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Rainbow matchings for 3-uniform hypergraphs

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ABSTRACT

Kühn, Osthus, and Treglown and, independently, Khan proved that if H is a 3-uniform hypergraph with n vertices, where $n \in 3\mathbb{Z}$ and large, and $\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$, then H contains a perfect matching. In this paper, we show that for $n \in 3\mathbb{Z}$ sufficiently large, if $F_1, \ldots, F_{n/3}$ are 3-uniform hypergraphs with a common vertex set and $\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$ for $i \in [n/3]$, then $\{F_1, \ldots, F_{n/3}\}$ admits a rainbow matching, i.e., a matching consisting of one edge from each F_i. This is done by converting the rainbow matching problem to a perfect matching problem in a special class of uniform hypergraphs.

1. Introduction

For any positive integer k and any set S, let $[k] := \{1, \ldots, k\}$ and $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq 2^{V(H)}$, and we write $e(H) := |E(H)|$ and often identify $E(H)$ with H. For a positive integer k,

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a hypergraph H is said to be *k-uniform* if $E(H) \subseteq \binom{V(H)}{k}$, and a k-uniform hypergraph is also called a *k-graph*.

A *matching* in a hypergraph H is a set of pairwise disjoint edges in H, and we use $\nu(H)$ to denote the maximum size of a matching in H. The problem for finding maximum matchings in hypergraphs is NP-hard, even for 3-graphs [17]. It is of interest to find good sufficient conditions that guarantee large matchings.

Erdős [8] conjectured in 1965 that, for positive integers k, n, t, if H is a k-graph on n vertices and $\nu(H) < t$ then $e(H) \leq \max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}$. This bound is tight because of the complete k-graph on $kt-1$ vertices and the k-graph on n vertices in which every edge intersects a fixed set of $t-1$ vertices. For recent progress on this conjecture, see [4,5,9,10,13,16,22]. In particular, Frankl [9] proved that if $n \geq (2t-1)k-(t-1)$ and $\nu(H) < t$ then $e(H) \leq \binom{n}{k} - \binom{n-t+1}{k}$. This result was further improved by Frankl and Kupavskii [11].

There has been extensive study on degree conditions for large matchings in uniform hypergraphs. Let H be a hypergraph and $T \subseteq V(H)$. The *degree* of T in H, denoted by $d_H(T)$, is the number of edges in H containing T. For any integer $l \geq 0$, let $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}\}$ denote the minimum *l-degree* of H. Hence, $\delta_0(H) = e(H)$. Note that $\delta_1(H)$ is often called the minimum *vertex* degree of H. For $u \in V(H)$, let $N_H(u) := \{e : e \subseteq V(H) \setminus \{u\} \text{ and } e \cup \{u\} \in E(H)\}$. When there is no confusion, we also view $N_H(u)$ as a hypergraph with vertex set $V(H) \setminus \{u\}$ and edge set $N_H(u)$.

For integers n, k, d satisfying $0 \leq d \leq k-1$ and $n \in k\mathbb{Z}$, let $m_d(k, n)$ denote the minimum integer m such that every k-graph H on n vertices with $\delta(H) \geq m$ has a perfect matching. Kühn, Osthus and Treglown [20] and, independently, Khan [18] determined $m_1(k, n)$ for $k = 3$ and large n. Khan [19] also determined $m_1(k, n)$ for $k = 4$ and large n. For $d = k-1$, $m_{k-1}(k, n)$ was determined for large n by Rödl, Ruciński, and Szemerédi [24]. This result was generalized by Treglown and Zhao [26] to the range $k/2 \leq d \leq k-1$, where they also determined the extremal families.

There are attempts to extend the above conjecture of Erdős to a family of hypergraphs. Let $\mathcal{F} = \{F_1, \ldots, F_t\}$ be a family of hypergraphs. A set of pairwise disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$. (In this situation, we also say that $\mathcal{F}$ or $\{F_1, \ldots, F_t\}$ *admits* a rainbow matching.) Aharoni and Howard [3] made the following conjecture, which first appeared in Huang, Loh, and Sudakov [16]: Let t be a positive integer and $\mathcal{F} = \{F_1, \ldots, F_t\}$ such that, for $i \in [t]$, $F_i \subseteq \binom{[n]}{k}$ and $e(F_i) > \max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}$; then $\mathcal{F}$ admits a rainbow matching. Huang, Loh, and Sudakov [16] showed that this conjecture holds when $n > 3k^2t$, and Frankl and Kupavskii [12] showed that this conjecture holds when $n \geq 12tk\log(e^2t)$.

In this paper, we prove a degree version of the above conjecture for rainbow matchings, which extends the results of Kühn, Osthus, and Treglown [20] and, independently, of Khan [18] for 3-graphs to families of 3-graphs.

Theorem 1.1. *Let $n \in 3\mathbb{Z}$ be positive and sufficiently large and let $\mathcal{F} = \{F_1, \ldots, F_{n/3}\}$ be a family of n-vertex 3-graphs such that $V(F_i) = V(F_1)$ for $i \in [n/3]$. If $\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$ for $i \in [n/3]$, then $\mathcal{F}$ admits a rainbow matching.*

The bound on $\delta_1(F_i)$ in Theorem 1.1 is sharp. To see this, let $m \leq n/3$ and let $H(n,m)$ denote a 3-graph that is isomorphic to the 3-graph with vertex set $[n]$ and edge set

$$\left\{ e \in \binom{[n]}{3} : e \not\subseteq [m] \text{ and } e \cap [m] \neq \emptyset \right\}.$$

Note that for $n \in 3\mathbb{Z}$, $\delta_1(H(n, n/3-1)) = \binom{n-1}{2} - \binom{2n/3}{2}$ and $H(n, n/3-1)$ has no perfect matching. Hence, the family of $n/3$ copies of $H(n, n/3-1)$ admits no rainbow matching.

To prove Theorem 1.1, we convert this rainbow matching problem to a perfect matching problem for a special class of hypergraphs. For any integer $k \geq 2$, a k-graph H is $(1, k-1)$-*partite* if there exists a partition of $V(H)$ into sets V_1, V_2 (called *partition classes*) such that for any $e \in E(H)$, $|e \cap V_1| = 1$ and $|e \cap V_2| = k-1$. A $(1, k-1)$-partite k-graph with partition classes V_1, V_2 is *balanced* if $(k-1)|V_1| = |V_2|$.

Let $n \in 3\mathbb{Z}$, let P and Q be disjoint sets such that $|P| = n$ and $|Q| = n/3$, and let $Q = \{v_1, \ldots, v_{n/3}\}$. Let $\mathcal{F} = \{F_1, \ldots, F_{n/3}\}$ be a family of 3-graphs on the same vertex set P. We use $H_{1,3}(\mathcal{F})$ to represent the balanced $(1,3)$-partite 4-graph with partition classes Q, P and edge set $\bigcup_{i=1}^{n/3} E_i$, where $E_i = \{e \cup \{v_i\} : e \in E(F_i)\}$ for $i \in [n/3]$. If $E(F_i) = E(H(n, n/3))$ and $V(F_i) = V(H(n, n/3))$ for all $i \in [n/3]$, then we write $H_{1,3}(n, n/3)$ for $H_{1,3}(\mathcal{F})$. The following observations will be useful:

- (*i*) $E(F_i)$ is the neighborhood of v_i in $H_{1,3}(\mathcal{F})$ for $i \in [n/3]$, and $\mathcal{F}$ admits a rainbow matching if, and only if, $H_{1,3}(\mathcal{F})$ has a perfect matching.
- (*ii*) $e(F_i) \geq \frac{n}{3}\delta_1(F_i)$ for all $i \in [n/3]$, and $d_{H_{1,3}(\mathcal{F})}(v) \geq \sum_{i=1}^{n/3} \delta_1(F_i)$ for $v \in P$.
- (*iii*) $d_{H_{1,3}(\mathcal{F})}(\{u,v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$, provided $\delta_1(F_i) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for $i \in [n/3]$.
- (*iv*) $\delta_1(H_{1,3}(\mathcal{F})) \geq \frac{n}{3}\left(\binom{n-1}{2} - \binom{2n/3}{2} + 1\right)$, provided $d_{H_{1,3}(\mathcal{F})}(\{u,v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$.

By observations (*i*) and (*iii*), Theorem 1.1 follows from the following result.

Theorem 1.2. *Let $n \in 3\mathbb{Z}$ be positive and sufficiently large, and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $|P| = n$ and $Q = n/3$. Suppose $d_H(\{u,v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$. Then H has a perfect matching.*

To prove Theorem 1.2, we take the usual approach by considering whether or not H is close to some $H_{1,3}(n, n/3)$ on the same vertex set. Given $\varepsilon > 0$ and two k-graphs H_1, H_2 with $V(H_1) = V(H_2)$, we say that H_2 is ε-close to H_1 if $|E(H_1) \setminus E(H_2)| < \varepsilon |V(H_1)|^k$.

In Section 2, we prove Theorem 1.2 when H is close to some $H_{1,3}(n, n/3)$, using the structure of $H_{1,3}(n, n/3)$ to find a perfect matching in H greedily. This is the extremal case, as $H_{1,3}(n, n/3)$ is an extremal graph for Theorem 1.2.

In the non-extremal case, H is not close to any $H_{1,3}(n, n/3)$ on $V(H)$. We first find a small matching M' in H that can be used to "absorb" small sets of vertices, then find an almost perfect matching M'' in $H - V(M')$, and finally use M' to absorb $V(H) \setminus V(M' \cup M'')$. A more detailed account is given in Section 6.

In Section 3, we prove an absorbing lemma for (1, 3)-partite 4-graphs, which can be used to find the absorbing matching M'. In Section 5, we find the almost perfect matching M'' in $H - V(M')$. For this, we use the approach of Alon et al. in [4] to find random subgraphs with desired properties (including the existence of perfect fractional matchings). However, we need to modify this approach to make it work, which is done in Section 4. First, we need the random subgraphs to be balanced. Second, in the non-extremal case, the (1, 3)-partite 4-graphs do not have large sparse sets; so we also need to control the independence number of those random subgraphs, for which we use the hypergraph container result of Balogh et al. [7].

We remark that most of the work in this paper may be used to extend Theorem 1.2 to 4-graphs. However, one of the key results needed is a stability result similar to Lemma 4.4 for 3-graphs. For additional references on matchings in hypergraphs, see [1,2,14,15].

2. The extremal case

In this section, we prove Theorem 1.2 for the case when H is close to some $H_{1,3}(n, n/3)$ on $V(H)$. First, we prove a result on rainbow matchings for a small family of hypergraphs, which will serve as induction basis for our proof.

Lemma 2.1. *Let n, t, k be positive integers such that $n > 2k^4t$. Let F_i, $i \in [t]$, be n-vertex k-graphs with a common vertex set. If $\delta_1(F_i) > \binom{n-1}{k-1} - \binom{n-t}{k-1}$ for $i \in [t]$ then $\{F_1, \dots, F_t\}$ admits a rainbow matching.*

Proof. We apply induction on t. Note that the assertion is trivial when $t = 1$. So assume $t > 1$ and the assertion holds for $t - 1$. Then, since $\delta_1(F_i) > \binom{n-1}{k-1} - \binom{n-t}{k-1} > \binom{n-1}{k-1} - \binom{n-(t-1)}{k-1}$, $\{F_1, \dots, F_{t-1}\}$ admits a rainbow matching, say M.

Suppose for a contradiction that $\{F_1, \dots, F_t\}$ does not admit a rainbow matching. Then every edge of F_t must intersect M. So there exists $v \in V(M)$ such that $d_{F_t}(v) > e(F_t)/(kt)$. Note that

$$\delta_1(F_t) > \binom{n-1}{k-1} - \binom{n-t}{k-1} > \binom{n-1}{k-1}\left(1 - \left(1 - \frac{k-1}{n-1}\right)^{k-1}\right) > \frac{t(k-1)}{2(n-1)}\binom{n-1}{k-1},$$

since $n > 2k^4t$. So we have

$$d_{F_t}(v) > \frac{\delta_1(F_t)n/k}{kt} > \frac{t(k-1)n}{2(n-1)k^2t}\binom{n-1}{k-1} > \frac{1}{2k^2}\binom{n-1}{k-1}.$$

Let $F_i' = F_i - v$ for $i \in [t-1]$. Since

$$\delta_1(F_i') \geq \delta_1(F_i) - \binom{n-2}{k-2} > \binom{n-1}{k-1} - \binom{n-t}{k-1} - \binom{n-2}{k-2} = \binom{n-2}{k-1} - \binom{n-t}{k-1},$$

it follows from induction hypothesis that $\{F_1', \dots, F_{t-1}'\}$ admits a rainbow matching, say M'.

Note that the number of edges in F_t containing v and intersecting M' is at most

$$k(t-1)\binom{n-2}{k-2} < \frac{1}{2k^2}\binom{n-1}{k-1} < d_{F_t}(v),$$

as $n \geq 2k^4t$. Hence, v is contained in some edge of $F_t - V(M')$, say e. Now $M' \cup \{e\}$ is a rainbow matching for $\{F_1, \dots, F_t\}$, a contradiction. □

Next, we prove Theorem 1.2 for the case when, for every vertex v, most of the edges of $H_{1,3}(n, n/3)$ containing v also lie in H. More precisely, given $\alpha > 0$, $H_{1,3}(n, n/3)$, and a $(1,3)$-partite 4-graph H with $V(H) = V(H_{1,3}(n, n/3))$, we say that a vertex $v \in V(H)$ is *α-good* with respect to $H_{1,3}(n, n/3)$ if $|N_{H_{1,3}(n,n/3)}(v) \setminus N_H(v)| \leq \alpha n^3$. Otherwise we say that v is *α-bad* with respect to $H_{1,3}(n, n/3)$.

Lemma 2.2. *Let n be positive integer and H be a balanced $(1,3)$-partite 4-graph on $4n/3$ vertices, and let α be a constant with $0 < \alpha < 2^{-12}$. If all vertices of H are α-good with respect to some $H_{1,3}(n, n/3)$ on $V(H)$, then H has a perfect matching.*

Proof. Let Q, P be the partition classes of H, and let $U \cup W$ be partition classes of $H(n, n/3)$ such that $|Q| = |W| = n/3$, $|U| = 2n/3$, and $V(H(n, n/3)) = P$.

Let M be a matching in H that only uses edges consisting of two vertices from U and one vertex from each of Q and W, and choose such M that $|M|$ is maximum. Let $Q' := Q \setminus V(M)$, $U' = U \setminus V(M)$, and $W' = W \setminus V(M)$. Then $|U'|/2 = |W'| = |Q'|$.

Note that $|M| \geq n/4$. For, otherwise, $|U'|/2 = |W'| = |Q'| = n/3 - |M| > n/12$. Then, by the maximality of M, we have, for any $u \in U'$,

$$|N_{H_{1,3}(n,n/3)}(u) \setminus N_H(u)| \geq |Q'||W'|(|U'| - 1) > n^3/12^3 > \alpha n^3,$$

a contradiction.

Now suppose M is not a perfect matching in H. Then Q', U', W' are all non-empty. Let $v \in Q'$, $u_1, u_2 \in U'$ be distinct, and $w \in W'$.

Let $\{e_1, e_2, e_3\}$ be an arbitrary set of three pairwise distinct edges from M. By the maximality of M, no matching of size 4 in H is contained in $e_1 \cup e_2 \cup e_3 \cup \{v, w, u_1, u_2\}$ and uses only edges with two vertices from U and one vertex from each of Q and W.

Hence, there exists $S \in E(H_{1,3}(n, n/3)) \setminus E(H)$ such that $S \subseteq e_1 \cup e_2 \cup e_3 \cup \{v, w, u_1, u_2\}$, $|S \cap e_i| = 1$ for $i \in [3]$, $|S \cap \{v, w, u_1, u_2\}| = 1$, and S has two vertices from U and one vertex from each of Q and W.

Note that there are $\binom{m}{3}$ choices for $\{e_1, e_2, e_3\}$, which result in distinct choices for S. So the number of edges in $E(H_{1,3}(n, n/3)) \setminus E(H)$ containing exactly one vertex from $\{v, w, u_1, u_2\}$ is at least

$$\binom{m}{3} \geq \binom{n/4}{3} > n^3/(2^{10}).$$

This implies that for some $u \in \{v, w, x_1, x_2\}$,

$$|N_{H_{1,3}(n,n/3)}(u) \setminus N_H(u)| > n^3/(2^{12}) > \alpha n^3,$$

a contradiction. $\square$

Having proved the above two results, we are ready to complete the proof of Theorem 1.2 in the case when H is close to some $H_{1,3}(n, n/3)$.

Lemma 2.3. *Let $n \in 3\mathbb{Z}$ be positive and $\varepsilon > 0$ sufficiently small, and let H be a balanced $(1, 3)$-partite 4-graph with partition classes Q, P and $3|Q| = |P| = n$. Suppose H is ε-close to some $H_{1,3}(n, n/3)$ with $P = V(H(n, n/3))$. If $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for all $u \in P$ and $v \in Q$, then H has a perfect matching.*

Proof. Let U, W denote the partition of $P = V(H(n, n/3))$ such that $|W| = |U|/2 = n/3$. Note that $|Q| = n/3$. Let B denote the set of $\sqrt{\varepsilon}$-bad vertices of H with respect to $H_{1,3}(n, n/3)$. Since H is ε-close to $H_{1,3}(n, n/3)$, we have $|B| \leq 4\sqrt{\varepsilon}n$. Let $Q \cap B = \{v_1, \ldots, v_q\}$ and $Q = \{v_1, \ldots, v_{n/3}\}$, and let $W' \subseteq W \setminus B$ such that $|W'| = n/3 - (q + |W \cap B|) \geq n/3 - 4\sqrt{\varepsilon}n$.

First, we find a matching M_0' in $H - W'$ covering $Q \cap B$. For this, let $F_i = N_H(v_i) - W'$ for $i \in [n/3]$. Note that, for $i \in [n/3]$, $\delta_1(N_H(v_i)) = \min\{d_H(\{u, v_i\}) : u \in P\} \geq \binom{n-1}{2} - \binom{2n/3}{2} + 1$. Hence,

$$\begin{aligned} \delta_1(F_i) &\geq \delta_1(N_H(v_i)) - \left(\binom{n-1}{2} - \binom{n-|W'|-1}{2}\right) \\ &> \binom{n-|W'|-1}{2} - \binom{2n/3}{2} \\ &= \binom{n-|W'|-1}{2} - \binom{n-|W'|-(q+|W \cap B|)}{2}. \end{aligned}$$

Since $|B| \leq 4\sqrt{\varepsilon}n$, $|W'| \geq n/3 - 4\sqrt{\varepsilon}n$ and $q + |W \cap B| < (n - |W'|)/(2 \cdot 3^4)$. Hence by Lemma 2.1, $\{F_1, \ldots, F_{q+|W \cap B|}\}$ admits a rainbow matching, say M_0. Let $M_0 = \{e_i \in$

$E(F_i) : i \in [q + |W \cap B|]\}$, and let $M'_0 = \{e_i \cup \{v_i\} : i \in [q + |W \cap B|]\}$. Then M'_0 is a matching in H and $Q \cap B \subseteq V(M'_0)$.

Next, we find a matching in $H_1 := H - V(M'_0)$ covering $B \setminus V(M'_0)$, in two steps. Since ε is very small, we can choose η such that $\sqrt{\varepsilon} \ll \eta \ll 1$. We divide $B \setminus V(M'_0)$ to two disjoint sets B_1, B_2 such that, for each $x \in B \setminus V(M'_0)$, $x \in B_1$ if, and only if, H_1 has at least ηn^3 edges each of which contains x and exactly one vertex in W'.

We greedily pick a matching M_1 in H_1 such that $B_1 \subseteq V(M_1)$ and every edge of M_1 contains at least one vertex from B_1 and exactly one vertex from W'. This can be done since each time we pick an edge e for a vertex $x \in B_1$, we have at least ηn^3 choices and at most $4(4\sqrt{\varepsilon}n)n^2$ ($\ll \eta n^3$ as $\sqrt{\varepsilon} \ll \eta$) of which intersect a previously chosen edge.

Now we find a matching M_2 in $H_2 := H_1 - V(M_1)$ such that $B_2 \subseteq V(M_2)$. Note that

$$\delta_1(H_2) \geq \delta_1(H) - 4|M'_0 \cup M_1|n^2 \geq \frac{n}{3}\left(\binom{n-1}{2} - \binom{2n/3}{2} + 1\right) - 16\sqrt{\varepsilon}n^3.$$

Hence, for any $x \in B_2$, the number of edges containing x and disjoint from W' is at least

$$\delta_1(H_2) - \eta n^3 - |Q|\binom{|W'|}{2} > \eta n^3,$$

as $\sqrt{\varepsilon} \ll \eta \ll 1$ and $n/3 = |Q| \geq |W'|$. Thus, since $\sqrt{\varepsilon} \ll \eta$, we greedily find a matching M_2 in $H_1 - V(M_1)$ such that $B_2 \subseteq V(M_2)$, M_2 is disjoint from W', and every edge of M_2 contains at least one vertex from B_2.

Thus, $M_1 \cup M_2$ gives the desired matching in $H_1 := H - V(M'_0)$ covering $B \setminus V(M'_0)$. Note that $|M'_0 \cup M_1 \cup M_2| \leq (q+|W \cap B|)+|B_1|+|B_2| \leq 2|B| \leq 8\sqrt{\varepsilon}n$. Also note that each vertex of $H - V(M'_0 \cup M_1 \cup M_2)$ is $\sqrt{\varepsilon}$-good in H (with respect to $H_{1,3}(n, n/3)$). Thus, for every vertex $u \in U - V(M'_0 \cup M_1 \cup M_2)$, the number of edges of $H - V(M'_0 \cup M_1 \cup M_2)$ containing u and exactly two vertices of $W - V(M'_0 \cup M_1 \cup M_2)$ is at least

$$\frac{n}{3}\binom{n/3}{2} - \sqrt{\varepsilon}n^3 - 4|M'_0 \cup M_1 \cup M_2|n^2 > \eta n^3,$$

as $\sqrt{\varepsilon}\eta \ll 1$. Hence, we may greedily find a matching M'_2 in $H - V(M'_0 \cup M_1 \cup M_2)$ such that $|M'_2| = |M_2|$ and every edge of M'_2 contains exactly two vertices of W'.

Let $M = M'_0 \cup M_1 \cup M_2 \cup M'_2$ and $m = |M|$. Then $m \leq 8\sqrt{\varepsilon}n$. Let $H_3 = H - V(M)$. Let $H_{1,3}(n - 3m, n/3 - m)$ be obtained from $H_{1,3}(n, n/3)$ by removing $V(M)$. Then, for any $x \in V(H_3)$,

$$\begin{aligned} &|N_{H_{1,3}(n-3m,n/3-m)}(x) \setminus N_{H_3}(x)| \\ &\leq |N_{H_{1,3}(n,n/3)}(x) \setminus N_H(x)| \\ &\leq \sqrt{\varepsilon}n^3 \\ &\leq 2\sqrt{\varepsilon}(n - 3m)^3. \end{aligned}$$

Thus, every vertex of H_3 is $2\sqrt{\varepsilon}$-good with respect to $H_{1,3}(n-3m, n/3-m)$. By Lemma 2.2, H_3 contains a perfect matching, say M_3. Now $M_3 \cup M$ is a perfect matching in H. $\square$

3. Absorbing lemma

Our strategy to prove Theorem 1.2 is to find a small matching M' in H that can be used to "absorb" small sets of vertices, find an almost perfect matching M'' in $H - V(M')$, and then use M' to absorb $V(H) \setminus V(M' \cup M'')$. In this section, we prove such an absorbing lemma for $(1,3)$-partite 4-graphs. Our proof follows along the same lines as in [24].

Lemma 3.1. *Let $n \in 3\mathbb{Z}$ be large enough and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P|$ and $\delta_1(H) \geq (n/3)\left(\binom{n-1}{2} - \binom{2n/3}{2} + 1\right)$. Let ρ, ρ' be constants such that $0 < \rho' \ll \rho \ll 1$. Then H has a matching M' such that $|M'| \leq \rho n$ and, for any subset $S \subseteq V(H) \setminus V(M')$ with $|S| \leq \rho' n$ and $3|S \cap Q| = |S \cap P|$, $H[S \cup V(M')]$ has a perfect matching.*

Proof. We call a balanced 12-element set $A \subseteq V(H)$ an *absorbing set* for a balanced 4-element set $T \subseteq V(H)$ if $H[A]$ has a matching of size 3 and $H[A \cup T]$ has a matching of size 4. Denote by $\mathcal{L}(T)$ the collection of all absorbing sets for T. Then

(1) for every balanced $T \in \binom{V(H)}{4}$, $|\mathcal{L}(T)| > 10^{-8} n^{12}/12!$.

Let $T = \{u_0, u_1, u_2, u_3\} \in \binom{V(H)}{4}$ be balanced, with $u_0 \in Q$ and $u_1, u_2, u_3 \in P$. We form an absorbing set for T by choosing four pairwise disjoint 3-sets U_0, U_1, U_2, U_3 in order.

First, we choose a 3-set $U_0 \subseteq P \setminus T$ such that $U_0 \cup \{u_0\} \in E(H)$. The number of choices for U_0 is at least

$$d_H(u_0) - 3\binom{n-3}{2} > \delta_1(H) - 3\binom{n-1}{2} > \frac{n}{9}\binom{n-1}{2}.$$

Now fix a choice of U_0, and let $U_0 = \{w_1, w_2, w_3\}$. Note that, for each $x \in P$, $N_H(x)$ is a subset of the union of $\{\{x_0, x_1, x_2\} : x_0 \in Q, x_1, x_2 \in P\}$. Hence, $|N_H(u_i) \cup N_H(w_i)| \leq \frac{n}{3}\binom{n}{2}$. Thus, for $i \in [3]$,

$$|N_H(u_i) \cap N_H(w_i)| \geq \frac{2n}{3}\left(\binom{n-1}{2} - \binom{2n/3}{2} + 1\right) - \frac{n}{3}\binom{n}{2} \geq \frac{n}{30}\binom{n-1}{2}.$$

For $i \in [3]$, we choose 3-sets U_i from $(V(H) \setminus T) \setminus \bigcup_{j=0}^{i-1} U_j$ such that $U_i \cup \{u_i\}$ and $U_i \cup \{w_i\}$ are both edges of H. For each choice of U_j, $0 \leq j \leq i-1$, the number of choices for U_i is at least

$$|N_H(u_i) \cap N_H(w_i)| - 13(n/3)n \geq \frac{n}{30}\binom{n-1}{2} - 13n^2/3 > \frac{n}{50}\binom{n-1}{2}.$$

Let $A = \bigcup_{i=0}^{3} U_i$. Then $\{U_i \cup \{w_i\} : i \in [3]\}$ is a matching in $H[A]$, and $\{U_i \cup \{u_i\} : i \in [3] \cup \{0\}\}$ is a matching in $H[A \cup T]$. Thus A is an absorbing set for T. Since there are more than $10^{-8} n^{12}$ choices of (U_0, U_1, U_2, U_3), there are more than $10^{-8} n^{12}/12!$ absorbing sets for T.

Now, form a family $\mathcal{F}$ of subsets of $V(H)$ by selecting each of the $\binom{n/3}{3}\binom{n}{9}$ possible balanced 12-sets independently with probability

$$p = \frac{\rho n}{2\binom{n/3}{3}\binom{n}{9}}.$$

Then, it follows from Chernoff's bound that, with probability $1 - o(1)$ (as $n \to \infty$),

(2) $|\mathcal{F}| \leq \rho n$, and
(3) $|\mathcal{L}(T) \cap \mathcal{F}| \geq p|\mathcal{L}(T)|/2 \geq 10^{-10} \rho n$ for all balanced $T \in \binom{V(H)}{4}$.

Furthermore, the expected number of intersecting pairs of sets in $\mathcal{F}$ is at most

$$\binom{n/3}{3}\binom{n}{9}\left[3\binom{n/3-1}{2}\binom{n}{9} + 9\binom{n-1}{8}\binom{n/3}{3}\right] p^2 < \rho^{1.5} n.$$

Thus, using Markov's inequality, we derive that, with probability at least 1/2,

(4) $\mathcal{F}$ contains at most $2\rho^{1.5} n$ intersecting pairs.

Hence, with positive probability, $\mathcal{F}$ satisfies (2), (3), and (4). Let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by removing one set from each intersecting pair and deleting all non-absorbing sets. Then $\mathcal{F}'$ consists of pairwise disjoint absorbing sets, such that, for each $T \in \binom{V(H)}{4}$,

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq 10^{-10} \rho n/2.$$

Since $\mathcal{F}'$ consists only of pairwise disjoint absorbing sets, $H[V(\mathcal{F}')]$ has a perfect matching, say M'. Then $|M'| \leq \rho n$. To complete the proof, take an arbitrary $S \subseteq V(H) \setminus V(M')$ with $|S| \leq \rho' n$ and $3|S \cap Q| = |S \cap P|$, where $\rho' \leq 10^{-10}\rho/2$. Note that S can be partitioned into t balanced 4-sets, say $T_1, \ldots, T_t$, for some $t \leq \rho' n/4 < 10^{-10} \rho n/2$. We can greedily choose distinct absorbing sets $A_i \in \mathcal{F}'$ in order for $i = 1, \ldots, t$, such that $H[A_i \cup T_i]$ has a perfect matching. Hence, $H[S \cup V(M')]$ has a perfect matching as required. □

4. Perfect fractional matchings

When H is not close to any $H_{1,3}(n, n/3)$ we will show that H contains a $(1,3)$-partite 4-graph H' in which no independent set is too large (see Lemma 4.3) and we then use this property of H' to show that H' has a perfect fractional matching (see Lemma 4.5).

To obtain H', we use the hypergraph container method developed by Balogh, Morris, and Samotij [7] and, independently, by Saxton and Thomason [25]. A family $\mathcal{F}$ of subsets of a set V is said to be *increasing* if, for any $A \in \mathcal{F}$ and $B \subseteq V$, $A \subseteq B$ implies $B \in \mathcal{F}$. Let H be a hypergraph. We use $v(H), e(H)$ to denote the number of vertices, number of edges in H, respectively. We also use $\Delta_l(H)$ to denote the maximum l-degree of H, and $\mathcal{I}(H)$ to denote the collection of all independent sets in H. Let $\varepsilon > 0$ and let $\mathcal{F}$ be a family of subsets of $V(H)$. We say that H is $(\mathcal{F}, \varepsilon)$-*dense* if $e(H[A]) \geq \varepsilon e(H)$ for every $A \in \mathcal{F}$. We use $\overline{\mathcal{F}}$ to denote the family consisting of subsets of $V(H)$ not in $\mathcal{F}$.

Lemma 4.1 *(Balogh, Morris, and Samotij, 2015). For every $k \in \mathbb{N}$ and all positive c and ε, there exists a positive constant C such that the following holds. Let H be a k-graph and let $\mathcal{F}$ be an increasing family of subsets of $V(H)$ such that $|A| \geq \varepsilon v(H)$ for all $A \in \mathcal{F}$. Suppose that H is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0, 1)$ is such that, for every $l \in [k]$,*

$$\Delta_l(H) \leq cp^{l-1}\frac{e(H)}{v(H)}.$$

Then there exist a family $\mathcal{S} \subseteq \binom{V(H)}{\leq Cpv(H)}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(H) \to \mathcal{S}$ such that, for every $I \in \mathcal{I}(H)$,

$$g(I) \subseteq I \quad \text{and} \quad I \setminus g(I) \subseteq f(g(I)).$$

In order to apply Lemma 4.1 we need a family $\mathcal{F}$ of subsets of $V(H)$ so that H is $(\mathcal{F}, \varepsilon)$-dense, which is possible when H is not close to any $H_{1,3}(n, n/3)$.

Lemma 4.2. *Let ρ, ε be reals such that $0 < \rho \leq \varepsilon/4 \ll 1$, let $n \in 3\mathbb{Z}$ be large, and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$ and $d_H(\{u, v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for any $v \in Q$ and $u \in P$. If H is not ε-close to any $H_{1,3}(n, n/3)$ with $V(H_{1,3}(n, n/3)) = P$, then H is $(\mathcal{F}, \varepsilon/6)$-dense, where $\mathcal{F} = \{A \subseteq V(H) : |A \cap Q| \geq (1/3 - \varepsilon/8)n$ and $|A \cap P| \geq (2/3 - \varepsilon/8)n\}$.*

Proof. Suppose to the contrary that there exists $A \subseteq V(H)$ such that $|A \cap Q| \geq (1/3 - \varepsilon/8)n$, $|A \cap P| \geq (2/3 - \varepsilon/8)n$, and $e(H[A]) \leq \varepsilon e(H)/6$. Choose such A that $|P \setminus A| \geq n/3$ and let $W \subseteq P \setminus A$ such that $|W| = n/3$. Let $A_1 = A \cap P$ and $A_2 = A \cap Q$, and let $B_1 = (P \setminus W) \setminus A_1$, $B_2 = Q \setminus A_2$, and $B = B_1 \cup B_2$. Then $|A_1| \leq 2n/3$ and, by the choice of A, $|B_1| \leq \varepsilon n/8$ and $|B_2| \leq \varepsilon n/8$.

Let $U = P \setminus W = A_1 \cup B_1$ and let H_0 denote the $H_{1,3}(n, n/3)$ with partition classes Q, U, W. We derive a contradiction by showing that $|E(H_0) \setminus E(H)| < \varepsilon n^4$. By the definition of $H(n, n/3)$, each $f \in E(H_0) \setminus E(H)$ intersects U. So

$$|E(H_0) \setminus E(H)| \leq |\{f \in E(H_0) : f \cap B_1 \neq \emptyset\}| + |\{f \in E(H_0) \setminus E(H) : f \cap A_1 \neq \emptyset\}|.$$

Since $|B_1| \leq \varepsilon n/8$, we have $|\{f \in E(H_0) : f \cap B_1 \neq \emptyset\}| \leq |B_1||Q||P|^2/2 \leq \varepsilon n^4/48$. To bound $|\{f \in E(H_0) \setminus E(H) : f \cap A_1 \neq \emptyset\}|$, we note that, for each fixed $u \in A_1$,

$$|\{f \in E(H) : u \in f,\ f \cap B \neq \emptyset\}| \leq |B_1||P||Q| + |B_2||P|^2/2 < \varepsilon n^3/8,$$

and that, for each $f \in E(H)$ with $u \in f$, we have $f \cap B \neq \emptyset$, or $f \subseteq A$, or $f \in E(H_0)$. So for any $u \in A_1$,

$$\begin{aligned} &|\{f \in E(H) : u \in f,\ f \in E(H_0)| \\ \geq & d_H(u) - |\{f \in E(H) : u \in f,\ f \cap B \neq \emptyset\}| - |\{f \in E(H) : u \in f,\ f \subseteq A\}| \\ \geq & d_H(u) - \varepsilon n^3/8 - d_{H[A]}(u). \end{aligned}$$

Hence,

$$\begin{aligned} &|\{f \in E(H_0) \setminus E(H) : f \cap A_1 \neq \emptyset\}| \\ \leq &\sum_{u \in A_1} |\{f \in E(H_0) \setminus E(H) : u \in f\}| \\ \leq &\sum_{u \in A_1} (d_{H_0}(u) - |\{f \in E(H) : u \in f,\ f \in E(H_0)|) \\ \leq &\sum_{u \in A_1} \left(d_{H_0}(u) - d_H(u) + \varepsilon n^3/8 + d_{H[A]}(u)\right). \end{aligned}$$

Since for $u \in A_1$, $d_{H_0}(u) = \frac{n}{3}\left(\binom{n-1}{2} - \binom{2n/3-1}{2}\right)$ and $d_H(u) = \sum_{v \in Q} d_H(\{u,v\}) \geq \frac{n}{3}\left(\binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2\right)$, we have $d_{H_0}(u) - d_H(u) \leq \rho n^3/3$ (for large n). Hence,

$$\begin{aligned} |E(H_0) \setminus E(H)| &\leq \varepsilon n^4/48 + |A_1|\,(\rho/3 + 3\varepsilon/8)\,n^3 + \sum_{u \in A_1} d_{H[A]}(u) \\ &\leq (\varepsilon/48 + 4\rho/9 + \varepsilon/4)\,n^4 + 3e(H[A]) \quad (\text{since } |A_1| \leq 2n/3) \\ &\leq (1/48 + 1/9 + 1/4)\,\varepsilon n^4 + 3\varepsilon n^4/6 \quad (\text{since } e(H[A]) \leq \varepsilon e(H)/6) \\ &< \varepsilon n^4, \end{aligned}$$

a contradiction. □

We now use Lemma 4.1 to control the independence number of a random subgraph.

Lemma 4.3. *Let $c, \varepsilon', \alpha_1, \alpha_2$ be positive reals, let $\gamma > 0$ with $\gamma \ll \min\{\alpha_1, \alpha_2\}$, let k, n be positive integers with $n \in 3\mathbb{Z}$, and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$, $e(H) \geq cn^4$, and $e(H[F]) \geq \varepsilon' e(H)$ for all $F \subseteq V(H)$ with $|F \cap P| \geq \alpha_1 n$ and $|F \cap Q| \geq \alpha_2 n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then, with probability at least $1 - n^{O(1)} e^{-\Omega(n^{0.1})}$, every independent set J in $H[R]$ satisfies $|J \cap P| \leq (\alpha_1 + \gamma + o(1))n^{0.1}$ or $|J \cap Q| \leq (\alpha_2 + \gamma + o(1))n^{0.1}$.*

Proof. Define $\mathcal{F} := \{A \subseteq V(H) : e(H[A]) \geq \varepsilon' e(H) \text{ and } |A| \geq \varepsilon' n\}$. Then $\mathcal{F}$ is an increasing family, and H is $(\mathcal{F}, \varepsilon')$-dense. Let $p = n^{-1}$ and $v(H) = 4n/3$. Then, for $l \in [4]$,

$$\Delta_l(H) \leq \binom{4n/3}{4-l} \leq (4n/3)^{4-l} \leq (4/3)^{4-l} c^{-1} n^{-l} e(H) = (4/3)^{4-l+1} c^{-1} p^{l-1} \frac{e(H)}{v(H)}.$$

Thus by Lemma 4.1, there exist constant C, family $\mathcal{S} \subseteq \binom{V(H)}{\leq C}$, and function $f : \mathcal{S} \to \overline{\mathcal{F}}$, such that every independent set in H is contained in some $T \in \mathcal{T} := \{F \cup S : F \in f(\mathcal{S}), S \in \mathcal{S}\}$. Since $\mathcal{S} \subseteq \binom{V(H)}{\leq C}$, $|\mathcal{S}| \leq C(4n/3)^C$ and, hence,

$$|\mathcal{T}| = |\mathcal{S}||f(\mathcal{S})| \leq |\mathcal{S}|^2 \leq C^2(4n/3)^{2C}.$$

Since for $T \in \mathcal{T}$ it is possible that $|T \cap P| < \alpha_1 n + C$ or $|T \cap Q| < \alpha_2 n + C$, we need to make the sets in $\mathcal{T}$ slightly larger in order to apply Chernoff's inequality. For each $T \in \mathcal{T}$, let T' be a set obtained from T by adding vertices such that $|T' \cap P| = \max\{|T \cap P|, \lceil \alpha_1 n + C \rceil\}$ and $|T' \cap Q| = \max\{|T \cap Q|, \lceil \alpha_2 n + C \rceil\}$. (We choose one such T' for each T.) Let $\mathcal{T}' := \{T' : T \in \mathcal{T}\}$. Then

$$|\mathcal{T}'| \leq |\mathcal{T}| \leq C^2(4n/3)^{2C}.$$

Note that for each fixed $T' \in \mathcal{T}'$, we have $|R \cap T' \cap P| \sim Bi\left(|T' \cap P|, n^{-0.9}\right)$ and $|R \cap T' \cap Q| \sim Bi\left(|T' \cap Q|, n^{-0.9}\right)$. Hence, $\mathbb{E}(|R \cap T' \cap P|) = n^{-0.9}|T' \cap P|$ and $\mathbb{E}(|R \cap T' \cap Q|) = n^{-0.9}|T' \cap Q|$. Applying Chernoff's bound to $|R \cap T' \cap P|$ and $|R \cap T' \cap Q|$ by taking $\lambda = \gamma n^{0.1}$, we have,

$$\mathbb{P}\left(\big||R \cap T' \cap P| - n^{-0.9}|T' \cap P|\big| \geq \lambda\right) \leq e^{-\Omega(\lambda^2/(n^{-0.9}|T' \cap P|)} \leq e^{-\Omega(n^{0.1})}, \text{ and}$$
$$\mathbb{P}\left(\big||R \cap T' \cap Q| - n^{-0.9}|T' \cap Q|\big| \geq \lambda\right) \leq e^{-\Omega(\lambda^2/(n^{-0.9}|T' \cap Q|)} \leq e^{-\Omega(n^{0.1})}.$$

So with probability at most $2e^{-\Omega(n^{0.1})}$, $|R \cap T' \cap P| \geq n^{-0.9}|T' \cap P| + \lambda \geq (\alpha_1 + \gamma + C/n)n^{0.1}$ and $|R \cap T' \cap Q| \geq n^{-0.9}|T' \cap Q| + \lambda \geq (\alpha_2 + \gamma + C/n)n^{0.1}$.

Therefore, with probability at most $2C^2 n^{2C} e^{-\Omega(n^{0.1})}$, there exists some $T' \in \mathcal{T}'$ such that $|R \cap T' \cap P| \geq (\alpha_1 + \gamma + C/n)n^{0.1}$ and $|R \cap T' \cap Q| \geq (\alpha_2 + \gamma + C/n)n^{0.1}$. Hence, with probability at least $1 - 2C^2 n^{2C} e^{-\Omega(n^{0.1})}$, $|R \cap T' \cap P| < (\alpha_1 + \gamma + C/n)n^{0.1}$ or $|R \cap T' \cap Q| < (\alpha_2 + \gamma + C/n)n^{0.1}$ for all $T' \in \mathcal{T}'$.

Now let J be an independent set in $H[R]$. Then J is also an independent set in H; so there exist $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $J \subseteq T \subseteq T'$. Thus $J \subseteq R \cap T'$; so $|J \cap P| \leq |R \cap T' \cap P|$ and $|J \cap Q| \leq |R \cap T' \cap Q|$. Hence, with probability at least $1 - 2C^2 n^{2C} e^{-\Omega(n^{0.1})}$, for all independent set J in $H[R]$, $|J \cap P| \leq (\alpha_1 + \gamma + C/n)n^{0.1}$ or $|J \cap Q| \leq (\alpha_2 + \gamma + C/n)n^{0.1}$. □

To show that a $(1,3)$-partite 4-graph with no large independent set has a perfect fractional matching, we need a result from [21] about stable 2-graphs. A 2-graph G is *stable* with respect to a labeling $u_1, \dots, u_n$ of its vertices if, for any $i, j, k, l \in [n]$ with $k \le i$ and $l \le j$, $u_i u_j \in E(G)$ implies $u_k u_l \in E(G)$.

Lemma 4.4 *(Lu, Yu, and Yuan 2019). Let c, ρ be constants such that $0 < \rho \ll 1$ and $0 < c < 1/2$, let m, n be positive integers such that n is sufficiently large and $cn \le m \le n/2 - 1$, and let G be a 2-graph with $\nu(G) \le m$. Suppose G is stable with respect to the ordering of its vertices $u_1, \dots, u_n$. If $e(G) > \binom{n}{2} - \binom{n-m}{2} - \rho n^2$, then G is $2\sqrt{\rho}$-close to the graph with vertex $V(G)$ and edge set $\{e \in \binom{V(G)}{2} : e \cap \{u_i : i \in [n/3 - 1]\} \neq \emptyset$.*

We now prove the main result of this section. A *fractional matching* in a k-graph H is a function $w : E \to [0, 1]$ such that for any $v \in V(H)$, $\sum_{\{e \in E: v \in e\}} w(e) \le 1$. A fractional matching w is *perfect* if $\sum_{e \in E} w(e) = |V(H)|/k$.

Lemma 4.5. *Let ρ, ε be constants with $0 < \varepsilon \ll 1$ and $0 < \rho < \varepsilon^{12}$, and let H be a $(1, 3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$. Suppose $d_H(\{u, v\}) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for any $v \in Q$ and $u \in P$. If H contains no independent set S with $|S \cap Q| \ge n/3 - \varepsilon^2 n$ and $|S \cap P| \ge 2n/3 - \varepsilon^2 n$, then H contains a perfect fractional matching.*

Proof. Let $\omega : V(H) \to \mathbb{R}^+ \cup \{0\}$ be a minimum fractional vertex cover of H, i.e., $\sum_{x \in e} \omega(x) \ge 1$ for $e \in E(H)$ and, subject to this, $\sum_{x \in V(H)} \omega(x)$ is minimum. Let $P = \{u_1, \dots, u_n\}$ and $Q = \{v_1, \dots, v_{n/3}\}$, such that $\omega(v_1) \ge \cdots \ge \omega(v_{n/3})$ and $\omega(u_1) \ge \cdots \ge \omega(u_n)$. Let H' be the $(1, 3)$-partite 4-graph with vertex set $V(H)$ and edge set $E(H') = E'$, where

$$E' = \left\{ e \in \binom{V(H)}{4} \ : \ |e \cap Q| = 1 \text{ and } \sum_{x \in e} \omega(x) \ge 1 \right\}.$$

We claim that ω is a minimum fractional vertex cover of H'. Since ω is fractional vertex cover of H, $e \in E(H)$ implies that $e \in E(H')$; so $E(H) \subseteq E(H')$ and ω is also a fractional vertex cover of H'. Let ω' be a minimum fractional vertex cover of H'. Then $\omega(H) \ge \omega'(H')$, where $\omega(H) := \sum_{v \in V(H)} \omega(v)$ and $\omega'(H') := \sum_{v \in V(H')} \omega'(v)$. On the other hand, ω' is also a vertex cover of H; so $\omega'(H') \ge \omega(H)$. Hence, $\omega(H) = \omega'(H')$, i.e., ω is a minimum fractional vertex cover of H'.

Let $\nu_f(H)$ and $\nu_f(H')$ denote the maximum fractional matching numbers of H and H', respectively; then by the Strong Duality Theorem of linear programming, $\nu_f(H) = \omega(H)$ and $\nu_f(H') = \omega(H')$. Thus $\nu_f(H) = \nu_f(H')$ and, hence, it suffices to show that H' has a perfect matching.

Next, we observe that the edges of H' form a stable family with respect to the above ordering of vertices in P and Q: for any $e_1 = \{v_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}\}$ and $e_2 = \{v_{j_1}, u_{j_2}, u_{j_3}, u_{j_4}\}$ with $i_l \ge j_l$ for $1 \le l \le 4$, $e_2 \in E(H')$ implies $e_1 \in E(H')$. To see this, note that, since $i_l \ge j_l$ for $1 \le l \le 4$, we have $\omega(v_{i_1}) \ge \omega(v_{j_1})$ and $\omega(u_{i_l}) \ge \omega(u_{j_l})$ for $2 \le l \le 4$. If $e_2 \in E(H')$ then $\sum_{x \in e_2} \omega(x) \ge 1$; so $\sum_{x \in e_1} \omega(x) \ge 1$ and, hence, $e_1 \in E(H')$.

Let G denote the graph with vertex set P and edge set formed by $N_{H'}(\{v_{n/3}, u_n\})$. Then G is stable with respect to $u_1, \ldots, u_n$. Note that $e(G) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ (by assumption). Since the edges of H' form a stable family, $\{u, v\} \cup e \in E(H')$ for all $u \in P, v \in Q$, and $e \in E(G)$. Thus, if G contains a matching $M := \{e_1, \ldots, e_{n/3}\}$ then let $x_1, \ldots, x_{n/3} \in P \setminus V(M)$; we see that $\{\{v_i, x_i\} \cup e_i \in E(H') : i \in [n/3]\}$ is a perfect matching in H'.

Thus, we may assume $\nu(G) < n/3$. Recall that $e(G) > \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ and G is a stable 2-graph. Hence, by Lemma 4.4, G is $2\sqrt{\rho}$-close to the graph with vertex $V(G)$ and edge set $\{e \in \binom{V(G)}{2} : e \cap \{u_i : i \in [n/3-1]\}\} \neq \emptyset$. Therefore, G has at most $\sqrt{2\sqrt{\rho}}n$ vertices in $\{u_j : j \in [n/3-1]\}$ of degree less than $n-1-\sqrt{2\sqrt{\rho}}n$. Since G is stable with respect to $u_1, \ldots, u_n$, we have $d_G(u_{n/3-\sqrt{2\sqrt{\rho}}n}) \geq n-1-\sqrt{2\sqrt{\rho}}n$.

Since $\rho < \varepsilon^{12}$ and H contains no independent set S such that $|S \cap Q| \geq n/3 - \varepsilon^2 n$ and $|S \cap P| \geq 2n/3 - \varepsilon^2 n$, we may form a matching M_0 of size $\sqrt{2\sqrt{\rho}}n$ in $H - \{u_1, \ldots, u_{n/3}\}$ by greedily choosing edges.

Since $d_G(u_{n/3-\sqrt{2\sqrt{\rho}}n}) \geq n-1-\sqrt{2\sqrt{\rho}}n$, $G - V(M_0)$ has a matching M of size $n/3 - \sqrt{2\sqrt{\rho}}n$ which can be found by greedily choosing distinct neighbors of u_i, $1 \leq i \leq n/3 - \sqrt{2\sqrt{\rho}}n$, in $V(G) \setminus V(M_0)$. Since $\{u, v\} \cup e \in E(H')$ for $u \in P, v \in Q$, and $e \in M$, we may extend M to a matching M' of size $|M|$ in $H' - M_0$. Then $M' \cup M_0$ gives a perfect matching in H'. $\square$

5. Almost perfect matchings

In this section, we use Lemmas 4.5 and 5.2 to find a "near regular" spanning subgraph of H. The discussion here follows that in [4]. We need to find a sequence of random subgraphs of a balanced $(1,3)$-partite 4-graph and use them to find a subgraph on which a "Rödl nibble" result can be applied.

First, we show how to find such a sequence. The following result is a lemma in [21], which was essentially the first of the two round randomization in [4].

Lemma 5.1. *Let $n > k > d > 0$ be integers with $k \geq 3$ and let H be a k-graph on n vertices. Take $n^{1.1}$ independent copies of R and denote them by R^i, $1 \leq i \leq n^{1.1}$, where R is chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$ and then deleting less than k vertices uniformly at random so that $|R| \in k\mathbb{Z}$. For each $X \subseteq V(H)$, let $Y_X := |\{i : X \subseteq R^i\}|$ and $\mathrm{DEG}^i_X := |\{e \setminus X : X \subseteq e$ and $e \setminus X \subseteq R^i\}|$. Then, with probability at least $1 - o(1)$, we have*

(*i*) $Y_{\{v\}} = (1 + o(1))n^{0.2}$ *for* $v \in V(H)$,
(*ii*) $Y_{\{u,v\}} \leq 2$ *for distinct* $u, v \in V(H)$,
(*iii*) $Y_e \leq 1$ *for* $e \in E(H)$,
(*iv*) $|R^i| = (1 + o(1))n^{0.1}$ *for* $i = 1, \ldots, n^{1.1}$, *and*
(*v*) *if* μ, ρ' *are constants with* $0 < \mu \ll \rho'$, $n/k - \mu n \leq m \leq n/k$, *and* $\delta_d(H) \geq \binom{n-d}{k-d} - \binom{n-d-m}{k-d} - \rho' n^{k-l}$, *then for any positive real* $\xi \geq 2\rho'$, *we have*

$$\mathrm{DEG}^i_D > \binom{|R^i| - d}{k-d} - \binom{|R^i| - d - |R^i|/k}{k-d} - \xi |R^i|^{k-d}$$

for all $i = 1, \ldots, n^{1.1}$ *and all* $D \in \binom{V(H)}{d}$.

Since we work with balanced $(1,3)$-partite 4-graphs, we need to make sure each random subgraph taken is also balanced. So we slightly modify the randomization process in the above lemma. We first fix an arbitrary small set $S \subseteq V(H)$. Each time we obtain a random copy R, we delete some vertices in $R \cap S$ so that the resulting graph is balanced. We can do so in a way that, with high probability, all properties in Lemma 5.1 remain (approximately) true.

Lemma 5.2. *Let n be a sufficiently large positive integer, and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$. Let $S \subseteq V(H)$ be a set of vertices such that $|S \cap Q| = n^{0.99}/3$ and $|S \cap P| = n^{0.99}$. Take $n^{1.1}$ independent copies of R_+ and denote them by R^i_+, $1 \le i \le n^{1.1}$, where R_+ is chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$. Define $R^i_- = R^i_+ \setminus S$ for $1 \le i \le n^{1.1}$.*

Then, with probability $1 - o(1)$, for any sequence R^i, $1 \le i \le n^{1.1}$, satisfying $R^i_- \subseteq R^i \subseteq R^i_+$, all of the following hold:

- *(i)* $|R^i| = (4/3 + o(1))n^{0.1}$ *for all* $i = 1, \ldots, n^{1.1}$.
- *(ii)* *For each* $X \subseteq V(H)$, *let* $Y_X := |\{i : X \subseteq R^i\}|$, *then,*
  - *(iia)* $Y_{\{v\}} \le (1 + o(1))n^{0.2}$ *for* $v \in V(H)$,
  - *(iib)* $Y_{\{v\}} = (1 + o(1))n^{0.2}$ *for* $v \in V(H) \setminus S$,
  - *(iic)* $Y_{\{u,v\}} \le 2$ *for distinct* $u, v \in V(H)$, *and*
  - *(iid)* $Y_e \le 1$ *for* $e \in E(H)$.
- *(iii)* *For each* $X \in \binom{V(H)}{2}$, *let* $\mathrm{DEG}^i_X = |N_H(X) \cap \binom{R^i}{2}|$. *If* $\rho > 0$ *is a constant and* $d_H(\{u,v\}) \ge \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ *for all* $v \in Q$ *and* $u \in P$, *then for any constant* $\xi \ge 5\rho$, *we have*

$$\mathrm{DEG}^i_{\{u,v\}} > \binom{|R^i \cap P| - 1}{2} - \binom{2|R^i \cap P|/3}{2} - \xi |R^i \cap P|^2,$$

for all $i = 1, \ldots, n^{1.1}$, $v \in Q$, *and* $u \in P$.

Proof. Note that $\mathbb{E}(|R^i_+|) = (4n/3) \cdot n^{-0.9} = 4n^{0.1}/3$, and

$$\mathbb{E}(|R^i_-|) = (4n/3 - 4n^{0.99}/3) \cdot n^{-0.9} = 4n^{0.1}/3 - 4n^{0.09}/3.$$

By Chernoff's inequality,

$$\mathbb{P}(|R^i_+| - 4n^{0.1}/3 \ge n^{0.095}) \le e^{-\Omega(n^{0.09})}$$

and

$$\mathbb{P}(|R^i_-| - (4n^{0.1}/3 - 4n^{0.09}/3) \leq -n^{0.095}) \leq e^{-\Omega(n^{0.09})}.$$

In particular, (i) holds with probability at least $1 - e^{-\Omega(n^{0.09})}$.

Let $Y^+_X := |\{i : X \subseteq R^i_+\}|$ for $X \subseteq V(H)$. Then $Y^+_X \sim Bi(n^{1.1}, n^{-0.9|X|})$ and $Y_X \leq Y^+_X$ for all $X \subseteq V(H)$, and $Y_X = Y^+_X$ for all $X \subseteq V(H) \setminus S$. Then by Lemma 5.1, (iic) and (iid) hold with probability $1 - o(1)$.

For each $v \in V(H)$, $\mathbb{E}(Y^+_{\{v\}}) = n^{0.2}$, thus by Chernoff's inequality,

$$\mathbb{P}\left(\left|Y^+_{\{v\}} - n^{0.2}\right| \geq n^{0.15}\right) \leq e^{-\Omega(n^{0.1})}.$$

Thus (iia) and (iib) hold with probability at least $1 - e^{-\Omega(n^{0.1})}$.

Let $\deg^i_X = \left|N_H(X) \cap \binom{R^i_-}{2}\right|$. To prove (iii), since n is sufficiently large, it suffices to show that for all $v \in Q$ and $u \in P$,

$$\deg^i_{\{u,v\}} > \binom{n^{0.1} - 1}{2} - \binom{2n^{0.1}/3}{2} - \xi n^{0.2}/2.$$

Conditioning on $|R^i_+| < 4n^{0.1}/3 - n^{0.095}$ and $|R^i_-| > (4n^{0.1}/3 - 4n^{0.01}/3) - n^{0.095}$ for all i, we have, for all $v \in Q$ and $u \in P$,

$$\begin{aligned}\mathbb{E}(\deg^i_{\{u,v\}}) &= d_{H-S}(\{u,v\})(n^{-0.9})^2 \\ &\geq (1 - o(1))\left(\binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2\right)(n^{-0.9})^2 \\ &\geq \binom{n^{0.1} - 1}{2} - \binom{2n^{0.1}/3}{2} - 2\rho n^{0.2},\end{aligned}$$

where the first inequality holds because $|S| = 4n^{0.99}/3$ (and, hence, $d_{H-S}(\{u,v\}) = (1 - o(1))d_H(\{u,v\})$). In particular, $\mathbb{E}\left(\deg^i_{\{u,v\}}\right) = \Omega(n^{0.2})$. Next, we apply Janson's Inequality (Theorem 8.7.2 in [6]) to bound the deviation of $\deg^i_{\{u,v\}}$. Write $\deg^i_{\{u,v\}} = \sum_{e \in N_H(\{u,v\})} X_e$, where $X_e = 1$ if $e \subseteq R^i_-$ and $X_e = 0$ otherwise. Then

$$\Delta := \sum_{e \cap f \neq \emptyset} \mathbb{P}(X_e = X_f = 1) \leq \binom{n-1}{2}\binom{2}{1}\binom{n-3}{1}(n^{-0.9})^3$$

and, thus, $\Delta = O(n^{0.3})$. By Janson's inequality, for any constant $\gamma > 0$,

$$\mathbb{P}\left(\deg^i_{\{u,v\}} \leq (1 - \gamma)\mathbb{E}(\deg^i_{\{u,v\}})\right) \leq e^{-\gamma^2 \mathbb{E}(\deg^i_{\{u,v\}})/(2 + \Delta/\mathbb{E}(\deg^i_{\{u,v\}}))} = e^{-\Omega(n^{0.1})}.$$

Since $\xi \geq 5\rho$ (and taking γ sufficiently small), the union bound implies that, with probability at least $1 - n^{2+1.1}e^{-\Omega(n^{0.1})}$, for all $v \in Q$ and $u \in P$ and for all $i \in [n^{1.1}]$,

$$\deg^i_{\{u,v\}} > \binom{n^{0.1}-1}{2} - \binom{2n^{0.1}/3}{2} - \xi n^{0.2}/2.$$

Thus, (iii) holds with probability at least

$$(1 - n^{1.1}e^{-\Omega(n^{0.09})})(1 - n^{2+1.1}e^{-\Omega(n^{0.1})}) > 1 - n^4 e^{-\Omega(n^{0.09})}.$$

Hence, it follows from union bound that, with probability at least $1 - o(1)$, (i)-(iii) hold for any sequence R^i, $1 \le i \le n^{1.1}$, satisfying $R^i_- \subseteq R^i \subseteq R^i_+$. $\square$

In order to apply Lemma 4.5, we need an additional requirement that the induced subgraphs R_i be balanced.

Lemma 5.3. *Let n, H, P, Q, S and R^i_+, R^i_-, $i \in [n^{1.1}]$, be given as in Lemma 5.2. Then, with probability $1 - o(1)$, for every $i \in [n^{1.1}]$, there exist subgraphs R_i such that $R^i_- \subseteq R^i \subseteq R^i_+$ and R^i is balanced.*

Proof. Recall that $|P| = n$, $|Q| = n/3$, $|S \cap P| = n^{0.99}$, and $|S \cap Q| = n^{0.99}/3$, and that R^i_+ is formed by taking each vertex of H independently and uniformly at random with probability $n^{-0.9}$. So for $i \in [n^{1.1}]$,

$$\begin{aligned}
&\mathbb{E}(|R^i_+ \cap P|) = n^{0.1},\\
&\mathbb{E}(|R^i_+ \cap P \cap S|) = n^{0.09},\\
&\mathbb{E}(|R^i_+ \cap Q|) = n^{0.1}/3, \quad \text{and}\\
&\mathbb{E}(|R^i_+ \cap P \cap S|) = n^{0.09}/3.
\end{aligned}$$

By Chernoff's inequality,

$$\begin{aligned}
&\mathbb{P}\left(||R^i_+ \cap P| - n^{0.1}| \ge n^{0.08}\right) \le e^{-\Omega(n^{0.06})},\\
&\mathbb{P}\left(||R^i_+ \cap P \cap S| - n^{0.09}| \ge n^{0.08}\right) \le e^{-\Omega(n^{0.07})},\\
&\mathbb{P}\left(||R^i_+ \cap Q| - n^{0.1}/3| \ge n^{0.08}\right) \le e^{-\Omega(n^{0.06})}, \quad \text{and}\\
&\mathbb{P}\left(||R^i_+ \cap Q \cap S| - n^{0.09}/3| \ge n^{0.08}\right) \le e^{-\Omega(n^{0.07})}.
\end{aligned}$$

Thus, with probability $1 - o(1)$, for all $i \in [n^{1.1}]$,

$$\begin{aligned}
&|R^i_+ \cap P| \in [n^{0.1} - n^{0.08}, n^{0.1} + n^{0.08}],\\
&|R^i_+ \cap P \cap S| = (1 + o(1))n^{0.09},\\
&|R^i_+ \cap Q| \in [n^{0.1}/3 - n^{0.08}, n^{0.1}/3 + n^{0.08}], \quad \text{and}\\
&|R^i_+ \cap Q \cap S| = (1 + o(1))n^{0.09}.
\end{aligned}$$

Therefore,

$$\big||R^i_+ \cap P| - 3|R^i_+ \cap Q|\big| \leq 4n^{0.08} < \min\{|R^i_+ \cap P \cap S|, |R^i_+ \cap Q \cap S|\}.$$

Hence, with probability $1 - o(1)$, R^i can be taken to be balanced for all $i \in [n^{1.1}]$. □

Another small difference between here and [4] is that condition (ii) in Lemma 5.2 is slightly weaker than the corresponding condition in [4]. In [4] all vertices have almost the same degree, but here a small portion of the vertices could have smaller degree. The following lemma reflects a slightly weaker conclusion due to this difference, and the proof mainly follows that of Claim 4.1 in [4].

Lemma 5.4. *Let n, H, S, R^i, $i = 1, \dots, n^{1.1}$ be given as in Lemma 5.3 such that each $H[R^i]$ is a balanced $(1,3)$-partite 4-graph and has a perfect fractional matching w^i. Then there exists a spanning subgraph H'' of $H' := \bigcup_{i=1}^{n^{1.1}} H[R^i]$ such that*

(i) $d_{H''}(u) \leq (1 + o(1))n^{0.2}$ *for* $u \in S$,
(ii) $d_{H''}(v) = (1 + o(1))n^{0.2}$ *for* $v \in V(H) \setminus S$, *and*
(iii) $\Delta_2(H'') \leq n^{0.1}$.

Proof. Let $H' = \bigcup_{i=1}^{n^{1.1}} H[R^i]$. By (iid) of Lemma 5.2, each edge of H is contained in at most one R^i. Let i_e denote the index i such that $e \subseteq R^i$ (if exists); and let $w^{i_e}(e) = 0$ when i_e is not defined. Let H'' be a spanning subgraph of H' obtained by independently selecting each edge e at random with probability $w^{i_e}(e)$.

For $v \in V(H'')$, let $I_v = \{i : v \in R^i\}$, $E_v = \{e \in H' : v \in e\}$, and $E^i_v = E_v \cap H[R^i]$. Then E^i_v, $i \in I_v$, form a partition of E_v. Hence, for $v \in V(H'')$,

$$d_{H''}(v) = \sum_{e \in E_v} 1 = \sum_{i \in I_v} \sum_{e \in E^i_v} X_e,$$

where $X_e \sim Be(w^{i_e}(e))$ is the Bernoulli random variable with $X_e = 1$ if $e \in E(H'')$ and $X_e = 0$ otherwise. Thus, since $\sum_{e \in E^i_v} w^i(e) = 1$ (as w^i is a perfect fractional matching in $H[R^i]$),

$$\mathbb{E}(d_{H''}(v)) = \sum_{i \in I_v} \sum_{e \in E^i_v} w^i(e) = \sum_{i \in I_v} 1.$$

Hence, $\mathbb{E}(d_{H''}(v)) = (1 + o(1))n^{0.2}$ for $v \in V(H) \setminus S$ (by (iib) of Lemma 5.2), and $\mathbb{E}(d_{H''}(v)) \leq (1 + o(1))n^{0.2}$ for $v \in S$ (by (iia) of Lemma 5.2). Now by Chernoff's inequality, for $v \in V(H) \setminus S$,

$$\mathbb{P}(|d_{H''}(v) - n^{0.2}| \geq n^{0.15}) \leq e^{-\Omega(n^{0.1})},$$

and for $v \in S$,

$$\mathbb{P}(d_{H''}(v) - n^{0.2} \geq n^{0.15}) \leq e^{-\Omega(n^{0.1})}.$$

Thus by taking union bound over all $v \in V(H)$, we have that, with probability $1-o(1)$, $d_{H''}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H) \setminus S$ and $d_{H''}(v) \leq (1+o(1))n^{0.2}$ for all $v \in S$.

Next, note that for distinct $u, v \in V(H)$,

$$d_{H''}(\{u,v\}) = \sum_{e \in E_u \cap E_v \cap E(H'')} 1 = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} X_e$$

and $\mathbb{E}(d_{H''}(\{u,v\})) = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} w^i(e)$. By (iic) in Lemma 5.2, $|I_u \cap I_v| \leq 2$. So $\mathbb{E}(d_{H''}(\{u,v\})) \leq |I_u \cap I_v| \leq 2$. Thus by Chernoff's inequality,

$$\mathbb{P}(d_{H''}(\{u,v\}) \geq n^{0.1}) \leq e^{-\Omega(n^{0.2})}.$$

Hence by a union bound $\Delta_2(H'') \leq n^{0.1}$ with probability $1-o(1)$.

Therefore, with probability $1-o(1)$, H'' satisfies (i), (ii), and (iii). $\square$

We also need the following result attributed to Pippenger [23], stated as Theorem 4.7.1 in [6]. A *cover* in a hypergraph H is a set of edges whose union is $V(H)$.

Lemma 5.5 *(Pippenger and Spencer, 1989). For every integer $k \geq 2$ and reals $r \geq 1$ and $a > 0$, there are $\gamma = \gamma(k,r,a) > 0$ and $d_0 = d_0(k,r,a)$ such that for every n and $D \geq d_0$ the following holds: Every k-uniform hypergraph $H = (V,E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:*

(1) For all vertices $x \in V$ but at most γn of them, $d(x) = (1 \pm \gamma)D$;
(2) For all $x \in V$, $d(x) < rD$;
(3) For any two distinct $x, y \in V$, $d(x,y) < \gamma D$;

contains a cover of at most $(1+a)(n/k)$ edges.

Note that H contains a cover of at most $(1+a)(n/k)$ edges implies that H contains a matching of size at least $(1-(k-1)a)(n/k)$ (see, for example, [23]). Now we are ready to state and prove the main result of this section, which will be used to find an almost perfect matching after deleting an absorber.

Lemma 5.6. *Let $\sigma > 0$ and $0 < \rho \leq \varepsilon/4 \ll 1$, let n be a sufficiently large positive integer, and let H be a $(1,3)$-partite 4-graph with partition classes Q, P such that $3|Q| = |P| = n$. Suppose H is not ε-close to any $H_{1,3}(n, n/3)$ with $V(H_{1,3}(n,n/3))$ and $d_H(\{u,v\}) \geq \binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2$ for all $v \in Q$ and $u \in P$. Then H contains a matching covering all but at most σn vertices.*

Proof. By Lemmas 5.2 and 5.3, we have the random subgraphs R^i, $i \in [n^{1.1}]$, such that, with probability $1 - o(1)$, all R^i satisfies the properties in Lemmas 5.2 and 5.3. In particular, $H[R_i]$ is balanced with respect to the partition classes Q, P.

Next, by Lemma 4.2, H is $(\mathcal{F}, \varepsilon/6))$-dense, where

$$\mathcal{F} = \{A \subseteq V(H) : |A \cap Q| \geq (1/3 - \varepsilon/8)n \text{ and } |A \cap P| \geq (2/3 - \varepsilon/8)n\}.$$

Note that

$$e(H) = \sum_{v \in Q} \sum_{u \in P} d_H(\{u, v\})/3 \geq (n/3)(n/3) \left(\binom{n-1}{2} - \binom{2n/3}{2} - \rho n^2 \right) \geq n^4/100.$$

Hence by Lemma 4.3 (and choosing suitable $\alpha_1, \alpha_2, \gamma$), we see that, with probability $1 - o(1)$, for all $i \in [n^{1.1}]$ and for all independent sets J in $H[R^i]$, $|J \cap P| \leq (\alpha_1 + \gamma + o(1))n^{0.1} < n/3 - \varepsilon^2 n$ or $|J \cap Q| \leq (\alpha_2 + \gamma + o(1))n^{0.1} < 2n/3 - \varepsilon^2 n$.

Moreover, by *(iii)* of Lemma 5.2, with probability $1 - o(1)$, $d_{H[R^i]}(\{u, v\}) > \binom{|R^i \cap P| - 1}{2} - \binom{2|R^i \cap P|/3}{2} - \xi |R^i \cap P|^2$ for all $u \in P$ and $v \in Q$. Hence, by Lemma 4.5, $H[R^i]$ contains a perfect fractional matching for all $i \in [n^{1.1}]$.

Thus by Lemma 5.4, there exists a spanning subgraph H'' of $\bigcup_{i=1}^{n^{1.1}} H[R^i]$ such that $d_{H''}(u) \leq (1 + o(1))n^{0.2}$ for each $u \in S$, $d_{H''}(v) = (1 + o(1))n^{0.2}$ for each $v \in V(H) \setminus S$, and $\Delta_2(H'') \leq n^{0.1}$. Hence, by Lemma 5.5 (by setting $D = n^{0.2}$), H'' contains a cover of at most $(1 + a)(n/3)$ edges, where a is a constant satisfying $0 < a < \sigma/3$.

Now by greedily deleting intersecting edges, we obtain a matching of size at least $(1 - 3a)(n/3)$. Hence H contains a matching covering all but at most σn, provided n is sufficiently large. □

6. Conclusion

Proof of Theorem 1.2. By Lemma 2.3, we may assume H is not ε-close to any $H_{1,3}(n, n/3)$, where $\varepsilon \ll 1$. By Lemma 3.1, $H_{1,3}(\mathcal{F})$ has a matching M' such that, for some $0 < \rho' \ll \rho \ll \varepsilon$, $|M'| \leq \rho n/4$ and, for any $S \subseteq V(H_{1,3}(\mathcal{F}))$ with $|S| \leq \rho' n$ and $3|S \cap Q| = |S \cap P|$, $H_{1,3}(\mathcal{F})[S \cup V(M')]$ has a perfect matching.

Let $H_1 = H - V(M')$. Then $d_{H_1}(\{u, v\}) \geq \binom{n'-1}{2} - \binom{2n'/3}{2} - \rho(n')^2$ for all $v \in Q \cap V(H_1)$ and $u \in P \cap V(H_1)$, and H_1 is not (2ε)-close to $H_{1,3}(n', n'/3)$, where $n' = (1 - o(1))n$.

By Lemma 5.6, H_1 contains a matching M_1 covering all but at most σn vertices, where we choose σ so that $0 < \sigma < \rho'$. Now $H[(V(H_1) \setminus V(M_1)) \cup V(M)]$ has a perfect matching M_2. Clearly, $M_1 \cup M_2$ forms a perfect matching in H. □

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