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ABSTRACT

We present a simple method, not based on the transfer matrices, to prove vanishing of dynamical transport exponents. The method is applied to long-range quasiperiodic operators.

I. INTRODUCTION

Bourgain, partially with collaborators, developed a powerful method to prove Anderson localization for ergodic Schrödinger operators (see Ref. 1 and references therein). The method relies heavily, in both perturbative and nonperturbative settings, on the subharmonic function theory and the theory of semi-algebraic sets and has turned out to be quite robust. While the precursor was the non-perturbative approach of Ref. 16 that initiated the emphasis on obtaining off-diagonal Green's function decay using bulk features rather than individual eigenfunctions, Bourgain's method has crystallized and developed the key ideas that did not require transfer matrices/nearest-neighbor Laplacians, thus allowing, in particular, the extension to Toeplitz matrices as well as multidimensional localization results. See also Refs. 12 and 18 for streamlining and simplification of Bourgain's multidimensional method and the non-self-adjoint version.

Discrete quasiperiodic operators with the Laplacian replaced by a Toeplitz operator appear naturally in the context of Aubry duality and have been studied by several authors. Let $H_{\theta,\alpha,\epsilon}$, with $(\theta,\alpha)\in\mathbb{T}^2$, act on $\ell^2(\mathbb{Z})$ by

$$(H_{\theta,\alpha,\epsilon}u)_n := \epsilon\left(\sum_{k\in\mathbb{Z}} a_{n-k}u_k\right) + v(\theta+n\alpha)u_n, \tag{1}$$

where $|a_n|\leq A_1e^{-a|n|}$ for some $a, A_1 > 0$ and $a_{-n}=\overline{a_n}$. Bourgain's main localization result for the long-range case is as follows:

Theorem 1.1 (Ref. 1, Theorem 11.20). *If v is analytic non-constant on $\mathbb{T}$, then for $|\epsilon|\leq\epsilon_0$, $\epsilon_0=\epsilon_0(A_1,a,v)$, $H_{\theta,\alpha,\epsilon}$ satisfies Anderson localization for a full measure set of $(\theta,\alpha)\in\mathbb{T}^2$.*

We note that this theorem is *non-perturbative*, that is, ϵ_0 does not depend on α. There is also a stronger *arithmetic* (that is, with an arithmetic full measure condition on the frequency and phase) localization result for $v(\theta)=\cos 2\pi\theta$,[2] and recently, an arithmetic multidimensional result was obtained as a corollary of dual quantitative reducibility in Ref. 8, but for the general function v, Bourgain's non-arithmetic Theorem 1.1 remains the strongest available. We note that the perturbative multidimensional version appears in Ref. 12; however, in the multidimensional case, there is no essential difference between the nearest neighbor and long-range Laplacians.

At the same time, Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) is extremely fragile. Indeed, it was shown by Gordon[9] and del Rio, Makarov, and Simon[7] that a generic rank one perturbation of an operator with an interval in the spectrum even in the regime of *dynamical* localization leads to a singular continuous spectrum and, therefore, by the RAGE theorem, growth of the moments. However, it was shown in Ref. 6 that under the condition of SULE, present in many models, this growth can be at most logarithmic and thus preserves vanishing of the dynamical exponents. Thus, one can argue that it is vanishing of the dynamical exponents $\beta(q)$ [see (3) for the definition] that captures the physically relevant effect of localization.

Indeed, such localization-type results [vanishing of $\beta(q)$] have been obtained, in increasing generality, for random and quasiperiodic operators as a corollary of positive Lyapunov exponents in Refs. 3, 4, 13, and 14, with Ref. 10 covering the entire class of ergodic operators with base dynamics of zero topological entropy, a class that includes shifts and skew-shifts on higher-dimensional tori. Clearly, those techniques are transfer-matrix based and thus do not extend to long-range operators.

In this article, we present a very simple method to obtain such quantum dynamical upper bounds for the long-range case and show that one part of Bourgain's localization proof can serve as an input to obtain an arithmetic result: vanishing of quantum dynamical exponents for all long-range quasiperiodic operators with Diophantine frequencies, all phases, and sufficiently large analytic potentials (see Corollary 1.6). This should be contrasted with the non-arithmetic Theorem 1.1. We note also that Anderson localization for *all* phases does not even hold.[11,15]

Bourgain's method consists of multiple parts, and the one in question is establishing the sublinear bound (33) for the number of boxes of size N^c in a box of size N that do not have the off-diagonal Green function decay. Our method requires only the presence of *one* box of size N^c *with* the off-diagonal Green function decay, in a box of size N; thus, Bourgain's sublinear bound is even an overkill for a needed input.

We note that, while suitable for long-range, our method is still one-dimensional, as only in dimension 1 does one box create a barrier and thus a good estimate for the Green's function in a bigger box. Yet, it does provide the first departure from the Lyapunov exponent/transfer-matrix based methods and leads to a strong corollary. In addition, it extends easily to the (not necessarily uniform) band, requiring only one "good box" to apply Theorem 1.4.

Let us now introduce the main concepts. We restrict here to dimension one, although many of the statements and definitions are easily extendable to higher dimensions. For a fixed self-adjoint operator H on $\ell^2(\mathbb{Z})$, $\phi \in \ell^2(\mathbb{Z})$ and $p, T > 0$, let

$$\langle |X|_\phi^p \rangle (T) = \frac{2}{T} \int_0^\infty e^{-2t/T} \sum_{n\in\mathbb{Z}} |n|^p |(e^{-itH}\phi, \delta_n)|^2. \tag{2}$$

The growth rate of $\langle |X|_\phi^p \rangle (T)$ characterizes how fast does $e^{-itH}\phi$ spread out. The power law bounds for $\langle |X|_\phi^p \rangle (T)$ are naturally characterized by the following upper transport exponents $\beta_\phi^+(p)$:

$$\beta_\phi^+(p) = \limsup_{T\to\infty} \frac{\ln \langle |X|_\phi^p \rangle (T)}{p \ln T}. \tag{3}$$

Here, we study Schrödinger operators on $\ell^2(\mathbb{Z})$ of the form

$$H = A + V,$$

where $V = \{V_n\}_{n\in\mathbb{Z}}$ is real bounded and A is a long-range operator of the form

$$(Au)_n = \sum_{k\in\mathbb{Z}} a_{n-k} u_k,$$

where $|a_n| \le A_1 e^{-a|n|}$ for some $a, A_1 > 0$ and $a_{-n} = \overline{a_n}$.

More precisely,

$$(Hu)_n = \left(\sum_{k\in\mathbb{Z}} a_k u_{n-k} \right) + V_n u_n. \tag{4}$$

Just like Schrödinger operators, such operators admit a ballistic bound on the transport exponents.

Theorem 1.2. *Let H be given by (4). Assume that ϕ is compactly supported. Then, the upper transport exponent $\beta_\phi^+(q) \le 1$ for any $q > 0$.*

Remark 1.3. In fact, sufficiently fast decay works equally well, but we restrict in all results, to the compactly supported ϕ, for simplicity.

Theorem 1.2 is probably well known, but we did not find the proof in the literature. The proof, following the ideas of Refs. 19 and 20, is presented in the Appendix.

Let R_Λ be the operator of restriction to $\Lambda \subset \mathbb{Z}$. Define the Green's function by

$$G_\Lambda(z) = (R_\Lambda(H - zI)R_\Lambda)^{-1}. \tag{5}$$

Set $G(z) = (H - zI)^{-1}$. Clearly, both $G_\Lambda(z)$ and $G(z)$ are always well defined for $z \in \mathbb{C}_+ \equiv \{z \in \mathbb{C} : \Im z > 0\}$. Sometimes, we drop the dependence on z for simplicity. Since the operator H given by (4) is bounded, there exists $K > 0$ such that $\sigma(H) \subset [-K + 1, K - 1]$. Our main general result is as follows:

Theorem 1.4. *Let H be given by (4). Suppose there exist $\delta > 0$ and $N_0 > 0$ such that the following is true. Let $z = E + i\varepsilon$ with $|E| \le K$ and $0 < \varepsilon \le \delta$. Suppose that for $N > N_0$, there exists an interval $I \subset [-\frac{N}{2}, -\frac{N}{4}]$ or $I \subset [\frac{N}{4}, \frac{N}{2}]$ such that $|I| \ge N^\delta$ and that for any $n, n' \in I$ and $|n - n'| \ge \frac{1}{20}|I|$, we have*

$$|G_I(z)(n; n')| \le e^{-|I|^\delta}.$$

Assume ϕ is compactly supported. Then, the upper transport exponent $\beta_\phi^+(p) = 0$ for any $p > 0$.

Remark 1.5. For the Schrödinger case, the existence of such an interval I (in fact, a stronger statement, but this is not important) can be deduced from the positive Lyapunov exponents and Cramer's rule by the method going back to Ref. 17.

We say $\alpha \in \mathbb{R}$ is Diophantine if there exist κ and $\tau > 0$ such that for any $k \in \mathbb{Z}\backslash\{0\}$,

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{\tau}{|k|^\kappa},$$

where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \operatorname{dist}(x, \mathbb{Z})$.

Let $H_{\alpha,\theta,\epsilon}$ be as in (1). Fixing α and ϵ, we denote the $\beta_\phi^+(p)$ for operator $H_{\alpha,\theta,\epsilon}$ by $\beta_{\phi,\theta}^+(p)$. Our main application is as follows:

Corollary 1.6. There exists an $\epsilon_0 = \epsilon_0(v, A_1, a) > 0$ such that for any compactly supported ϕ and Diophantine α, $\beta_{\phi,\theta}^+(p) = 0$ for any $|\epsilon| \le \epsilon_0$, $\theta \in \mathbb{R}$, and $p > 0$.

It immediately implies also the following corollary:

Corollary 1.7. There exists an $\epsilon_0 = \epsilon_0(v, A_1, a) > 0$ such that for any $\phi \in \ell^2(\mathbb{Z})$, the spectral measure μ_ϕ of operator $H_{\theta,\alpha,\epsilon}$ is zero dimensional for any $\theta \in \mathbb{R}$, Diophantine α, and any $|\epsilon| \le \epsilon_0$.

II. PROOF OF THEOREM 1.4

For the Schrödinger case, the proof would be just a double application of the resolvent identity,

$$\begin{aligned} G = {} & G_I + G_{I^c} - (G_I + G_{I^c})(H - H_I - H_{I^c})(G_I + G_{I^c}) \\ & + (G_I + G_{I^c})(H - H_I - H_{I^c})(G_I + G_{I^c})(H - H_I - H_{I^c})G, \end{aligned}$$

ensuring the decay of $|G(0, n)|$ based on the "barrier" box I. The problem with the long-range case is that such an expansion for $G(0, N)$ will contain terms all grouped nearby, thus neither incorporating the decay coming from the barrier box nor from $|a_n|$. In order to tackle this difficulty we introduce several extra steps, all involving applications of the resolvent identity but with different boxes.

Since ϕ has a compact support, there exists K_1 such that $\phi(n) = 0$ for $|n| \ge K_1$.

Assume $T > \frac{1}{\delta}$. Fix $z = E + i\frac{1}{T}$ with $|E| \le K$. Below, C (c) is a large (small) constant that may depend on δ, K, A_1, a, ϕ, and $V = \{V_n\}$. Let $I = [b - \ell, b + \ell]$ with $\ell > 0$ and b such that $|b \pm \ell|$ is large. Suppose

$$|G_I(m, n)| \le Ce^{-c\ell^c} \tag{6}$$

for any $m \in I, n \in I$ and $|m - n| \ge \frac{1}{20}\ell$.

Recall that if

$$\Lambda = \Lambda_1 \cup \Lambda_2, \Lambda_1 \cap \Lambda_2 = \emptyset,$$

then

$$G_\Lambda = G_{\Lambda_1} + G_{\Lambda_2} - (G_{\Lambda_1} + G_{\Lambda_2})(H_\Lambda - H_{\Lambda_1} - H_{\Lambda_2})G_\Lambda$$

[provided the relevant matrices $R_\Lambda(H - zI)R_\Lambda$ and $R_{\Lambda_i}(H - zI)R_{\Lambda_i}$ are invertible], where $H_\Lambda = R_\Lambda H R_\Lambda$.

If $m \in \Lambda_1$ and $n \in \Lambda$, we have

$$G_\Lambda(m,n) = G_{\Lambda_1}(m,n)\chi_{\Lambda_1}(n) - \sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} G_{\Lambda_1}(m,n_1)a_{n_1-n_2}G_\Lambda(n_2,n). \tag{7}$$

Therefore,

$$|G_\Lambda(m,n)| \le |G_{\Lambda_1}(m,n)\chi_{\Lambda_1}(n)| + C\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(m,n_1)|e^{-c|n_1-n_2|}|G_\Lambda(n_2,n)|. \tag{8}$$

Lemma 2.1. Assume that for some interval $I = [b - \ell, b + \ell]$ and $z = E + \frac{i}{T}$, (6) holds. Then,

$$|G_\Lambda(m,n)| \le CT^2 e^{-c\ell^c} \tag{9}$$

for any $n \in I$, $m \in [b - \ell + \frac{\ell}{10}, b + \ell]$, and $|m - n| \ge \frac{1}{10}\ell$, where $\Lambda = (-\infty, b + \ell]$.

Proof. Let $\Lambda_1 = I = [b - \ell, b + \ell]$ and $\Lambda_2 = (-\infty, b - \ell - 1]$. Clearly, $\Lambda = \Lambda_1 \cup \Lambda_2$. By (6) and (8), one has that

$$|G_\Lambda(m,n)| \le Ce^{-c\ell^c} + C\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(m,n_1)|e^{-c|n_1-n_2|}|G_\Lambda(n_2,n)|. \tag{10}$$

It suffices to bound the second term on the right of (10).

For any $n_1 \in \Lambda_1$,

$$\sum_{n_2\in\Lambda_2} e^{-c|n_1-n_2|} \le C. \tag{11}$$

If $n_1 \in [b - \ell, b - \ell + \frac{\ell}{20}]$, by the fact that $m \in [b - \ell + \frac{\ell}{10}, b + \ell]$ and (6), one has

$$|G_{\Lambda_1}(m,n_1)| \le Ce^{-c\ell^c}. \tag{12}$$

If $n_1 \in [b - \ell + \frac{\ell}{20}, b + \ell]$, one has

$$\sum_{n_2\in\Lambda_2} e^{-c|n_1-n_2|} \le Ce^{-c\ell}. \tag{13}$$

Since $\Im z = \frac{1}{T}$, one has that

$$|G_{\Lambda_1}(m,n_1)| \le T, |G_\Lambda(n_2,n)| \le T. \tag{14}$$

By (11)–(14), we have

$$\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(m,n_1)|e^{-c|n_1-n_2|}|G_\Lambda(n_2,n)| \le CT^2 e^{-c\ell^c}.$$

□

Lemma 2.2. Assume $b - \ell$ is large. Under the conditions of Lemma 2.1, we have that for any j with $|j| \le K_1$ and $n \in [b + \ell - \frac{\ell}{10}, b + \ell]$,

$$|G_\Lambda(j,n)| \le CT^4 e^{-c\ell^c}, \tag{15}$$

where $\Lambda = (-\infty, b + \ell]$.

Proof. Let $\Lambda_2 = [b-\ell, b+\ell]$, $\Lambda_1 = (-\infty, b-\ell-1]$, and $\Lambda = (-\infty, b+\ell]$. By (8), one has that for any j with $|j| \le K_1$,

$$|G_\Lambda(j,n)| \le C \sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(j,n_1)| e^{-c|n_1-n_2|} |G_\Lambda(n_2,n)|. \tag{16}$$

For any $n_2 \in \Lambda_2$,

$$\sum_{n_1\in\Lambda_1} e^{-c|n_1-n_2|} \le C. \tag{17}$$

If $n_2 \in [b-\ell, b+\ell-\frac{\ell}{5}]$, by the fact that $n \in [b+\ell-\frac{\ell}{10}, b+\ell]$ and (9), one has

$$|G_\Lambda(n_2,n)| = |G_\Lambda(n,n_2)| \le CT^2 e^{-c\ell^c}. \tag{18}$$

If $n_2 \in [b+\ell-\frac{\ell}{5}, b+\ell]$, by the fact that $n_1 \le b-\ell$, one has

$$\sum_{n_1\in\Lambda_1} e^{-c|n_1-n_2|} \le Ce^{-c\ell}. \tag{19}$$

By (14) and (17)–(19), we have

$$\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(j,n_1)| e^{-c|n_1-n_2|} |G_\Lambda(n_2,n)| \le CT^4 e^{-c\ell^c}.$$

This implies (15). □

Lemma 2.3. Let $z = E + \frac{i}{T}$. Assume that $\ell \ge |\tilde{N}|^\delta$ and for some interval $I = [b-\ell, b+\ell]$ with $I \subset \left[\frac{|\tilde{N}|}{4}, \frac{|\tilde{N}|}{2}\right]$ or $I \subset \left[-\frac{|\tilde{N}|}{2}, -\frac{|\tilde{N}|}{4}\right]$, (6) holds. Then, for any j with $|j| \le K_1$,

$$|G_\Lambda(j,\tilde{N})| \le CT^6 e^{-c|\tilde{N}|^c}, \tag{20}$$

where $\Lambda = (-\infty, \infty)$.

Proof. Without loss of generality, assume $\tilde{N} > 0$. Let $\Lambda_1 = (-\infty, b+\ell]$, $\Lambda_2 = [b+\ell+1, \infty)$, and $\Lambda = (-\infty, \infty)$. By (8), one has

$$|G_\Lambda(j,\tilde{N})| \le C \sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(j,n_1)| e^{-c|n_1-n_2|} |G_\Lambda(n_2,N)|. \tag{21}$$

First, one has

$$\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} e^{-c|n_1-n_2|} \le C. \tag{22}$$

If $n_1 \in [b+\ell-\frac{\ell}{10}, b+\ell]$, By (15), one has

$$|G_{\Lambda_1}(j,n_1)| \le CT^4 e^{-c\ell^c}. \tag{23}$$

If $n_1 \in (-\infty, b+\ell-\frac{\ell}{10}]$ and $n_2 \in \Lambda_2$, one has

$$e^{-c|n_1-n_2|} \le Ce^{-c\ell}. \tag{24}$$

By (14) and (22)–(24), we have

$$\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(j,n_1)| e^{-c|n_1-n_2|} |G_\Lambda(n_2,\tilde{N})| \le CT^6 e^{-c\ell^c}.$$

This implies (20). □

Proof of Theorem 1.4. This is standard. For any j with $|j| \le K_1$, let

$$a(j,n,T) = \frac{2}{T}\int_0^\infty e^{-2t/T}|(e^{-itH}\delta_j,\delta_n)|^2 dt. \tag{25}$$

By the Parseval formula,

$$a(j,n,T) = \frac{1}{T\pi}\int_{-\infty}^{\infty}\left|\left(\left(H-E-\frac{i}{T}\right)^{-1}\delta_j,\delta_n\right)\right|^2 dE. \tag{26}$$

Recall that $\sigma(H) \subset [-K+1, K-1]$. For any $E \in (-\infty,-K)\cup(K,\infty)$, $\eta = \text{dist}(E+\frac{i}{T}, \text{spec}(H)) \ge 1$. The well-known Combes–Thomas estimate yields for large n,

$$\left|\left(\left(H-E-\frac{i}{T}\right)^{-1}\delta_j,\delta_n\right)\right| \le Ce^{-c|n|}. \tag{27}$$

By (26) and (27), one has that

$$a(j,n,T) \le Ce^{-c|n|} + \frac{1}{T\pi}\int_{-K}^{K}\left|\left(\left(H-E-\frac{i}{T}\right)^{-1}\delta_j,\delta_n\right)\right|^2 dE. \tag{28}$$

By Lemma 2.3, we have for any $|E| \le K$,

$$\left|\left(\left(H-E-\frac{i}{T}\right)^{-1}\delta_j,\delta_n\right)\right| \le CT^6 e^{-c\ell^c} \le CT^6 e^{-c|n|^c}. \tag{29}$$

By (28) and (29), one has that

$$a(j,n,T) \le CT^{11}e^{-c|n|^c}. \tag{30}$$

Therefore,

$$\begin{aligned}\langle |X|_\phi^p\rangle(T) &\le C\sum_{|j|\le K_1}\sum_{n\in\mathbb{Z}}|n|^p a(j,n,T)\\ &\le \sum_{n\in\mathbb{Z}} CT^{11}|n|^p e^{-c|n|^c}\\ &\le CT^{11}.\end{aligned} \tag{31}$$

It implies

$$\beta_\phi^+(p) \le \frac{11}{p}.$$

Since $\beta_\phi^+(p)$ are nondecreasing, we have that for every $p > 0$,

$$\beta_\phi^+(p) \le \lim_{p\to\infty}\beta_\phi^+(p) = 0. \tag{32}$$

□

III. PROOF OF COROLLARY 1.6

Under the assumption of Corollary 1.6, one has that when $|\epsilon| \le \epsilon_0$, the following holds for some $\delta_0 > 0$:

$$\#\{b \in \mathbb{Z} : |b| \le N, I_b \text{ does not satisfy } (6)\} \le N^{1-\delta_0}. \tag{33}$$

This was proved in Ch. 11 of Ref. 1 for $E \in \mathbb{R}$ and also holds for complex energies (this is mentioned already in Ref. 1).[18] Therefore, Corollary 1.6 follows from Theorem 1.4. □

DEDICATION

This paper is devoted to the memory of Jean Bourgain. Both authors have been profoundly influenced by Jean. S.J. was fortunate to experience direct influence (see Ref. 5). W.L. believes he became a mathematician through detailed reading of Ref. 1.

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APPENDIX: PROOF OF THEOREM 1.2

Proof. By an easy application of the Hölder inequality, $\beta_\phi^+(q)$ is nondecreasing with respect to q. Therefore, it suffices to show that for any $N \in \mathbb{N}$, $\beta_\phi^+(2N) \le 1$.

Define the free long-range Schrödinger operator by

$$(H_0 u)_n = \sum_{k\in\mathbb{Z}} a_{n-k} u_k = \sum_{k\in\mathbb{Z}} a_k u_{n-k}.$$

For any sequence $\gamma = \{\gamma_k\}$ with $|\gamma_k| \le Ce^{-c|k|}$, we define the momentum operator X_{2p}^γ,

$$(X_{2p}^\gamma u)_n = n^p \sum_k \gamma_k u_{n-k}$$

and $\hat{X}_{2p}^\gamma = -i[H_0, X_{2p}^\gamma]$, where $[B_1, B_2] = B_2B_1 - B_1B_2$.

Direct computations imply

$$\begin{aligned}(\hat{X}_{2p}^\gamma u)_n &= -i\left(n^p \sum_{k_1\in\mathbb{Z}} \gamma_{k_1} (H_0 u)_{n-k_1} - \sum_{k\in\mathbb{Z}} a_k (X_{2p}^\gamma u)_{n-k}\right)\\ &= -i\left(n^p \sum_{k_1\in\mathbb{Z},k\in\mathbb{Z}} \gamma_{k_1} a_k u_{n-k_1-k} - \sum_{k\in\mathbb{Z},k_1\in\mathbb{Z}} a_k (n-k)^p \gamma_{k_1} u_{n-k-k_1}\right)\\ &= -i \sum_{k\in\mathbb{Z},k_1\in\mathbb{Z}} (n^p - (n-k)^p) a_k \gamma_{k_1} u_{n-k-k_1}\\ &= -i \sum_{m\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} (n^p - (n-k)^p) a_k \gamma_{m-k}\right) u_{n-m}.\end{aligned}$$

Therefore, $\hat{X}_{2p}^\gamma$ can be rewritten as

$$\hat{X}_{2p}^\gamma = \sum_{j=0}^{p-1} X_{2j}^{\gamma^j} \tag{A1}$$

for some new sequences $\{\gamma_k^j\}$ with $|\gamma_k^j| \le C_j e^{-c_j|k|}$, $j = 0, 1, \cdots, p-1$.

Let

$$X_{2p}^\gamma(t) = e^{itH} X_{2p}^\gamma e^{-itH}, \hat{X}_{2p}^\gamma(t) = e^{itH} \hat{X}_{2p}^\gamma e^{-itH}.$$

Differentiating $X_{2p}^\gamma(t)$, one has that

$$\frac{dX_{2p}^\gamma(t)}{dt} = \hat{X}_{2p}^\gamma(t). \tag{A2}$$

We will show inductively that

$$(X_{2N}^{\gamma}(t)\phi, X_{2N}^{\gamma}(t)\phi) \le C_{\gamma,\phi,N} t^{2N} \text{ for large } t. \tag{A3}$$

We first prove (A3) for $N = 1$. Differentiating $X_2^{\gamma}(t)$, one has that

$$\frac{dX_2^{\gamma}(t)}{dt} = \hat{X}_2^{\gamma}(t), \tag{A4}$$

where $\hat{X}_2^{\gamma}(t)$ is a bounded selfadjoint operator by (A1). By (A2), one has

$$X_2^{\gamma}(t) = X_2^{\gamma} + \int_0^t \hat{X}_2^{\gamma}(s) ds. \tag{A5}$$

This implies

$$\begin{aligned}
&(X_2^{\gamma}(t)\phi, X_2^{\gamma}(t)\phi) \\
&\quad = (X_2^{\gamma}\phi + \int_0^t \hat{X}_2^{\gamma}(s)\phi \mathrm{d}s, X_2^{\gamma}\phi + \int_0^t \hat{X}_2^{\gamma}(s)\phi ds) \\
&\quad \le \|X_2^{\gamma}\phi\|^2 + 2\|X_2^{\gamma}\phi\| \quad \int_0^t \|\hat{X}_2^{\gamma}(s)\phi\| ds + \left(\int_0^t \|\hat{X}_2^{\gamma}(s)\phi\| \mathrm{d}s\right)^2 \\
&\quad \le C_{\gamma,\phi} t^2 + C_{\gamma,\phi} t + C_{\gamma,\phi},
\end{aligned}$$

since ϕ has compact support and $\hat{X}_2^{\gamma}(t)$ is bounded.

Assume that (12) holds for $p \le N - 1$. This means that for any sequence $\{\gamma_k\}$ and $p = 1, 2, \cdots, N - 1$,

$$(X_{2p}^{\gamma}(t)\phi, X_{2p}^{\gamma}(t)\phi) \le C_{\gamma,\phi,p} t^{2p} \text{ for large } t. \tag{A6}$$

By (A2), one has

$$X_{2N}^{\gamma}(t) = X_{2N}^{\gamma} + \int_0^t \hat{X}_{2N}^{\gamma}(s) \mathrm{d}s. \tag{A7}$$

By (A1) and (A6), we have

$$\|\hat{X}_{2N}^{\gamma}(t)\phi\| \le C_{\gamma,\phi,N} t^{N-1} \text{ for large } t. \tag{A8}$$

This implies, for large t,

$$\begin{aligned}
&(X_{2N}^{\gamma}(t)\phi, X_{2N}^{\gamma}(t)\phi) \\
&\quad = \left(X_{2N}^{\gamma}\phi + \int_0^t \hat{X}_{2N}^{\gamma}(s)\phi \mathrm{d}s, X_{2N}^{\gamma}\phi + \int_0^t \hat{X}_{2N}^{\gamma}(s)\phi \mathrm{d}s\right) \\
&\quad \le \|X_{2N}^{\gamma}\phi\|^2 + 2\|X_{2N}^{\gamma}\phi\| \quad \int_0^t \|\hat{X}_{2N}^{\gamma}(s)\phi\| \mathrm{d}s + \left(\int_0^t \|\hat{X}_{2N}^{\gamma}(s)\phi\| \mathrm{d}s\right)^2 \\
&\quad \le C_{\gamma,\phi,N} t^{2N}.
\end{aligned}$$

Let $\{\gamma_k\}$ be the sequence such that $\gamma_0 = 1$ and $\gamma_k = 0$ for $k \ne 0$. Therefore, one has

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |n|^{2N} |(e^{-itH}\phi, \delta_n)|^2 &= (X_{2N}^{\gamma} e^{-itH}\phi, X_{2N}^{\gamma} e^{-itH}\phi) \\
&= (e^{itH} X_{2N}^{\gamma} e^{-itH}\phi, e^{itH} X_{2N}^{\gamma} e^{-itH}\phi) \\
&= (X_{2N}^{\gamma}(t)^{\gamma}\phi, X_{2N}^{\gamma}(t)\phi) \\
&\le C_{\phi,N} t^{2N}.
\end{aligned} \tag{A9}$$

By (2) and (A9), one has

$$\langle |\hat{X}|_\phi^{2N} \rangle(T) \le \frac{2}{T}\int_0^\infty e^{-2t/T} C_{\phi,N} t^{2N} \mathrm{d}t$$
$$\le C_{\phi,N} T^{2N}.$$

Thus, $\beta_\phi^+(q) \le 1$ for any $q > 0$. □

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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