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Peak Infection Time for a Networked SIR Epidemic with Opinion Dynamics

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Abstract— **We propose an SIR epidemic model coupled with opinion dynamics to study an epidemic and opinions spreading in a network of communities. Our model couples networked SIR epidemic dynamics and opinions towards the severity of the epidemic. We develop an epidemic-opinion based threshold condition to capture the moment when a weighted average of the epidemic states starts to decrease exponentially fast over the network, namely the peak infection time. We define an effective reproduction number to characterize the behavior of the model through the peak infection time. We use both analytical and simulation-based results to illustrate that the opinions reflect the recovered levels within the communities after the epidemic dies out.**

I. Introduction

The COVID-19 pandemic has caused severe suffering across the world in both public health and the economy. These hardships have motivated researchers from various backgrounds to study the viral pathogenesis, epidemic spreading models, mitigation strategies [1], [2], etc. Besides the COVID-19 pandemic, it is relevant to build dynamic models to study viral spreading processes to predict future outbreaks and to design control algorithms to mitigate the epidemic [3]. One of the popular ways to capture viral spreading processes is by using network-based compartmental models [4]. In networked epidemic models, the infection rates, healing rates, and network structures all play important roles in determining the behaviors of the epidemic spreading processes. Recently, social factors such as human awareness [5], opinion interactions [6], etc., are being taken into consideration when modeling epidemic spreading over networks. In this work, we will consider the classical networked Susceptible-Infected-Recovered (SIR) model coupled with opinion dynamics.

In previous works, researchers have studied the networked SIR models from different perspectives. In [4], the authors study the dynamical behavior of the networked SIR model, and analyze the threshold conditions for an epidemic to increase or decrease. In [7], the authors leverage testing data to estimate the key parameters of the networked SIR model to design resource allocation methods to mitigate the epidemic. Recall that people's beliefs in the seriousness of the epidemic is one important social factor that will have an impact on the spreading process. For example, [8] studies the correlations between the awareness-driven behaviors during the COVID-19 pandemic and the spreading of the COVID-19. Further, [9] constructs a multiplex network with a networked SEIV model coupled with opinion dynamics, then explores the disease-free equilibrium. Inspired by the health-belief model developed by social scientists [10], where people's behavior in the pandemic will be influenced by their beliefs in the seriousness of the epidemic, [6] and [11] develop a networked SIS model with cooperative opinion dynamics, and both cooperative and antagonistic opinion dynamics, respectively. The authors in [6] study both the disease-free and non-disease-free equilibria of the model. Our previous work, [11], defines an opinion-dependent reproduction number to explore further the effect of the antagonistic opinions in SIS epidemic spreading. Based on the health-belief model, we will develop a networked SIR model coupled with cooperative opinion dynamics.

The main contributions of this work can be summarized as follows: we define a networked SIR epidemic model coupled with opinion dynamics. Then, we develop two concepts: an effective reproduction number and a peak infection time. We utilize the effective reproduction number to explore epidemic spreading by studying the peak infection time. In particular, different from the previous works [6], [11], where stability and convergence of the equilibria are the main focuses, this work emphasizes more on exploring the transient behavior of the epidemic, characterized by the effective reproduction number and the peak infection time. Additionally, we further analyze the opinion states via the behavior of the epidemic.

We organize the paper as follows: In Section II, we introduce the networked SIR model coupled with opinion dynamics and formulate the problems of interest. Section III studies the equilibrium of the developed model. Based on the model, Section III defines the effective reproduction number and peak infection time. Section III further explores the epidemic's dynamical behavior by relating the effective reproduction number and the peak infection time. Section IV illustrates the results of the paper through simulations. Section V concludes the paper and outlines research directions. Note that all the proofs are included in [12].

Notation

For any positive integer n, we use $[n]$ to denote the index set $\{1, 2, \dots, n\}$. We view vectors as column vectors and write $x^\top$ to denote the transpose of a column vector x. For a vector x, we use x_i to denote the ith entry. For any matrix $M \in \mathbb{R}^{n\times n}$, we use $[M]_{i,:}$, $[M]_{:,j}$, $[M]_{ij}$, to denote its ith row, jth column, and ijth entry, respectively. We use $\tilde{M} = \text{diag}\{m_1, \dots, m_n\}$ to represent a diagonal matrix

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$\tilde{M} \in \mathbb{R}^{n \times n}$ with $[\tilde{M}]_{ii} = m_i$, $\forall i \in [n]$. We use $\mathbf{0}_n$ and $\mathbf{1}_n$ to denote the vectors whose entries all equal 0 and 1, respectively, and I_n to denote the $n \times n$ identity matrix.

For a real square matrix M, we use $\rho(M)$ and $\sigma(M)$ to denote its spectral radius and spectral abscissa (the largest real part among its eigenvalues), respectively. For any two vectors $v, w \in \mathbb{R}^n$, we write $v \geq w$ if $v_i \geq w_i$, and $v \gg w$ if $v_i > w_i$, $\forall i \in [n]$. The comparison notations between vectors are used for matrices as well, for instance, for $A, B \in \mathbb{R}^{n \times n}$, $A \gg B$ indicates that $A_{ij} > B_{ij}$, $\forall i, j \in [n]$.

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let matrix $A \in \mathbb{R}^{n \times n}$, $[A]_{ij} = a_{ij}$, denote the adjacency matrix of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $a_{ij} \in \mathbb{R}_{>0}$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Graph $\mathcal{G}$ does not allow self-loops, i.e., $a_{ii} = 0$, $\forall i \in [n]$. Let $k_i = \sum_{j \in \mathcal{N}_i} |a_{ij}|$, where $\mathcal{N}_i = \{v_j | (v_j, v_i) \in \mathcal{E}\}$ denotes the neighbor set of v_i and $|a_{ij}|$ denotes the absolute value of a_{ij}. The graph Laplacian of $\mathcal{G}$ is defined as $L \triangleq \tilde{K} - A$, where $\tilde{K} \triangleq \text{diag}\{k_1, \ldots, k_n\}$.

II. Modeling and Problem Formulation

In this section, we introduce the networked SIR model coupled with opinion dynamics. We also formulate the problem to be analyzed in this work.

We start by defining a *disease transmission network* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as a weighted directed graph with a node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ representing n disjoint communities and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ representing disease-transmitting contacts over $\mathcal{V}$. We denote the weight of each edge (v_j, v_i) as β_{ij}. Then, a basic continuous-time networked SIR model on graph $\mathcal{G}$, which was studied in [4], can be defined as:

$$\dot{s}_i(t) = -s_i(t) \sum_{j \in \mathcal{N}_i} \beta_{ij} x_j(t), \tag{1a}$$

$$\dot{x}_i(t) = s_i(t) \sum_{j \in \mathcal{N}_i} \beta_{ij} x_j(t) - \gamma_i x_i(t), \tag{1b}$$

$$\dot{r}_i(t) = \gamma_i x_i(t), \tag{1c}$$

where $(s_i(t), x_i(t), r_i(t)) \in [0, 1]$, $\forall i \in [n]$ are the states indicating the proportion of susceptible, infected, and recovered population in community $i \in [n]$ at time $t \geq t_0$, respectively. Moreover, $\beta_{ij} \in \mathbb{R}_{\geq 0}$ is the *transmission rate* from community j to i, and $\gamma_i \in \mathbb{R}_{\geq 0}$ is the *recovery rate* of community i. Note that (1) satisfies $s(t) + x(t) + r(t) = 1$ $\forall t \geq t_0 \in \mathbb{R}_{\geq 0}$ as a result of the assumption that for any initial condition we have $s(t_0) + x(t_0) + r(t_0) = 1$, and $\dot{s}(t) + \dot{x}(t) + \dot{r}(t) = 0$ $\forall t \in \mathbb{R}_{\geq 0}$.

Similarly, we define the *opinion spreading network* as a directed graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$, where the edge set $\bar{\mathcal{E}} \subseteq \mathcal{V} \times \mathcal{V}$ represents the opinion-disseminating interactions over the same n communities. Each edge in the graph is weighted by $\bar{a}_{ij} \in \mathbb{R}_{\geq 0}$ indicating the opinion-disseminating influence from node j to node i. Let $o_i(t) \in [0, 1]$, $\forall i \in [n]$, $t \geq t_0$, denote the belief of community i on the severity of the epidemic at time t, where $o_i(t) = 1$ indicates community i considers the epidemic to be extremely serious, while $o_i(t) = 0$ implies community i believes the epidemic is not serious at all. We adapt Abelson's models of opinion dynamics from [13, Equation (10)], where $\bar{\mathcal{N}}_i = \{v_j | (v_j, v_i) \in \bar{\mathcal{E}}\}$:

$$\dot{o}_i(t) = \sum_{j \in \bar{\mathcal{N}}_i} \bar{a}_{ij}(o_j(t) - o_i(t)). \tag{2}$$

We assume that the n communities share a homogeneous minimum incoming transmission rate $\beta_{\min}$ and a homogeneous minimum recovery rate $\gamma_{\min}$, where $\beta_{\min}$ corresponds to the strongest belief of a community in the severity of the epidemic $o_i(t) = 1$, while $\gamma_{\min}$ corresponds to the weakest belief of a community in the severity of the epidemic $o_i(t) = 0$. To couple the networked SIR model with the opinion dynamics, we employ the health-belief model, which is the best known and most widely used theory in health behavior research [10]. We define a networked SIR model influenced by the opinion dynamics as:

$$\dot{s}_i(t) = -s_i(t) \sum_{j \in \mathcal{N}_i} (\beta_{ij} - (\beta_{ij} - \beta_{\min}) o_i(t)) x_j(t), \tag{3a}$$

$$\begin{aligned} \dot{x}_i(t) = s_i(t) &\sum_{j \in \mathcal{N}_i} (\beta_{ij} - (\beta_{ij} - \beta_{\min}) o_i(t)) x_j(t) \\ &- (\gamma_{\min} + (\gamma_i - \gamma_{\min}) o_i(t)) x_i(t). \end{aligned} \tag{3b}$$

In (3), the transmission rate of community i, is obtained through the linear interpolation between β_{ij} and β_{min} scaled by the level of community i's belief in the seriousness of the epidemic, $o_i(t)$. A higher $o_i(t)$ will lead to lower transmission rates for community i. A similar interpretation can apply to the healing rates.

Notice that $(1 - s_i(t)) = x_i(t) + r_i(t)$, $t \geq t_0$, $\forall i \in [n]$, captures the proportion of the population that are infected/have been infected with the epidemic. Hence, $(1 - s_i(t))$ captures the total *infection level* within community i. By modifying the opinion dynamics in (3) via cooperating the infection level:

$$\dot{o}_i(t) = (1 - s_i(t) - o_i(t)) + \sum_{j \in \bar{\mathcal{N}}_i} (o_j(t) - o_i(t)), \tag{4}$$

where a higher proportion of the infected plus recovered population within community i will lead to a stronger belief in the seriousness of the epidemic, and vice versa.

We have presented the epidemic-opinion model in (3)-(4), then we can state the problem of interest in this work. We are interested in exploring the mutual influence between the epidemic spreading over the graph $\mathcal{G}$ of n communities in (3), and the opinions of the n communities about the epidemic captured by graph $\bar{\mathcal{G}}$ in (4). In this paper, we will:

1) analyze the equilibria of the system in (3)-(4). In particular, we connect the opinion states at the equilibrium to the infection level of the communities;
2) define an effective reproduction number to characterize the spreading of the disease. In particular, we explore the transient behavior of the epidemic-opinion model by leveraging peak infection time;
3) illustrate the results through simulations.

The analysis presented in this work can provide insights for decision-makers who aim to analyze disease spreading and its coupling with the public's opinion towards the epidemic.

III. Main Results

This section examines the mutual influence between the epidemic dynamics in (3), and the opinion dynamics in (4). Particularly, we construct the compact form of the incorporated system to define an effective reproduction number to explore the peak infection time of the model. We also analyze the evolution of the epidemic by using the effective reproduction number and peak infection time.

We write (3) and (4) in a compact form as follows:

$$\dot{s}(t) = -(\tilde{S}(t)(B - \tilde{O}(t)(B - B_{\min})))x(t), \tag{5a}$$

$$\dot{x}(t) = (\tilde{S}(t)(B - \tilde{O}(t)(B - B_{\min})))x(t) - (G_{\min} + (G - G_{\min})\tilde{O}(t))x(t), \tag{5b}$$

$$\dot{o}(t) = (\mathbf{1}_n - s(t)) - (\bar{L} + I_n)o(t), \tag{5c}$$

where $\tilde{S}(t) = \text{diag}(s(t))$, $\tilde{O}(t) = \text{diag}(o(t))$, $G_{\min}$ and G are diagonal matrices, with $[G_{\min}]_{ii} = \gamma_{\min}$, and $[G]_{ii} = \gamma_i$, $\forall i \in [n]$. Note that $\bar{L}$ is the Laplacian matrix of the opinion spreading graph $\bar{\mathcal{G}}$. By defining $B(o(t)) = (B - \tilde{O}(t)(B - B_{\min}))$, $G(o(t)) = (G_{\min} + (G - G_{\min})\tilde{O}(t))$,

$$\dot{s}(t) = -(\tilde{S}(t)B(o(t)))x(t), \tag{6a}$$

$$\dot{x}(t) = \tilde{S}(t)B(o(t))x(t) - G(o(t))x(t). \tag{6b}$$

For the epidemic spreading process, we assume that community i can pass the virus to community j through at least one directed path in the network $\mathcal{G}$, $\forall i, j \in [n]$, $i \neq j$. For the opinion spreading process, we assume that community i can affect community j's opinion through at least one directed path in $\bar{\mathcal{G}}$. Therefore, we have the following assumption for the epidemic and opinion spreading over the communities:

Assumption 1. *Suppose* $\forall i \in [n]$, $s_i(0), x_i(0), o_i(0) \in [0, 1]$, $s_i(0) + x_i(0) + r_i(0) = 1$, $\gamma_i \geq \gamma_{\min} > 0$, *and* $\beta_{ij} \geq \beta_{\min} > 0$, $\forall j \in \mathcal{N}_i$. *Further, both* $\mathcal{G}$ *and* $\bar{\mathcal{G}}$ *are strongly connected.*

A. Equilibrium

First we show the model in (5) is well-defined.

Lemma 1. *For the epidemic-opinion model in* (5)*, if* $(s_i(0), x_i(0), o_i(0)) \in [0, 1]$, *and* $s_i(0) + x_i(0) + r_i(0) = 1$, *then* $(s_i(t), x_i(t), o_i(t)) \in [0, 1]$, $\forall t > 0$, $\forall i \in [n]$.

Lemma 2. *If* $s_i(0), x_i(0), o_i(0) \in [0, 1]$, $\forall i \in [n]$, *the susceptible states,* $s_i(t)$, *are monotonically decreasing.*

After considering the monotonicity of the susceptible population, we next study the equilibria of the epidemic-opinion model.

Lemma 3. *The equilibria of the epidemic-opinion model in* (5) *take the form* $(s_e, \mathbf{0}_n, (\bar{L}+I_n)^{-1}(\mathbf{1}_n - s_e))$, *where* $[s_e]_i \in [0, 1]$ *and* $[(\bar{L} + I_n)^{-1}(\mathbf{1}_n - s_e))]_i \in [0, 1]$, $\forall i \in [n]$.

Lemma 3 shows that there are infinite equilibria for the epidemic-opinion model captured by (5). In particular, the lemma indicates that the opinion states of the communities at the equilibrium can be uniquely evaluated as a function of the steady-state susceptible population in the communities. The following lemma further characterizes the condition that the communities reach a consensus on their opinions, i.e., the opinion states are the same when the epidemic disappears.

Lemma 4. *The communities will reach consensus on their opinions if and only if all the communities have the same proportion of infections, captured by the equilibria* $(s_e, \mathbf{0}_n, \mathbf{1}_n - s_e)$, *where* $[s_e]_i = [s_e]_j$ $\forall i \neq j$.

Lemma 3 and Lemma 4 summarize the equilibria of the epidemic-opinion model in (5). In particular, the lemmas show that the communities' beliefs in the seriousness of the epidemic can reflect the infection level. More importantly, the communities will reach consensus on the seriousness of the epidemic if and only if the epidemic caused the same proportion of infected population in all communities. Under this situation, the belief on the seriousness of the epidemic is proportional to the proportion of the recovered population in all communities, characterized by $o_e = \mathbf{1}_n - s_e$.

Remark 1. *The communities can rarely reach a consensus of their opinions on the epidemic's severity since it will be implied by Lemma 4 that every community has the same infection level, which is unusual. However, one exception is when every community is fully infected, then all communities will agree that the epidemic is utterly severe* ($o_e = \mathbf{1}_n$)*.*

B. Effective Reproduction Number

The effective reproduction number of the model characterizes the dynamical behavior of the system. We introduce the following lemmas before formally defining this notion.

Definition 1. *[Effective Reproduction Number* $R_o(t)$*] Let* $R_o(t) = \rho(G^{-1}(o(t))\tilde{S}(t)B(o(t)))$, $\forall t \geq t_0$, *denote the effective reproduction number, where* $G(o(t))$, $\tilde{S}(t)$, *and* $B(o(t))$ *are defined in* (5b)*.*

Note that the effective reproduction number $R_o(t)$ depends not only on the proportion of the susceptible population $s(t)$, but also on the evolving of the opinion states $o(t)$.

Proposition 1. *The effective reproduction number* $R_o(t)$ *has the following properties:*

1) *If*
$$G^{-1}(o(t_2))\tilde{S}(t_2)B(o(t_2)) \geq G^{-1}(o(t_1))\tilde{S}(t_1)B(o(t_1)),$$
then $R_o(t_2) \geq R_o(t_1)$;
2) $R_o(t)$ *is strictly monotonically decreasing with respect to* $s(t)$, $\forall t \geq t_0$;
3) *If* $o(t_1) \leq o(t_2)$, $\forall t_1 < t_2$, *then* $R_o(t_2) \leq R_o(t_1)$.

The effective reproduction number is influenced by both the opinion states and the proportion of the susceptible population. In particular, when the opinions are fixed, the susceptible proportion will always ensure that the effective reproduction number decreases, since the recovered population will not be infected again. The opinion states will also have an influence on the change of the effective reproduction number in both directions: higher opinion states will lead to a lower effective reproduction number, and vice versa. As we mentioned in Section II, communities with stronger beliefs in the seriousness of the epidemic will take actions to avoid infections, leading to a lower effective reproduction number, and vice versa. Further, when all communities think the epidemic is extremely serious, $o(t) = o_{\max} = \mathbf{1}_n, \forall t \geq t_0$. When all communities think the epidemic is not worth treating at all during the pandemic, $o(t) = o_{\min} = \mathbf{0}_n$, $\forall t \geq t_0$. Under the two extreme situations, the effective reproduction number satisfies the following result.

Corollary 1. *For all $t \geq t_0$, the effective reproduction number $R_o(t)$ satisfies $R_{\min}(t) \leq R_o(t) \leq R_{\max}(t)$, where*

$$R_{\min}(t) = \rho(G^{-1}(o_{\max})\tilde{S}(t)B(o_{\max})),$$

$$R_{\max}(t) = \rho(G^{-1}(o_{\min})\tilde{S}(t)B(o_{\min})).$$

Corollary 1 indicates that, given any time t, if the proportion of the susceptible population of each community are the same, the effective reproduction number is determined by the opinion states, where stronger beliefs in the seriousness of the epidemic lead to a lower effective reproduction number, and vice versa. Compared to the classical SIR model [14], where the effective reproduction number is monotonically decreasing with respect to the decreasement of the proportion of the susceptible population, under the influence of the opinions, $R_o(t)$ defined in this work may not monotonically decrease. Therefore, $R_o(t)$ can lead to more diverse behaviors in the epidemic spreading process. In order to analyze the dynamical behavior of the epidemic-opinion model, we define a concept called peak infection time to characterize the influence of the effective reproduction number $R_o(t)$ in determining the behavior of the epidemic.

C. Peak Infection Time

To connect the effective reproduction number $R_o(t)$ to the behavior of the epidemic-opinion model, we denote

$$\sigma(t) = \sigma(\tilde{S}(t)B(o(t)) - G(o(t))), \tag{7}$$

as the spectral abscissa of $(\tilde{S}(t)B(o(t)) - G(o(t)))$ and $p(t)$ as the corresponding normalized left eigenvector $\forall t \geq t_0$. From Assumption 1 and [15, Sec. 2.1 and Lemma 2.3], $(\tilde{S}(t)B(o(t)) - G(o(t)))$ is an irreducible Metzler matrix, thus $\sigma(t)$, $\forall t \geq t_0$, is a positive real eigenvalue. Additionally, the normalized left eigenvector $p(t)$ satisfies $p(t) \gg \mathbf{0}_n$ and $p^\top(t)\mathbf{1}_n = 1$, $\forall t \geq t_0$. Then, we define a weighted average of the epidemic states, for a given $t_1 \in [t_0, t_2]$, $p^\top(t_1)x(t)$ as a metric to reflect the trend of the epidemic over the time interval $[t_0, t_2]$. Based on the properties of $\sigma(t)$ and $p(t)$, we have $p^\top(t_1)x(t) \geq 0$, $\forall t \geq t_0$ and $p^\top(t_1)x(t) = 0$ if and only if $x(t) = \mathbf{0}_n$. Therefore, $p^\top(t_1)x(t)$ reflects the overall trend of the epidemic spreading over the time interval $[t_0, t_2]$, and $p^\top(t_1)x(t) = 0$ if and only if the epidemic has died out.

Definition 2. *[Peak Infection Time t_p] A peak infection time t_p is defined as a turning point in $[t_0, t_1]$, where $p^\top(t_p)x(t)$ is increasing for all $t \in [t_0, t_p)$ and $p^\top(t_p)x(t)$ is decreasing for all $t \in (t_p, t_1]$, for sufficiently small time intervals $(t_p - t_0) > 0$ and $(t_1 - t_p) > 0$.*

The peak infection time describes a point where the weighted average of the infected proportions $p^\top(t_p)x(t)$ over the communities reaches a local peak value over $[t_0, t_1]$.

Theorem 1. *Given a peak infection time t_p, we have $R_o(t_p) = 1$, $R_o(t) > 1$, for $t \in [t_0, t_p)$ and $R_o(t) < 1$, for $t \in (t_p, t_1]$, for $t_p - t_0 > 0$ and $t_1 - t_p > 0$ sufficiently small.*

Note that $R_o(t) = 1$ is a necessary condition for the peak infection time, thus the condition does not guarantee that t is the peak infection time. From Proposition 1, $R_o(t)$ is not a monotonic function with respect to t. Consider the case that $R_o(t_1) = 1$, if, for $\epsilon > 0$, $R_o(t_1 - \epsilon) < 1$ and $R_o(t_1 + \epsilon) > 1$, the time t_1 is not the peak infection time. Additionally, from Lemma [15, Sec. 2.1 and Lemma 2.3], $p(t_p)$ is unique for a peak infection time t_p.

For $\forall t \in [t_1, t_2]$, from Lemma 2 and (6b), we have

$$(\tilde{S}(t_1)B(o_{\min}) - G(o_{\min})) \geq (\tilde{S}(t)B(o(t)) - G(o(t))). \tag{8}$$

Based on Corollary 1, $R_{\min} \leq R_o(t) \leq R_{\max}(t)$, $\forall t \geq t_0$, we define $\sigma_{\max}(t) = \sigma(\tilde{S}(t)B(o_{\min}) - G^{-1}(o_{\min}))$, corresponding to $R_{\max}(t)$. Since $(\tilde{S}(t)B(o(t)) - G^{-1}(o(t))$ is a Metzler matrix $\forall t$, from [16, Sec. 1, Lemma 2], we have $\sigma_{\max}(t_1) \geq \sigma_{\max}(t) \geq \sigma(t)$, $\forall t \geq t_1$. Then, we define $p_{\max}(t_1)$ corresponding to $\sigma_{\max}(t_1)$, and multiplying $p_{\max}(t_1)$, on both sides of (6b),

$$\frac{d}{dt}(p^\top_{\max}(t_1)x(t)) = p^\top_{\max}(t_1)((\tilde{S}(t)B(o(t)) - G(o(t)))x(t)).$$

Then, based on (8),

$$\begin{aligned}\frac{d}{dt}(p^\top_{\max}(t_1)x(t)) &\leq p^\top_{\max}(t_1)((\tilde{S}(t_1)B(o_{\min}) - G(o_{\min}))x(t)) \\ &= \sigma_{\max}(t_1)p^\top_{\max}(t_1)x(t),\end{aligned} \tag{9}$$

which leads to

$$p^\top_{\max}(t_1)x(t) \leq p^\top_{\max}(t_1)x(t_1)e^{(\sigma_{\max}(t_1))t},$$

for any $t \geq t_1$. The inequality listed above indicates that, when $\sigma_{\max}(t_1) < 0$, the weighted average $p^\top_{\max}(t_1)x(t)$ will decrease exponentially fast to zero, $\forall t \geq t_1$, implying that $x(t)$ will decrease exponentially fast to zero. From [17, Prop. 1] and Corollary 1, $\sigma_{\max}(t_1) < 0$ leads to $R_{\max}(t_1) < 1$, which guarantees $R_o(t) < 1$, $\forall t \geq t_1$. Hence, we have the following corollary, where we define t_1 as t_f.

Corollary 2. *If $R_{\max}(t_f) < 1$, there will exist no peak infection time in (t_f, ∞), and $p^\top_{\max}(t_f)x(t)$ $\forall t \geq t_f$ will monotonically decrease to zero exponentially fast, indicating that the epidemic will die out exponentially fast.*

Corollary 2 connects $R_o(t)$ to the behavior of the epidemic process. In particular, for $t_f = 0$, at the beginning stages of the epidemic, even with every community ignoring the epidemic, we still have $R_{\max}(0) < 1$ which means that the epidemic is serious, and will disappear quickly.

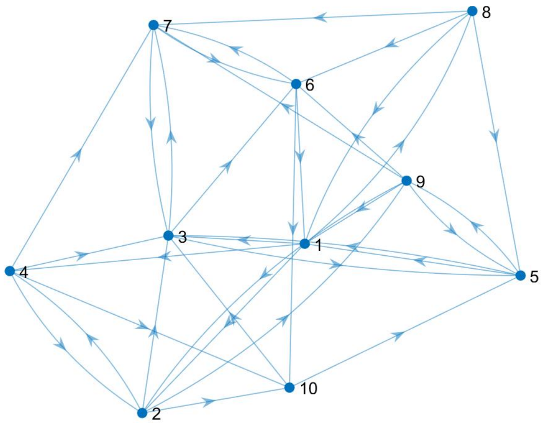

Fig. 1: The graph topology $\mathcal{G}$ of the simulated epidemic and opinion interactions

Theorem 1 also implies that, if the effective reproduction number $R_{\min}(t)$ at the beginning stages of the epidemic is greater than 1, before disappearing, there must exist at least one peak infection time. This phenomenon is captured by the following corollary.

Corollary 3. *If $R_{\min}(0) > 1$, then*

1) *there will be at least one peak infection time t_p;*
2) *$p_{\min}^{\top}(t_0)x(t)$ will increase exponentially fast before reaching a peak infection time t_p.*

By combining Corollaries 2 and 3 with Theorem 1, we can connect the behavior of the system in (5) to the peak infection time of the system in the following theorem.

Theorem 2. *For the epidemic-opinion model in* (5)*, the system will converge to an equilibrium $(s_e, \mathbf{0}_n, (\bar{L}+I_n)^{-1}(\mathbf{1}_n - s_e))$, and the convergence is exponentially fast.*

Combined with Theorem 1, Corollaries 2, and 3, Theorem 2 implies that the epidemic will die out eventually, but the effective reproduction number $R_o(t)$ will determine whether there will be an outbreak or the epidemic will die out directly.

IV. Simulation

In the section, we will illustrate the main results developed in this work via simulations. Consider the epidemic coupled with opinions spreading over ten communities. The epidemic and opinion spreading network satisfies Assumption 1, and share the same graph topology $\mathcal{G}$ as shown in Fig. 1.

We set the initial condition $x(0) = 0.01 \times \mathbf{1}_n$, $s(0) = 0.99 \times \mathbf{1}_n$, and $o(0) = \mathbf{0}_n$. We also set the parameters $\beta_{\min} = 0.2$, $\gamma_{\min} = 0.07$, and each β_{ij} is uniformly sampled from $[0.2, 1]$. Similarly, each γ_i is uniformly sampled from $[0.07, 0.1]$. We apply only unit edge weights to the opinion graph in all simulations.

Fig. 2(a) shows that the proportion of the susceptible population in all communities decreases monotonically as claimed in Lemma 2, and Fig. 2(b) shows the evolution of the epidemic states, with the weighted average state $x_w(t) = p^{\top}(t_p)x(t)$ $\forall t \geq t_0$ being captured by the dashed line. Note that we use $p^{\top}(t_p)$ for the entire time interval. Furthermore,

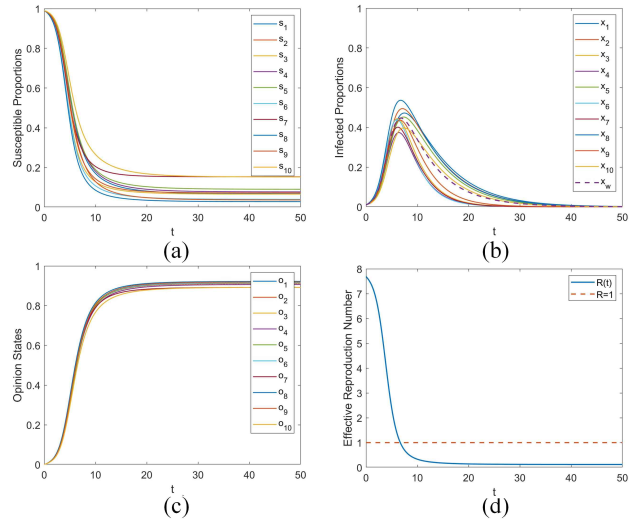

Fig. 2: Typical evolution of epidemics and opinions

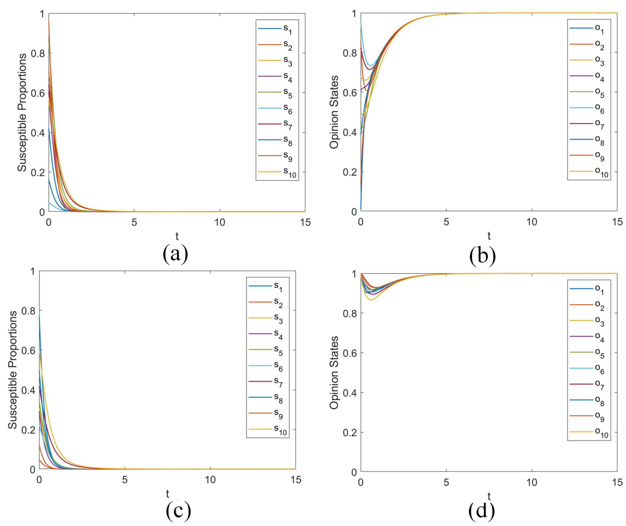

Fig. 3: States convergence with wide-spread initial opinions

the trend of the weighted average of the epidemic states follows the changes of the effective reproduction number $R_o(t)$ in Fig. 2(d): $x_w(t) = p^{\top}(t_p)x(t)$ is increasing when $R_o(t) > 1$; $x_w(t) = p^{\top}(t_p)x(t)$ is decreasing when $R_o(t) < 1$. Then, $x_w(t) = p^{\top}(t_p)x(t)$ reaches a local peak when $R_o(t) = 1$. Fig. 2 (a)-(d) illustrate the behavior of the system in (5) based on the effective reproduction number $R_o(t)$ and the peak infection time t_p, as we proved in Theorem 1, Corollaries 2, and 3. Additionally, Fig. 2(c) shows that, at the beginning stages of the epidemic, when no community considers the epidemic as a threat, the beliefs in the seriousness of the epidemic will increase with the decreasement of the susceptible population. Meanwhile, as the susceptible population decreases and the opinion states increase, the effective reproduction number $R_o(t)$ decreases, which aligns with Proposition 1. As stated in Theorem 2, the states of the system converge to zero exponentially fast.

Next, we will show the special case where the opinions reach consensus. As mentioned in Lemma 4, the opinion

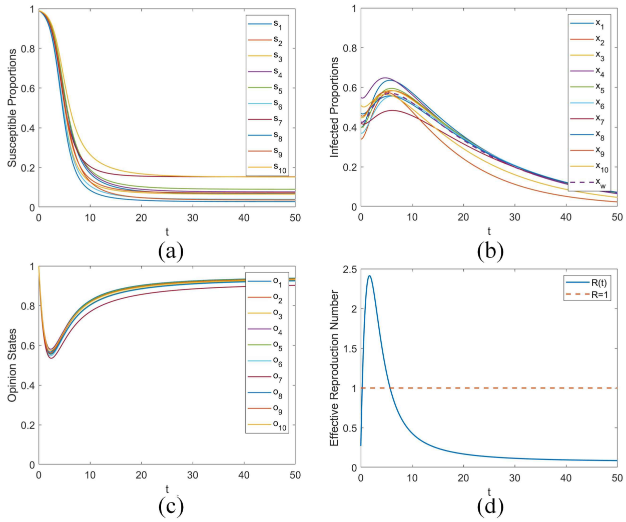

Fig. 4: State dynamics with non-monotonic $R_o(t)$

states will reach consensus at the equilibrium if and only if all the communities have the same infection level. We set $\beta_{\min} = 0.1$ and $\gamma_{\min} = 0.14$, while β_{ij} and γ_i are uniformly sampled from $[0.1, 0.6]$ and $[0.14, 0.30]$, respectively to generate plots in Fig. 3. In Fig. 3(a) and (c), the initial conditions of the epidemic states are uniformly sampled from $[0, 1]$. In Fig. 3(b) we randomly sample the initial opinion states from $[0, 1]$, and In Fig. 3(d) we set the initial opinion states as $o(0) = \mathbf{1}_n$. Both Fig. 3(a) and (c) capture the extreme case where everyone in the population becomes infected, i.e., where the susceptible states converge to $s_e = \mathbf{0}_n$. Therefore, based on Lemma 4, the opinion states at the equilibrium will take the form $o_e = \mathbf{1}_n - s_e = \mathbf{1}_n$, captured by Fig. 3(b) and (d). The simulations demonstrate that, when reaching agreement after the epidemic dies out, the evaluations on the seriousness of the epidemic can reflect the infection level.

Fig. 4 aims to show that the effective reproduction number $R_o(t)$ may not decrease monotonically, unlike the classical SIR model, as stated before. We set $\beta_{\min} = 0.01$ and $\gamma_{\min} = 0.05$, while β_{ij} and γ_i are uniformly sampled from $[0.01, 0.4]$ and $[0.05, 0.1]$, respectively. We assume initial opinions $o(0) = \mathbf{1}_n$ as shown in Fig. 4(c), and we sample the initial infected proportion for each community from $[0.3, 0.6]$ randomly. In Fig. 4(d), since $R_{\min}(0) < 1$, the weighted average $p^\top(t_p)x(t)$ decreases at the beginning stages of the outbreak. However, the communities soon realize the epidemic is not as severe as they have evaluated as captured in Fig. 4(c). As the opinion states decrease, $R_o(t)$ increases, causing the weighted average $p^\top(t_p)x(t)$ to increase again, captured by Fig. 4(b)-(d). In Fig. 4(d), we observe that there are two peak candidates where $R_o(t) = 1$; we rule out the first by Theorem 1, which states that peak infection time must satisfy $R_o(t) > 1$, for $t \in [t_0, t_p)$ and $R_o(t) < 1$, for $t \in (t_p, t_1]$, for $t_p - t_0 > 0$ and $t_1 - t_p > 0$ sufficiently small. However, the second point where $R_o(t) = 1$ is a peak infection time, consistent with Fig. 4(b) and (d). Unlike the classical SIR model [14], Fig. 4(d) illustrates that $R_o(t)$ is not monotonically decreasing.

V. Conclusion

In this work, we develop a networked SIR model coupled with opinion dynamics to study epidemic spreading processes over multiple communities. We define the effective reproduction number and peak infection time to characterize the transient behavior of the epidemic. We also study the convergence time to the equilibria. Additionally, we discover that the opinion states at the equilibria can reflect the infection level of each community to some degree. The current work can be further extended to study the influence of the structures of the opinion spreading networks on the behavior of the system. Another potential future research direction is to design control algorithms that influence the opinions to change the behavior of the epidemic.

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