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ORIGINAL PAPER

The Two-Eyes Lemma: A Linking Problem for Table-Top Necklaces

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Abstract
In this note, we answer a combinatorial question that is inspired by cusp geometry of hyperbolic 3-manifolds. A table-top necklace is a collection of sequentially tangent beads (i.e. spheres) with disjoint interiors lying on a flat table (i.e. a plane) such that each bead is of diameter at most one and is tangent to the table. We analyze the possible configurations of a necklace with at most 8 beads linking around two other spheres whose diameter is exactly 1. We show that all the beads are forced to have diameter one, the two linked spheres are tangent, and that each bead must be tangent to at least one of the two linked spheres. In fact, there is a 1-parameter family of distinct configurations.

Keywords Packing · Spheres · Horoball · Hyperbolic

Mathematics Subject Classification 52C17 · 57K32

1 Introduction

Start with a disc D of radius r in the Euclidean plane. What is the maximal number of discs of radius r with disjoint interiors that each *kiss* D? We say two discs *kiss* if they intersect on their boundaries but not in their interiors. The answer is 6, as can be seen by noting that the visual angle (as measured from the center of D) of a

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kissing disc is 60°. Further, all such configurations are the same up to rotation about D, and the centers of the 6 discs are the vertices of a regular hexagon.

This leads to the classical kissing problem: what is the maximal number of equal radius spheres that simultaneously kiss a base sphere of the same radius? This question was the subject of a correspondence between Isaac Newton and James Gregory in the 17th century. Newton thought the answer was 12 but Gregory wondered whether 13 might work. Newton was correct, with the first correct proof given by Schütte and van der Waerden in 1953 [8]. One could also ask about how many essentially distinct 12-kissings there are. It turns out that there are infinitely many that are fundamentally different and then one could ask for a description of this parameter space. Similarly, this question is of interest in higher dimensions. Good references for this material are the classic text "Sphere Packings, Lattices and Groups" by Conway and Sloane (Chapter 2) [1] and the semi-expository paper "The Twelve Spheres Problem" by Kusner, Kusner, Lagarias, and Shlosman [4].

In the course of our work on low-volume hyperbolic 3-manifolds [3], we faced a different generalization of the kissing problem. Here, we came upon a cycle (or necklace) of kissing spheres (or beads) of diameter at most one lying on a flat table. Many questions about the topology of such necklaces are studied by Maehara in [5]. Related results can also be found in the work of Maehara and Oshiro in [6] and Ramírez Alfonsín and Rasskin in [7]. Such results have interesting applications, for example in [3], the authors make use of existence and configurations of short necklaces to prove strong theorems about exceptional Dehn fillings and volumes of hyperbolic 3-manifolds. Here, we focus on a special linking necklace configuration that answers a kissing problem for such table-top beads, also known as horoballs in hyperbolic geometry.

Suppose a necklace of ≤ 8 such beads winds around beads C_1 and C_2, also on the table, with disjoint interiors and of height (i.e. diameter) exactly one. As a consequence of the *Two-Eyes Lemma* (see below), we are able to prove that C_1 and C_2 must kiss, that each bead must kiss C_1 or C_2, and that each necklace bead must be of height one. An example of this is obtained by taking a hexagonal packing of height-one spheres, labeling two abutters as C_1 and C_2, and then observing the cycle of 8 spheres encircling them. In fact, there is a 1-parameter family of essentially different solutions that is gotten by sliding one sphere along C_1 (or C_2) and then all other sphere positions are forced. Further, these are the only possible solutions. See Figs. 1 and 2. We note that when all the beads are assumed to be of height one, our result reduces to a planar problem that is quite easy to address. Additionally, in the context of encircling just one bead, the answer is well-known and arises uniquely from the hexagonal packing.

We are naturally led to the following question, which we simply pose, but do not address. Given two abutting spheres of radius r in $\mathbb{R}^3$, what is the kissing number for these two spheres? That is, what is the maximal number of (non-overlapping) radius r spheres that each kiss either of the two abutting spheres?

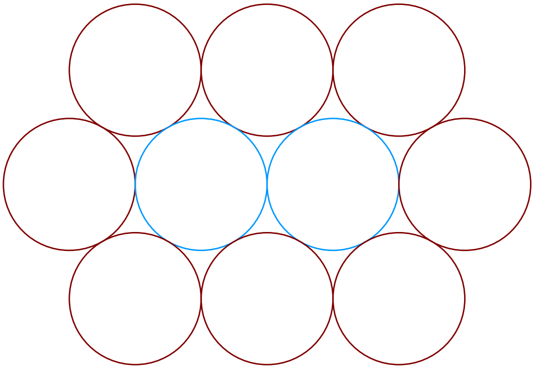


Fig. 1 Hexagonal configuration

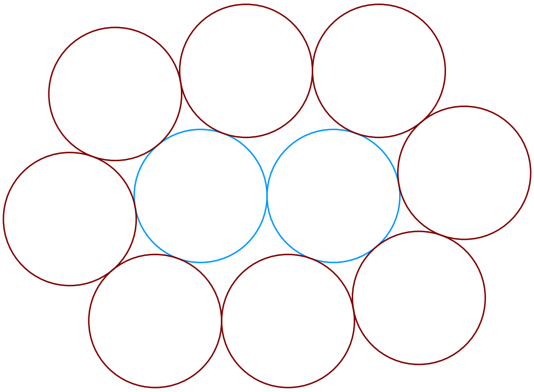


Fig. 2 Non-hexagonal configuration

2 Set-Up and Statement of Main Proposition

Let $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ denote the upper half-space of $\mathbb{R}^3$. Throughout this note, a *bead* will be a ball in $\mathbb{H}^3$ tangent to the boundary plane $z = 0$ and of *diameter at most 1*. We call a bead *full-sized* if it has diameter exactly 1. We will often refer to the diameter as *height*. Let $\pi : \mathbb{H}^3 \to \mathbb{R}^2$ be the orthogonal projection onto the plane $z = 0$. The *center of a bead B* will be the Euclidean center of the disk $\pi(B)$ in the plane.

Definition 1 A *k-necklace* $\eta = N_1 \cup \cdots \cup N_k$ is a cyclicly ordered set of k beads with disjoint interiors such that one is tangent to the next. In what follows indices for a k-necklace are always modulo k. The number k is called the *necklace* or *bead number* of η.

Sequentially connecting the centers of beads in a necklace η by straight line segments gives a piecewise linear loop L_η in the plane. Given a full-sized bead C, we say that a necklace η *winds around*, *encircles*, or *links* with C if the winding number of L_η around the center of C is nonzero. The main result of this note answers a question about how necklaces can link with two beads at the same time. See Fig. 3.

Proposition 1 *If C_1 and C_2 are full-sized beads with disjoint interiors then the minimum bead number of a necklace η with less-than-or-equal-to full-sized beads encircling both C_1 and C_2 is 8. If the bead number is 8, then all beads in η must be full-sized. Further, all 8-necklaces arise by taking N_1 tangent to C_1 or C_2 and then placing the remaining beads cyclically, making sure that each N_i abuts C_1 and/or C_2. See Figs. 1 and 2.*

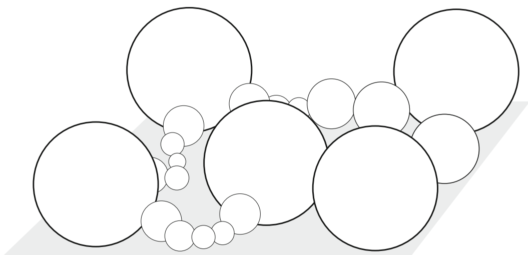
3 Motivation

One motivation from this question arises from the geometry of orientable, non-compact, finite-volume hyperbolic 3-manifolds, see [9] for a reference. Such manifolds include large families of knot and link complements on $\mathbb{S}^3$. Geometrically, these manifolds arise as quotients of hyperbolic 3-space $\mathbb{H}^3$ by a lattice Γ in $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ acting by isometries. Here, our upper-half space $\mathbb{H}^3$ is equipped with the hyperbolic metric. The ends of these manifolds always have a neighborhood homeomorphic to $T^2 \times (0,1)$ and are called cusps. Every cusp contains an embedded neighborhood whose lift in the universal cover $\mathbb{H}^3$ is a collections of disjoint beads tangent to the boundary of $\mathbb{H}^3$, called *horoballs*. If one takes a maximally embedded neighborhood, these beads become tangent and form necklaces. Arrangements and linking of such necklaces play a key role in understanding low-complexity hyperbolic 3-manifolds. For example, in [3] the authors are able to show that 7-necklaces that arise in this way are never knotted and never link each other, allowing them to classify large families of hyperbolic 3-manifolds. For 8-necklaces, it is unclear if linking can happen in this context. A subtle point is that only tangencies in a fixed Γ-orbit are considered valid when looking at such necklaces. Experimentally, linking of one bead in this setting requires a 9-necklace, while linking two beads can be realized by 12-necklaces, see Fig. 4. In this note, we classify, without restricting to the context of hyperbolic 3-manifolds, all configuration of 8-necklaces linking two beads.

4 The Two-Eyes Lemma

Since we will be working with beads lying on a table, we will need the following useful observation that is easy to derive with basic Euclidean geometry:

Fig. 3 A side-view sketch of a bead necklace



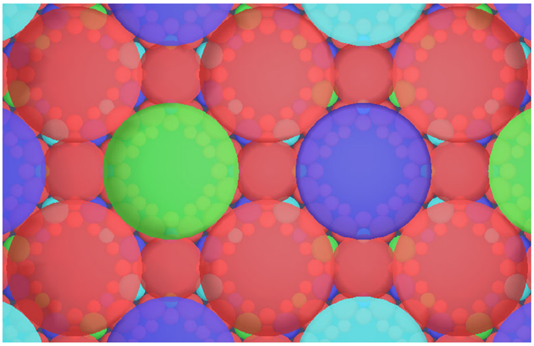


Fig. 4 A (red) 12-necklace linking two beads in the top-view of the cusp neighborhood of the 8_2^4-link compliment in $\mathbb{S}^3$. Image made using SnapPy [2]

Lemma 1 *Let B_1, B_2 be two beads with disjoint interiors, centers b_1, b_2 and of heights h_1, h_2. Then $|b_1 - b_2|^2 \geq h_1 h_2$ with equality if and only if B_1 and B_2 are tangent.*

A direct corollary is a statement about visual angles.

Corollary 1 (*Visual Angle*) *Let C be a full-sized bead and let B be a bead tangent to C, then the visual angle of $\pi(B)$ from center(C) is $\leq \pi/3$ with equality if and only if B is full-sized.*

We now turn to the Two-Eyes Lemma, which is depicted in Figs. 5, 6, 7 and 8.

Lemma 2 (*Two-Eyes Lemma*) *Let C_1 and C_2 be full-sized beads with disjoint interiors. Let B_1 and B_2 be tangent beads with heights $h_1 \leq 1$ and $h_2 \leq 1$, respectively, with interiors disjoint from $C_1 \cup C_2$. Let L be the line through center(C_1) and center(C_2) and let v_1 and v_2 be lines orthogonal to L passing*

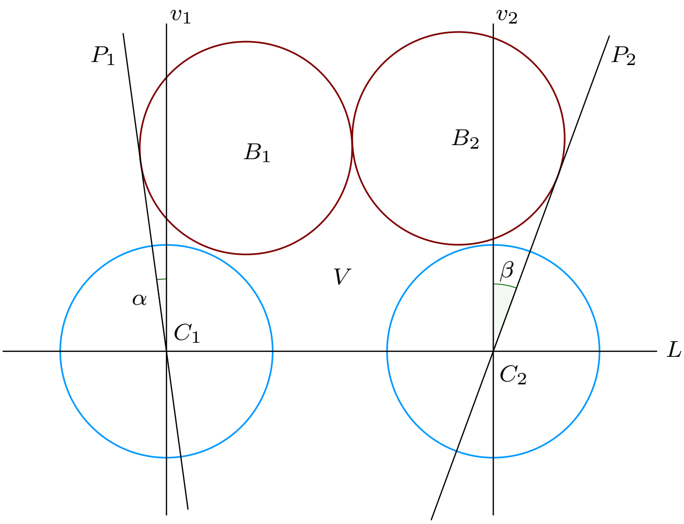


Fig. 5 $\alpha + \beta \leq \pi/3$

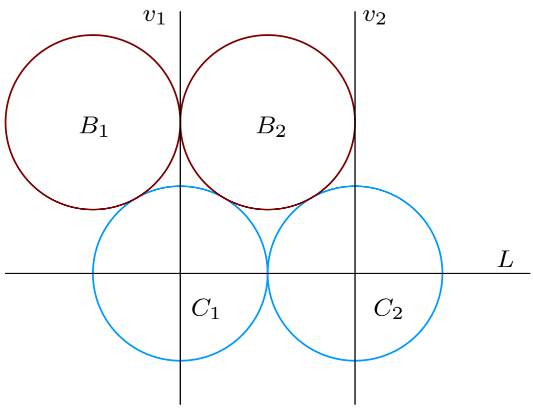


Fig. 6 $\alpha = \pi/3, \beta = 0$

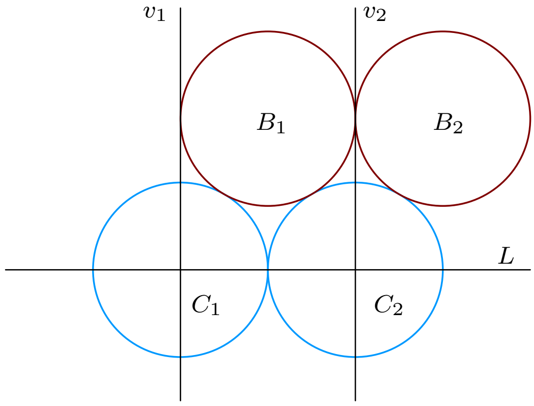


Fig. 7 $\alpha = 0, \beta = \pi/3$

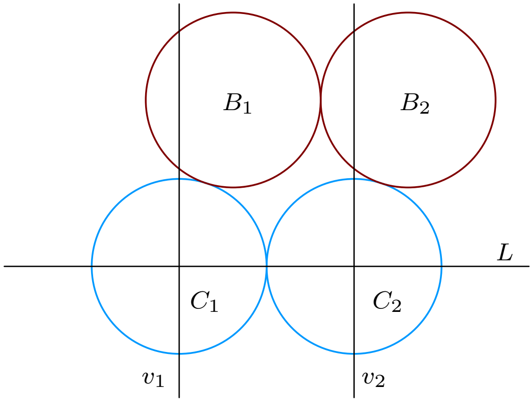


Fig. 8 $0 < \alpha, \beta < \pi/3, \alpha + \beta = \pi/3$

through center(C_1) and center(C_2), respectively. Let V_1 and V_2 be the planes in $\mathbb{H}^3$ with boundaries containing v_1 and v_2 and let V be the closure of the region bounded by $V_1 \cup V_2$ in $\mathbb{H}^3$. Suppose that for each $i, B_i \cap V_i \neq \emptyset$. Let P_i denote the line tangent to $\pi(B_i)$ through center(C_i) such that $\pi(B_1 \cup B_2)$ lies to one side. Finally let α (resp. β) be the acute angle between P_i and v_i. Then,

1. $\alpha + \beta \leq \pi/3$
2. If $\alpha + \beta = \pi/3$, then
   (a) C_1 is tangent to C_2
   (b) B_1 and B_2 are full-sized
   (c) for $i = 1, 2$ we have that B_i is tangent to C_i and the line J through center (B_1) and center (B_2) is parallel to L.

Proof To start with, we can assume that L is parallel to the x-axis. The proof involves a series of steps whereby the positions of B_1, B_2, C_1, C_2 are repeatedly *improved*. The reader should note that any *improvement* strictly increases $\alpha + \beta$. In the end $\alpha + \beta = \pi/3$ and the various beads satisfy the equality conclusions. We repeatedly use the fact that an operation that moves center (B_2) infinitesimally closer to P_2 is β increasing with the analogous fact holding for α.

Let

$$\begin{aligned} b &= |\text{center}(B_1) - \text{center}(B_2)|, \\ c &= |\text{center}(C_1) - \text{center}(C_2)|, \text{ and} \\ d_{ij} &= |\text{center}(B_i) - \text{center}(C_j)| \text{ for } i,j \in \{1,2\}. \end{aligned}$$

We can assume that center$(C_1) = (0,0)$, center$(C_2) = (c,0)$, center$(B_1) = (x_1, y_1)$ and center$(B_2) = (x_2, y_2)$. Note that $c \geq 1$, $-h_1/2 \leq x_1 \leq h_1/2$ and $-h_2/2 \leq x_2 - c \leq h_2/2$. By Lemma 1, we also have that $b = \sqrt{h_1 h_2}$ and $d_{ij} \geq \sqrt{h_i}$, with equality if and only if B_i is tangent to C_j.

Step 1. At the cost of possibly increasing $\alpha + \beta$ we can assume that either B_1 is tangent to C_1 or B_2 is tangent to C_2.

Proof If both $B_1 \cap C_1 = \emptyset$ and $B_2 \cap C_2 = \emptyset$, then we can translate $B_1 \cup B_2$ in the $(0, -1)$ direction until a first tangency occurs. Note that both α and β increase. If $B_1 \cap C_2 \neq \emptyset$ but $B_1 \cap C_1 = \emptyset$, then we can obtain a contradiction as follows: we have $(x_1 - c)^2 + y_1^2 = d_{12}^2 = h_1$ and $x_1^2 + y_1^2 = d_{11}^2 > h_1$. However, since $x_1 \leq h_1/2$, we obtain $1 \leq c < h_1 \leq 1$, a contradiction. A similar fact holds for B_2, thus the tangency is of the type claimed. □

Step 2. At the cost of possibly increasing $\alpha + \beta$ we can additionally assume that either $C_1 \cap C_2 \neq \emptyset$ or each of B_1 and B_2 are respectively tangent to C_1 and C_2.

Proof It suffices to consider the case where B_1 is tangent to C_1. If C_2 is disjoint from B_2, then translate C_2 in the $(-1, 0)$ direction until a first tangency occurs. Note that β increases. If C_2 becomes tangent to B_1 first, then by the computation in Step 1, $c = 1$ and C_2 is also tangent to C_1. Lastly, we observe that $B_2 \cap V_2 \neq \emptyset$ remains true as we translate by computation: if $-h_2/2 \leq x_2 - c \leq h_2/2$ fails as we decrease c, we have that $x_2 > c + h_2/2$. But $x_2 \leq x_1 + b = x_1 + \sqrt{h_1 h_2} \leq h_1/2 + (h_1 + h_2)/2$, so we obtain $1 \leq c < h_1 \leq 1$, a contradiction. □

Step 3. At the cost of possibly increasing $\alpha + \beta$ we can further assume that for each $i, B_i \cap C_i \neq \emptyset$.

Proof It suffices to consider the case that $B_1 \cap C_1 \neq \emptyset$ and $B_2 \cap C_2 = \emptyset$. Let J denote the ray from center (B_1) through center (B_2). First assume that $J \cap P_2 \neq \emptyset$. For each $t \geq 0$ we translate B_2 away from B_1 by moving its center Euclidean distance t along J away from center (B_2) to obtain $B_2'(t)$. We then expand $B_2'(t)$ keeping its center fixed until it first hits B_1 to obtain $B_2(t)$. Let B_2(new) be the first $B_2(t)$ that is either full-sized or satisfies $B_2(t) \cap C_2 \neq \emptyset$. Note that if B_2 (new) $\neq B_2$, then β increases. We now abuse notation by denoting B_2 (new) by B_2. Thus, if $B_2 \cap C_2 = \emptyset$, then B_2 is full-sized and by Step 2, $C_1 \cap C_2 \neq \emptyset$.

If $J \cap P_2 = \emptyset$, then apply a clockwise rotation about the line between center(B_1) and ∞ until either $B_2 \cap C_2 \neq \emptyset$ or $J \cap P_2 \neq \emptyset$. This operation is strictly β increasing. If now $J \cap P_2 \neq \emptyset$, then argue as in the first paragraph to conclude that either Step 3 holds or B_2 is full sized and $C_1 \cap C_2 \neq \emptyset$.

We have now reduced to the case that B_2 is full-sized, $C_1 \cap C_2 \neq \emptyset$ and $B_2 \cap C_2 = \emptyset$. Observe that $y_2 \geq y_1$. This is immediate if B_1 is full-sized. In general, center (B_1) lies on the line perpendicular to the midpoint of the segment between center (C_1) and center (B_2) since B_1 is tangent to the full-sized beads C_1 and B_2. Since $x_1 \leq 1/2 \leq x_2$, the maximal y_1 is obtained when B_1 is full-sized and hence $y_2 \geq y_1$. Since P_2 has non-negative slope, a clockwise rotation about γ both transforms B_2 to a bead tangent to C_2 and increases β. □

We can now conclude the proof of the Two-Eyes Lemma. This argument is a simplified version suggested by the referee in place of one using hyperbolic geometry.

Step 4. Either $\alpha + \beta < \pi/3$ or $\alpha + \beta = \pi/3$ and the quadrilateral with corners b_1, b_2, c_1, and c_2 must be a rhombus (Fig. 9).

Proof Following Fig. 10, since B_1, B_2 are tangent, at most full-sized, and tangent to C_1, C_2, respectively, by Step 3, we have the following inequalities on the lengths of the edges:

$$|b_1 b_2| \leq |b_1 c_1|, \quad |b_1 b_2| \leq |b_2 c_2| \quad \text{and} \quad |b_1 c_1| \leq |c_1 c_2|, \quad |b_2 c_2| \leq |c_1 c_2|.$$

With the angles at the vertices ψ_{b_i} and ψ_{c_i}, the inequalities above tell us that $\gamma_{c_1} \leq \gamma_{b_2}$ and $\gamma_{c_2} \leq \gamma_{b_1}$. Additionally, since $|b_1 b_2| \leq |c_1 c_2|$, it follows that $\gamma_{c_1} + \gamma_{c_2} \leq \pi$. Now, since the angle between P_i and the ray $c_i b_i$ is at most $\pi/6$ by the visual angle restriction on beads, we see that $\alpha + \beta \leq \pi/3$. Equality would force $\gamma_{b_1} = \gamma_{c_2}$ and $\gamma_{b_2} = \gamma_{c_1}$, meaning that our quadrilateral is a rhombus. In particular,

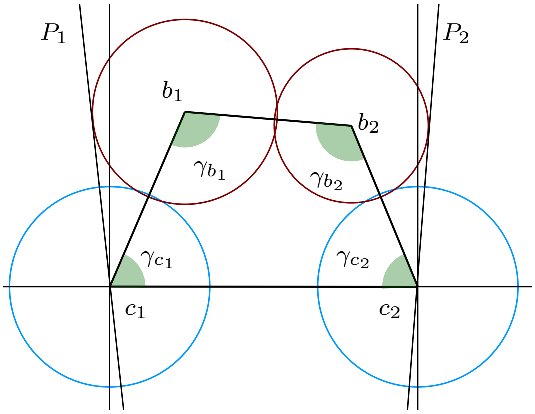


Fig. 9 Quadrilateral in the proof of Lemma 2

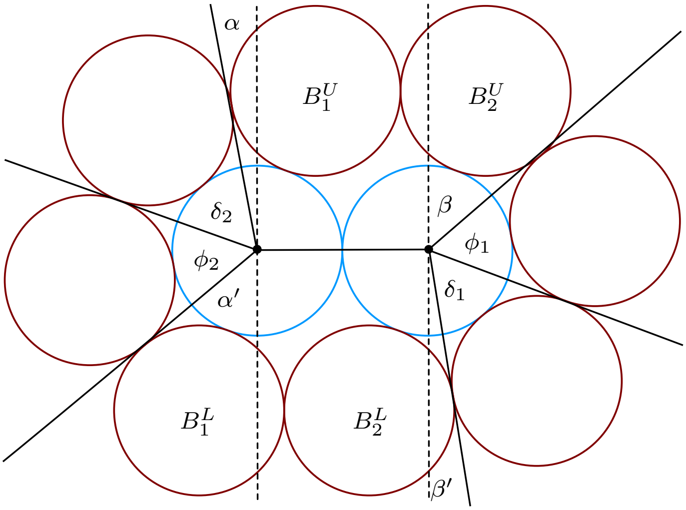


Fig. 10 Arrangement of 8 beads and visual angles. V is the region between the dotted planes

all the edges are of equal length. Since $|c_1c_2| \geq 1$ and $|b_1b_2| \leq 1$, it follows that all edges have length 1 and all beads are full size. □

This completes the proof of the Two-Eyes Lemma. □

5 Proof of Proposition 1

The proof of the main proposition is now just a counting argument.

Proof of Proposition 1 As in the proof of the Two-Eyes Lemma, consider the planes V_1, and V_2. Since the necklace η winds around C_1 and C_2, it follows that V_1 and V_2 each intersect at least two beads of η. For $i = 1, 2$, let B_i^U, B_i^L be these beads intersecting V_i with centers in the upper and lower half-planes, respectively. These four beads are distinct. Further, we can assume that B_2^U, B_2^L have the largest x-coordinates and B_1^U, B_1^L are the smallest x-coordinates amongst all choices in η satisfying the non-empty intersection conditions. Since all beads are at most full-sized, visual angle around center(C_i) tells us that, away from the critical case where *both* B_i^U and B_i^L are tangent to V_i, we need at least two more beads to connect B_1^L to B_1^U and at least two more to connect B_2^U to B_2^L in the clockwise direction along η. Away from this critical case, the necklace must have at least 8 beads.

Assume we are in the critical case where B_i^U and B_i^L are tangent to V_i for some i. Without loss of generality, we can take $i = 1$. By the minimality of the x-coordinates and the fact that necklace beads are sequently tangent, we can assume that B_1^U and B_1^L lie entirely to the left of V_1, aside from the points of tangency. The region V, between V_1 and V_2, will then contain at least two more beads, but these cannot be B_2^U, B_2^L by the maximality of the x-coordinate and because all the beads are at most full-sized. Therefore, we need at least 1 bead to join B_1^L to B_1^U, 2 more beads in V, and at least 1 more bead to join B_2^U to B_2^L, giving us a total of 8.

We turn to the case where η has exactly 8 beads. It remains to show that all are full sized and the configuration is obtained by sliding the hexagonal example. For this, we will use the Two-Eyes Lemma and visual angle arguments. Assume that in each of the pairs $\{B_1^U, B_2^U\}$ and $\{B_2^L, B_1^L\}$ at least one of the beads is not tangent to the associated V_i. In this setting, our counting argument in the first paragraph gives that the beads B_1^U and B_2^U are tangent. Similarly for B_1^L and B_2^L. Let α, β be the angles from the Two-Eyes Lemma applied to the pair $\{B_1^U, B_2^U\}$ and α', β' be the angles for the pair $\{B_2^L, B_1^L\}$. It follows that $\alpha + \beta \leq \pi/3$ and $\alpha' + \beta' \leq \pi/3$. For each i, we have exactly two beads in η connecting B_i^L to B_i^U with centers in the complement of V. Let δ_i, φ_i be the visual angles from center(C_i) of these beads. Then, cutting out V, we have that the sum of the angles satisfies

$$2\pi \leq (\beta + \alpha) + \delta_1 + \varphi_1 + (\beta' + \alpha') + \delta_2 + \varphi_2 \leq \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = 2\pi.$$

See Fig. 10. Note that this figure is slightly simplified in that a pair of tangents to a bead shadow can overlap with neighboring pairs. It follows that $\delta_i = \varphi_i = \pi/3$ and $\alpha + \beta = \alpha' + \beta' = \pi/3$. Thus, all beads in η are full-sized and tangent to C_1 or C_2. Hence, all the beads in η are tangent to C_i are part of a hexagonal packing around C_i. This allows us to compute $\alpha = \pi - \delta_1 - \varphi_1 - \beta' = \pi/3 - \beta'$ and, similarly, $\beta = \pi/3 - \alpha'$. Since $\alpha + \beta = \pi/3$ and $\alpha' + \beta' = \pi/3$, we obtain a one-parameter family of beads parametrized by, say, α. □

Without loss of generality, the remaining case is where B_1^U is tangent to V_1 and B_2^U is tangent to V_2 (with the x coordinate max/min condition above). There is then at least one bead from B_1^U to B_2^U in the clockwise direction along η. Let $D_{1,1}, D_{1,2}$ be the next two beads in the counter-clockwise direction from B_1^U and $D_{2,1}, D_{2,2}$ the next two beads in the clockwise direction from B_2^U along η. If the visual angle of at least one of $D_{i,j}$ from center(C_i) is $< \pi/3$, then $D_{i,2} \neq B_i^L$. Counting the beads tells us that at least one of B_i^L has to be tangent to V_i. In fact, both must. Indeed, if B_2^L is tangent to V_2 then it cannot be tangent to B_1^L and one more bead is required in the clockwise direction. Similarly, if B_1^L is tangent to V_1. Thus, $D_{i,2} = B_i^L$ for $i = 1, 2$ and $D_{i,j}$ have visual angle $\pi/3$, which means they are full-sized and tangent to C_i. The beads that connect B_1^U to B_2^U and B_2^U to B_1^U in the clockwise direction must also be full-sized and tangent to both C_1 and C_2 to bridge the "width" of V. Thus, we are in the configuration above where $\alpha = \pi/3$.

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Availability of Data and Material Note relevant.

Code Availability Note relevant.

Declarations

Conflict of interest None.

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