

# On a Networked SIS Epidemic Model with Cooperative and Antagonistic Opinion Dynamics

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**Abstract**—We propose a mathematical model to study coupled epidemic and opinion dynamics in a network of communities. Our model captures SIS epidemic dynamics whose evolution is dependent on the opinions of the communities toward the epidemic, and vice versa. In particular, we allow both cooperative and antagonistic interactions, representing similar and opposing perspectives on the severity of the epidemic, respectively. We propose an Opinion-Dependent Reproduction Number to characterize the mutual influence between epidemic spreading and opinion dissemination over the networks. Through stability analysis of the equilibria, we explore the impact of opinions on both epidemic outbreak and eradication, characterized by bounds on the Opinion-Dependent Reproduction Number. We also show how to eradicate epidemics by reshaping the opinions, offering researchers an approach for designing control strategies to reach target audiences to ensure effective epidemic suppression.

## I. INTRODUCTION

### A. Motivation

Epidemiological models have been extensively studied for the purpose of understanding the spread of infectious diseases through societies [1]–[3]. A key epidemiologic metric in these models is the basic reproduction number ( $R_0$ ), which describes the expected number of cases directly generated by one case in an infection-free population; similarly, the effective reproduction number ( $R_t$ ) characterizes the average number of new infections caused by a single infected individual at time  $t$  in the partially susceptible population [4]. These reproduction numbers can be affected by a variety of factors, including biological properties of the epidemic, environmental conditions, and the opinions and behaviors of the population [5]. Communities believing that an outbreak is severe will react quickly to suppress the epidemic, in terms of policy-making, rule-following, etc [6]. On the other hand, misinformation or disbelief in an epidemic could potentially lead communities to react incorrectly, perpetuating outbreaks [7]. Further, social media significantly increases the rate at which opinions and misinformation spread through communities. During the ongoing SARS-CoV-2 pandemic, the adoption of masks, social-distancing,

and vaccination has been significantly influenced by highly polarized attitudes about the virus, as well as misinformation. The polarized opinions toward the seriousness of the epidemic shift people's reactions toward the virus in different ways, leading to drastic differences in infections in different communities/areas [8]. Given the significant link between opinion polarization and pandemic-related outcomes, it is of great importance to provide a foundational understanding of epidemic spreading processes under the effects of polarized opinions. Motivated by the critical role that opinion dynamics plays in the epidemic spreading (and vice versa), in this work we study a networked epidemic model coupled with a networked opinion model possessing both cooperative and antagonistic interactions to understand the mutual influence between epidemic spreading and opinion dissemination over communities. Below, we discuss related literature and then describe our contributions.

### B. Literature Review

Researchers have developed and studied threshold conditions and equilibria of networked SIS epidemic models to characterize virus spreading over communities [9], as well as the connections between the networked SIS epidemic model and Markov chain models [10]. Networked epidemic models have also attracted considerable attention [2], [3], [11], [12] for modeling the spread of malware in computer networks [13] and attacks in cyber-physical systems [14]. On the opinions side, the Altafini model [15] lays a foundation for various extensions on modeling and control of opinions spreading on signed networks, capturing both cooperative and antagonistic interactions. Therefore, to capture the co-existence of rival opinions among different populations, we follow the Altafini model by using networks with both positive and negative edges to characterize the cooperative and antagonistic interactions, respectively, of opinion exchange dynamics between communities, as studied in [15]–[17]. Recent work has combined disease spreading models with human awareness models [18]–[22], where the infection rates scale with the human awareness toward the epidemics. However, these models lack an explicit dynamical model to represent the change of the population's perception on the severity of the epidemics over time [23]–[25].

To couple the networked SIS model with opinion dynamics, we employ the health belief model, which is the most widely used theory in health behavior research [26]. The health belief model proposes that people's beliefs<sup>1</sup> about health problems, perceived benefits of actions, and/or

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<sup>1</sup>In this article, beliefs, attitudes, and opinions are used interchangeably.

perceived barriers to actions can explain their engagement, or lack thereof, in health-promoting behavior. Therefore, people's beliefs in their perceived susceptibility and/or in their perceived severity of the illness affect how susceptible they are and/or how effective they will be at healing from these epidemics. In this article, we model the population's beliefs of the severity of the epidemic using opinion dynamics with both cooperative *and* antagonistic interactions.

### C. Contribution

The contributions of this work are the following.

- 1) We develop a networked SIS model coupled with both cooperative and antagonistic opinion dynamics. The opinion dynamics evolves on an opinion-dependent sign switching topology, which characterizes the change of the opinions of the communities toward the epidemic.
- 2) We define an Opinion-Dependent Reproduction Number ( $R_t^o$ ) of the epidemic model. We use  $R_t^o$  to characterize the severity of the epidemic (mild, moderate, severe), equilibria, and stability, which reveal the mutual influence between epidemic spreading and opinion dissemination over the communities.
- 3) We interpret our results in the context of real-world phenomena under the SARS-CoV-2 pandemic. We propose ways to guide control design and select target communities to shift the opinions of the communities in order to better control the epidemic.

### D. Outline of the article

This work is organized as follows. In Section II, we state the motivation and present the Epidemic-Opinion network model. In Section III, we study the equilibria of the Epidemic-Opinion network model. Section IV defines the Opinion-Dependent Reproduction Number, which characterizes the behavior of the epidemic spreading process. Section IV also explores how to influence the opinions in order to suppress the outbreak. Section V and Section VI present simulation and conclusion/future works, respectively. Note that the proofs are included in the Appendix. Due to the page limit, a more detailed version is available in [27].

### E. Notations

For any positive integer  $n$ , we use  $[n]$  to denote the index set  $\{1, 2, \dots, n\}$ . We view vectors as column vectors and write  $x^\top$  to denote the transpose of a column vector  $x$ . We use  $x^n$  to denote the vector, of the same size as  $x$ , whose each entry equals the  $n$ th power of the corresponding entry of  $x$ . For a vector  $x$ , we use  $x_i$  to denote the  $i$ th entry. For any matrix  $M \in \mathbb{R}^{n \times n}$ , we use  $M_{i,:}$ ,  $M_{:,j}$ ,  $M_{ij}$  ( $[M]_{ij}$ ), to denote its  $i$ th row,  $j$ th column, and  $ij$ th entry, respectively. We use  $M = \text{diag}\{m_1, \dots, m_n\}$  to represent a diagonal matrix  $M \in \mathbb{R}^{n \times n}$  with  $M_{ii} = m_i$ ,  $\forall i \in [n]$ . We use  $\mathbf{0}$  and  $\mathbf{e}$  to denote the vectors whose entries all equal 0 and 1, respectively, and  $I$  to denote the identity matrix. The dimensions of the vectors and matrices are to be understood from the context. Let  $\partial[c, d]^n$  and  $\text{Int}[c, d]^n$  denote the boundary and interior of the cube  $[c, d]^n$ ,  $c, d \in \mathbb{R}$ , respectively.

For a real square matrix  $M$ , we use  $\rho(M)$  and  $s(M)$  to denote its spectral radius and spectral abscissa (the largest real part among its eigenvalues), respectively. For any two vectors  $v, w \in \mathbb{R}^n$ , we write  $v \geq w$  if  $v_i \geq w_i$ , and  $v \gg w$  if  $v_i > w_i$ ,  $\forall i \in [n]$ . The comparison notations between vectors are applicable for matrices as well, for instance, for  $A, B \in \mathbb{R}^{n \times n}$ ,  $A \gg B$  indicates that  $A_{ij} > B_{ij}$ ,  $\forall i, j \in [n]$ . For any two sets  $A$  and  $B$ , we use  $A \setminus B$  to denote the set of elements in  $A$  but not in  $B$ . We also employ a modified sign function:

$$\text{sgnm}(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$$

The Dirac delta function, which is the first derivative of  $\frac{1}{2}\text{sgnm}(\cdot)$ , is represented by  $\theta(\cdot)$ . Note that we use  $\text{sign}(\cdot)$  to represent the original sign function, where  $\text{sign}(x) = \text{sgnm}(x)$ ,  $\forall x \neq 0$ , and  $\text{sign}(0) = 0$ . We use the modified sign function  $\text{sgnm}(\cdot)$  to classify non-negative and negative opinion states, and use the original sign function  $\text{sign}(\cdot)$  to distinguish between positive and negative edge weights.

Consider a directed graph  $G = (\mathcal{V}, \mathcal{E})$ , with the node set  $\mathcal{V} = \{v_1, \dots, v_n\}$  and the edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ . Let matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  denote the adjacency matrix of  $G = (\mathcal{V}, \mathcal{E})$ , where  $a_{ij} \in \mathbb{R}$  if  $(v_j, v_i) \in \mathcal{E}$  and  $a_{ij} = 0$  otherwise. Graph  $G$  does not allow self-loops, i.e.,  $a_{ii} = 0$ ,  $\forall i \in [n]$ . Let  $k_i = \sum_{j \in \mathcal{N}_i} |a_{ij}|$ , where  $\mathcal{N}_i = \{v_j | (v_j, v_i) \in \mathcal{E}\}$  denotes the neighbor set of  $v_i$  and  $|a_{ij}|$  denotes the absolute value of  $a_{ij}$ . The graph Laplacian of  $G$  is defined as  $L \triangleq K - A$ , where  $K \triangleq \text{diag}\{k_1, \dots, k_n\}$  denotes the degree matrix of the graph  $G$ .

## II. MODELING AND PROBLEM FORMULATION

In this section, we introduce an SIS model, coupled with an opinion dynamics model. In particular, we assume an epidemic is spreading over a group of communities, where the interactions between the communities facilitate the spreading. Furthermore, the opinion of each community about the epidemic evolves as a function of the community's infected proportion, the community's own opinion, and the opinions of other communities.

### A. Epidemic Dynamics

Consider an epidemic spreading over  $n$  connected communities represented by a directed graph  $G = (\mathcal{V}, \mathcal{E})$ , where the node set  $\mathcal{V} = \{v_1, \dots, v_n\}$  and the edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  represent the communities and the epidemic spreading interactions, respectively. The epidemic spreading interactions over  $n$  communities are captured by an adjacency matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ , where  $a_{ij} \in \mathbb{R}_{\geq 0}$ . A directed edge  $(v_j, v_i)$  indicates that community  $j$  can infect community  $i$ .

We use a networked SIS model to capture the epidemic dynamics of the  $n$  communities, such that  $x_i \in [0, 1]$  represents the proportion of the infected population in community  $i$ ,  $i \in [n]$ . Note that  $x_i \times 100\% \in [0\%, 100\%]$ . In this work,  $x_i$  and  $x_i(t)$  are used interchangeably. We leverage the epidemic dynamics proposed in [9] to capture the evolution of the  $n$  communities:

$$\dot{x}_i(t) = -\delta_i x_i(t) + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} \beta_j a_{ij} x_j(t), \quad (1)$$

where  $\delta_i \in \mathbb{R}_{\geq 0}$  is the average curing rate of community  $i$ , and  $\beta_i a_{ij} \in \mathbb{R}_{\geq 0}$  is the average infection rate of community  $j$  to community  $i$ .

### B. Opinion Dynamics

We let the opinions disseminate over a graph  $\bar{G} = (\mathcal{V}, \bar{\mathcal{E}})$ . The opinion graph  $\bar{G}$  allows both positive and negative edge weights, representing cooperative and antagonistic interactions on opinions exchanged between the communities, respectively. Let  $\bar{A}_u \in \mathbb{R}^{n \times n}$  with  $[\bar{A}_u]_{ij} = |\bar{a}_{ij}|$ , where  $\bar{a}_{ij}$  is the  $ij$ th entry of the adjacency matrix of  $\bar{G}$ . Therefore,  $\bar{A}_u$  captures the coupling strength of the opinion dynamics without considering the signs. Similar to the definition of graph  $G$ , we define the neighbor set of node  $v_i$  in graph  $\bar{G}$  as  $\bar{\mathcal{N}}_i$ , and use  $\bar{K}$  and  $\bar{L}$  to denote the degree matrix and Laplacian matrix of  $\bar{G}$ , respectively.

For all  $t \geq 0$ ,  $o(t) \in \mathbb{R}^n$  is the opinion vector of the  $n$  communities,  $o_i(t) \in [-0.5, 0.5]$ ,  $i \in [n]$ . The range  $o_i(t) \in [-0.5, 0.5]$  denotes the belief of community  $i$  about the severity of the epidemic at time  $t$ . Note that  $o_i(t)$  and  $o_i$  are used interchangeably in this work. The opinion  $o_i(t) = 0.5$  indicates that community  $i$  considers the epidemic to be extremely serious, while  $o_i(t) = -0.5$  implies that community  $i$  thinks the epidemic is not worth addressing. We assume that communities with a neutral opinion  $o_i(t) = 0$  marginally lean toward treating the epidemic as a threat.

It is natural to consider that communities with the same attitude toward the epidemic exchange their opinions cooperatively, while communities with different attitudes exchange their opinions antagonistically. Hence, based on the communities' beliefs toward the epidemic at any given time, we allow the edge signs of the opinion graph  $\bar{G}$  to switch. We achieve this behavior by partitioning the node set of the communities  $\mathcal{V}$  into two groups,  $\mathcal{V}_1(o(t)) = \{v_i \in \mathcal{V} \mid \text{sgnm}(o_i(t)) = 1, i \in [n]\}$  and  $\mathcal{V}_2(o(t)) = \{v_i \in \mathcal{V} \mid \text{sgnm}(o_i(t)) = -1, i \in [n]\}$ . Then we construct the adjacency matrix of graph  $\bar{G}$  as  $\bar{A}(o(t)) = \Phi(o(t))\bar{A}_u\Phi(o(t))$ , where  $\Phi(o(t)) = \text{diag}\{\text{sgnm}(o_1(t)), \dots, \text{sgnm}(o_n(t))\}$  is a gauge transformation matrix. Through the construction of  $\bar{A}(o(t))$ ,  $|\bar{a}_{ij}|$  is fixed and  $\text{sign}(\bar{a}_{ij})$  is switchable. In particular, for all non-zero entries in  $\bar{A}(o(t))$ ,  $\text{sign}(\bar{a}_{ij}) = 1$  if  $\text{sgnm}(o_j(t))$  and  $\text{sgnm}(o_i(t))$  are the same, otherwise,  $\text{sign}(\bar{a}_{ij}) = -1$ . Therefore,  $\bar{A}(o(t))$  captures the switch of the opinion interactions through the attitude changing toward the epidemic. An opinion graph  $\bar{G}$  constructed in this manner is always structurally balanced by the following definition and lemma.

**Definition 1.** [Structural Balance [15]] A signed graph  $\bar{G} = (\mathcal{V}, \bar{\mathcal{E}})$  is structurally balanced if the node set  $\mathcal{V}$  can be partitioned into  $\mathcal{V}_1$  and  $\mathcal{V}_2$  with  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$  and  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ , where  $\bar{a}_{ij} \geq 0$  if  $v_i, v_j \in \mathcal{V}_q$ ,  $q \in \{1, 2\}$ , and  $\bar{a}_{ij} \leq 0$  if  $v_i \in \mathcal{V}_q$  and  $v_j \in \mathcal{V}_r$ ,  $q \neq r$ , and  $q, r \in \{1, 2\}$ .

**Lemma 1.** [15] A connected signed graph  $\bar{G}$  is structurally balanced if and only if there exists a gauge transformation matrix  $\Phi = \text{diag}\{\phi_1, \dots, \phi_n\} \in \mathbb{R}^{n \times n}$ , with  $\phi_i \in \{\pm 1\}$ , such that  $\Phi \bar{A} \Phi \in \mathbb{R}^{n \times n}$  is non-negative, where  $\bar{A}$  is the adjacency matrix of  $\bar{G}$ .

After defining the opinion interaction matrix  $\bar{A}(o(t))$ , a variant of the opinion dynamics model in [15] evolving over the  $n$  communities with both cooperative and antagonistic interactions is given by

$$\dot{o}_i(t) = \sum_{j \in \bar{\mathcal{N}}_i} |\bar{a}_{ij}(o(t))| (\text{sign}(\bar{a}_{ij}(o(t)))o_j(t) - o_i(t)), \quad (2)$$

with the compact form

$$\dot{o}(t) = -\Phi(o(t))\bar{L}_u\Phi(o(t))o(t), \quad (3)$$

where  $\bar{L}_u$  represents the Laplacian matrix of  $\bar{A}_u$ .

*Example of Eq. (3):* Consider opinion interactions captured by the graph with three nodes in Fig. 1. The positive and negative opinion states are represented by unshaded and shaded nodes, respectively. At time  $t_0$ , the unsigned Laplacian matrix  $L_u$  of the graph in Fig. 1 (a) and the time-varying gauge transformation matrix  $\Phi(o(t_0))$  defined in Lemma 1 are given by

$$\bar{L}_u = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 0 & -3 & 3 \end{bmatrix}, \quad \Phi(o(t_0)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

respectively. Since node 1 and node 2 share their opinions cooperatively, while node 3 shares its opinion with node 1 and node 2 antagonistically, we have  $[\Phi(o(t_0))]_{11} = [\Phi(o(t_0))]_{22}$ ,  $[\Phi(o(t_0))]_{33} = -1$ . Based on Lemma 1, the signed Laplacian matrix of opinion formation in Fig. 1 (a) is  $\bar{L}(o(t_0)) = \Phi(o(t_0))\bar{L}_u\Phi(o(t_0))$ . At time  $t_1$ , the opinion of node 2 drops below zero, which enables node 2 to share a positive edge with node 3, and a negative edge with node 1. Thus, the signed Laplacian matrix of the graph in Fig. 1 (b) is given by  $\bar{L}(o(t_1)) = \Phi(o(t_1))\bar{L}_u\Phi(o(t_1))$ , where the entries of the gauge transformation matrix at  $t_1$  are given by  $[\Phi(o(t_1))]_{11} = 1$ ,  $[\Phi(o(t_1))]_{22} = [\Phi(o(t_1))]_{33} = -1$ .

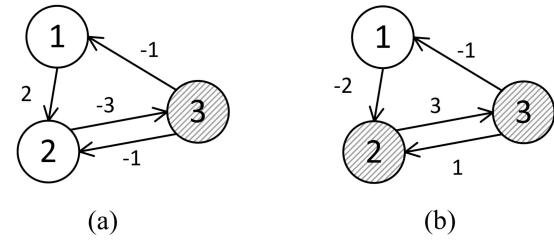


Fig. 1: Opinion spreading networks: (a) Opinion formation and the corresponding network structure at  $t_0$ ; (b) Opinion formation and the corresponding network structure at  $t_1$ .

### C. Coupled Epidemic-Opinion Dynamics

Having introducing the networked SIS epidemic model spreading over the  $n$  communities, and opinions spreading over the same  $n$  communities, we now introduce network dynamical models that couple the epidemic dynamics with the opinion dynamics. Assume that a community's opinion/attitude toward the severity of an epidemic will affect its actions, which leads to the variation of the community's average healing rate and infection rate. For instance, a community being very cautious about the epidemic will broadcast the influence of the epidemic more frequently, and

make policies to suppress the epidemic, and the people in that community will be more likely to follow the instructions given by scientific institutions, and seek treatments in a timely manner. These actions will result in the community having a lower average infection rate and a higher average healing rate. To better describe the situation, we use  $\delta_{\min}$  and  $\beta_{\min}$  to denote the possible minimum average healing rate and infection rate for all communities, respectively.<sup>2</sup> To incorporate the opinion dynamics in (2) into the epidemic dynamics in (1), we consider

$$\begin{aligned} \dot{x}_i(t) = & -[\delta_{\min} + (\delta_i - \delta_{\min})o'_i(t)]x_i(t) \\ & + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} [\beta_{ij} - (\beta_{ij} - \beta_{\min})o'_i(t)]x_j(t), \end{aligned} \quad (4)$$

where  $o'_i(t) = o_i(t) + 0.5$  shifts the opinion into the range  $[0, 1]$ . In particular, the term  $o'_i(t)$  scales the average healing and infection rates between their maximum and minimum values. In the case when  $o_i(t) = -0.5$ , which implies that community  $i$  does not consider the epidemic a threat at time  $t$ , community  $i$  will take no action to protect itself and thus is maximally exposed to the infection. In the case when  $o_i(t) = 0.5$ , which implies that community  $i$  believes the epidemic is extremely serious, it will implement policies and limitations to decrease the infection rate and seek out all possible medical treatment options to improve its healing rate. Therefore, the model allows the communities' opinions to affect how susceptible they are and how effectively they heal from the virus, capturing the health belief model [26], as explained in the Introduction.

The proportion of the infected population of a community  $i$  can also have an effect on its opinion/attitude toward the epidemic. Consider the epidemic dynamics in (1) incorporated with (2) as follows:

$$\begin{aligned} \dot{o}_i(t) = & (x_i(t) - o'_i(t)) \\ & + \sum_{j \in \mathcal{N}_i} |\bar{a}_{ij}(o(t))| (\text{sign}(\bar{a}_{ij}(o(t)))o_j(t) - o_i(t)). \end{aligned} \quad (5)$$

The first term on the right hand side of (5) captures how the infected proportion of a community affects its own opinion. If  $o'_i(t)$  is small but the community is heavily infected, i.e.,  $x_i(t)$  is large,  $o'_i(t)$  will increase. If  $o'_i(t)$  is large but the community has few infections, i.e.,  $x_i(t)$  is small,  $o'_i(t)$  will decrease. This behavior is sensible since a community's infection level should affect its belief in the severeness of the virus, which is consistent with the health belief model in [26]. The second term on the right hand side of (5) is from (2). The neighbors of community  $i$  affect its opinion cooperatively ( $\text{sign}(\bar{a}_{ij}(o(t))) = 1$ ) or antagonistically ( $\text{sign}(\bar{a}_{ij}(o(t))) = -1$ ).

#### D. Problem Statements

Now that we have presented the Epidemic-Opinion model in (4) and (5), we can state the problem of interest in this

<sup>2</sup>We assume homogeneous minimum healing and infection rates for simplicity. The results in this work can be extended to the heterogeneous case.

work. We are interested in exploring the mutual influence between the epidemic spreading over  $n$  communities captured by graph  $G$  in (4) and the opinions of the  $n$  communities about the epidemic captured by graph  $\bar{G}$  in (5). We will analyze the equilibria of the system in (4)-(5) under different settings. In particular, we will define an Opinion-Dependent Reproduction Number to characterize the spreading of the virus. We will study the stability of the equilibria of the system to infer the behaviors of the epidemic and opinions spreading over the  $n$  communities. Finally, we will generate strategies to eradicate the epidemic by affecting the opinion states of the system, which could potentially guide the use of social media to broadcast the severity of the pandemic to appropriate communities to suppress the epidemic.

### III. EQUILIBRIA OF EPIDEMIC-OPINION DYNAMICS

This section considers the mutual influence between the epidemic dynamics in (4) and opinion dynamics in (5), and lays a foundation for analyzing the stability and convergence to the equilibria of the coupled dynamics under different conditions in the next section. We impose the following natural restrictions on the parameters of the models throughout the work.

**Assumption 1.** Let  $x_i(0) \in [0, 1]$ ,  $o_i(0) \in [-0.5, 0.5]$ ,  $\delta_i \geq \delta_{\min} > 0$ , and  $\beta_{ij} \geq \beta_{\min} > 0$ ,  $\forall i \in [n]$  and  $\forall j \in \mathcal{N}_i$ . The epidemic spreading graph  $G$  and opinion dissemination graph  $\bar{G}$  are strongly connected.

Note that the adjacency matrix of a strongly connected graph is irreducible. Therefore, the adjacency matrix  $A$  of the graph  $G$  is a nonnegative irreducible matrix. A real square matrix  $M$  is called a Metzler matrix if  $M_{ij} \geq 0$ ,  $\forall i, j \in [n]$  and  $i \neq j$ , which implies that the adjacency matrix  $A$  is also an irreducible Metzler matrix.

We write (4) and (5) in a compact form as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{o}(t) \end{bmatrix} = \begin{bmatrix} W(o(t)) & \mathbf{0} \\ I & -\Phi(o(t))\bar{L}_u\Phi(o(t)) - I \end{bmatrix} \times \begin{bmatrix} x(t) \\ o(t) \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ 0.5\mathbf{e} \end{bmatrix}, \quad (6)$$

where

$$\begin{aligned} W(o(t)) = & -(D_{\min} + (D - D_{\min})(O(t) + 0.5I)) \\ & + (I - X(t))(B - (O(t) + 0.5I)(B - B_{\min})), \end{aligned}$$

$O(t) = \text{diag}\{o_1(t), \dots, o_n(t)\}$ ,  $X(t) = \text{diag}\{x_1(t), \dots, x_n(t)\}$ ,  $D = \text{diag}\{\delta_1, \dots, \delta_n\}$ ,  $D_{\min} = \delta_{\min}I$ ,  $B = [\beta_{ij}] \in \mathbb{R}^{n \times n}$  and  $B_{\min} = \beta_{\min}\bar{A}$ , with  $\bar{A} \in \mathbb{R}^{n \times n}$  being the unweighted adjacency matrix of graph  $G$  (with  $\bar{A}_{ij} \in \{0, 1\}$ ,  $\forall i, j \in [n]$ ). Note that the Epidemic-Opinion model in (6) is a nonlinear system, due to the nonlinearity brought by  $W(o(t))$  and  $-\Phi(o(t))\bar{L}_u\Phi(o(t))$ . The system in (6) follows:

$$\dot{x}(t) = -D(o(t))x(t) + (I - X(t))B(o(t))x(t) \quad (7)$$

$$\dot{o}(t) = Ix(t) - (\Phi(o(t))\bar{L}_u\Phi(o(t)) + I)o(t) - 0.5\mathbf{e}, \quad (8)$$

where  $D(o(t)) = D_{\min} + (D - D_{\min})(O(t) + 0.5I)$ , and  $B(o(t)) = B - (O(t) + 0.5I)(B - B_{\min})$  are the opinion-dependent healing and infection matrices, respectively.

**Remark 1.** Based on Assumption 1,  $B(o(t))$  is an irreducible non-negative matrix and  $D(o(t))$  is a positive definite diagonal matrix,  $\forall t$ . It can be verified that the matrix  $W(o(t))$  is a Metzler matrix. Since graph  $\bar{G}$  is strongly connected and structurally balanced, the Laplacian matrix  $\Phi(o(t))\bar{L}_u\Phi(o(t))$  of graph  $\bar{G}$  has only one zero eigenvalue, and the rest of the eigenvalues have positive real parts [15, Lemma 2]. Note that the gauge transformation defined in Lemma 1 does not change the spectra of a matrix [15], and thus,  $\Phi(o(t))\bar{L}_u\Phi(o(t))$  and  $\bar{L}_u$  share the same spectra. Further, the matrix  $(\Phi(o(t))\bar{L}_u\Phi(o(t))+I)$  has one eigenvalue located at one, and the rest of its eigenvalues have positive real parts larger than one, which implies that the matrix  $-(\Phi(o(t))\bar{L}_u\Phi(o(t))+I)$  is Hurwitz,  $\forall t$ . Note that the opinion dynamics in (8) is a state-based switching system. For each subsystem, the matrix  $\Phi(o(t))\bar{L}_u\Phi(o(t))$  is the standard signed Laplacian matrix.

Now that we have introduced the Epidemic-Opinion model, in order to analyze the behavior of the system in (6), we need to show that the model is well-defined.

**Lemma 2.** For the system defined in (6), if  $x_i(t) \in [0, 1]$  and  $o_i(t) \in [-0.5, 0.5]$ ,  $\forall i \in [n]$ , then  $x_i(t+\tau) \in [0, 1]$  and  $o_i(t+\tau) \in [-0.5, 0.5]$ ,  $\forall i \in [n]$ , and  $\forall \tau > 0$ .

To explore the equilibria of (6), let  $z(t) = [x^\top o^\top]^\top$  denote the states of the system in (6),  $z(t) \in \mathbb{R}^{2n}$ , and  $z^* = [x^*]^\top [o^*]^\top$  denote an equilibrium of (6). We say  $x^*$  and  $o^*$  are the equilibria of (7) and (8), respectively.

**Definition 2.** Let the state  $(x^*, o^*)$  denote an equilibrium of (6), where  $(x^*, o^*)$  is

- 1) a consensus-healthy state if  $(x^*, o^*) = (\mathbf{0}, o^*)$ , and  $o_i^* = o_j^*$ ,  $\forall i, j \in [n]$ ;
- 2) a dissensus-healthy state if  $(x^*, o^*) = (\mathbf{0}, o^*)$ , and  $\exists i, j \in [n]$ , s.t.  $o_i^* \neq o_j^*$ ;
- 3) a consensus-endemic state if  $x^* \geq \mathbf{0}$ ,  $x^* \neq \mathbf{0}$ , and  $o_i^* = o_j^*$ ,  $\forall i, j \in [n]$ ;
- 4) a dissensus-endemic state if  $x^* \geq \mathbf{0}$ ,  $x^* \neq \mathbf{0}$ , and  $\exists i, j \in [n]$ , s.t.  $o_i^* \neq o_j^*$ .

In this work, we use the term *healthy state* to describe both case 1) and case 2) in Definition 2, and the term *endemic state* for both case 3) and case 4). It is obvious that the Epidemic-Opinion model in (6) has a consensus-healthy state  $(\mathbf{0}, -0.5e)$  as its trivial equilibrium. Further, from (8), when  $o^* = -0.5e$ ,

$$\begin{aligned} \mathbf{0} &= x^* + (\bar{L}_u + I) \times 0.5e - 0.5e \\ &= x^* + \bar{L}_u \times 0.5e + 0.5e - 0.5e \\ &= x^*, \end{aligned}$$

which indicates  $x^* = \mathbf{0}$ . Therefore, the consensus state  $o^* = -0.5e$  must pair with the healthy state  $x^* = \mathbf{0}$ . The following theorem summarizes the healthy equilibria of (6).

**Theorem 1.** For the Epidemic-Opinion model in (6), a healthy equilibrium  $z^*$  is either the unique consensus-healthy state  $(x^* = \mathbf{0}, o^* = -0.5e)$ , or a dissensus-healthy state  $(x^* = \mathbf{0}, o^*)$ , with  $o^* = (\Phi(o^*)\bar{L}_u\Phi(o^*)+I)^{-1}(-0.5e)$  with

both positive and negative entries, and  $o_i^* \in [-0.5, 0.5]$ ,  $\forall i \in [n]$ .

Theorem 1 states that the dissensus opinion equilibrium  $o^* = (\Phi(o^*)\bar{L}_u\Phi(o^*)+I)^{-1}(-0.5e)$  is always in the set  $[-0.5, 0.5]^n$ , i.e., its value is always consistent with its physical meaning.

**Corollary 1.** For the Epidemic-Opinion model in (6), the consensus-healthy state  $(x^* = \mathbf{0}, o^* = -0.5e)$  is the unique equilibrium with consensus in opinions, that is,  $o^* = \alpha e$ ,  $\alpha \in [-0.5, 0.5]$ .

Note that if  $\exists i \in [n]$ , s.t.  $o_i^* = 0$ , from (2), it must be true that  $o_i^* = 0$ ,  $\forall i \in [n]$ . However, Corollary 1 states that  $z^*$  does not include the case that  $o^* = \mathbf{0}$ . Therefore, we have the next corollary.

**Corollary 2.** For the Epidemic-Opinion model in (6), the equilibrium  $z^*$  cannot include  $o_i^* = 0$ ,  $\forall i \in [n]$ .

**Remark 2.** Theorem 1 indicates the possible equilibria when the epidemic dies out. The consensus-healthy state  $(x^* = \mathbf{0}, o^* = -0.5e)$  implies that, at the stage that the epidemic disappears, all communities agree the epidemic is not a threat. However, the existence of dissensus-healthy states  $(x^* = \mathbf{0}, o^*)$ , with  $o^*$  having both positive and negative entries, describes the scenario when communities hold different beliefs toward the epidemic at the time when the epidemic is about to disappear, and thus the epidemic will still cause possible contention between different communities. Corollary 1 states that the only way to ensure all communities reach agreement is that they all agree that the epidemic is not a threat at the moment the epidemic dies out. In other words, it is impossible for all communities to agree that the epidemic is not worth treating seriously while the epidemic is still spreading.

After analyzing the healthy equilibria, the following lemma further explores the endemic equilibria of (6).

**Lemma 3.** If  $(x^*, o^*)$  is an endemic equilibrium of the system in (6), then  $\mathbf{0} \ll x^* \ll e$ ,  $-0.5e \ll o^* \ll 0.5e$ .

**Remark 3.** Lemma 3 states that if an endemic state exists, no community can be completely infection free or completely infected. Further, the equilibrium of a community cannot be equal to one of the extreme beliefs.

#### IV. STABILITY ANALYSIS OF EPIDEMIC-OPINION DYNAMICS

In this section, we analyze the properties of the equilibria of the Epidemic-Opinion model in (6), to reveal the mutual influence between disease spreading and opinion formation during an epidemic.

As mentioned in [4], the reproduction number  $R$  of an epidemic is critical in determining the spreading of the epidemic. In line with the expression on the reproduction number  $R$ , we define an Opinion-Dependent Reproduction Number  $R_t^o$  to characterize the performance of the Epidemic-Opinion model in (6).

**Definition 3.** [Opinion-Dependent Reproduction Number] Let  $R_t^o = \rho(D(o(t))^{-1}B(o(t)))$  denote the Opinion-Dependent Reproduction Number, where  $D(o(t))$  and  $B(o(t))$  are defined in (7).

Note that the Opinion-Dependent Reproduction Number  $R_t^o$  depends on the variation of the opinion states  $o(t)$ . When all communities think the epidemic is extremely serious, by defining  $o_{\max} = 0.5e$ , we have

$$R_{\min} = \rho(D(o_{\max})^{-1}B(o_{\max})) = \rho(D^{-1}B_{\min}).$$

Instead, when all communities believe that the epidemic is not serious at all, by defining  $o_{\min} = -0.5e$ , we have

$$R_{\max} = \rho(D(o_{\min})^{-1}B(o_{\min})) = \rho(D_{\min}^{-1}B).$$

**Proposition 1.** The Opinion-Dependent Reproduction Number  $R_t^o$  has the following properties:

- 1) For all  $t_0, t_1 \in \mathbb{R}_{\geq 0}$ , if  $o(t_0) \leq o(t_1)$ , then  $R_{t_0}^o \geq R_{t_1}^o$ , and vice versa;
- 2) For all  $t_0, t_1 \in \mathbb{R}_{\geq 0}$ ,  $R_{\min} \leq R_t^o \leq R_{\max}$ .

**Remark 4.** Recall from the Introduction that reproduction numbers of epidemic models are important metrics to capture the seriousness of epidemics. The reproduction number of an SIS model is determined by its transmission and healing rates. In reality, multiple factors such as social distancing, vaccination, and hospital capacity have an impact on transmission and/or healing rates. Compared to previous works that study time-invariant SIS networked models [9], and time-varying SIS networked models [28], Definition 3 and Proposition 1 analyze the reproduction number based on the influence of opinions on the transmission and healing rates.

Proposition 1 indicates that the opinions toward the epidemic provide bounds on the Opinion-Dependent Reproduction Number  $R_t^o$ . The more seriously community  $i$  treats the epidemic, i.e., with a higher  $o(t)$ , the lower  $R_t^o$  is, and vice versa. In the following sections, we interpret  $R_t^o$  as the severity of an epidemic, and explore the behavior of (6) through bounds on  $R_t^o$ .

#### A. Mild Viruses

In this section we explore the behavior of viruses that are only slightly contagious, that is, where  $R_{\max}^o \leq R_{\max} \leq 1$ . First we analyze the equilibrium of the system in (6) under the condition that  $R_{\max} \leq 1$ .

**Proposition 2.** If  $R_{\max} \leq 1$ , then every equilibrium of (6) is a healthy state.

Proposition 2 claims that when  $R_{\max} \leq 1$ , i.e., the severity of the epidemic is mild, the Epidemic-Opinion model in (6) has only healthy equilibria, despite the evolution of the opinions. To further explore the behavior of (6), we study the stability of the healthy equilibria.

**Proposition 3.** If  $R_{\max} < 1$ , then all the healthy equilibria  $(\mathbf{0}, o^*)$  of (6) are locally exponentially stable.

**Theorem 2.** If  $R_{\max} \leq 1$ , then for any initial condition, the system in (6) will asymptotically converge to a healthy

equilibrium. If  $R_{\max} < 1$ , the convergence is exponentially fast.

**Remark 5.** Proposition 3 and Theorem 2 reveal that, under the condition that  $R_{\max} \leq 1$ , the initial conditions of the epidemic (i.e., the level of the infection in each community) and/or the opinion states (i.e., how much the communities underestimate the severity of the epidemic), will not hinder the epidemic from disappearing quickly. Hence, it is unnecessary to broadcast the severity of the epidemic publicly, since the epidemic will die out quickly regardless of the opinions.

The initial condition may affect the opinions after the epidemic disappears. Imagine the case where few infections appear in each community and no community believes the epidemic is serious at the beginning. Then, the epidemic will disappear and all communities will reach a consensus that the epidemic is not a threat, captured by the unique consensus-healthy equilibrium. Alternatively, consider the case where some communities are heavily infected at the beginning, hence they believe the epidemic is a threat. Even after the epidemic dies out quickly, disagreement will linger between communities, corresponding to the dissensus-healthy equilibria.

#### B. Severe Viruses

In this section we explore the behavior of viruses that are very contagious, that is, where  $R_{\min} > 1$ . Note that Theorem 2 demonstrates that the healthy states  $(\mathbf{0}, o^*)$ , are equilibria for (6) under any  $R_t^o$ , s.t.  $R_{\max} \leq 1$ . Therefore, we consider the stability properties of the healthy equilibria.

**Lemma 4.** Under the condition that  $R_{\min} > 1$ , all the healthy equilibria  $(\mathbf{0}, o^*)$  are unstable.

**Remark 6.** Lemma 4 states that, when the Opinion-Dependent Reproduction Number is large,  $R_{\min} > 1$ , i.e., the epidemic is highly contagious, even if all communities have zero infection, one infected person appearing in any community will result in an outbreak, leading to a pandemic. Further, with all communities being extremely cautious about the epidemic ( $o^* = 0.5e$ ), taking every action suggested by the scientific institutions, public health officials, and the media to protect themselves, the epidemic will continue spreading. Therefore, it would be impossible to eradicate the epidemic by relying only on non-pharmaceutical interventions (NPIs) through social media.

After studying the healthy equilibria in Lemma 4, we explore the existence of the endemic equilibria under the condition that  $R_{\min} > 1$ . Lemma 3 claims that, if it exists, the endemic state  $(x^*, o^*)$  must satisfy  $e \gg x^* \gg \mathbf{0}, 0.5e \gg o^* \gg -0.5e$ . Since  $(-D + B_{\min})$  is an irreducible Metzler matrix,  $\rho(D^{-1}B_{\min}) > 1$  implies  $s(-D + B_{\min}) > 0$ . From [29, Sec. 2.1 and Lemma 2.3], let  $\phi \triangleq s(-D + B_{\min})$  be the spectral abscissa of  $(-D + B_{\min})$  with an associated right eigenvector  $y \gg \mathbf{0}$ . Without loss of generality, assume  $\max_{i \in [n]} y_i = 1$ . Now, define for any  $\epsilon \in [0, 1)$ , a convex and compact subset of  $\chi$  as

$$\Xi_\epsilon \triangleq \{z \in \chi : z_i \geq \epsilon y_i, \forall i \in [n]\}, \quad (9)$$

where

$$\chi = \{z \in \mathbb{R}^{2n} \mid z_i \in [0, 1], i = 1, \dots, n; \\ z_i \in [-0.5, 0.5], i = n+1, \dots, 2n\}.$$

Note that  $\Xi_0 = \chi$  and  $\forall \epsilon > 0, \Xi_\epsilon \subset \chi$ . From the proof of Lemma 2 and the piece-wise continuity of the system in (6), we have the following results.

**Lemma 5.** *Consider the system in (6). If  $z(t) \in \partial\chi \setminus (\mathbf{0}, o^*)$ , where  $(\mathbf{0}, o^*)$  indicates the set of healthy equilibria of (6), then  $z(t + \tau) \in \text{Int}\chi, \forall \tau > 0$ .*

**Theorem 3.** *Suppose that  $R_{\min} > 1$ . Then, there exists a sufficiently small  $\bar{\epsilon}$  such that  $\Xi_\epsilon$  defined in (9) for every  $\epsilon \in (0, \bar{\epsilon}]$  is a positive invariant set for the system in (6). Moreover, (6) has at least one endemic equilibrium in  $\chi$ .*

Combined with Theorem 1, Theorem 3 states that, when  $R_{\min} > 1$ , the system in (6) has both healthy and endemic equilibria. Lemma 4 shows the healthy equilibria are unstable. Note that previous work shows the existence of a limit cycle for time-varying networked SIS models under different switching conditions [28]. In our work, the coupling of opinion dynamics changes the parameters of the networked SIS model, and the opinion network can switch. Proving the attractiveness of this set and the existence of limit cycles or lack thereof remains a research direction for future work.

### C. Moderate Viruses

The previous sections show that the opinion states have little impact on the behavior of epidemics when the epidemic is either highly contagious ( $R_{\min} > 1$ ) or very mild ( $R_{\max} < 1$ ). In this section we explore the behavior of viruses that are moderately contagious, that is, where  $R_{\min} < 1$  and  $R_{\max} > 1$ . We show that the moderate viruses cases include properties from both mild and severe viruses, where the behavior of the epidemic is affected by the opinion dynamics. We propose one strategy to analyze how the influence of stubborn communities on opinion states can impact the behavior of the epidemic.

1) *Healthy State:* Recall from Definition 3 that the Opinion-Dependent Reproduction Number is determined by the opinion states of the system, which are continuous. Based on Proposition 1,  $R_{\min} \leq R_t^o \leq R_{\max}$ . If  $R_{\min} < 1$  and  $R_{\max} > 1$ , there must be at least one opinion  $\bar{o}$  such that  $R_t^{\bar{o}} = 1$ . Therefore, the system in (6) may contain properties that both cases  $R_{\min} > 1$  and  $R_{\max} < 1$  have. To study eradication strategies of the epidemic, we need to explore the behavior of healthy equilibria. Since the range of  $R_t^o$  depends on the opinion state  $o(t)$ , for healthy equilibria, we evaluate  $R_t^{o^*}$  regarding the opinion state  $o^*$  at the healthy equilibrium. Thus, we have the following results.

**Theorem 4.** *For the system in (6), if  $R_{\min} < 1$ , and  $R_{\max} > 1$ , the following statements hold:*

- 1) *All the dissensus-healthy equilibria  $(\mathbf{0}, o^*)$  satisfying  $R_t^{o^*} < 1$  are locally exponentially stable;*
- 2) *All the dissensus-healthy equilibria  $(\mathbf{0}, o^*)$  satisfying  $R_t^{o^*} > 1$  are unstable;*

- 3) *The consensus-healthy equilibrium  $(\mathbf{0}, -0.5e)$  is unstable.*

Theorem 4 implies that the local stability of the healthy equilibria depends on  $R_t^{o^*}$ . Moreover, one might never find a locally stable healthy equilibria, if no opinion state of the dissensus-equilibria satisfies Case 1) in Theorem 4. Further, from Corollary 1, that the consensus-healthy state is the unique equilibrium for opinion consensus, we have that in the moderate virus case, the communities will not reach consensus in the absence of control strategies. Moreover, Theorem 4 states that the epidemic cannot be eradicated while all the communities are ignoring the epidemic. Hence, with a moderate virus, in order to ensure the epidemic is eradicated, control strategies can be leveraged to reshape the opinions of the communities to maintain the opinion states above a certain threshold, such that the Opinion-Dependent Effective Reproduction Number is always below one.

To analyze the stability of the healthy equilibria under interference on opinion states, we introduce threshold opinion vectors. Since  $R_t^o$  is related to  $o(t)$ , we can always find an opinion vector in  $[-0.5, 0.5]^n$  such that  $R_t^o \leq 1$  or  $R_t^o > 1$ , based on the definition of  $R_t^o$ , and the moderate virus assumption that  $R_{\max} > 1$  and  $R_{\min} < 1$ . The threshold opinion vectors are defined to analyze the system under the condition that all the opinion vectors are either smaller or greater than the threshold opinion vector. From the property of  $R_t^o$  that  $o(t_0) \leq o(t_1)$  leads to  $R_{t_0}^o \geq R_{t_1}^o$  given by Proposition 1, we have the following corollary.

**Corollary 3.** *If  $R_{\min} < 1$  and  $R_{\max} > 1$ , there must exist one threshold opinion vector  $\bar{o}$  such that  $R_t^{\bar{o}} = 1$ .*

Corollary 3 is a direct result of Proposition 1; note that there could be more than one  $\bar{o}$  satisfying Corollary 3. Further, if  $o(t) > \bar{o}, \forall t$ , then  $R_t^o < 1$ . Instead, if  $o(t) < \bar{o}, \forall t$ , then  $R_t^o > 1$ . From Corollary 3, we can capture the stability of the healthy equilibria in Theorem 4 through  $\bar{o}$ .

**Corollary 4.** *For the system in (6), if  $R_{\min} < 1$  and  $R_{\max} > 1$ , the dissensus-healthy equilibria  $(\mathbf{0}, o^*)$  satisfying  $o^* \gg \bar{o}$  are locally exponentially stable, while the equilibria  $(\mathbf{0}, o^*)$  satisfying  $o^* \ll \bar{o}$  are unstable.*

**Remark 7.** *Case 3) of Theorem 4 and Corollary 4 imply that, when the epidemic is moderate, the epidemic cannot be eradicated if all communities ignore it ( $o^* = -0.5e$ ) or do not treat it seriously enough ( $o^* \ll \bar{o}$ ). Further, Case 1) of Theorem 4 and Corollary 4 indicate that the epidemic will disappear when all communities believe that the epidemic is severe past a certain degree ( $o^* \gg \bar{o}$ ).*

2) *Stubborn Communities:* After bridging the gap between the behavior of the healthy equilibria and the opinion threshold vector  $\bar{o}$ , we propose eradication strategies by leveraging the idea of reshaping the opinion formation of the communities through control strategies. Note that the system in (6) might have no stable healthy equilibrium. Assuming the system in (6) has at least one locally stable healthy equilibrium, from Corollary 3, there must exist an  $\bar{o}$ , s.t.  $o^* \geq \bar{o}$ . Therefore, to eradicate the epidemic, we

can employ external influence on the communities to drive  $o(t)$  above  $\bar{o}$ . One method is to consider the existence of stubborn communities in (8), i.e., the opinions of the stubborn communities are not influenced by their neighbors [30, Eq. (7)]. Then, the following theorem captures the behavior of (6) with stubborn communities.

**Theorem 5.** *For the system in (6), if  $R_{\min} < 1$  and  $R_{\max} > 1$ , the stubborn communities driving  $o(t) \gg \bar{o}, \forall t \geq 0$ , will ensure that the system in (6) converges to the set of healthy equilibria.*

The proof of Theorem 5 is similar to the proof of Theorem 2, except that the opinion states are maintained above the threshold vector  $\bar{o}$ .

**Remark 8.** *Theorem 5 states, when the epidemic is moderate, we can eradicate it by selecting stubborn communities to drive the opinions of all the communities above the threshold vector  $\bar{o}$ . The situation implies that, by broadcasting the severity of the epidemic to some target communities, we can influence all the communities' beliefs toward the seriousness of the epidemic. In particular, when all communities' opinions are driven above a threshold vector, i.e., all the communities consider the epidemic somewhat serious, they will take the proper actions to end the epidemic.*

Theorem 5 shows the role of stubborn communities in epidemic suppression. However, optimally selecting the proper stubborn communities and designing external control signals to influence the stubborn communities is challenging. Hence, by Proposition 1, we explore a particular method that considers stubborn communities with fixed opinion states equaling to 0.5,  $\forall t$ . The following result provides a way of selecting extreme stubborn communities for a particular case.

**Corollary 5.** *Consider an opinion vector  $\vec{o}$  with both positive and negative entries,  $\vec{o}_i \in \{-0.5, 0.5\}, \forall i \in [n]$ . If  $\exists \vec{o}$ , s.t.  $R_t^{\vec{o}} < 1$ , the system in (6) can reach a healthy state by setting  $o_i(t) = 0.5, \forall t \geq 0, \forall i$  satisfying  $\vec{o}_i = 0.5$ .*

Corollary 5 reveals that, if we can find an  $R_t^{\vec{o}} < 1$ , under the condition that the opinions of all communities are controlled at the extreme beliefs, then  $R_t^{\vec{o}}$  will not exceed one after letting the communities with negative extreme opinions evolve freely, while maintaining the communities with positive extreme opinions the same. Corollary 5 offers a way of selecting communities to make them stubborn in order to suppress the epidemic. The elements of the opinion vector  $\vec{o}_i \in \{-0.5, 0.5\}$  can be adjusted to generate different combinations of stubborn communities and opinions, e.g., exploring stubborn communities through Corollary 5 with the condition  $\vec{o}_i \in \{-0.5, o\}$ , with  $o = 0.5\alpha, \alpha \in (-1, 1]$  being the stubborn state.

**Remark 9.** *Corollary 5 reveals the role of stubborn communities in determining the behavior of the epidemic. As discussed in this section, when viruses are either mild or severe, without changing the opinions of all communities through control strategies, the epidemic caused by mild*

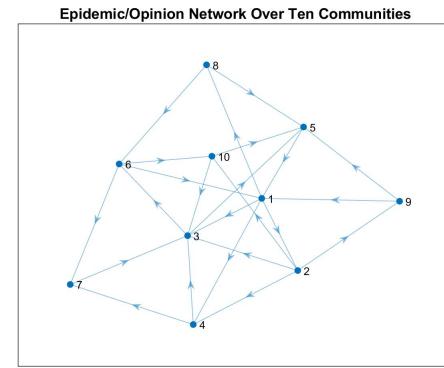


Fig. 2: The graph for the epidemic and opinion interactions.

viruses will disappear even with communities ignoring the virus, while the epidemic caused by severe viruses will not be eradicated even when all communities take various actions caused by the highest awareness toward the epidemic. In the moderate virus case, where extreme opinions from both sides (ignoring the epidemics or treating the epidemic extremely seriously) play important roles in determining the spread of the epidemic, positive stubborn opinions will drive the overall opinion states higher, leading to lower  $R_t^{\vec{o}}$ . On the other hand, stubborn communities with negative opinions will increase the  $R_t^{\vec{o}}$ , making it harder for the communities to eradicate epidemics. Additionally, the existence of the epidemic may lead to dissensus on the seriousness of the epidemic, which is modeled by the signed switching opinion network structure.

## V. SIMULATIONS

In this section, we illustrate the main results through the following examples. Consider an epidemic process spreading over ten communities, with the epidemic and opinion spreading through the same network satisfying Assumption 1, captured by the graph  $G$  in Fig. 2. Note that we use the same graph structure in  $G$  to capture the epidemic and opinion graphs to simplify the simulation, and our results still apply to communities with different epidemic and opinion interactions.

First, we consider the case that the epidemic is mild, which indicates  $R_{\max} \leq 1$ . By generating parameters of the infection and healing rates randomly, we obtain  $R_{\min} = 0.174$ ,  $R_{\max} = 0.381$ . Consistent with Proposition 2 and Theorem 2, when  $R_{\max} \leq 1$ , i.e., the epidemic is mild, all the communities reach healthy states with zero infections, illustrated in Fig. 3 (a) and (c). Further, under the condition that the epidemic will eventually disappear, Theorem 2 states that opinions of the communities either reach the consensus-healthy state  $(\mathbf{0}, -0.5e)$ , where all the communities agree that the epidemic is not serious, illustrated in Fig. 3 (a) and (b), or a dissensus-healthy state  $(\mathbf{0}, o^*)$ , with the communities holding both positive and negative opinion states toward the epidemic, illustrated in Fig. 3 (c) and (d). Additionally, it is implied from the simulation that opinion consensus

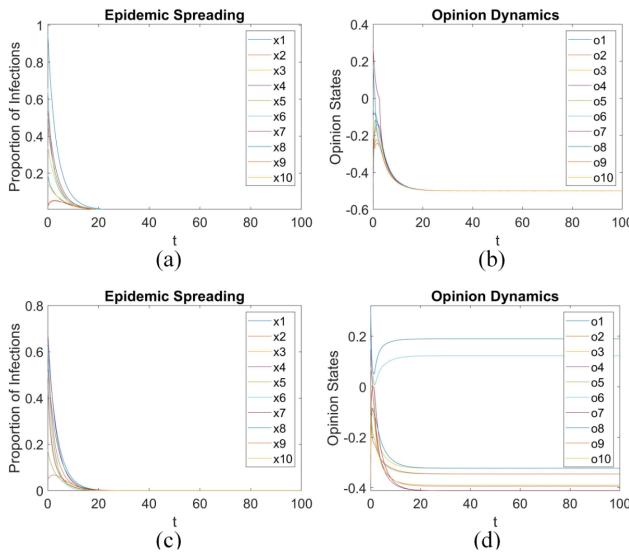


Fig. 3: Under the condition  $R_{\min} = 0.174$ ,  $R_{\max} = 0.381$ , with the graph of ten communities from Fig. 2, the communities reach the healthy state: (a) epidemic states converge to zero, (b) opinion states reach consensus, (c) epidemic states converge to zero, (d) opinion states reach dissensus.

or dissensus after the epidemic disappears is dependent on the initial infected proportion and initial belief toward the seriousness of each community. By comparing Fig. 3 (b) and (d), if the opinion of each community drops below zero early, all the communities will reach consensus on the seriousness of the epidemic. The conditions for reaching consensus and dissensus are worth exploring in future work.

Fig. 4 illustrates the situation where the epidemic is severe, characterized by  $R_{\min} > 1$ . Through randomly generating parameters satisfying the condition, we have  $R_{\min} = 1.63$ ,  $R_{\max} = 2.84$ . Consistent with Lemma 4, none of the communities can ever reach a healthy state. Fig. 4 (a) also implies the existence of an endemic equilibrium, illustrating Theorem 3. Fig. 4 (b) shows that all the ten communities reach dissensus, since as shown in Corollary 1, the unique case that all communities reach consensus on the seriousness of the epidemic is dependent on the condition that the epidemic dies out. Therefore, when the epidemic reaches an element in the set of endemic states, as shown in Fig. 4 (a), the opinion must reach dissensus as in Fig. 4 (b). Further, when the epidemic becomes endemic, the opinion of each community at the equilibria can be obtained by solving (4) and (5). Note that the continuity at the opinion-switching points and the Lipschitz continuity between the opinion-switching points can be observed from the example.

Lastly, we consider the case where the epidemic is moderate. Thus, we generate parameters leading to  $R_{\min} = 0.51 < 1$ ,  $R_{\max} = 2.61 > 1$  to characterize this situation. As described in Theorem 4, the stability of the healthy equilibria depends on the opinion states. More importantly, from Theorem 5 and Corollary 5, we can appropriately select stubborn communities to eradicate the epidemic. Fig. 5 illustrates the role of stubborn communities in epidemic

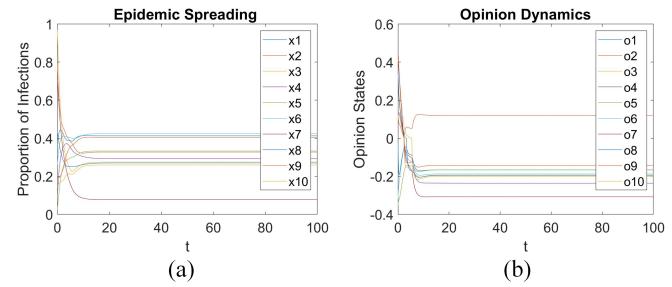


Fig. 4: The communities under the condition  $R_{\min} = 1.63$ ,  $R_{\max} = 2.84$ , with the graph of ten communities from Fig. 2, reach a dissensus-endemic state: (a) epidemic states reach endemic equilibrium, (b) opinion states reach dissensus.

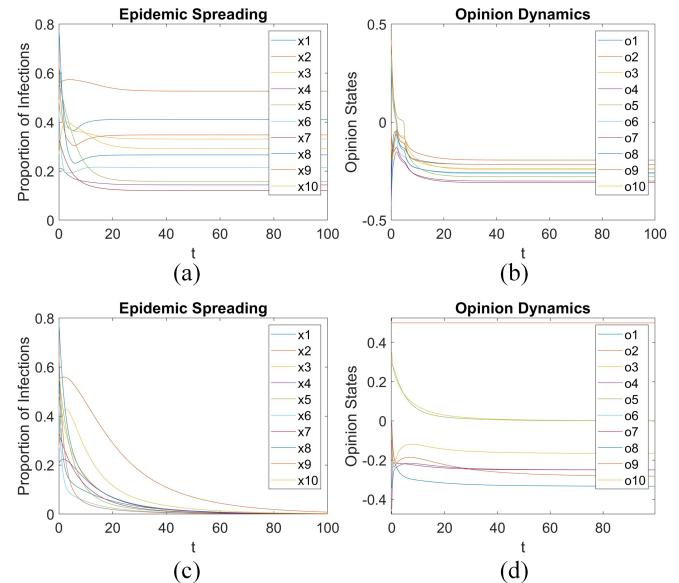


Fig. 5: All the communities under the condition  $R_{\min} = 0.51$ ,  $R_{\max} = 2.61$ , with the graph of ten communities from Fig. 2: (a) epidemic states reach endemic equilibrium, (b) opinion states without stubborn opinions, (c) epidemic states converge to zero, (d) opinion states with stubborn opinions.

suppression, showing the same system under two different settings. The system captured by Fig. 5 (a) and (b) reaches a dissensus-endemic state. With the exact same conditions, we fix the opinion states of the communities 1, 6, and 9 as  $[o(t)]_1 = [o(t)]_6 = [o(t)]_9 = 0.5$ ,  $\forall t \geq 0$ , which means communities 1, 6, and 9 always believe that the epidemic is extremely severe, and the opinions of other communities will not impact their beliefs in the severity of the epidemic. Meanwhile, communities 1, 6, and 9 keep broadcasting the information that the epidemic is very severe to their neighbors. Through this setting, compared to the same system captured by Fig. 5 (a) and (b), Fig. 5 (c) and (d) show that all the communities reach the healthy state, illustrating the results derived in Theorem 4, Theorem 5, and Corollary 5, that appropriate selection of stubborn communities, broadcasting their cautious opinions to other communities, can help suppress the epidemic.

## VI. CONCLUSION

This work studies the mutual influence between epidemic spreading and opinion dissemination over connected communities. By defining an Opinion-Dependent Reproduction Number, our work reveals the behavior of the Epidemic-Opinion model in (6). Our work also illustrates the role of stubborn communities in epidemic eradication. The results of this work pave the way for a more detailed analysis of the extended models. In this work, the stability analysis of the endemic states under the condition that  $R_{\min} > 1$  still needs to be further explored. The study of the endemic states could reveal the impact of opinions on the most serious epidemics. Next, except for using stubborn opinions, one could consider control design to shape the opinions of the communities to eradicate the epidemic. Further extensions of the results in this work consist of validating the opinion-dependent sign switching network model with real-world data and extending the ideas to couple opinion dynamics with the SIR (susceptible-infected-recovered) model. Additionally, diseases including SARS-CoV-2 have an incubation period that cannot be ignored [31], [32], which is not considered in our current model. The incubation period, usually captured by the exposed compartment in a compartmental model, will bring new challenges in analyzing opinion formation on epidemic spreading, since the infected population is not aware of its infection during the incubation period. Future work on studying SEIR (susceptible-exposed-infected-recovered) epidemic models coupled with opinion dynamics would therefore be of value.

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## APPENDIX

*Proof of Lemma 2:* Please see the Appendix of [27].  $\square$

*Proof of Theorem 1:* We first show that the healthy state  $x^* = \mathbf{0}$  can only pair with the unique consensus state  $o^* = -0.5\mathbf{e}$ . Recall the definition of  $\Phi(o(t))$  in Section II-B; if  $o_i^* = o_j^*$ ,  $\forall i, j \in [n]$ , then  $\Phi(o^*) = \pm I$ . From (8), the equilibria of the opinion dynamics satisfy

$$-(\bar{L}_u + I)o^* = 0.5\mathbf{e}. \quad (10)$$

Thus we have that  $-0.5\mathbf{e}$  is an eigenvector of the matrix  $-(\bar{L}_u + I)$  paired with the largest eigenvalue  $-1$ . From Remark 1,  $-(\bar{L}_u + I)$  is nonsingular, and thus  $o^* = -0.5\mathbf{e}$  is the unique solution of (10). Therefore,  $(x^* = \mathbf{0}, o^* = -0.5\mathbf{e})$  is the unique consensus-healthy equilibrium of (6).

For dissensus-healthy states,  $(x^* = \mathbf{0}, o^*)$ , if  $o^* \gg \mathbf{0}$  or  $o^* \ll \mathbf{0}$ , which implies  $\Phi(o^*) = \pm I$ , the equilibrium of the opinion dynamics in (8) becomes  $-(\bar{L}_u + I)o^* = 0.5\mathbf{e}$ , which has only  $-0.5\mathbf{e}$ , the consensus state, as its solution. Therefore,  $o^*$  in  $(x^* = \mathbf{0}, o^*)$  must have both positive and negative entries. Based on the fact that  $(\Phi(o(t))\bar{L}_u\Phi(o(t)) + I)$  is a nonsingular matrix, the equation

$$(\Phi(o^*)\bar{L}_u\Phi(o^*) + I)o^* = x^* - 0.5\mathbf{e}, \quad (11)$$

has a unique solution for each  $(\Phi(o^*))$ , given by  $o^* = (\Phi(o^*)\bar{L}_u\Phi(o^*) + I)^{-1}(-0.5\mathbf{e})$ , when  $x^* = \mathbf{0}$ . Now we show that each solution must satisfy  $0.5\mathbf{e} \geq o^* \geq -0.5\mathbf{e}$ .

Assume that  $[\Phi(o^*)]_{ii} = -1$ ,  $\forall i \in \{1, \dots, m\}$  and  $[\Phi(o^*)]_{jj} = 1$ ,  $\forall j \in \{m+1, \dots, n\}$ . Let  $\bar{L} = \Phi(o^*)\bar{L}_u\Phi(o^*)$ . Without loss of generality, suppose to the contrary that  $o_1^* < -0.5$ . Based on the assumption,  $o_i^* < 0$ ,  $\forall i \in \{2, \dots, m\}$ , while  $o_j^* \geq 0$ ,  $\forall j \in \{m+1, \dots, n\}$ . Considering the first row of (11), we have

$$-|[\bar{L}]_{11}|o_1^* - o_1^* + \sum_{i=2}^m |[\bar{L}]_{1i}|o_i^* - \sum_{j=m+1}^n |[\bar{L}]_{1j}|o_j^* = 0.5.$$

For  $|[\bar{L}]_{11}| = \sum_{k=2}^n |[\bar{L}]_{1k}|$ , we have

$$\begin{aligned} & -\sum_{i=2}^m |[\bar{L}]_{1i}|o_i^* - \sum_{j=m+1}^n |[\bar{L}]_{1j}|o_j^* \\ & + \sum_{i=2}^m |[\bar{L}]_{1i}|o_i^* - \sum_{j=m+1}^n |[\bar{L}]_{1j}|o_j^* - o_1^* = 0.5. \end{aligned}$$

Note that if  $o_i^* \geq o_1^*$ ,  $\forall i \in \{2, \dots, m\}$ , and  $o_j^* \leq -o_1^*$ ,  $\forall j \in \{m+1, \dots, n\}$ , the left side of the equation above must be greater than 0.5. Therefore, there must exist at least one  $o_i^*$ ,  $i \in \{2, \dots, m\}$ , satisfying  $o_i^* < o_1^*$  and/or at least one  $o_j^*$ ,  $j \in \{m+1, \dots, n\}$ , satisfying  $o_j^* > -o_1^*$ . Suppose that  $o_2^* < o_1^*$ , then for the second row of (11), the same statement holds, that there must exist at least one  $o_i^*$ ,  $i \in \{3, \dots, m\}$ , satisfying  $o_i^* < o_2^* < o_1^*$ , and/or at least one  $o_j^*$ ,  $j \in \{m+1, \dots, n\}$ , satisfying  $o_j^* > -o_2^* > -o_1^*$ . Following this procedure, for the last row corresponding to the last entry in  $o^*$ , we can no longer find any entries satisfying the condition. Therefore, no solution of (11) can be smaller than  $-0.5$ . A similar process can be applied to show that

no solution of the equation (11) can have an element larger than  $0.5$ .

Therefore, for each sign pattern of  $\Phi(o^*)$ , the system in (6) has one unique dissensus-healthy state  $o^* = (\Phi(o^*)\bar{L}_u\Phi(o^*) + I)^{-1}(-0.5\mathbf{e})$ .  $\square$

*Proof of Corollary 1:* Please see the Appendix of [27].  $\square$

*Proof of Lemma 3:* Please see the Appendix of [27].  $\square$

*Proof of Proposition 1:* 1) Based on Assumption 1,  $D(o(t))$  is a positive definite diagonal matrix and  $B(o(t))$  is an irreducible nonnegative matrix,  $\forall t$ . Hence,  $D(o(t))^{-1}B(o(t))$  is an irreducible nonnegative matrix. Without loss of generality, consider the case where,  $\exists o_i(t_0) < o_i(t_1)$ ,  $i \in [n]$ ,  $t_1 > t_0 > 0$ , while  $o_j(t_0) = o_j(t_1)$ ,  $\forall i, j \in [n]$ ,  $i \neq j$ . Recall that  $D(o(t)) = D_{\min} + (D - D_{\min}I)(O(t) + 0.5I)$  and  $B(o(t)) = B - (O(t) + 0.5I)(B - B_{\min})$ . Based on  $o_i(t_0) < o_i(t_1)$ , we have  $D_{ii}(o(t_0)) < D_{ii}(o(t_1))$ , leading to  $D_{ii}^{-1}(o(t_0)) > D_{ii}^{-1}(o(t_1))$ , and  $B_{i,:}(o(t_0)) > B_{i,:}(o(t_1))$ , while the rest of  $D^{-1}(o(t_1))$  and  $B(o(t_1))$  are equal to  $D^{-1}(o(t_0))$  and  $B(o(t_1))$ , respectively. Hence,  $o_i(t_0) < o_i(t_1)$  leads to

$$[D(o(t_0))^{-1}B(o(t_0))]_{i,:} > [D(o(t_1))^{-1}B(o(t_1))]_{i,:}.$$

From [29, Thm. 2.7, and Lemma 2.4], we have

$$\rho[D(o(t_0))^{-1}B(o(t_0))] > \rho[D(o(t_1))^{-1}B(o(t_1))],$$

which means  $R_{t_0}^o > R_{t_1}^o$ . The proof holds for the situations where more than one opinion states in  $o(t_0)$  are smaller than  $o(t_1)$ . The same method can verify the case that  $o(t_0) \geq o(t_1)$ , then  $R_{t_0}^o \leq R_{t_1}^o$ .

2) This statement is two special cases of 1). Since  $-0.5\mathbf{e} \leq o(t) \leq 0.5\mathbf{e}$ , when  $o(t) = o_{\min} = -0.5\mathbf{e}$ , based on the first statement of Proposition 1,

$$\begin{aligned} \rho(D(o(t))^{-1}B(o(t))) & \leq \rho[D(o_{\min})^{-1}B(o_{\min})] \\ & = \rho(D_{\min}^{-1}B) = R_{\max}. \end{aligned}$$

When  $o(t) = o_{\max} = 0.5\mathbf{e}$ ,

$$\begin{aligned} \rho(D(o(t))^{-1}B(o(t))) & \geq \rho(D(o_{\max})^{-1}B(o_{\max})) \\ & = \rho(D^{-1}B_{\min}) = R_{\min}. \end{aligned} \quad \square$$

*Proof of Proposition 2:* Please see the Appendix of [27].  $\square$

*Proof of Proposition 3:* We derive the Jacobian matrix  $df_{x,o}$  of (6) evaluated at  $(x, o)$  as follows:

$$\begin{bmatrix} W(o) - \tilde{V}(x, o) & -(D - D_{\min})X - (I - X)\tilde{B} \\ I & -(\Phi(o)\bar{L}_u\Phi(o) + I) - \tilde{\Delta} \end{bmatrix},$$

where  $\tilde{V}(x, o)$  and  $\tilde{B}$  are diagonal matrices with the  $i$ th diagonal entries being the  $i$ th entries of the vectors  $(B - (O + 0.5I)(B - B_{\min}))x$  and  $(B - B_{\min})x^2$ , respectively, and  $\tilde{\Delta} = (\Delta(\bar{L}_u + I)\Phi(o)o + \Phi(o)(\bar{L}_u + I)\Delta o)$ , with the Dirac delta function  $\theta(\cdot)$  and  $\Delta = \text{diag}\{2\theta(o_1), \dots, 2\theta(o_n)\}$ . From Corollary 2, we have  $o_i^* \neq 0$ ,  $\forall i \in [n]$ , for all equilibria of (6). Therefore,  $\tilde{\Delta} = \mathbf{0}$  when evaluated at all the equilibria,

due to  $\theta(o_i) = 0$ , when  $o_i \neq 0, \forall i \in [n]$ . We evaluate the Jacobian matrix at all healthy equilibria, ( $x = \mathbf{0}, o^*$ ),

$$df_{\mathbf{0}, o^*} = \begin{bmatrix} W(o) & \mathbf{0} \\ I & -(\Phi(o^*)\bar{L}_u\Phi(o^*) + I) \end{bmatrix}. \quad (12)$$

Note that for each equilibrium with its opinion formation  $o^*$ , when all of the opinion states are evolving closely enough to  $o^*$ , the gauge transformation matrix  $\Phi(o^*)$  is fixed. From Remark 1, the spectrum of  $-(\Phi(o^*)\bar{L}_u\Phi(o^*) + I)$  is the same as  $-(\bar{L}_u + I)$ . Further, for any opinions  $o^*$ , the matrices are Hurwitz. Hence, the stability of the system depends on the spectrum of  $W(o(t))$ . From [33, Prop. 1],

$$s(-(D_{\min} + (D - D_{\min})(O^* + 0.5I)) \\ + (B - (O^* + 0.5I)(B - B_{\min}))) < 0$$

if and only if

$$\rho((D_{\min} + (D - D_{\min})(O^* + 0.5I))^{-1} \\ \times (B - (O^* + 0.5I)(B - B_{\min}))) < 1,$$

Further, by Proposition 1,

$$\rho((D_{\min} + (D - D_{\min})(O^* + 0.5I))^{-1} \\ \times (B - (O^* + 0.5I)(B - B_{\min}))) \leq R_{\max} \leq 1,$$

we have  $s(W(o(t))) \leq 0$ . Hence, the Jacobian matrices evaluated at healthy equilibria are Hurwitz if  $R_{\max} < 1$ , leading to the results, by Lyapunov's indirect method.  $\square$

*Proof of Theorem 2:* Please see the Appendix of [27].  $\square$

*Proof of Theorem 3:* Please see the Appendix of [27].  $\square$

*Proof of Lemma 4:* Through the Jacobian matrix evaluated at  $(\mathbf{0}, o^*)$  in (12), where the spectrum of  $W(o(t))$  determines the stability of the Jacobian matrix, it can be shown that  $R_{\min} > 1$  leads to,

$$s(-(D_{\min} + (D - D_{\min})(O^* + 0.5I)) \\ + (B - (O^* + 0.5I)(B - B_{\min}))) > 0,$$

following the same process as in Proposition 3. Therefore, the Jacobian matrix evaluated at  $(\mathbf{0}, o^*)$  in (12) has at least one positive eigenvalue. Hence, all the healthy equilibria  $(\mathbf{0}, o^*)$ , under the condition that  $R_{\min} > 1$ , are unstable.  $\square$

*Proof of Theorem 4:* Under the condition that  $R_t^{o^*} < 1$ , proof of the local stability of all the healthy equilibria in Theorem 4 is similar to the proof of local stability of the healthy equilibria in Theorem 2. By switching the condition  $R_{\max} < 1$  to  $R_t^{o^*} < 1$ , the Jacobian matrix in (12) evaluated at  $(\mathbf{0}, o^*)$  is Hurwitz, which completes the proof of Case 1) of the theorem. The proof of case 2) follows the same procedure under the condition that  $R_t^{o^*} > 1$ ; showing that  $R_t^{o^*} > 1$  implies that the Jacobian matrix in (12) evaluated at  $(\mathbf{0}, o^*)$  is not Hurwitz. Therefore, the dissensus-healthy equilibria are unstable. For Case 3),  $R_t^{o^*}$  at  $(\mathbf{0}, -0.5e)$  is  $\rho(D(o_{\min})^{-1}B(o_{\min})) = R_{\max}$ . Since  $R_{\max} > 1$  for moderate virus, from Lemma 4, the consensus-healthy equilibrium is unstable.  $\square$

*Proof of Corollary 5:* By selecting stubborn communities with opinion states fixed at 0.5 from Corollary (5), based on Proposition 1, the system in (6) satisfies  $R_t^{\bar{o}} < 1$ . Since all non-stubborn communities will have their opinion states greater or equal than  $-0.5$ , from Proposition 1,  $R_t^o < 1$ . Thus, the system in (6) converges to a healthy state.  $\square$



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