On singularities in the quaternionic Burgers equation

Vladimir Sverak

University of Minnesota

Dedicated to Professor Alexander Shnirelman on the occasion of his 75th birthday.

Abstract

We consider the equation $q_t + qq_x = q_{xx}$ for $q\colon \mathbf{R}\times(0,\infty)\to\mathbf{H}$ (the quaternions), and show that while singularities can develop from smooth compactly supported data, such situations are non-generic. The singularities will disappear under an arbitrary small "generic" smooth perturbation of the initial data. Similar results are also established for the same equation in $\mathbf{S}^1\times(0,\infty)$, where $\mathbf{S}^1$ is the standard one-dimensional circle.

Résumé. L'équation $q_t + qq_x = q_{xx}$ pour $q\colon \mathbf{R}\times(0,\infty)\to\mathbf{H}$ (les quaternions) est considérée. Nous montrons que bien que des singularités peuvent se développer en temps fini à partir de données initiales lisses à support compact, cette situation n'est pas générique. Les singularités disparaissent après une perturbation générique lisse arbitrairement petite de la donnée initiale. Des résultats similaires sont également établis pour la même équation dans $S^1\times(0,\infty)$, où S^1 est le cercle unidimensionnel.

1 Introduction

Singularities can form from smooth initial data for a number of PDEs. The PDE in the back of our mind in this article will be the 3d Navier-Stokes equation. In this case, the possibility of singularity formation from "nice" initial data is still open at the time of this writing. Leray [12] introduced a notion of weak solutions that can pass through potential singularities and showed that all singularities must happen only in a relatively small closed set of times. This result was later significantly extended by [16, 4, 13] to the smallness of the singular set in space-time. If one could prove uniqueness in a suitable class of weak solutions, the situation would be quite satisfactory — the equation would (globally) predict the future from the information about the current state, as we expect from a Newtonian system. The singularities might or might not be present, but their potential presence would perhaps not be too disturbing — they could be considered as an acceptable price one has to pay for the various idealizations made in the derivation of the equations.

If the presence of singularities would result in non-uniqueness, as suggested by [9, 10, 6], the mathematical model provided by the Navier-Stokes equations would be incomplete. It would not predict the future of the system from its current state, in sharp contrast to what we expect from the Newtonian models. Such a scenario, if realized, would cast doubt on the predictive power of the equation.[1] The importance of uniqueness has always been emphasized by Ladyzhenskaya. We refer the reader for example to [11].

There is still a possibility where all the phenomena of singularities, non-uniqueness, and physically meaningful predictability could coexist — namely in the scenario where all the singularities are unstable or "non-generic". If this was the case, then "typical solutions" would be free of singularities and the evolution defined by their initial conditions would be uniquely determined. The singularities could still form from "non-generic" data, but the probability of encountering such data in the physical world would be zero and hence the equation would not lose its predictive power for most practical purposes.

There are no examples of evolution equations known to the author where similar scenarios would occur. Of course, there are many examples of unstable singularities for various evolution equations, but here we are interested in a situation where *all* singularities are unstable (and the set of solutions that develop singularities is non-empty, of course). As we speculated above, perhaps the Navier-Stokes equation might belong to this category. Among other evolution equations, the 2d harmonic map heat flow could be a good candidate, see [1].

Here we give an example of an equation for which all singularities are unstable. We will consider a parabolic system for functions $q = q(x,t)$ with values in the quaternions $\mathbf{H} \sim \mathbf{R}^4$, where the identification is realized as usual by $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$, with (q_0, q_1, q_2, q_3) considere as an element of $\mathbf{R}^4$. The multiplication is given by the usual multiplication table for $\mathrm{i}, \mathrm{j}, \mathrm{k}$, specified by $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1\,,\ \mathrm{ij} = \mathrm{k}\,,\ \mathrm{ji} = -\mathrm{k}$ and the fact that $\mathbf{H}$ is an (associative) algebra. The variable x will be taken one-dimensional, so that $(x,t) \to q(x,t)$ will be $\mathbf{H}-$valued functions on $\mathbf{R} \times (0,\infty)$. For such functions we consider the equation

$$q_t + qq_x = q_{xx} \tag{1.1}$$

and the corresponding Cauchy problem in $\mathbf{R} \times (0,\infty)$,

$$\begin{aligned} q_t + qq_x &= q_{xx} \\ q(x,0) &= a(x)\,, \end{aligned} \tag{1.2}$$

where $a\colon \mathbf{R} \to \mathbf{H}$ is the initial condition.

[1] The model certainly loses predictive power at the level of low-regularity weak solutions. In the inviscid case this was first shown by Scheffer[17]. Schnirelman [18] studied this phenomenon from a new angle. In recent years, applications of convex integration lead to further important developments, such as [5, 8, 2].

There are many ways to see that for (reasonable) $\mathbf{R}$-valued functions a the problem is globally well posed. When $q(x,t) = u(x,t)$ for a real-valued u, the solutions satisfy the energy estimate

$$\int_{\mathbf{R}} \frac{1}{2} u^2(x,t_2)\,dx + \int_{t_1}^{t_2} \int_{\mathbf{R}} u_x^2\,dx\,dt = \int_{\mathbf{R}} \frac{1}{2} u^2(x,t_1)\,, \tag{1.3}$$

which, together with standard local well-posedness results, is sufficient to show the global well-posedness.[2]

Many more estimates are available in the $\mathbf{R}-$valued case, including the maximum principle. These estimates do not survive when passing from the real-valued case to the quaternionic case. In fact, they already fail for the complex-valued case (which can, of course, be considered as a special case of the quaternionic-valued situation). In the complex-valued case, singularities can develop from smooth compactly supported initial data, as shown in [15]. Generically, the singularities in the complex-valued case are stable, as follows easily from the analysis in [15], and as we also will see below.

When we take the range of q to be the quaternions $\mathbf{H}$, one still has singularities, but they will all become unstable. To make the statement more precise, let us consider some space X of initial condition $a\colon \mathbf{R} \to \mathbf{H}$ for which the equation is locally well-posed. For example, $X = L^1(\mathbf{R},\mathbf{H})$ is such a space as we will see below. We could also take for X the (Frechet space) of the smooth rapidly decaying Schwartz functions with values in $\mathbf{H}$. Let $X_{\rm sing}$ be a subset of the functions a in X for which the solution of (1.1) develops a singularity in finite time. For $a \in X \setminus X_{\rm sing}$ the solutions q will be smooth in $\mathbf{R} \times (0,\infty)$.

In this note we will discuss variations of the following observation:

Observation

The set $X \setminus X_{\rm sing}$ contains a subset X_0 which is open and dense in X and has the property that the solutions starting X_0 behave well as $t \to \infty$. Analogous results are true also in the space-periodic case.[3]

For more precise statements see Theorem 1 and Theorem 2 below.

Remarks

1. We will see that, in some sense, $X_{\rm sing}$ is of co-dimension 2 in X (although it is not a manifold and can have non-smooth points).
2. We will see that a perturbation from $a \in X_{\rm sing}$ to $a + \varphi \notin X_{\rm sing}$ can be achieved by an arbitrarily small smooth function φ with support in any predetermined subinterval of $\mathbf{R}$.
3. The space $X = L^1(\mathbf{R},\mathbf{H})$ is "critical" for the Cauchy problem (1.2), in the sense

[2] One of the many options is to use (a subset of) techniques developed in a 1934 paper by Leray [12] for the Navier-Stokes equation.

[3] In this case we can take for example $X = L^1(\mathbf{S}^1,\mathbf{H})$, or the smooth functions on $\mathbf{S}^1$.

that the norm in X is invariant under the scaling of the initial data $a(x) \to \lambda a(\lambda x)$ corresponding to the scaling symmetry $q(x,t) \to \lambda q(\lambda x, \lambda^2 t)$ of the equation. As we will see, our equation is locally well-posed in this space.[4] Similarly to results in [15], in $X = L^1(\mathbf{R}, \mathbf{H})$ the Cauchy problem (1.2) is globally well-posed in X with $q(\cdot,t) \to 0$ for $t \to \infty$ when $\int_R |a - \bar{a}|\, dx < \pi$. It easily follows from our considerations below that a necessary and sufficient condition for $(0,T)$ with $T < \infty$ to be the maximal interval of existence of the local solution of 1.2 with $a \in X$ is[5] $\lim_{t \to T_-} \int_{\mathbf{R}} |q(x,t)| dx = \infty$. The results in [15] already imply that this scenario can occur for a suitably chosen a.

4. There may be interesting questions concerning uniqueness. The solutions which we construct can be analytically continued beyond the singularities (although the equation may not be satisfied weakly at the singularities). It is not clear if there is another reasonable way to continue the solution after the singularity. In some other non-linear parabolic equations which develop singularities (such as the complex Ginzburg-Landau equation, or the harmonic maps heat flow) the analyticity in time can be lost everywhere in space at the singular times,[6] which opens the possibility of non-uniqueness.

2 The Cole-Hopf transformation

2.1 From the heat equation to the Burgers equation

We will rely on the Cole-Hopf transformation, which essentially makes the equation explicitly solvable. Its use for the real-valued Burgers equation goes back to [3, 7]. The calculation in the real case can be taken without change to the complex-valued case, see, for example, [15]. Here we use it for quaternionic-valued functions, which requires some care due to the non-commutativity of the quaternions.

Let $v\colon \mathbf{R} \times [0,\infty) \to \mathbf{H}$ be a solution of the heat equation

$$v_t = v_{xx} \tag{2.1}$$

which satisfies $v(x,t) \neq 0$ in $\mathbf{R} \times (0,\infty)$. Let us write

$$v_t = vA\,,\; v_x = vB\,, \tag{2.2}$$

where A, B are quaternionic-valued functions. We can also write $A = v^{-1}v_t\,, B = v^{-1}v_x$. The expressions are well-defined as v does not vanish by our assumptions.

[4]Our method for establishing this will use the special structure of the equation, it will not be based on finding a good functional-analytic setup for applying a standard fixed-point theorem by establishing a contraction property of the Picard iteration. It might be interesting to see what is the lowest regularity threshold for a proof via Picard interation.

[5]This will again be seen from the special structure of the equation and not via more general methods by which such statements are usually proved.

[6]See, for example, [14].

Differentiating (2.2) we obtain

$$v_{xx} = vB^2 + vB_x\,, \quad v_{tx} = vBA + vA_x\,, \quad v_{xt} = vAB + vB_t\,. \tag{2.3}$$

Using the heat equation (2.1) and the identity $v_{xt} = v_{tx}$, we have

$$A = B^2 + B_x\,, \qquad BA + A_x = AB + B_t\,. \tag{2.4}$$

Substituting $B^2 + B_x$ for A into the second equation, we obtain

$$B^3 + BB_x + B_xB + BB_x + B_{xx} = B^3 + B_xB + B_t\,, \tag{2.5}$$

which simplifies to

$$B_t = 2BB_x + B_{xx}\,. \tag{2.6}$$

Setting

$$q = -2B = -2v^{-1}v_x\,, \tag{2.7}$$

we see that

$$q_t + qq_x = q_{xx}\,. \tag{2.8}$$

The formula

$$q = -\,2v^{-1}v_x \tag{2.9}$$

maps the $(\mathbf{H}\setminus\{0\})$-valued solutions of the heat equation (2.1) into the solutions of the quaternionic Burgers equation (1.1).

The mapping $v \to q = -2v^{-1}v_x$ is not one-to-one, as q does not change if we replace v by $c(t)v$ for any ($\mathbf{H}-$valued) function $c(t)$. If w satisfies the heat equation, then $v = c(t)w(x,t)$ satisfies

$$v_t = v_{xx} + \gamma(t)v\,, \qquad \gamma(t) = c'(t)c^{-1}(t)\,. \tag{2.10}$$

As one can expect, if we repeat the calculation above with (2.1) replaced by (2.10), we still get that $q = -2v^{-1}v_x$ satisfies (2.8).

Remark: The calculations above are not really tied to quaternions or their subalgebras. One can work with functions with values in an associative algebra with a unit element.

2.2 From the Burgers equation to the heat equation

Let us assume that $q\colon \mathbf{R}\times(t_1,t_2) \to \mathbf{H}$ is a smooth function that solves the equation

$$q_t + qq_x = q_{xx}. \tag{2.11}$$

Our assumption of smoothness is only "qualitative", we do not assume any bounds.[7] By regularity theory for parabolic equations, the smoothness of the solution of equation (2.11) follows from weaker assumptions. For example, to make sense of (2.11)

[7] Of course, for a given smooth function some non-effective bounds are implied by the fact that a continuous function on a compact set is bounded.

in distributions, one can assume $q \in L^3_{x,t\,\mathrm{loc}}$ and $q_x \in L^{\frac{3}{2}}_{x,t\,\mathrm{loc}}$. It is not hard to conclude from standard regularity theory that such distributional solutions are actually smooth. There are many other sufficient conditions for smoothness, but this will not be at the center of our interest. We will simply assume that our solutions are smooth.

For the Cauchy problem in $\mathbf{R}\times(0,T)$ it may be restrictive to assume smoothness up to $t=0$ (although this would still be enough to illustrate our main point). At the same time, we do need to assume some global bound to ensure uniqueness. We will work with the following definition[8]:

Definition 2.1 *A local-in-time solution of the Cauchy problem (1.2) is a function q defined for some $T>0$ on $\mathbf{R}\times(0,T)$ and satisfying the following assumptions:*

(i) q *is smooth in* $\mathbf{R}\times(0,T)$*;*

(ii) $q \in L^\infty(0,T;L^1(\mathbf{R}))$*;*

(iii) $q(\cdot,t)\to a$ *in* L^1 *when* $t\searrow 0$.

The justification for Definition 2.1 comes from the fact that for this class of solutions we have both the local-in-time existence and the uniqueness of the solutions of the Cauchy problem.

One can use the Cole-Hopf transformation in the previous section to prove the local-in-time existence of solutions to the Cauchy problem (1.2).

Proposition 2.1 *For any $a\in L^1(\mathbf{R},\mathbf{H})$ the Cauchy problem (1.2) has a unique local-in-time solution as defined by Definition (2.1).*

Proof: As we already mentioned, the existence of at least one solution is easily seen from subsection (2.1). Let us now consider any solution v satisfying the assumptions of Definition (2.1). Let us fix $x_1\in\mathbf{R}$ and define $v\colon \mathbf{R}\times(0,T)\to\mathbf{H}$ by

$$v_x(x,t) = -\frac{1}{2}v(x,t)q(x,t)\ , \qquad v(x_1,t)=1. \tag{2.12}$$

Using that q satisfies (2.11), we have

$$(-2v^{-1}v_x)_t + (-2v^{-1}v_x)(-2v^{-1}v_x)_x = (-2v^{-1}v_x)_{xx}\,. \tag{2.13}$$

Using the formula $(v^{-1})_x = -v^{-1}v_xv^{-1}$, it is easy to check that for $v\neq 0$ the equation (2.13) is equivalent to

$$(v_t - v_{xx})_x = -\frac{1}{2}(v_t-v_{xx})q\,. \tag{2.14}$$

[8]Here and below we will slightly abuse notation by writing $L^1(\mathbf{R})$ also for vector valued functions (such as $L^1(\mathbf{R},\mathbf{H})$, for example.

In other words, the quantity $f = v_t - v_{xx}$ satisfies the same equation $f_x = -\frac{1}{2}fq$ as the quantity v. This means that there is an $\mathbf{H}-$valued function $c(t)$ such that

$$v_t - v_{xx} = c(t)v\,. \tag{2.15}$$

Given that q is smooth in $\mathbf{R}\times(0,T)$, the function $c(t)$ can easily seen to be a smooth function of t in $(0,T)$ (perhaps "blowing up" as $t\to 0$ or $t\to T$, although we will rule out the first possibility later). Indeed, evaluating $v_t - v_{xx}$ at (x_1,t) and using $v(x_1,t)=1$ together with $v_x = -\frac{1}{2}vq$, we see that $c(t) = -v_{xx}(x_1,t) = \frac{1}{2}q_x(x_1,t) - \frac{1}{4}q^2(x_1,t)$. Replacing $v(x,t)$ by $\gamma(t)v(x,t)$ for an $\mathbf{H}-$valued function $\gamma\colon(0,T)\to\mathbf{H}$ satisfying $\gamma_t(t) = c(t)\gamma(t)$ we see that we can assume without loss of generality

$$v_t = v_{xx}\,, \tag{2.16}$$

together with

$$v_x(x,t) = -\frac{1}{2}v(x,t)q(x,t)\,. \tag{2.17}$$

We claim that the condition

$$\int_{\mathbf{R}} |q(x,t)|\,dx \le C \tag{2.18}$$

together with (2.16) and (2.17) implies that the limits

$$\lim_{x\to\pm\infty} v(x,t) = V_{\pm} \tag{2.19}$$

exist and are independent of t. To verify the claim, we first note that the solutions of the ODE satisfy

$$|(v\bar{v})_x| = |-\frac{1}{2}v(q+\bar{q})\bar{v}| \le |v\bar{v}||q|\,. \tag{2.20}$$

As v is smooth in $\mathbf{R}\times(0,T)$, we see from (2.17) and (2.18) that $\int_{\mathbf{R}}|v_x(x,t)|\,dx$ is uniformly bounded on each compact sub-interval of $(0,T)$. Therefore v is bounded in $\mathbf{R}\times(t_1,t_2)$ for each $0<t_1<t_2<T$ and the limits $V_{\pm}(t) = \lim_{x\to\pm\infty} v(x,t)$ exist for each t. For any $0<t_1<t<T$ we have

$$v(x,t) = \int_{\mathbf{R}} v(y,t_1)\Gamma(x-y,t-t_1)\,dy\,, \tag{2.21}$$

where $\Gamma(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ is the heat kernel. We see from this formula and the above estimates that $V_{\pm}(t) = V_{\pm}(t_1)$. Since t_1 was an arbitrary point of $(0,T)$, we see that $V_{\pm}(t)$ are independent of t. As v does not identically vanish, we have $V_{\pm}\neq 0$. We conclude that $v(x,t)$ is uniformly bounded in $\mathbf{R}\times(0,T)$ and satisfies the uniform bound $\int_{\mathbf{R}}|v_x(x,t)|\,dx \le C_1 < \infty$ for $t\in(0,T)$. Hence, due to (2.16), $v(x,0) = \lim_{t\to 0_+} v(x,t)$ is well-defined as a BV function. Using the L^1 continuity of the map $t\to q(\cdot,t)$ at $t=0$ and the assumption $q(\cdot,0)\in L^1$, we can pass to the limit $t \to 0_+$ in equation (2.17) and we see that $v_x(\cdot, 0) \in L^1(\mathbf{R})$. We have $\lim_{x\to\pm\infty} v(x,0) = V_\pm$ and $v_x(\cdot,t) \to v_x(\cdot,0)$ in L^1. We see that function q is given by $q = -2v^{-1}v_x$ with v solving (2.16) and with the initial condition $v(x,0)$ satisfying (2.17) at $t = 0$. Changing v to cv for $c \in \mathbf{H}$ does not affect q. We see that the solution q is unique and is determined by the construction in Subsection 2.1. ■

Remark: If we only assume 2.18, without any assumptions on the behavior of $q(\cdot,t)$ for $t \to 0_+$, we can still conclude that $v(\cdot,t) \to v(\cdot,0)$ in BV, but the functions $v_x(\cdot,0)$ and $q(\cdot,0)$ are only defined as ($\mathbf{H}-$valued) measures, they may not be in L^1. It may be interesting to look at this situation in more detail. One possible result might be that if we change the strong convergence in L^1 in point (iii) of Definition 2.1 to distributional convergence (while still assuming $a \in L^1$), the definition could be equivalent. The case when a is an $\mathbf{H}-$valued measure should also be interesting, although the Cauchy problem may not always be well-posed in this class.

3 Singularities of solutions in $\mathbf{R} \times (0, \infty)$

Let $a \in L^1(\mathbf{R}, \mathbf{H})$ and let $q = -2v^{-1}v_x$ by the (unique) solution of the Cauchy problem (1.2) constructed in the previous section via the Cole-Hopf tansformation. As before, we let

$$V_- = \lim_{x\to-\infty} v(x,0)\,, \qquad V_+ = \lim_{x\to\infty} v(x,0) \tag{3.1}$$

As we have seen above, for each $t > 0$ we still have

$$\lim_{x\to\pm\infty} v(x,t) = V_\pm\,. \tag{3.2}$$

More precisely, letting $\phi(x) = \int_{-\infty}^x \Gamma(y,1)\,dy$, it is easy to see that we have

$$v(x,t) = V_-\left(1 - \phi\left(\frac{x}{\sqrt{t}}\right)\right) + V_+\phi\left(\frac{x}{\sqrt{t}}\right) \; + \; O\left(\frac{1}{\sqrt{t}}\right)\,, \qquad t \to \infty\,. \tag{3.3}$$

We see that, as $t \to \infty$, the image of the mapping $x \to v(x,t)$ is deformed from a possibly complicated curve to the affine segment joining V_- and V_+.

We recall a result proved in [15]:

Lemma 1 *Let $v\colon \mathbf{R} \times [0,\infty) \to \mathbf{C}$ be a bounded complex-valued solution of the heat equation $v_t = v_{xx}$ that does not vanish in some neighborhood of $\mathbf{R} \times \{0\}$. Then all zeros of v are isolated.*

The following statement will also be useful.

Lemma 2 *Let $V\colon \mathbf{R} \to \mathbf{H}$ be a $W_0^{1,1}$ function with $|V(x)| \geq \epsilon$ for some $\varepsilon > 0$. Then we can "rotate" V by $V \to W = bV = W_0 + W_1\mathrm{i} + W_2\mathrm{j} + W_3\mathrm{k}$ for some $b \in \mathbf{H}$ with $|b| = 1$ so that the values of the functions $W_0 + W_1\mathrm{i}$ and $W_2\mathrm{j} + W_3\mathrm{k}$ stay at a positive distance from zero.*

Proof: We can identify $\mathbf{H}$ with $\mathbf{C}^2$ by writing $q = z_1 + \mathrm{j}z_2$ and consider the action of $\mathbf{C}\setminus\{0\}$ on $\mathbf{H}$ defined by $q \sim (z_1, z_2) \to qc \sim (z_1c, z_2c)$. The manifold of the orbits of this action is the complex projective space $P^1(\mathbf{C}) \sim \mathbf{S}^2$ (the standard sphere), with the natural projection $\pi\colon \mathbf{H} \to P^1(\mathbf{C})$ given by $q \to \pi(q) = \{qc\,,\, c \in \mathbf{C}\setminus\{0\}\}$. The left multiplication $z_1 + \mathrm{j}z_2 \to q(z_1 + \mathrm{j}z_2)$ generates a standard action of the group of the unit quaternions (which is equivalent to the action of SU(2) on $P^1(\mathbf{C})$, which in turn is equivalent to the standard action of SO(3) on $\mathbf{S}^2$). Hence the question becomes as follows: given a $W_0^{1,1}$ curve $\gamma = \pi \circ V$ in $P^1(\mathbf{C})$ and two points p', p'' in $P^1(\mathbf{C})$, can we move γ by a suitable conformal isometry Q of $P^1(\mathbf{C})$ so that $Q\gamma$ contains neither p' nor p''. Since the image of γ is a rectifiable set of finite 1-dimensional Hausdorff measure that can be compactified in $P^1(\mathbf{C})$ by adding two points (the limits of $\pi \circ V$ for $x \to \pm\infty$), this is clearly possible. ■

Let us now consider (1.2) for some $a\colon \mathbf{R} \to \mathbf{H}$ belonging to L^1 and let V be a solution of $V_x = Va$ in $\mathbf{R}$. By Lemma 2 we can assume (possibly after multiplying V by a suitable fixed quaternion) that the function $V_0 + V_1\mathrm{i}$ and $V_2\mathrm{j} + V_3\mathrm{k}$ do not vanish near $\mathbf{R} \times \{0\}$. Let v be the solution of the Cauchy problem for the heat equation with the initial condition V. Using Lemma 1 we see that the zeros of $v_0 + v_1\mathrm{i}$ and $v_2\mathrm{j} + v_3\mathrm{k}$ are isolated. Let us denote by (x_k, t_k) the zeros of $v_0 + v_1\mathrm{i}$ and by (y_l, s_l) the zeros of $v_2\mathrm{j} + v_3\mathrm{k}$. Typically these sequences are finite. They can be infinite in non-generic cases (see [15]), although this may not be clear if a is compactly supported. The condition for (x_k, t_k) to be a zero of v is, of course,

$$y_l = x_k\,,\ s_l = t_k \qquad \text{for some } l\,, \tag{3.4}$$

which is, roughly speaking, a codimension 2 condition, as the zeros (x_k, t_k), (y_l, s_l) can be (and typically are) stable (i.e. survive a small perturbation only with a small shift).

Without going to precise definitions, another way to see that the codimension of the set of initial conditions leading to a singularity should be, in some sense, two is the following. For any $\tau > 0$ the image $\Sigma = v(\mathbf{R} \times (\tau, \infty) \subset \mathbf{H}$ is an object of dimension ≤ 2 and the condition singularity formation is $0 \in \Sigma$.

One can easily see that a suitable small smooth perturbation of the initial data can remove all existing zeros of v. A simple perturbation that achieves this is for example a (small) shift of $v_0(x,t) + v_1(x,t)\mathrm{i}$ to $v_0(x-\xi, t-\tau) + v_1(x-\xi, t-\tau)\mathrm{i}$, where $\xi \in \mathbf{R}$ and $\tau \leq 0$ are suitable small parameters. The zeros of

$$v_0(x-\xi, t-\tau) + v_1(x-\xi, t-\tau)\mathrm{i} + v_2(x,t)\mathrm{j} + v_3(x,t)\mathrm{k} \tag{3.5}$$

are determined by

$$\xi = x_k - y_l \qquad \tau = t_k - s_l \qquad \text{for some } k, l \tag{3.6}$$

and hence for all (ξ, τ) outside of a countable set the function (3.5) will have no zeros.

Another way to do the perturbation is the following. Let us first assume that the segment between V_+ and V_- (where we use the notation (3.1)) does not contain $0 \in \mathbf{H}$. This is, of course, generically the case. The situation when the segment contains zero is, roughly speaking, co-dimension three, and it is easy to see that a suitable smooth perturbation of the initial datum with a small smooth norm and small support can bring us to the generic situation.

Assuming the segment between V_+ and V_- does not contain zero, we note that sufficiently large t the solution $v(x,t)$ cannot vanish, and hence only a finite number of the points (x_k, t_k), (y_l, s_l) can be candidates for zeros of v. Let us assume these are

$$(x_1, t_1), (x_2, t_2), \dots, (x_m, t_m), \qquad (y_1, s_1), (y_2, s_2), \dots (y_n, s_n). \tag{3.7}$$

Let us assume our numbering is such that for some $p \le \min(m,n)$ the points

$$(x_1, t_1) = (y_1, s_1), \dots, (x_p, t_p) = (y_p, s_p), \tag{3.8}$$

are exactly all zeros of v. Let φ and ψ be non-trivial non-negative functions supported respectively in a given intervals I, J and let us replace $V_0 + V_1\mathrm{i}$ with $V_0 + \epsilon_1\varphi + (V_1 + \epsilon_2\psi)\mathrm{i}$ while leaving $V_2\mathrm{j} + V_3\mathrm{k}$ unchanged. It is easy to see that by maximum principle and linearity, the solution $v^{(\epsilon_1,\epsilon_2)}$ corresponding to this new initial condition will have no zeros whenever (ϵ_1, ϵ_2) is sufficiently small and not $(0,0)$, pointing again towards co-dimension 2 for the set of initial data leading to a singularity. We will summarize the main point we wish to make as follows, leaving out some of the technical details that the interested reader can easily fill in from the discussion above.

Theorem 1 *The Cauchy problem (1.2) has the following features:*

(i) It is locally-in-time well-posed for initial data $a \in L^1$.

(ii) Finite-time singularities can develop for smooth, compactly supported initial data a. A necessary and sufficient condition for $(0,T)$ with $0 < T < \infty$ to be the maximal time-interval on which the local solution with $a \in L^1$ can be extended is that $\int_{\mathbf{R}} |q(x,t)|\,dx$ stay bounded on $[0,t_1]$ for any $t_1 < T$ and

$$\lim_{t \to T_-} \int_{\mathbf{R}} |q(x,t)|\,dx = \infty.$$

(iii) The set of the initial data for which the solution is globally well-posed and approaches zero for $t \to \infty$ is open and dense in L^1. In fact, the perturbation to initial data leading to a global smooth solution can always be achieved by a smooth function with small smooth norm and small support. Moreover, the set of initial data for which we do not have a global smooth solution is, in the sense discussed above, of co-dimension 2.

Proof: See above. ∎

4 Singularities of solutions in $\mathbf{S}^1 \times (0, \infty)$

4.1 Local-in-time well-posedness

We now turn our attention to the Cauchy problem for $q\colon \mathbf{S}^1 \times (0, \infty) \to \mathbf{H}$:

$$
\begin{aligned}
q_t + qq_x &= q_{xx} \qquad \text{in } \mathbf{S}^1 \times (0, \infty)\,, && (4.1)\\
q|_{t=0} &= a \qquad \text{in } \mathbf{S}^1. && (4.2)
\end{aligned}
$$

We will think of $\mathbf{S}^1$ as $\mathbf{R}/2\pi\mathbf{Z}$ and, as usual, we will identify the functions on $\mathbf{S}^1$ with 2π-periodic functions on $\mathbf{R}$. Our approach will again be based on the Cole-Hopf transformation and we will represent the solutions q of (4.1) as $q = -2v^{-1}v_x$, where v is a solution of the heat equation. However, this transformation does typically not lead to a periodic v from a periodic q. This is already seen in case of $q \equiv$ const. The simple lemmata below will be useful in dealing with this small complication.

Lemma 3 *Let $b\colon \mathbf{R} \to \mathbf{H}$ be $L-$periodic (i.e., $b(x+L) = b(x)$ for each $x \in \mathbf{R}$), and let $v\colon \mathbf{R} \to H$ be a solution of the linear equation $v_x = vb$ in $\mathbf{R}$. Then there exists $c \in \mathbf{H}$ such that $v(x+L) = cv(x)$ for each $x \in \mathbf{R}$.*

Proof: Set $f(x) = v(x+L)v^{-1}(x)$. We compute

$$
\begin{aligned}
f'(x) &= v_x(x+L)v^{-1}(x) - v(x+L)v^{-1}(x)v_x(x)v^{-1}(x)\\
&= v(x+L)b(x+L)v^{-1}(x) - v(x+L)v^{-1}(x)v(x)b(x)v^{-1}(x) \qquad (4.3)\\
&= v(x+L)b(x)v^{-1}(x) - v(x+L)b(x)v^{-1}(x) = 0\,. \blacksquare
\end{aligned}
$$

Lemma 4 *Assume $v\colon \mathbf{R} \times (t_1, t_2) \to \mathbf{H}$ satisfies the heat equation $v_t - v_{xx} = 0$ and that for some $t_0 \in (t_1, t_2)$ there exists $c \in \mathbf{H}$ such that $v(x+L, t_0) = cv(x, t_0)$ for each $x \in \mathbf{R}$. If there exists $\kappa > 0$ such that $|v(x,t)| \le \kappa \cosh \kappa x$ in $\mathbf{R} \times (t_1, t_2)$, then $v(x+L, t) = cv(x,t)$ in $\mathbf{R} \times (t_1, t_2)$.*

Proof: Let $f(x,t) = v(x+L,t) - cv(x,t)$. Then $f_t - f_{xx} = 0$ in $\mathbf{R} \times (t_1, t_2)$. Moreover $f(x, t_0) = 0$ for each $x \in \mathbf{R}$. This implies $f \equiv 0$ in $\mathbf{R} \times (t_1, t_2)$ by standard results about the heat equation. $\blacksquare$

Let us now consider a smooth periodic (in space) solution $q\colon \mathbf{R} \times (t_1, t_2) \to \mathbf{H}$ with period L, *i.e.*, $q(x+L, t) = q(x,t)$ in $\mathbf{R} \times (t_1, t_2)$. As in Section (2), we can define $v\colon \mathbf{R} \times (t_1, t_2)$ by

$$
v_x(x,t) = -\frac{1}{2}v(x,t)q(x,t) \qquad v(x_1, t) = 1\,, \tag{4.4}
$$

where x_1 is some fixed element of $\mathbf{R}$. As in the proof of Proposition (2.1), we can derive that v satisfies $v_t - v_{xx} = c(t)v$ for some function $\mathbf{H}-$valued smooth function

$c(t)$ on (t_1, t_2). Following the next step in the same proof, we see that by changing v to $\gamma(t)v$ and removing the condition $v(x_1, t) = 1$, we can find a non-trivial function $v\colon \mathbf{R} \times (t_1, t_2) \to \mathbf{H}$ such that

$$v_t - v_{xx} = 0\,, \qquad v_x = -\frac{1}{2}vq\,, \qquad \text{in } \mathbf{R} \times (t_1, t_2) \tag{4.5}$$

and $v(x_0, t_0) \neq 0$ where (x_0, t_0) is some fixed point of $\mathbf{R} \times (t_1, t_2)$. By Lemma 3 and Lemma 4 we have $v(x + L, t) = cv(x, t)$ in $\mathbf{R} \times (t_1, t_2)$ for some fixed $c \in \mathbf{H}$.

Lemma 5 *In the situation above, we can assume without loss of generality that $v(x + L, t) = cv(x, t)$ in $\mathbf{R} \times (t_1, t_2)$ for some $c \in \mathbf{C} \subset \mathbf{H}$.*

Proof: We note that when a non-trivial pair q, v satisfies (4.5), then for any $\beta \in \mathbf{H} \setminus \{0\}$ the pair $q, \beta v$ is non-trivial and also satisfies (4.5). It is easy to see that the change $v \to \beta v$ changes c to $\beta c \beta^{-1}$. Given $c \in \mathbf{H}$, we can always find $\beta \in \mathbf{H}$ so that $\beta c \beta^{-1} \in \mathbf{C} \subset \mathbf{H}$. ∎

Lemma 6 *Assume $v\colon \mathbf{R} \times (t_1, t_2) \to \mathbf{H}$ satisfies the conditions (4.5) with and $L-$periodic $q(x, t)$ satisfying*

$$\int_0^L |q(x,t)|\, dx \leq C\,, \qquad t \in (t_1, t_2)\,, \tag{4.6}$$

for some finite constant C. In addition, assume that

$$v(x + L, t) = cv(x, t) \qquad \textit{in } \mathbf{R} \times (t_1, t_2) \tag{4.7}$$

for some $c \in \mathbf{H}$. Then v is uniformly bounded in $[0, L] \times (t_1, t_2)$.

Proof: By Lemma (5) we can assume that $c \in \mathbf{C}$. Let $a \in \mathbf{C}$ with $-\pi < \arg a \leq \pi$ be such that $e^{aL} = c$. Let us consider the function

$$w(x, t) = e^{-ax} v(x, t). \tag{4.8}$$

The function w satisfies $w(x + L, t) = w(x, t)$ and hence can be considered as a function of $\mathbf{S}^1$. (Here we assume that $L = 2\pi$, which can be assumed without loss of generality). The function w satisfies

$$w_t = (\partial_x + a)^2 w \tag{4.9}$$

in $\mathbf{R} \times (t_1, t_2)$. We let

$$N(t) = ||w(\cdot, t)||_{L^2(\mathbf{S}^1)}\,, \qquad W(x, t) = \frac{w(x, t)}{N(t)}\,. \tag{4.10}$$

In view of the equation

$$v_x = -\frac{1}{2} v q \tag{4.11}$$

satisfied by v for each $t \in (t_1, t_2)$, the assumptions on q and the obvious identity $||W(\cdot,t)||_{L^2} = 1$, the set $\mathcal{W} = \{W(\cdot,t),\, t \in (t_1,t_2)$ is uniformly bounded in $W^{1,1}$ for $t \in (t_1,t_2)$ and hence pre-compact in the unit sphere $\{u \in L^2(\mathbf{S}^1, \mathbf{H})\,, ||u||_{L^2} = 1\}$ of the space $L^2(\mathbf{S}^1, \mathbf{H})$. In particular, any of its accumulation points must be a non-zero function. On the other hand, for a fixed $t_0 \in (t_1, t_2)$ one can write (assuming $L = 2\pi$)

$$w(x,t) = \sum_{k \in \mathbf{Z}} e_k(x) e^{(ik+a)^2(t-t_0)} w_k^{(t_0)}\,, \qquad e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}\,, \tag{4.12}$$

and

$$N(t) = ||w(\cdot,t)||_{L^2} = \left(\sum_k |e^{(ik+a)^2(t-t_0)} w_k^{(t_0)}|^2 \right)^{\frac{1}{2}}\,, \tag{4.13}$$

where $w_k^{(t_0)}$ are the Fourier coefficients of $w(\cdot, t_0)$. If $N(t_j) \to \infty$ for $t_j \to 0$, we see from (4.12) and the definition of $W(x,t)$ that for any function in the accumulation point of the family $W(\cdot, t_j)$ all of its Fourier coefficients have to vanish, which gives a contradiction with the pre-compactness of $\mathcal{W}$. We see that $N(t)$ has to be bounded by as $t \to 0$ and one easily completes the proof. ■

Corollary 4.1 *The quaternionic Burgers equation on $\mathbf{S}^1$ is locally-in-time well-posed for the initial conditions in $L^1(\mathbf{S}^1, \mathbf{H})$.*

This follows from Lemma 6 by considerations similar to those in the proof of Proposition 2.1.

4.2 Singularities

We consider the Cauchy problem (4.1). The singularities of $q(x,t)$ will again be related to the zeros of the function $v(x,t)$ defined at first by (4.4) and then "re-calibrated" to satisfy (4.5) and the conclusion of Lemma (5). We will assume that $v(\cdot, 0) \in W^{1,1}$. We define w by (4.8). The function w is $L-$periodic in x (and we can think of it as a function on $\mathbf{S}^1$. It satisfies the equation (4.9). Moreover, by Lemma (2) we can assume without loss of generality that the both functions $w_0 + w_1\mathrm{i}$ and $w_2\mathrm{j} + w_3\mathrm{k}$ do not vanish anywhere on $\mathbf{S}^1$ at time $t = 0$. We note that we can write

$$w_2\mathrm{j} + w_3\mathrm{k} = (w_2 + w_3\mathrm{i})\mathrm{j} \tag{4.14}$$

and hence the family of the functions of the form $w_0 + w_1\mathrm{i}$ as well as the family of the functions of the form $w_2\mathrm{j} + w_3\mathrm{k}$ are invariant under multiplication by complex-valued functions. In particular, both families are invariant under the operator $\partial_x + a$

when $a \in \mathbf{C} \subset \mathbf{H}$. We can hence treat the quaternionic equation

$$w_t = (\partial_x + a)^2 w \tag{4.15}$$

as two independent complex-valued equations.

Let us take $a \in \mathbf{C}$ and consider the initial-value problem

$$w_t = (\partial_x + a)^2 w\,, \quad w\colon \mathbf{S}^1 \times (0,\infty) \to \mathbf{H}\,, \quad w(x,0) \neq 0 \text{ for each } x \in \mathbf{S}^1\,. \tag{4.16}$$

As we have seen above, a natural regularity assumption on the initial data in our context is $w(\cdot,0) \in W^{1,1}$. Using the notation introduced in (4.12), let write the Fourier series of $w(\cdot,0)$ as

$$w(x,0) = \sum_k e_k(x) w_k\,. \tag{4.17}$$

Note that we should distinguish $e_k(x)w_k$ from $w_k e_k(x)$. Our convention at the moment is to represent quaternions by $z_1 + z_2\mathrm{j}$ with $z_1, z_2 \in \mathbf{C}$. We should hence write $e_k(x)w_k$. The solution of (4.16) is then

$$w(x,t) = \sum_k e_k(x) e^{(ik+a)^2 t} w_k\,. \tag{4.18}$$

For a generic $a \in \mathbf{C}$ the map $k \to \operatorname{Re}(ik+a)^2$ is injective. Let us first assume that we are dealing with this generic situation. Then, for any subset $S \subset \mathbf{Z}$ the function Re $k \to (ik+a)^2$ attains its maximum at exactly one point of S. Letting $S = \{k\,,\, w_k \neq 0\}$ and $k_0 \in \mathbf{Z}$ the point where the maximum on S is attained, it is easy to see that the mode

$$e_{k_0}(x) e^{(ik_0+a)^2 t} w_{k_0} \tag{4.19}$$

will dominate the Fourier series as $t \to \infty$, with the contribution $r(x,t)$ from rest of the series satisfying

$$\frac{r(x,t)}{|e^{(ik_0+a)^2 t}|} \to 0 \quad \text{as } t \to \infty. \tag{4.20}$$

In particular, in view of our assumptions on $w(x,0)$, the function $w(x,t)$ will have a finite number of isolated zeros in $\mathbf{S}^1 \times (0,\infty)$.

There are non-generic cases where the function $k \to \operatorname{Re}(ik+a)^2$ can attain maximum on S at two points k_1, k_2. Then the contribution to the Fourier series with the slowest decay (or the fastest growth) will be

$$e_{k_1}(x) e^{(ik_1+a)^2 t} w_{k_1} + e_{k_2}(x) e^{(ik_2+a)^2 t} w_{k_2} \tag{4.21}$$

In general, this function can have zeros which do not disappear as $t \to \infty$, but it easy to see that a suitable small perturbation of the coefficients w_{k_1}, w_{k_2} can remove these zeros. After this adjustment, the term (4.21) will dominate the remainder of the series for large t and the function $w(x,t)$ will again only have finitely many zeros in $\mathbf{S}^1 \times (0,\infty)$.

Writing the solution of the quaternionic equation for w as

$$w = w^{(1)} + w^{(2)}\mathrm{j} \tag{4.22}$$

where $w^{(1)} = w^{(1)}(x,t)$ and $w^{(2)} = w^{(2)}(x,t)$ are complex-valued, we see as in Section 3 that by slight shift or change of initial data for one of the complex-valued solutions $w^{(j)}$ we can remove the common zeros of the functions $w^{(1)}$ and $w^{(2)}$ and perturb to the situation where w has no zeros on $\mathbf{S}^1 \times (0,\infty)$. Let us look at the behavior for large t in the generic case when the map $k \to \mathrm{Re}(ik+a)^2$ is injective and the coefficient w_{k_0} at the point k_0 where $k \to \mathrm{Re}(ik+a)^2$ attains its minimum on $\mathbf{Z}$ does not vanish. Then the behavior for large t of w is dominated by

$$e_{k_0}(x)e^{(ik_0+a)^2 t}\beta \tag{4.23}$$

for some quaternion β. Recalling 4.8 and (4.5), we have

$$q = -2v^{-1}v_x = -2w^{-1}w_x - 2w^{-1}aw = -2\beta^{-1}(ik_0+a)\beta + o(1)\,, \qquad t \to \infty\,. \tag{4.24}$$

Recall that k_0 was obtained by maximizing the real part of $(ik+a)^2$. Writing $a = a_1 + a_2 i$ we see that k_0 is the integer minimizing $(k_0+a_2)^2$. Moreover, we assume that we are in the generic case where the minimum is assumed exactly at one integer k_0. In this case $ik_0 + a$ will be of the form $a_1 + a_2'$ with $|a_2'| < \frac{1}{2}$. It is easy to check that in this case the limiting value $-2\beta^{-1}(ik_0+a)\beta$ will represent a stable steady state of the quaternionic Burgers equation $q_t + qq_x = q_{xx}$ on $\mathbf{S}^1$. Here by stability we mean the standard linearized stability of the steady state.[9] Leaving a more detailed analysis of the non-generic cases to the interested reader, we will formulate the following statement relevant in the generic situation.

Theorem 2 *The Cauchy problem (4.1) for the quaternionic Burgers equation in* $\mathbf{S}^1 \times (0,\infty)$ *has the following features.*

(i) The problem is locally-in-time well-posed for initial data in $L^1(\mathbf{S}^1, \mathbf{H})$.

(ii) Finite-time singularities can develop from smooth initial data. A time-interval $(0,T)$ *with* $0 < T < \infty$ *is the maximal interval of existence of a local-in-time solution* q *if and only if*

$$\lim_{t\to T_-} \int_{\mathbf{S}^1} |q(x,t)|\,dx = \infty.$$

[9]The corresponding linearized equation at a constant steady state $\alpha \in \mathbf{H}$ is given by $q_t + \alpha q_x = q_{xx}$. The operator $q \to q_{xx} - \alpha q_x$ always has an eigenvalue $\lambda = 0$ (with the eigenvectors being constant functions), corresponding to the shift of the constant solution to another constant α'. By a simple change of variables $q \to \beta q \beta^{-1}$ we can assume that α is complex. The stability condition that — with the exception of the modes accounted for by the constant solutions — the spectrum of the linearized operator is in $\{\lambda\,,\, \mathrm{Re}\,\lambda < 0\}$ is easily seen to be $|\alpha - \bar{\alpha}| < 2$. This is satisfied for the constant $-2\beta^{-1}(ik_0+a)\beta$ in (4.24) in the generic case.

(iii) *For an open dense set of initial conditions* $a \in X = L^1(\mathbf{S}^1, \mathbf{H})$ *the Cauchy problem has a global smooth solution approaching a constant steady state that exhibits the stability properties discussed above.*[10]

Proof: See above.

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