ll \alpha$. Additionally, by (v) of Lemma 5.5, $\delta_d(H[R^i]) > \binom{|R^i|-d}{k-d} - \binom{|R^i|-d-|R^i|/k}{k-d} - \xi |R^i|^{(k-d)}$ for any $\xi \ge 2\rho'$. Thus by Lemma 4.3, with probability $1 - o(1)$, for each i, $H[R^i]$ has a perfect fractional matching.

Hence by Lemma 5.6, H has a spanning subgraph H'' such that $\mathrm{d}_{H''}(v) = (1 + o(1))n^{0.2}$ for each $v \in V$, and $\Delta_2(H'') \le n^{0.1}$. Thus we may apply Lemma 5.1 to find a matching covering all but at most σn vertices in H'' for sufficiently large n. ∎

6. Conclusion. In this section, we complete the proof of Theorem 1.1 and discuss some related work.

Proof of Theorem 1.1. By Lemmas 2.1 and 2.3, we may assume that for any $0 < \varepsilon < (8^{k-1} k^{5(k-1)} k!)^{-3}$, H is not ε-close to $H_k^{k-l}(U, W)$ for any partition of $V(H)$ into U, W with $|W| = m$.

By Lemma 3.2, there exist constants $c' = c'(k, l)$ and $\rho = \rho(c', k, l, \varepsilon)$ small enough, satisfying the following property: For positive integers a, h satisfying $h \le l$, $a \le k - l$, and $al \ge a(k-l) + (k-h)$, there exists a matching M_a such that $|M_a| \le 2k\rho n$ and, for any subset $S \subseteq V(H)$ with $|S| \le c' \rho n$, $H[V(M_a) \cup S]$ has a matching covering all but at most $al + h - 1$ vertices.

Now consider $H_1 = H - V(M_a)$. Then $\delta_l(H_1) \ge \delta_l(H) - (2k^2 \rho n) n^{k-l-1}$. Let $\rho_1 = 4k^2 \rho$ and $n_1 = n - k|M_a|$. Then, since n is large enough and $\rho \ll \varepsilon$,

$$\delta_l(H_1) \ge \binom{n_1 - l}{k - l} - \binom{n_1 - l - m}{k - l} - \rho_1 n_1^{k-l}$$

and H_1 is not $(\varepsilon/2)$-close to $H_k^{k-l}(U, W)$ for any partition of $V(H_1)$ into U, W with $|W| = m$.

By Lemma 5.7, H_1 has a matching M_1 such that $|V(H_1)\setminus V(M_1)| < c'\rho n_1 \le c'\rho n$. Then there exists a matching M_2 in $H_2 := H[V(M_a) \cup (V(H_1) \setminus V(M_1))]$ such that $|V(H_2)\setminus V(M_2)| \le al + h - 1$.

Now $M_1 \cup M_2$ is a matching in H covering all but at most $al + h - 1$ vertices of H. By taking $a = \lceil (k-l)/(2l-k) \rceil$ and $h = k - a(2l-k)$, which minimizes $al+h-1$, we see $M_1 \cup M_2$ is a matching in H of size $n/k - 1 - (1 - l/k)\lceil (k-l)/(2l-k) \rceil$. ☐

There are two places in the proof of Theorem 1.1 where we require $l > k/2$: Lemma 3.2 for absorbing matching and Lemma 4.3 for perfect fractional matchings. We do not know how to derive such results for $l \le k/2$. However, for $k = 3$ and $l = 1$, the absorbing part can be taken care of by the following result of Hàn, Person, and Schacht [12].

Lemma 6.1 (Hàn, Person, and Schacht, 2009). *Given any $\gamma > 0$, there exists an integer $n_0 = n_0(\gamma)$ such that the following holds. Suppose that H is a 3-graph on $n \ge n_0$ vertices such that $\delta_1(H) \ge (1/2 + 2\gamma)\binom{n}{2}$. Then there is a matching M in H of size $|M| \le \gamma^3 n/3$ such that for every subset $V' \subseteq V(H) \setminus V(M)$ with $|V'| \in 3\mathbb{Z}$ and $|V'| \le \gamma^6 n$, there is a matching in H covering precisely the vertices in $V' \cup V(M)$.*

For the perfect fractional matching part, we need a result of Berge [4] on maximum matchings. For a graph G, we use $c_o(G)$ to denote the number of odd components in G.

Lemma 6.2 (Berge, 1958). *Let G be a graph on n vertices. Then*

$$\nu(G) = \min\left\{(n - c_o(G - W) + |W|)/2 : W \subseteq V(G)\right\}.$$

Lemma 6.3. *Let c, ρ be constant with $0 < \rho \ll 1$ and $0 < c < 1/2$, and let m, n be positive integers with n sufficiently large and $cn \le m \le n/2 - 1$. Let G be a 2-graph with $V(G) = [n]$ such that $\nu(G) \le m$ and G is stable with respect to the natural order on $[n]$. If $e(G) > \binom{n}{2} - \binom{n-m}{2} - \rho n^2$, then G is $2\sqrt{\rho}$-close to $H_2^2([n] \setminus [m], [m])$.*

Proof. Since G is stable, we have

(1) $N_G(i) \setminus \{j\} \subseteq N_G(j) \setminus \{i\}$ for any $i, j \in [n]$ with $i > j$.

By Lemma 6.2, there exists $W \subseteq V(G)$ such that

$$\nu(G) = (n - c_o(G - W) + |W|)/2.$$

We choose the maximal such W, and let $C_1, \dots, C_q$ denote the components of $G - W$. Without loss of generality, assume $|V(C_1)| \ge \cdots \ge |V(C_q)|$, and let $c_i := |V(C_i)|$ for $i \in [q]$. Then

(2) $q = c_o(G - W)$, i.e., c_i is odd for all $i \in [q]$.

For, otherwise, suppose that c_i is even for some $i \in [q]$. Let $x \in V(C_i)$ and $W' := W \cup \{x\}$. Then $c_o(G - W') \ge c_o(G - W) + 1$. This forces $(n - c_o(G - W) + |W|)/2 = (n - c_o(G - W') + |W'|)/2$, as $\nu(G) = (n - c_o(G - W) + |W|)/2$. But then, W' contradicts the choice of W, completing the proof of (2).

Next, we claim that

(3) $c_i = 1$ for $i = 2, \dots, q$.

For, suppose $c_2 \ge 2$. Then $c_1 \ge c_2 \ge 2$, so there exist $a_1 b_1 \in E(C_1)$ and $a_2 b_2 \in E(C_2)$. If $a_1 > a_2$ then $a_1 b_2 \in E(G)$ by (1), and if $a_1 < a_2$ then $b_1 a_2 \in E(G)$ by (1). So there is an edge between C_1 and C_2, contradicting the fact that C_1 and C_2 are different components of $G - W$. This completes the proof of (3).

By (3), we have

$$\begin{aligned} m \geq \nu(G) &= (n - (c_o(G-W) - |W|))/2 \\ &= ((c_1 + |W| + q - 1) - (q - |W|))/2 \\ &= (c_1 - 1)/2 + |W|. \end{aligned}$$

Thus, $|W| \leq m - (c_1 - 1)/2$. Hence,

$$e(G) \leq \binom{n}{2} - \binom{n-|W|}{2} + \binom{c_1}{2} \leq \binom{n}{2} - \binom{n-m+(c_1-1)/2}{2} + \binom{c_1}{2}.$$

Since $e(G) > \binom{n}{2} - \binom{n-m}{2} - \rho n^2$, we have

$$\begin{aligned} \binom{n-m}{2} + \rho n^2 &> \binom{n-m+(c_1-1)/2}{2} - \binom{c_1}{2} \\ &= \binom{n-m}{2} + \frac{1}{8}(c_1-1)^2 + \frac{1}{4}(c_1-1)(2n-2m-1) - \binom{c_1}{2}, \end{aligned}$$

which gives

$$-\frac{3}{8}(c_1-1)^2 + \frac{1}{4}(c_1-1)(2n-2m-3) < \rho n^2.$$

Hence, $c_1 < \sqrt{\rho} n$, since $\rho \ll 1$ and $m \leq n/2 - 1$.

Note that every edge of G intersects $W \cup V(C_1)$. So by (1), every edge of G intersects $[|W|+c_1] \subseteq [m+(c_1+1)/2] \subseteq [m+\sqrt{\rho}n/2]$. Since $e(G) > \binom{n}{2} - \binom{n-m}{2} - \rho n^2$, we have

$$|E(H_2^2([n] \setminus [m], [m])) \setminus E(G)| \leq 2\sqrt{\rho} n^2.$$

This completes the proof of the lemma. ◻

Thus, using Lemma 6.3 instead of Lemma 4.2 in the end of the proof of (3) for Lemma 4.3, we see that Lemma 4.3 holds in the case when $k = 3$ and $l = 1$. Thus, our approach (using Lemma 6.1 instead of Lemma 3.2) gives an alternative proof of the following result of Kühn, Osthus, and Treglown [17] (and independently by Khan [14]) on perfect matchings in 3-graphs.

THEOREM 6.4 (Kühn, Osthus, and Treglown, 2013; Khan, 2013). *There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geq n_0$, $m \leq n/3$, and $\delta_1(H) > \binom{n-1}{2} - \binom{n-m}{2}$, then $\nu(H) \geq m$.*

For the general case, Hàn, Person, and Schacht [12] and, independently, Kühn, Osthus, and Treglown [17] conjectured that the asymptotic l-degree threshold for a perfect matching in a k-graph with n vertices is

$$\left(\max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-l}\right\} + o(1)\right)\binom{n-l}{k-l}.$$

The first term $(1/2 + o(1))\binom{n-l}{k-l}$ comes from a parity construction: Take disjoint nonempty sets A and B with $||A| - |B|| \leq 2$, form a hypergraph H by taking all k-subsets f of $A \cup B$ with $|f \cap A| \not\equiv |A| \pmod 2$. The second term is given by the hypergraph obtained from K_n^k (the complete k-graph on n vertices) by deleting all edges from a subgraph $K_{n-n/k+1}^k$.

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