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Square root p-adic L-functions I: Construction of a one-variable measure

Michael Harris

For Jacques Tilouine

The Ichino–Ikeda conjecture, and its generalization to unitary groups by N. Harris, gives explicit formulas for central critical values of a large class of Rankin–Selberg tensor products. The latter conjecture has been proved in full generality and applies to L-values of the form $L\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \mathrm{BC}(\pi')\right)$, where π and π' are cohomological automorphic representations of unitary groups $U(V)$ and $U(V')$, respectively. Here V and V' are hermitian spaces over a CM field, V of dimension n, V' of codimension 1 in V, and BC denotes the twisted base change to $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$.

This paper contains the first steps toward constructing a p-adic interpolation of the normalized square roots of these L-values, generalizing the construction in my paper with Tilouine on triple product L-functions. It will be assumed that the CM field is imaginary quadratic, π is a holomorphic representation and π' varies in an ordinary Hida family (of antiholomorphic forms). The construction of the measure attached to π uses recent work of Eischen, Fintzen, Mantovan, and Varma.

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1. Introduction

This paper is a continuation, after twenty years (!), of the author's project [Harris and Tilouine 2001] with Jacques Tilouine, whose official goal was the construction of one branch of the square root of the anticyclotomic p-adic L-function for a triple of classical modular forms. The unofficial goal of that paper was for this author to benefit from Jacques's patient instruction in Hida theory and p-adic L-functions. To the extent that the author does understand anything about the subject, it is largely a result of this collaboration.

Our paper was neither the first nor the last word on the topic of square root p-adic L-functions. The bibliography of [Harris and Tilouine 2001] included references to earlier work on anticyclotomic L-functions of Hecke characters of imaginary quadratic fields, and of classical L-functions of modular forms in Hida families, as well as a combination of the two that had been considered by Andrea Mori (finally published, more than 20 years after its discovery, in [Mori 2011]). The specific case of the triple product was vastly extended (and corrected) and put to good use by Darmon and Rotger in a series of difficult papers on Euler systems and the Birch–Swinnerton-Dyer conjecture over nonabelian extensions of $\mathbb{Q}$ (see [Darmon and Rotger 2014]).

More recently, the construction has been generalized to Shimura curves in [Barrera Salazar and Molina Blanco 2019]. In the meantime, Gan, Gross, and Prasad had identified a natural setting that includes all these special cases [Gan et al. 2012], and had formulated precise conjectures regarding the relative representation theory of certain pairs of reductive groups over local fields. These conjectures were completed by the global conjecture of Ichino and Ikeda (for orthogonal groups) and its analogue, due to N. Harris (for unitary groups) [Ichino and Ikeda 2010; Harris (R. N.) 2014]. In these conjectures, $G \supset H$ is a pair of groups — we consider the case where G is the special orthogonal or unitary group of a vector space V over a local or global field and H the stabilizer in G of an appropriate subspace of codimension 1. The conjectures of [Gan et al. 2012] classify the irreducible representations π of $G \times H$ over a *local field* that admit a linear form $\pi \to \mathbb{C}$ that is invariant under H, with respect to the diagonal embedding. The conjectures of [Ichino and Ikeda 2010; Harris (R. N.) 2014] concern the cuspidal automorphic representations π of $G \times H$ over a *number field*, and express the central (anticyclotomic) values of certain L-functions $L(s, \pi)$ as squares of periods of integrals over the adèle group of H of elements of π — we call them *Gan–Gross–Prasad periods* up to local and elementary factors.

The Gan–Gross–Prasad (GGP) conjectures have been proved by Waldspurger [2012] (for orthogonal groups over p-adic fields) and Beuzart-Plessis [2020] (for unitary groups, including the archimedean case). The Ichino–Ikeda–N. Harris

(IINH) conjecture has now been proved for unitary groups by Beuzart-Plessis et al. [2020; 2021] following earlier work of W. Zhang and Hang Xue. Ichino had already proved the conjecture for orthogonal groups in low dimensions, including a refinement of the result of [Harris and Kudla 1991] on triple products that was the starting point for [Harris and Tilouine 2001]. The main observation of [Harris and Tilouine 2001] is that the period integrals in [Harris and Kudla 1991] admit a p-adic interpolation over Hida families. The purpose of the present paper is to apply the same observation to the period integrals that arise in certain cases of the IINH conjecture when G and H are unitary groups.

Suppose G and H are the unitary groups of hermitian vector spaces V and V', respectively, over a fixed imaginary quadratic field $\mathcal{K}$,[1] with $\dim V = n = \dim V' + 1$. Stable quadratic base change from $G \times H$ to $\mathcal{G} := \mathrm{GL}(n)_{\mathcal{K}} \times \mathrm{GL}(n-1)_{\mathcal{K}}$ [Labesse 2011; Mok 2015; Kaletha et al. 2014] attaches to a (stable) π a cuspidal automorphic representation $\Pi = \Pi_n \boxtimes \Pi_{n-1}$ of $\mathcal{G}$, and $L(s, \pi)$ is then the Rankin–Selberg L-function $L(s, \Pi_n \otimes \Pi_{n-1})$. Moreover, as $G_\alpha \times H_\alpha$ varies over inner forms of $G \times H$, with $H_\alpha \subset G_\alpha$, there is a collection $\Phi(\Pi) = \{\pi_{\alpha,\beta}\}$, where each $\pi_{\alpha,\beta}$ is a cuspidal automorphic representation of $G_\alpha \times H_\alpha$, all of which have the same stable base change Π. We only consider the case where Π is cohomological for $\mathcal{G}$ — i.e., it contributes to the cuspidal cohomology of the locally symmetric space for $\mathcal{G}$ with appropriate local coefficients. Then the archimedean components $\pi_{\alpha,\beta,\infty}$ all belong to the respective discrete series of $G_\alpha(\mathbb{R}) \times H_\alpha(\mathbb{R})$ whose infinitesimal character corresponds to that of Π_∞. The combination of the GGP and IINH conjectures includes the assertion that if the central value $L\big(\tfrac{1}{2}, \Pi_n \otimes \Pi_{n-1}\big) \neq 0$, then it (or rather its ratio to a different special L-value) can be computed, up to elementary and local factors, as a ratio of a product of period integrals (over the adèles of H_α) of a *unique* $\pi_{\alpha,\beta} \in \Phi(\Pi)$. Specifically, the GGP conjecture, which is known in this case, asserts that there is a unique group $G_\alpha(\mathbb{R}) \times H_\alpha(\mathbb{R})$ and a unique discrete series $\pi_{\alpha,\beta,\infty}$ with the given infinitesimal character that admits a nontrivial linear form

$$\pi_{\alpha,\beta,\infty} \to \mathbb{C}$$

that is invariant under the diagonal embedding of $H_\alpha(\mathbb{R})$.

As Π varies in a p-adic family, the period integrals for the corresponding $\pi_{\alpha,\beta,\infty}$ can be seen as distinct branches of a hypothetical square root p-adic L-function, the relations between which have only begun to be explored; the example of [Darmon and Rotger 2014] shows that these relations are subtle even in low-dimensional cases. In this paper we treat the branch where the specialization of the p-adic interpolation at a classical point of the Hida family is a cup product of a pair of

[1] One can consider more general CM quadratic extensions of totally real fields; the restriction to the imaginary quadratic case is made for convenience.

automorphic forms, one of which is holomorphic, the other antiholomorphic — in other words, $\pi_{\alpha,\beta,\infty} = \pi_{n,\infty} \otimes \pi_{n-1,\infty}$, where $\pi_{n,\infty}$ (resp. $\pi_{n-1,\infty}$) is a holomorphic (resp. antiholomorphic) discrete series representation of $U(V)(\mathbb{R})$ (resp. $U(V')(\mathbb{R})$). Here we treat only the simplest case of a function of a single p-adic variable, which arises as a direct application of the construction of p-adic families of differential operators in [Eischen et al. 2018]. A planned sequel with Ellen Eischen should extend the results of the present paper to multidimensional Hida families. In subsequent work with Eischen and Pilloni, we hope to treat cases of cup products in coherent cohomology of higher degree. Most of the GGP periods, however, do not have such an interpretation. The corresponding p-adic L-functions should exist nonetheless, but we don't see how to construct them.

A construction of p-adic Rankin–Selberg L-functions for cohomological automorphic representations Π of $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ over $\mathbb{Q}$ has been known for some time [Kazhdan et al. 2000]; its generalization to arbitrary number fields is a more recent result of Januszewski [2016]. The method used there corresponds to the "branch," as in the previous paragraph, where the groups $G_\alpha(\mathbb{R})$ and $H_\alpha(\mathbb{R})$ are *definite* unitary groups. This is precisely the case in which the methods of the present paper give no p-adic variation at all, because there are no nontrivial differential operators. In unpublished notes, Eric Urban has sketched the beginning of a construction of a p-adic measure in this situation, again in the definite branch. In any case, there is little overlap between the results of [Kazhdan et al. 2000; Januszewski 2016], which treat general critical values of a single Rankin–Selberg L-function and its cyclotomic twists, and those of this paper, which treats only the central value but allows Π to vary in a p-adic family.

The hardest steps in the construction of any p-adic L-function are the computation of the local factors at archimedean and p-adic places. We deal with these steps in the present paper by avoiding them. The Ichino–Ikeda formula produces local factors at such places and we do not attempt to interpret them explicitly. It follows nevertheless from [Beuzart-Plessis 2020] that these factors can be computed in terms of local Rankin–Selberg zeta integrals for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$. These should be easier to compute than the Ichino–Ikeda local integrals. We expect to return to these computations in subsequent papers.

2. Unitary group Shimura varieties

We work over an imaginary quadratic field $\mathcal{K}$; most of our results go over without change to general CM fields, at the cost of more elaborate notation. The field $\mathcal{K}$ is given with a chosen embedding $\iota : \mathcal{K} \hookrightarrow \mathbb{C}$; the complex conjugate embedding is denoted c; with respect to ι, the group

$$U(1) = \ker N_{\mathcal{K}/\mathbb{Q}} : R_{\mathcal{K}/\mathbb{Q}}\,\mathrm{GL}(1) \to \mathrm{GL}(1)$$

can be attached to a Shimura datum $(U(1), Y_1)$, where $Y_1^{\pm}$ is the homomorphism

$$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} = \mathbb{C}^\times \ni z \mapsto z/\bar{z}$$

if the sign is -1 and is the *trivial* map if the sign is $+1$. The sign is $+1$ (resp. -1) if we consider $U(1)$ to be the unitary group of a 1-dimensional vector space over $\mathcal{K}$ endowed with a hermitian form of signature $(0, 1)$ (resp. $(1, 0)$); see the discussion in [Harris 2021, §2.2] for details.

Let V be an n-dimensional vector space over $\mathcal{K}$, endowed with a hermitian form of signature (r, s) (relative to ι). Let $U(V)$ be the unitary group of V. We define a Shimura datum $(U(V), Y_V)$ as in [Harris 2021]; see also Section A.1 in the Appendix. We choose a point $y \in Y_V$ corresponding to an embedding of Shimura data $(U(1), Y_1^{\pm}) \hookrightarrow (U(V), Y_V)$, and let $K_y \subset U(V)(\mathbb{R})$ denote its centralizer; in other words, the homomorphism y factors through a rational subgroup of $U(V)$ isomorphic to $U(1)$. Then there is an isomorphism $K_y \xrightarrow{\sim} U(r) \times U(s)$, where $U(d)$ is the compact unitary group of rank d for any d. We fix a maximal torus $T = T_y \subset K_y$ containing the chosen $U(1)$; without loss of generality we may assume $T \xrightarrow{\sim} U(1)^{r+s}$ as algebraic groups over $\mathbb{Q}$, with $U(1)^r \subset U(r)$ and $U(1)^s \subset U(s)$.

The Harish-Chandra decomposition of $\mathfrak{g} = \mathrm{Lie}(G_V)$ is given by

$$\mathfrak{g} = \mathfrak{p}_y^+ \oplus \mathfrak{p}_y^- \oplus \mathfrak{k}_y,$$

where $\mathfrak{k}_y = \mathrm{Lie}(K_y)$ and $\mathfrak{p}_y^+$ and $\mathfrak{p}_y^-$ are canonically isomorphic, respectively, to the holomorphic and antiholomorphic tangent spaces to Y_V at y. Then $\dim \mathfrak{p}_y^+ = \dim \mathfrak{p}_y^- = rs$.

2A. ***Conventions for holomorphic automorphic forms.*** Irreducible representations of $U(d)$ are parametrized by d-tuples $a_1 \geq a_2 \geq \cdots \geq a_d$, which are identified with characters of some chosen maximal torus. Thus irreducible automorphic vector bundles $\mathcal{E}_\kappa$ over the Shimura variety $\mathrm{Sh}_V := \mathrm{Sh}(U(V), Y_V)$ are parametrized by characters of T_y, and thus of (r, s)-tuples of integers

$$\big(b_1 \geq b_2 \geq \cdots \geq b_r; b_{r+1} \geq b_{r+2} \geq \cdots \geq b_n\big). \tag{2.1}$$

The $\mathcal{E}_\kappa$ whose global sections defined holomorphic automorphic forms in the discrete series correspond to κ of the form

$$\kappa = \big(a_{s+1} - s, \ldots, a_n - s; a_1 + r, \ldots, a_s + r\big), \tag{2.2}$$

where α is the dominant parameter

$$\alpha : a_1 \geq a_2 \geq \cdots \geq a_n \tag{2.3}$$

[Harris 1997, Proposition 2.2.7(iii)]. A κ satisfying (2.2) will be called *of holomorphic type*. We let M_κ denote the representation space of K_y with highest weight κ, and let

$$\mathbb{D}_\kappa = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}_y \oplus \mathfrak{p}^-)} M_\kappa$$

be the corresponding holomorphic discrete series.

A κ for which $\mathbb{D}_\kappa$ is the $(\mathfrak{g}, K_y)$-module attached to a discrete series representation will be called (for convenience) a *holomorphic discrete series parameter*. If κ is the parameter of (2.2), it is determined by (2.3), and we write $\kappa = \kappa_V(\alpha)$.

We consider a codimension 1 hermitian subspace $V' \subset V$, of signature $(r, s-1)$, and we assume that the base point $y \in Y_{V'} \subset Y_V$, so that its centralizer $K'_y \subset U(V')(\mathbb{R})$ is a maximal compact subgroup, isomorphic to $U(r) \times U(s-1)$. We write

$$\mathfrak{g}' = \mathfrak{p}_y^{+,\prime} \oplus \mathfrak{p}_y^{-,\prime} \oplus \mathfrak{k}'_y$$

for the Harish-Chandra decomposition of $\mathfrak{g}' = \mathrm{Lie}(U(V'))$. As representation of $K'_y = U(r) \times U(s-1)$, the r-dimensional quotient space

$$\mathfrak{n} = \mathfrak{p}_y^+ / \mathfrak{p}_y^{+,\prime}$$

is isomorphic to the representation $\mathrm{St}_r \otimes \mathrm{Triv}$, with parameter $(1, 0, \dots, 0; 0, \dots, 0)$. It follows from the recipe in [Harris 1986] that the restriction of $\mathbb{D}_\kappa$ to $U(\mathfrak{g}')$ can be written

$$\mathbb{D}_\kappa|_{U(\mathfrak{g}')} = \bigoplus_{i \geq 0} \bigoplus_{M_{\kappa'} \subset M_\kappa \otimes [\mathrm{Sym}^i \mathrm{St}_r \otimes \mathrm{Triv}]} \mathbb{D}_{\kappa'}. \tag{2.4}$$

Here the notation $\subset$ in the subscript means that the left-hand representation is an irreducible constituent of the restriction to $U(r) \times U(s-1)$ of the right-hand.

In what follows, we let $T' = T_y \cap U(V') = T_y \cap K'_y$. This is a maximal CM torus in $U(V')$ and the parameters in Lemma 2.5 below are relative to this torus. The inclusion $(U(V'), Y_{V'}) \subset (U(V), Y_V)$ is not an embedding of Shimura data, but this can be corrected by replacing $U(V')$ by $U(V') \times U(1)$, where $U(1)$ is the unitary group of the orthogonal complement to V' in V. We ignore this for the purposes of this paper.

Lemma 2.5. *If* $\kappa = \big(a_{s+1} - s, \dots, a_n - s; a_1 + r, \dots, a_s + r\big)$, *then as* $i \geq 0$ *varies, the set of irreducible representations of* K'_y *contained in the above sum is given by*

$$(b_1, \dots, b_r; c_1, \dots, c_{s-1}),$$

where

$$\delta_j := b_j + s - a_{s+j} \geq 0, \quad 1 \leq j \leq r;$$
$$a_1 + r \geq c_1 \geq a_2 + r \geq c_2 \cdots \geq c_{s-1} \geq a_s + r.$$

The parameter arises in degree i *exactly when* $\sum_{j=1}^r \delta_j = i$.

Proof. The assertion for the b_j follows from the Littlewood–Richardson rule [Goodman and Wallach 2009, §9.3] for the tensor product of an irreducible representation of $U(r)$ with $\mathrm{Sym}^i \mathrm{St}_r$, given our sign conventions; the assertion for the c_k follows from the usual branching formula for restriction from $U(s)$ to $U(s-1)$. □

This proposition thus follows from [Harris 1986, Lemma 7.2]:

Proposition 2.6. *Let κ be a holomorphic discrete series parameter. Let κ' be the highest weight of an irreducible representation of K'_y. Then there is a holomorphic differential operator*

$$\delta_{\kappa,\kappa'} : \mathcal{E}_\kappa \mid_{\mathrm{Sh}(V')} \to \mathcal{E}_{\kappa'}$$

if and only if κ' satisfies the inequalities of Lemma 2.5.

The following lemma is then obvious.

Lemma 2.7. *Suppose κ' satisfies the inequalities of Lemma 2.5. Then κ' is a holomorphic discrete series parameter for $G_{V'}$, and is of the form $\kappa' = \kappa_{V'}(\alpha')$ for the dominant parameter α' of $G_{V'}$ given by*

$$\alpha' = (a'_1 \geq \cdots \geq a'_{n-1}) = \big(c_1 - r \geq \cdots \geq c_{s-1} - r \geq b_1 + s - 1 \geq \cdots \geq b_r + s - 1\big).$$

Definition 2.8. We say $\delta^{\kappa,\kappa'}$ is of degree b if $\sum_{j=1}^{r} \delta_j = b$ in Lemma 2.5.

2A1. *Parameters and Hodge structures.* Let α be the dominant parameter in (2.3). Then α is the highest weight of an irreducible representation W_α of G_V, or of $\mathrm{GL}(n)$. As in [Harris 1997] we can attach to α a collection of Hodge numbers $(p_i, q_i) = (p_i(\alpha), q_i(\alpha))$ with $p_i = a_i + n - i$ and $p_i + q_i = n - 1$ for all i. We let $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. For each i, let $M_{\mathbb{C}}(p_i)$ denote the complex 1-dimensional vector space on which $\mathbb{S}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^\times \times \mathbb{C}^\times$ acts by the character

$$(z_1, z_2) \mapsto z_1^{-p_i(\alpha)} z_2^{-n+1+p_i(\alpha)},$$

and let $M_{\mathbb{C}}(\alpha) = \bigoplus_{i=1}^n M_{\mathbb{C}}(p_i)$. Similarly, let $M(p_i)$ denote $R_{\mathbb{C}/\mathbb{R}} M_{\mathbb{C}}(p_i)$; this is a 2-dimensional vector space with action of $\mathbb{S}(\mathbb{R})$. Then $M(\alpha) = \bigoplus_{i=1}^n M(p_i)$ is a real Hodge structure of dimension $2n$. We denote $M(\alpha)$ by the shorthand list of the p_i:

$$M(\alpha) = \big(a_1 + n - 1, \ldots, a_i + n - i, \ldots, a_n\big). \tag{2.9}$$

Let π be a cuspidal automorphic representation of G_V, and write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is an irreducible $(\mathfrak{g}_V, K_y)$-module and π_f is an irreducible representation of $G_V(\mathbf{A}_f)$. Suppose π contributes to the cohomology $H^0(\mathrm{Sh}(V), \mathcal{E}_{\kappa_V(\alpha)})$; in other words

$$H^0(\mathrm{Sh}(V), \mathcal{E}_{\kappa_V(\alpha)})[\pi] := \mathrm{Hom}_{G_V(\mathbf{A}_f)}\big(\pi_f, H^0(\mathrm{Sh}(V), \mathcal{E}_{\kappa_V(\alpha)})\big) \neq 0. \tag{2.10}$$

This is a property that depends only on π_∞; it says precisely that π_∞ is (depending on conventions) either isomorphic to or the contragredient of $\mathbb{D}_{\kappa_V(\alpha)}$. In the convention of [Eischen et al. 2020],

Hypothesis 2.11. *Assuming* (2.10), $\dim H^0(\mathrm{Sh}(V), \mathcal{E}_{\kappa_V(\alpha)})[\pi] = 1$.

This will be proved in the sequel to [Kaletha et al. 2014], and we will assume it here; the p-adic L-function can be constructed without the assumption of Hypothesis 2.11 but at the cost of additional notation. In any case it is known by [Labesse 2011] that, assuming (2.10), the base change $\Pi = \mathrm{BC}_{\mathcal{K}/\mathbb{Q}}(\pi)$ exists as a cuspidal cohomological automorphic representation of $\mathrm{GL}(n)_{\mathcal{K}}$. The compatible family of homomorphisms

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K}) \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell),$$

defined by many people (including in [Clozel et al. 2011], in most cases) is geometric in the sense of Fontaine–Mazur. In particular, the restriction of $\rho_{\pi,\ell}$ to a decomposition group at a prime dividing ℓ is de Rham with the Hodge numbers $(p_i(\alpha), q_i(\alpha))$ defined above.

Remark 2.12. Under hypothesis (2.10) we know that π_f has a model as an admissible representation of $G_V(\boldsymbol{A}_f)$ over a number field $E(\pi)$. We will be working with spaces of p-adic automorphic forms, so we will implicitly be assuming that the integer ring $\mathcal{O}_{E(\pi)}$, together with all the other integer rings that arise in the subsequent constructions, is embedded in a sufficiently large p-adic integer ring denoted $\mathcal{O}$. We will briefly need to work with models of (finite parts of) automorphic representations over the fraction field of $\mathcal{O}$, which we denote $\mathbb{L}$. The smooth representation theory of the finite adèles is indifferent to the topology of the fields of coefficients.

Suppose $\pi' = \pi'_\infty \otimes \pi'_f$ is an automorphic representation such that the contragredient $\pi'^{,\vee}$ contributes to the cohomology $H^0(\mathrm{Sh}(V'), \mathcal{E}_{\kappa_V(\alpha')})$, where α' is the highest weight of an irreducible representation $W_{\alpha'}$ of $\mathrm{GL}(n-1)$. In particular, π' is *antiholomorphic* — it contributes to the cohomology in degree $d' = \dim \mathrm{Sh}(V')$ of the automorphic vector bundle

$$\Omega^{d'}_{\mathrm{Sh}(V')} \otimes \mathcal{E}^\vee_{\kappa_V(\alpha')},$$

which is the Serre dual of $\mathcal{E}_{\kappa_V(\alpha')}$. For such a π', we assume the analogue of Hypothesis 2.11 holds for $\pi'^{,\vee}$, and we assume $\kappa' = \kappa_{V(\alpha')}$ satisfies the inequalities of Lemma 2.5; in other words, that α' is one of the parameters in Lemma 2.7.

2A2. *Parameters for the Hodge filtration.* Using the shorthand of (2.9), we have

$$M(\alpha')^\vee = \big(-b_r + r - 1, \dots, -b_1, -c_{s-1} + n - 2, \dots, -c_1 + r\big).$$

We consider the $2n(n-1)$-dimensional real Hodge structure

$$M(\alpha,\alpha')=R_{\mathbb{C}/\mathbb{R}}M_{\mathbb{C}}(\alpha)\otimes M_{\mathbb{C}}(\alpha')^{\vee}.$$

Then $M(\alpha,\alpha')_{\mathbb{C}}$ is the sum of eigenspaces of the form

$$(a_i+n-i-b_k+k-1,\bullet);\ (a_i+n-i-c_j+j-1+r,\bullet),$$

where in each case the two integers in the ordered pair add up to $2n-3$. The space $M(\alpha,\alpha')_{\mathbb{C}}$ contains an $n(n-1)$-dimensional subspace $F^{+}M(\alpha,\alpha')$, defined as in [Harris 2013]: it consists of pairs (x,y) as above with $x>y$.

2B. ***Igusa towers and pairings.*** Let p be a prime that splits in $\mathcal{K}$ as the product $\mathfrak{p}\cdot\mathfrak{p}'$. Identifying the algebraic closures of $\mathbb{Q}$ in $\mathbb{C}$ and in $\overline{\mathbb{Q}_p}$ places the embeddings of $\mathcal{K}$ in $\mathbb{C}$ and in $\overline{\mathbb{Q}_p}$ in bijection. We let $\mathfrak{p}$ be the prime above p associated to the fixed embedding $\iota:\mathcal{K}\hookrightarrow\mathbb{C}$ and identify

$$U(V)(\mathbb{Q}_p)\xrightarrow{\sim}\mathrm{GL}(n,\mathcal{K}_{\mathfrak{p}})\xrightarrow{\sim}\mathrm{GL}(n,\mathbb{Q}_p)\tag{2.13}$$

in such a way that $\mathfrak{p}_y^{-}\oplus\mathfrak{k}_y$ is identified with the Lie algebra of an upper triangular parabolic subalgebra of $\mathrm{Lie}(\mathrm{GL}(n))$. We also denote by $\mathrm{incl}_p:\mathcal{K}\hookrightarrow\overline{\mathbb{Q}_p}$ the embedding corresponding to $\mathfrak{p}$. We fix a neat level subgroup $K\subset U(V)(\boldsymbol{A}_f)$ with $K=K_p\times K^p$ with $K_p=\mathrm{GL}(n,\mathbb{Z}_p)$. The Shimura variety ${}_K\mathrm{Sh}(U(V))$ then has a smooth model ${}_KS(V)$ as a moduli space (Shimura variety of abelian type) over $\mathrm{Spec}(\mathcal{O})$ for some finite $\mathbb{Z}_p$-algebra $\mathcal{O}$. For each κ as above the vector bundle $\mathcal{E}_\kappa$ extends to a vector bundle over ${}_KS(V)$.

We choose K so that $K\cap U(V')(\boldsymbol{A}_f)=K'$ is neat and admits a factorization $K'=\mathrm{GL}(n-1,\mathbb{Z}_p)\times K'^{,p}$. We define ${}_{K'}S(V')$ as in the previous paragraph, and assume the embedding

$${}_{K'}S(V')\hookrightarrow{}_KS(V)\tag{2.14}$$

restricts (see Section 2B1 below) to an embedding of ordinary loci

$${}_{K'}S(V')^{\mathrm{ord}}\hookrightarrow{}_KS(V)^{\mathrm{ord}}\tag{2.15}$$

which lifts to a morphism of Igusa towers

$${}_{K'}\mathrm{Ig}(V')\hookrightarrow{}_K\mathrm{Ig}(V).\tag{2.16}$$

2B1. *Embeddings of Igusa towers.* As in [Eischen et al. 2020], we use the theory of ordinary Hida families developed in Hida's book [Hida 2004] (and completed by Kai-Wen Lan's verification of the necessary conditions: see the discussion in [Eischen et al. 2020, §2.9.6]). This theory is based on the study of analytic functions on Igusa towers. In this paper we use the conventions of [Eischen et al. 2020, §2]. We choose a p-adic embedding $\iota_p:\mathcal{K}\hookrightarrow\mathbb{C}_p$ as in [Eischen et al. 2020, §1.4.1], so that ι_p and the chosen inclusion $\iota:\mathcal{K}\hookrightarrow\mathbb{C}$ are associated as in [Harris et al. 2006, §1].

In order to define cohomological pairings between p-adic modular forms on the Shimura varieties ${}_{K'}S(V')$ and ${}_{K}S(V)$ we need to know that the map (2.15) actually exists. In the first place, strictly speaking there is a map of Shimura data

$$(U(V'), Y_{V'}) \times (U(1), Y_1^+) \hookrightarrow (U(V), Y_V). \tag{2.17}$$

The second factor on the left is a (pro)-finite set without any additional arithmetic structure — recall that with our conventions the homomorphism Y_1^+ is trivial. To understand the map (2.15) it is nevertheless better to start with the map of Shimura data of PEL type

$$\big(G(U(V') \times U(1)), X''_{V'}\big) \hookrightarrow (\mathrm{GU}(V), X_V)) \tag{2.18}$$

with $X''_{V'}$ defined as in [Harris 2021], §2.2. As a reminder: we can also embed $G(U(V') \times U(1))$ in $\mathrm{GU}(V') \times \mathrm{GU}(1)$, with $\mathrm{GU}(1) = R_{\mathcal{K}/\mathbb{Q}}(\mathbb{G}_m)_{\mathcal{K}}$. Then $X''_{V'}$ is a $G(U(V') \times U(1))(\mathbb{R})$-conjugacy class of homomorphisms whose image under the embedding in $\mathrm{GU}(V') \times \mathrm{GU}(1)$ lies in the product $X_{V'} \times X_{0,1}$, where $X_{0,1}$ is the homomorphism $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to \mathrm{GU}(1)(\mathbb{R})$ whose value on $\mathbb{R}$-valued points is given by $z \mapsto \bar{z}$.

Now the map (2.18) defines a morphism of PEL Shimura varieties, and thus of smooth models in level $K = K_p \times K^p$ as above:

$${}_{K'}S\big(G(U(V') \times U(1)), X''_{V'}\big) \hookrightarrow {}_{K}S(\mathrm{GU}(V), X_V), \tag{2.19}$$

with notation (and level subgroup K') defined by analogy with (2.14). We define ordinary loci

$$\begin{gathered} {}_{K_{V'}}S(\mathrm{GU}(V'), X_{V'})^{\mathrm{ord}} \subset {}_{K'}S(\mathrm{GU}(V'), X_{V'}); \\ {}_{K'_1}S(\mathrm{GU}(1), X_{0,1})^{\mathrm{ord}} \subset {}_{K'}S(\mathrm{GU}(1), X_{0,1}); \\ {}_{K'}S\big(G(U(V') \times U(1)), X''_{V'}\big)^{\mathrm{ord}} \subset {}_{K'}S\big(G(U(V') \times U(1)), X''_{V'}\big) \end{gathered}$$

as well as

$${}_{K}S(\mathrm{GU}(V), X_V)^{\mathrm{ord}} \subset {}_{K}S(\mathrm{GU}(V), X_V).$$

(Level subgroups are assumed compatible with all morphisms.)

We recall the discussion of the Igusa varieties in [Harris et al. 2006]. For any $n \geq 0$ we can define Igusa coverings (we omit the prime-to-p level structures from the notation)

$$\mathrm{GIg}(V')_n \to {}_{K_{V'}}S(\mathrm{GU}(V'), X_{V'})^{\mathrm{ord}}; \quad \mathrm{GIg}(0,1)_n \to {}_{K'_1}S(\mathrm{GU}(1), X_{0,1})^{\mathrm{ord}}$$

and

$$\mathrm{GIg}(V)_n \to {}_{K}S(\mathrm{GU}(V), X_V)^{\mathrm{ord}}.$$

(We reserve the notation $\mathrm{Ig}(V)$ for the Igusa towers over the unitary group Shimura varieties, and $\mathrm{Ig}(V)_n$ for the Igusa covering in level p^n.) When $n = 0$ this is the identity map. These correspond to pairs $(\underline{A}_{V'}, j^o_{V'})$, $(\underline{A}_{0,1}, j^o_{0,1})$, $(\underline{A}_V, j^o_V)$ as in [Harris et al. 2006, (2.1.6.2)]. Here for example, $\underline{A}_V$ is a quadruple $(A, \lambda, \iota, \alpha^p)$, with A an abelian scheme of dimension n, and

$$j^o_V : M(V)^0 \otimes \mu_{p^m} \hookrightarrow A[p^m]$$

is an embedding of finite flat group schemes with $\mathcal{O}_K/p^m\mathcal{O}_K$-action. The free $\mathcal{O}_K$-submodule $M(V)^0 \subset V$ (resp. $M(V')^0 \subset V'$, $M(0,1)^0 \subset \mathcal{K}$) has the property that the action of $\mathcal{O}_K$ is a sum of r copies (resp. r copies, 0 copies) of ι (or ι_p) and s copies (resp. $s-1$ copies, 1 copy) of $c\iota$ (or $c\iota_p$).

We let $\mathrm{GIg}(V', (0,1))_m$ denote the fiber product of $\mathrm{GIg}(V')_m \times \mathrm{GIg}(0,1)_m$ with ${}_{K'}S\big(G(U(V') \times U(1)), X''_{V'}\big)$ over ${}_{K_{V'}}S(\mathrm{GU}(V'), X_{V'})^{\mathrm{ord}} \times {}_{K'_1}S(\mathrm{GU}(1), X_{0,1})^{\mathrm{ord}}$. With these conventions, it follows as in the discussion in [Harris et al. 2006, §2.1.1] that

Lemma 2.20. *The morphism* (2.19) *defines canonical morphisms of Igusa towers*

$$\mathrm{GIg}(V', (0,1))_n \hookrightarrow \mathrm{GIg}(V)_n$$

for $n \geq 0$. *For* $n = 0$ *this defines a morphism*

$${}_{K'}S\big(G(U(V') \times U(1)), X''_{V'}\big)^{\mathrm{ord}} \hookrightarrow {}_{K}S(\mathrm{GU}(V), X_V)^{\mathrm{ord}}.$$

Finally, the maps (2.15) and (2.16) are obtained by twisting with the Igusa tower for the Shimura datum $(\mathrm{GU}(1), X_{0,1})$ as in [Harris 2021, §2]. We omit the details.

Remark 2.21. The local computations in [Eischen et al. 2020] make it clear that the Euler factors at p in the standard p-adic L-function for ordinary families depend strongly on the signatures at primes above p, in a way that is broadly consistent with the conjectures of Coates and Perrin-Riou on p-adic L-functions for motives. The same dependence on archimedean data is expected for p-adic L-functions constructed in the setting of the Ichino–Ikeda–N. Harris Conjecture 7.1. The signature enters in [Eischen et al. 2020] through a twist that guarantees the existence of embeddings of Igusa towers; see Remark 3.1.4 of [Eischen et al. 2020]. It is likely that similar twists will be needed in order to extend the constructions of the present paper to the setting of Pilloni's higher Hida theory [Pilloni 2020].

Let $(H_1, h_1) \subset (\mathrm{GU}(V), X(V))$ be a CM pair — in other words, H_1 is a torus. We say (H_1, h_1) is an *ordinary CM pair* if the image of the morphism

$${}_{K(H_1)}S(H_1, h_1) \to {}_{K}S(\mathrm{GU}(V), X_V)$$

consists of PEL abelian varieties with ordinary reduction at p, for appropriate level subgroups. Thus when $K \cap U(V)(\mathbf{A}) = K^p \times \mathrm{GL}(n, \mathbb{Z}_p)$, the morphism of ${}_{K(H_1)}S(H_1, h_1) \to {}_{K}S(\mathrm{GU}(V), X_V)$ extends to a finite morphism of integral models if (H_1, h_1) is an ordinary CM pair. We define an ordinary CM pair $(H, h) \subset (U(V), Y_V)$ analogously.

2B2. *Pairings.* Fix κ and $\delta_{\kappa,\kappa'} : \mathcal{E}_\kappa \mid_{\mathrm{Sh}(V')} \to \mathcal{E}_{\kappa'}$ as in Proposition 2.6. Let

$$d = rs = \dim \mathrm{Sh}(V), \quad d' = r(s-1) = \dim \mathrm{Sh}(V'),$$

and define

$$\mathcal{E}_{\kappa',\flat} = \Omega^{d'} \otimes \mathcal{E}_{\kappa'}^{\vee}$$

be the Serre dual of $\mathcal{E}_{\kappa'}$. Then there is a canonical Serre duality pairing

$$H^0({}_K S(V), \mathcal{E}_\kappa) \otimes H^{d'}({}_{K'} S(V'), \mathcal{E}_{\kappa',\flat}) \to \mathcal{O}.$$

More generally, if $K'_{p,r} \subset K'_p$ is the congruence subgroup defined in [Eischen et al. 2020], $K'_r = K'_{p,r} \times K'^{,p}$, we can define a finite flat $\mathcal{O}$-module

$$H^0({}_{K'_r} S(V'), \mathcal{E}_{\kappa'}) \subset H^0({}_{K'_r} \mathrm{Sh}(V'), \mathcal{E}_{\kappa'}) := H^0({}_{K'_r} S(V'), \mathcal{E}_{\kappa'}) \otimes_{\mathcal{O}} \mathcal{O}[1/p]$$

to be

$$H^0({}_{K'_r} S(V'), \mathcal{E}_{\kappa'}) = H^0({}_{K'_r} S(V'), \mathcal{E}_{\kappa'}) \otimes_{\mathcal{O}} \mathcal{O}[1/p] \cap \mathcal{V}_{V'}, \tag{2.22}$$

where $\mathcal{V}_{V'}$ is the algebra of p-adic modular forms on $\mathrm{Sh}_{V'}$ (see below). Then we let

$$H^{d'}({}_{K'} S(V'), \mathcal{E}_{\kappa',\flat}) = \mathrm{Hom}\big(H^0({}_{K'_r} S(V'), \mathcal{E}_{\kappa'}), \mathcal{O}\big). \tag{2.23}$$

and we obtain a Serre duality pairing

$$H^0({}_{K_r} S(V), \mathcal{E}_\kappa) \otimes H^{d'}({}_{K'_r} S(V'), \mathcal{E}_{\kappa',\flat}) \to \mathcal{O}, \tag{2.24}$$

where $K_r = K_{p,r} \times K^p$ is defined as before.

3. *p*-adic modular forms and differential operators

3A. ***Basic definitions.*** The algebra $\mathcal{V}_V$ of p-adic modular forms on Sh_V is defined as in [Eischen et al. 2018, §2.6], following [Hida 2004]. Specifically, we let $B, N, T \subset \mathrm{GL}(n)$ denote respectively the upper triangular Borel subgroup, its unipotent radical, and its diagonal torus. For any pair of nonnegative integers (n, m) we let

$$\mathrm{Ig}_{n,m,V} = \mathrm{Ig}(V)_n \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}/p^m)$$

where $\mathrm{Ig}(V)_n$ is the Igusa covering in level p^n, as above. In the notation of [Eischen et al. 2018] we let

$$V_{n,m,V} = H^0(\mathrm{Ig}_{n,m,V}, \mathcal{O}_{\mathrm{Ig}_{n,m,V}});\ V_{\infty,m,V} = \varinjlim_n V_{n,m,V};\ V_{\infty,\infty,V} = \varprojlim_m V_{\infty,m,V}.$$

and set

$$\mathcal{V}_V = V_{\infty,\infty,V}^{N(\mathbb{Z}_p)},$$

where N is the maximal unipotent subgroup of $U(V)$ defined in [Eischen et al. 2018, §2.1].

The group $T(\mathbb{Z}_p)$ acts on $\mathcal{V}_V$ and for any algebraic character α of T we let $\mathcal{V}_V[\alpha] \subset \mathcal{V}_V$ denote the corresponding eigenspace; the elements of $\mathcal{V}_V[\alpha]$ are called p-adic modular forms of weight α. There are canonical embeddings

$$\Psi = \Psi_\alpha : H^0(S_V, \mathcal{E}_\alpha) \hookrightarrow \mathcal{V}_V[\alpha]; \tag{3.1}$$

compatible with multiplication in the sense that

$$\Psi_\alpha \otimes \Psi_\beta = \Psi_{\alpha+\beta} : H^0(S_V, \mathcal{E}_\alpha) \otimes H^0(S_V, \mathcal{E}_\beta) \xrightarrow{\times} H^0(S_V, \mathcal{E}_{\alpha+\beta}) \hookrightarrow \mathcal{V}_V[\alpha+\beta];$$

the forms in the image of (3.1) are called *classical*. More generally, if

$$\alpha : T(\mathbb{Z}_p) \to \mathcal{O}_{\mathbb{C}_p}^\times$$

is any continuous character, we may define the space $\mathcal{V}_V[\alpha] \subset \mathcal{V}_V \otimes \mathcal{O}_{\mathbb{C}_p}$ of p-adic modular forms of weight α. In what follows, we use the embeddings incl_p and ι to identify the maximal tori T_y and T, so that B is contained in the maximal parabolic subgroup with Lie algebra $\mathfrak{p}_y^- \oplus \mathfrak{k}_y$. If α is a classical weight, we write

$$\Psi_\alpha : \mathcal{V}_V[\alpha] \subset \mathcal{V}_V \tag{3.2}$$

for the tautological inclusion, extending the inclusion of (3.1); the notation is consistent.

The embedding (2.16) determines a map

$$\mathrm{res}_{V'} : \mathcal{V}_V \to \mathcal{V}_{V'}. \tag{3.3}$$

This embedding is compatible with action of the torus T' on the two sides through its inclusion in T.

Let A be an algebraic torus over $\mathrm{Spec}(\mathbb{Z}_p)$. For any complete p-adic algebra $\mathcal{O}$, define the Iwasawa algebra

$$\Lambda_{\mathcal{O}}(A) = \mathcal{O}[\![A(\mathbb{Z}_p)]\!] = \varprojlim_{U \subset A(\mathbb{Z}_p)} \mathcal{O}[A/U],$$

where U runs over open compact subgroups of $A(\mathbb{Z}_p)$. Let $\mathcal{C}(A(\mathbb{Z}_p), \mathcal{O})$ denote the $\mathcal{O}$-algebra of continuous $\mathcal{O}$-valued functions on $A(\mathbb{Z}_p)$, endowed with the topology defined by the sup norm.

Definition 3.4. A $\mathcal{O}$-valued p-adic measure — more simply, an $\mathcal{O}$-valued measure — on A is a continuous $\mathcal{O}$-homomorphism from $\mathcal{C}(A(\mathbb{Z}_p), \mathcal{O})$ to $\mathcal{O}$.

It is well known that the set of $\mathcal{O}$-valued measures on A forms an $\mathcal{O}$-Banach module that is naturally identified with $\Lambda_{\mathcal{O}}(A)$. Multiplication in the $\mathcal{O}$-algebra

$\Lambda_{\mathcal{O}}(A)$ corresponds to *convolution* of measures. If $\phi \in \mathcal{C}(A(\mathbb{Z}_p), \mathcal{O})$ and $\mu \in \Lambda_{\mathcal{O}}(A)$, we write

$$\int_{A(\mathbb{Z}_p)} \phi \, d\mu := \mu(\phi).$$

For any torus A over $\operatorname{Spec}(\mathbb{Z}_p)$, and any $\mathbb{Z}_p$-algebra $\mathcal{O}$, let

$$\mathcal{W}_{\mathcal{O}}(A) = \operatorname{Hom}_{\text{cont}}(A(\mathbb{Z}_p), \mathcal{O}^\times) = \operatorname{Hom}_{\text{cont}}(\Lambda_{\mathcal{O}}(A), \mathcal{O}^\times).$$

The *weight space* for A is the rigid analytic space over $\mathbb{Q}_p$ attached to $\Lambda_{\mathcal{O}}(A)$. A weight for A is then an element of $\mathcal{W}_{\mathcal{O}}(A)$.

When $\mathcal{O} = \mathcal{V}_V$ we write $\operatorname{Meas}(A, \mathcal{V}_V)$ instead of $\Lambda_{\mathcal{V}_V}(A)$.

3B. *p-adic differential operators.* There is a quotient T^{Sym} of the torus T_y, of rank $\min(r, s)$, defined by a sublattice of the lattice of characters of T_y: the characters of T^{Sym} are spanned by the ones called *symmetric* in Definition 2.4.4 of [Eischen et al. 2018]. Symmetric characters are also assumed to be *dominant*; the precise condition is recalled below.

We recall the normalization of C^∞ differential operators (Maass operators) from [Eischen et al. 2018, §3.3.1]. For a weight κ of holomorphic type we let $\mathcal{E}_\kappa(C^\infty)$ denote the space of C^∞ global sections of $\mathcal{E}_\kappa$. Let λ be a symmetric character of T_y and let

$$D_\kappa^\lambda : \mathcal{E}_\kappa(C^\infty) \to \mathcal{E}_{\kappa+\lambda}(C^\infty) \tag{3.5}$$

be the differential operator introduced on pages 467–468 of [Eischen et al. 2018] (we are writing weights additively rather than multiplicatively). For any weight α of T let $[\alpha]'$ denote its restriction to the subtorus $T' \subset T$. Let

$$R_{V,V'}^\infty : \mathcal{E}_\alpha(C^\infty) \to \mathcal{E}_{[\alpha]'}(C^\infty)$$

denote the restriction of C^∞ sections (any α). We let

$$\operatorname{pr}_{[\alpha]'}^{\text{hol}} : \mathcal{E}_{[\alpha]'}(C^\infty) \to H^0(\operatorname{Sh}(V'), \mathcal{E}_{[\alpha]'})$$

denote the orthogonal projection on holomorphic sections (any α).

Let $\kappa' = [\kappa + \lambda]'$. The relation between the D_κ^λ and the holomorphic operator $\delta_{\kappa,\kappa'}$ is given by the following:

Lemma 3.6. *We write*

$$D^{\text{hol}}(\kappa, \kappa^\dagger) = \operatorname{pr}_{[\kappa^\dagger]'}^{\text{hol}} \circ R_{V,V'}^\infty \circ D_\kappa^{\kappa^\dagger - \kappa}$$

Then for all $\kappa^\dagger \le \kappa'$ there exist unique elements $\delta(\kappa', \kappa^\dagger) \in U(\mathfrak{p}^{+,\prime})$, defined over $\mathcal{K}$, such that

$$D_\kappa^\lambda = \sum_{\kappa^\dagger \le \kappa'} \delta(\kappa', \kappa^\dagger) \circ D^{\text{hol}}(\kappa, \kappa^\dagger).$$

The term $\delta(\kappa', \kappa')$ is a nonzero scalar in $\mathcal{K}$.

Proof. This is the analogue of Corollary 4.4.9 of [Eischen et al. 2020] and is proved in the same way. □

The idea of the proof is roughly the following. Write $E_{\kappa,y}$ for the fiber at y of the pullback of $\mathcal{E}_\kappa$ to the symmetric space Y_V; this is an irreducible representation of K_y. Then $D_\kappa^{\kappa^\dagger-\kappa}$ lifts, on automorphic forms, to a differential operator given in the enveloping algebra of $\mathfrak{p}_y^+$ by an explicitly normalized projection onto the $\kappa^\dagger$-isotypic subspace of

$$E_{\kappa,y}\otimes \mathrm{Sym}^{|\kappa^\dagger-\kappa|}(\mathfrak{p}_y^+),$$

where $|\kappa^\dagger-\kappa|$ is the degree of the weight $\kappa^\dagger-\kappa$. This isotypic subspace is the sum of its intersections with the irreducible constituents of the restriction to $U(\mathfrak{g}')$ of the discrete series $\mathbb{D}_\kappa$, as in (2.4). Only one of these intersections is the highest K'_y-type subspace of its corresponding constituent; this is the image of $\mathrm{pr}_{\kappa'}^{\mathrm{hol}}$. Each of the others is obtained from the highest K'_y-type of its irreducible $U(\mathfrak{g}')$-constituent $\mathbb{D}_{\kappa^\dagger}$. The existence of $\delta(\kappa',\kappa^\dagger)$ as in the lemma then follows from the obvious fact that $\mathbb{D}_{\kappa^\dagger}$ is generated over $U(\mathfrak{p}^{+,\prime})$ by its highest K'_y-type subspace.

The analogous p-adic differential operators are constructed in [Eischen et al. 2018, §3.3.2]. To preserve some of their notation while avoiding ambiguity we write

$$\mathcal{E}_\kappa(\mathrm{ord})=H^0(\mathrm{Ig}_V,\mathcal{E}_\kappa).$$

Then the operators are denoted

$$D_\kappa^{\lambda,\mathrm{ord}}:\mathcal{E}_\kappa(\mathrm{ord})\to\mathcal{E}_{\kappa+\lambda}(\mathrm{ord}). \tag{3.7}$$

We define a p-adic character χ of T^{Sym} to be a continuous group homomorphism $T^{\mathrm{Sym}}(\mathbb{Z}_p)\to\mathbb{Z}_p^\times$ that arises as the p-adic limit of dominant characters λ. The main results of [Eischen et al. 2018] are summarized in the following theorem:

Theorem 3.8. (*a*) *For any dominant character λ of T^{Sym}* (*or any symmetric character λ of T_y*) *there is a p-adic differential operator*

$$\Theta^\lambda:\mathcal{V}_V\to\mathcal{V}_V \tag{3.9}$$

characterized uniquely by either of the following properties:

(i) *For all classical weights α,*

$$\Theta^\lambda\circ\Psi_\alpha=\Psi_{\alpha+\lambda}\circ D_\alpha^{\lambda,ord}. \tag{3.10}$$

Here $\Psi_{\alpha+\lambda}$ is understood in the sense of (3.2).

(ii) *Let α be algebraic. Let $j:(H,h)\to(U(V),Y_V)$ be an ordinary CM pair, and for any κ let*

$$R_{H,h,j,\kappa}:H^0(S(V),\mathcal{E}_\kappa)\to H^0(S(H,h),j^*\mathcal{E}_\kappa)$$

denote the restriction map. Let

$$R^{p}_{H,h,j,\kappa} : \mathcal{E}^{\mathrm{ord}}_{\kappa} \to H^0(S(H,h), j^*\mathcal{E}_{\kappa}),$$

and

$$R^{\infty}_{H,h,j,\kappa} : \mathcal{E}_{\kappa}(C^{\infty}) \to H^0(S(H,h), j^*\mathcal{E}_{\kappa});$$

denote the analogous restrictions on p-adic and C^{∞} modular forms, respectively. Then for any $F \in H^0(S(V), \mathcal{E}_{\alpha})$,

$$R^{p}_{H,h,j,\alpha+\lambda} \circ \Theta^{\lambda} \circ \Psi_{\alpha}(F) = R^{\infty}_{H,h,j,\kappa} \circ D^{\lambda}_{\alpha}(F). \tag{3.11}$$

(*b*) *For any p-adic character χ of T^{Sym} there exists a p-adic differential operator*

$$\Theta^{\chi} : \mathcal{V}_V \to \mathcal{V}_V$$

characterized by the property: *whenever χ can be written as* $\lim_i \lambda_i$, *where λ_i are dominant algebraic characters*, *satisfying the inequalities of Theorems* 5.2.4 *and* 5.2.6 *of* [Eischen et al. 2018], *then*

$$\Theta^{\chi} = \lim_i \Theta^{\lambda_i}$$

(*limit in the operator norm*).

(*c*) *If $F \in \mathcal{V}_V$ is a p-adic modular form of weight $\alpha \in X^{\mathrm{an}}(T_y)$, then $\Theta^{\lambda}(F)$ is a p-adic modular form of weight $\alpha + \lambda$.*

Proof. Parts (a)(i), (b), and (c) are in Corollary 5.2.8 of [Eischen et al. 2018]. Part (a)(ii) can be proved by the arguments quoted in the proof of [Eischen et al. 2018, Proposition 7.2.3]. A complete proof will appear in forthcoming work. □

Remark 3.12. The inequalities cited in the statement of Theorem 3.8(b) guarantee that the characters λ_i tend to infinity in the positive chamber; indeed, that for every positive root α, $\lim_i \langle \alpha, \lambda_i \rangle = \infty$. In particular, when $\chi = 1$ is the trivial character, $\Theta^1 := \Theta^{\chi}$ is not the identity operator on $\mathcal{V}_V$, though it is an idempotent. This is familiar from Hida's theory in the case of elliptic modular forms: the p-adic differential operator of nonintegral weight χ multiplies the n-th Fourier coefficient of a classical modular form by the power n^{χ}, which is only defined if $(p, n) = 1$. A classical modular form whose n-th Fourier coefficient vanishes for every n divisible by p is called *p-depleted*. In our situation, the operation $F \mapsto \Theta^1(F)$ can be understood as p-depletion, even when the unitary group (over a general totally real field) is anisotropic.

4. One-dimensional *p*-adic measures defined by a holomorphic automorphic form

The differential operators defined in Section 3B give rise to a p-adic measure. We believe that they can be used to define a measure on the full space $T^{\mathrm{Sym}}(\mathbb{Z}_p)$, but for the purposes of this paper we restrict our attention to a 1-dimensional quotient torus, since the necessary definitions are already in [Eischen et al. 2018] in the form we need. First, we state a corollary to Theorem 3.8:

Corollary 4.1. *Let $F \in \mathcal{V}_V$ be a p-adic modular form of weight α. Then there exists a $\mathcal{V}_V$-valued measure μ_F^* on $T^{\mathrm{Sym}}(\mathbb{Z}_p)$ characterized by the property that, for any p-adic character χ of T^{Sym}, viewed as a symmetric character of T_y, we have*

$$\int_{T^{\mathrm{Sym}}(\mathbb{Z}_p)} \chi \, d\mu_F^* = \Theta^{\chi-\alpha}(F).$$

We recall that T_y is a maximal torus of the group $\mathrm{GL}(r) \times \mathrm{GL}(s) \xrightarrow{\sim} K_y$, and that the adjoint action of K_y on $\mathfrak{p}_y^+$ is equivalent to the natural conjugation action on the space of $r \times s$ matrices. This action is identified in [Eischen et al. 2018] with the representation $\mathrm{St}_r \otimes \mathrm{St}_s$, where St_a is the standard representation of $\mathrm{GL}(a)$ on a-dimensional space. Then the symmetric algebra

$$\begin{aligned} \mathrm{Sym}^*(\mathfrak{p}_y^+/\mathfrak{p}_y^{+,\prime}) \xrightarrow{\sim} \bigoplus_{i\geq 0} \mathrm{Sym}^*\big((\mathrm{St}_r \otimes \mathrm{St}_s)/(\mathrm{St}_r \otimes \mathrm{St}_{s-1})\big) \\ \xrightarrow{\sim} \bigoplus_{i\geq 0} \mathrm{Sym}^*(\mathrm{St}_r \otimes \mathrm{St}_1), \end{aligned} \tag{4.2}$$

where the last isomorphism is given by the isotypic decomposition $\mathrm{St}_s \xrightarrow{\sim} \mathrm{St}_1 \oplus \mathrm{St}_{s-1}$ as representation of the standard Levi subgroup $\mathrm{GL}(1) \times \mathrm{GL}(s-1) \subset \mathrm{GL}(s)$. The dominant characters λ of T^{Sym} can be written as parameters (2.1)

$$\big(b_1 \geq b_2 \geq \cdots \geq b_s \geq 0 \geq \cdots \geq 0; b_1 \geq b_2 \geq \cdots \geq b_s\big)$$

if $r \geq s$, and with the 0s in the second half of the parameter if $s > r$. Then the representations occurring in (4.2) have parameters

$$\lambda_b = (b \geq 0 \geq \cdots \geq 0; b; 0 \geq \cdots \geq 0), \tag{4.3}$$

where the two semicolons separate parameters for $\mathrm{GL}(r) \times \mathrm{GL}(1) \times \mathrm{GL}(s-1)$.

If $b \in \mathbb{Z}_p$, we write $\lambda_b = \lim_i b_i$, where $b_i = (b_{1,i}, \ldots, b_{\min(r,s)_i})$, where all the $b_{j,i}$ are nonnegative integers, $b = \lim_i b_{1,i}$ in the p-adic topology, $\lim_i b_{j,i} = 0$ in the p-adic topology for $j > 1$, and for all $1 \leq j \leq \min(r, s)$, $\lim_i b_{j,i} = \infty$ in the real topology.

Let $X(T^{\mathrm{Sym}})$ denote the character lattice of T^{Sym}. Let $X_1 \subset X(T^{\mathrm{Sym}})$ be the characters of the form λ_b as in (4.3). Then X_1 is the character group of a 1-dimensional quotient of T^{Sym}, which we identify with $\mathrm{GL}(1)$. Restricting the measure μ_F^* to characters of $\mathrm{GL}(1)$, we obtain the corollary:

Corollary 4.4. *Let $F \in \mathcal{V}_V$ be a p-adic modular form of weight α. Then there exists a $\mathcal{V}_V$-valued measure μ_F on $\mathrm{GL}(1, \mathbb{Z}_p)$ characterized by the property that,* *for any p-adic integer b, we have*

$$\int_{\mathrm{GL}(1,\mathbb{Z}_p)} x^b \, d\mu_F = \Theta^{\lambda_b}(F).$$

Definition 4.5. Let A be a torus over $\mathrm{Spec}(\mathbb{Z}_p)$. We say the $\mathcal{V}_V$-valued measure μ on $A(\mathbb{Z}_p)$ is equivariant of weight α if for any character χ of A, the integral $\int_{A(\mathbb{Z}_p)} \chi \, d\mu$ is a p-adic modular form of weight $\chi + \alpha$ for some fixed weight α.

The following corollary is then a consequence of Theorem 3.8(c).

Corollary 4.6. *Let $F \in \mathcal{V}_V$ be a p-adic modular form of weight α. Then the measures μ_F^* (resp. μ_F) on $T^{\mathrm{Sym}}(\mathbb{Z}_p)$ (resp. $\mathrm{GL}(1, \mathbb{Z}_p)$) are equivariant of weight α.*

We will be pairing the measure μ_F — or rather its restriction to $\mathrm{Sh}_{V'}$ — with Hida families of ordinary p-adic modular forms on $U(V')$. We could also pair the $\dim T^{\mathrm{Sym}}$-parameter measure with Hida families, but they will not give rise to more general special values, because the differential operators on Sh_V in directions parallel to $\mathrm{Sh}_{V'}$ do not change the automorphic representation of $U(V')$.

Suppose now that $F \in \mathcal{V}_V$ is a classical form of weight κ. Let κ' satisfy the inequalities of Lemma 2.5, so there is a holomorphic differential operator $\delta^{\kappa,\kappa'}$ as in Proposition 2.6.

Lemma 4.7. (*a*) *For all $\kappa^\dagger$ that satisfy the inequalities of Lemma 2.5, there is a differential operator*

$$\theta^{\mathrm{hol}}(\kappa, \kappa^\dagger) : \mathcal{V}_V \to \mathcal{V}_V$$

such that

$$\mathrm{res}_{V'} \circ \theta^{\mathrm{hol}}(\kappa, \kappa^\dagger)(F) = \delta^{\kappa,\kappa'}(F).$$

(*b*) *For all $\kappa^\dagger \leq \kappa'$, there are differential operators $\theta(\kappa, \kappa^\dagger) : \mathcal{V}_V \to \mathcal{V}_V$ such that*

$$\theta(\kappa, \kappa') = \sum_{\kappa^\dagger \leq \kappa'} \mathrm{res}_{V'} \circ \theta(\kappa, \kappa^\dagger) \circ \theta^{\mathrm{hol}}(\kappa, \kappa^\dagger),$$

with $\theta(\kappa, \kappa)$ a nonzero scalar. Here $\mathrm{res}_{V'}$ *is as in* (3.3).

Proof. Part (a) is the analogue of Proposition 8.1.1(d) of [Eischen et al. 2020]; it is derived in the same way from properties of restriction to CM points — in this case from Theorem 3.8(a)(ii). Part (b) is then the analogue of [Eischen et al. 2020, Corollary 8.1.2]. □

In what follows, the terms *antiordinary* and *antiholomorphic* are used as in [Eischen et al. 2020]; these are reviewed in Section A.2 of the Appendix.

Proposition 4.8. *There is a p-adic differential operator $\theta^{\kappa,\kappa'} : \mathcal{V}_V \to \mathcal{V}_{V'}$ with the property that, for any antiholomorphic antiordinary automorphic form g of weight*

κ' on $U(V')$ and any holomorphic automorphic form F of weight κ on $U(V)$, we have

$$[\theta^{\kappa,\kappa'}(F), g] = [\delta^{\kappa,\kappa'}(F), g].$$

Proof. This follows from Lemma 4.7 and from estimates on the denominators used to define the ordinary projector, as in the proof of [Eischen et al. 2020, Proposition 8.1.3]. □

In what follows we assume F to belong to a fixed holomorphic automorphic representation π.

5. Hida families

Recall the maximal torus $T' \subset U(V')$. Let $\Lambda' = \Lambda_{\mathcal{O}}(T')$ be the Iwasawa algebra of $T'(\mathbb{Z}_p)$; it is a noetherian local ring that is non canonically isomorphic to the tensor product over $\mathcal{O}$ of $n-1$ copies of $\mathcal{O}[[1+p\mathbb{Z}_p]]$.

5A. ***Ordinary parameters.*** We let $\mathbb{T}$ denote the ordinary p-adic Hecke algebra for cusp forms on the group $U(V')$. For the purposes of this paper, a Hida family is determined by a single antiordinary antiholomorphic automorphic representation τ of $U(V')$, and the completion $\mathbb{T}_\tau$ of $\mathbb{T}$ at the maximal ideal $\mathfrak{m}_\tau$ corresponding to τ. We write $\mathcal{O}$ for the coefficient ring denoted $\mathcal{O}_\tau$ as in [Eischen et al. 2020, §7.3], so that $\Lambda' = \Lambda_{\mathbb{Z}_p}(T') \otimes \mathcal{O}$, where $\Lambda_{\mathbb{Z}_p}(T')$ is the Iwasawa algebra of weights for T'_y. We let $\boldsymbol{T}^{\mathrm{ord}}_{K'_r,\kappa',\mathcal{O}}$ denote the ordinary Hecke algebra of weight κ' and level K'_r — notation as in [Eischen et al. 2020, §6.6.6] — and let $\mathbb{T}_{r,\kappa'}$ denote the completion of $\boldsymbol{T}^{\mathrm{ord}}_{K'_r,\kappa',\mathcal{O}}$ at $\mathfrak{m}_\tau$; the representation τ will be fixed for the remainder of the paper. We can write

$$\mathbb{T}_\tau = \varprojlim_r \mathbb{T}_{r,\kappa',\tau}$$

for any sufficiently regular κ', as in [Eischen et al. 2020, §7.1, Theorem 7.1]. For all claims regarding Hida families we refer to Hida's book [2004]. In particular, $\mathbb{T}_\tau$ is a finite flat Λ'-algebra.

5B. ***The Gorenstein condition.*** We introduce two versions of the Gorenstein hypothesis that was used in [Eischen et al. 2020] to define p-adic L-functions with values in Hida's ordinary Hecke algebra. The first is adapted to (holomorphic) automorphic forms of fixed weight κ'; compare [Eischen et al. 2020, Definition 6.7.9]:

Definition 5.1. The $\mathbb{T}_{r,\kappa'}$-module $S^{\mathrm{ord}}_{\kappa'}(K'_r, \mathcal{O})_\tau$ is said to satisfy the *Gorenstein hypothesis* if the following conditions hold.

- $\mathbb{T}_{r,\kappa'} \xrightarrow{\sim} \hat{\mathbb{T}}_{r,\kappa'} := \mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{r,\kappa'}, \mathcal{O})$ as $\mathcal{O}$-algebras.
- $S^{\mathrm{ord}}_{\kappa'}(K'_r, \mathcal{O})_\tau$ is free over $\mathbb{T}_{\kappa'}$.

The $\boldsymbol{T}^{\mathrm{ord}}_{K'_r,\kappa',\mathcal{O}}$-module $S^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})$ is said to satisfy the Gorenstein hypothesis if all its localizations at maximal ideals of $\boldsymbol{T}_{K'_r,\kappa',\mathcal{O}}$ satisfy the two conditions above.

The second version is a hypothesis on the big ordinary Hecke algebra, which is a finite Λ'-algebra; compare [Eischen et al. 2020, Hypothesis 7.3.2].

Hypothesis 5.2 (Gorenstein hypothesis). *Let* $\hat{\mathbb{T}}_\tau = \mathrm{Hom}_{\Lambda'}(\mathbb{T}_{\tau^\flat},\Lambda')$. *Then*

- $\hat{\mathbb{T}}_\tau$ *is a free rank-one* $\mathbb{T}_\tau$*-module via the isomorphism* $\flat : \mathbb{T}_\tau \xrightarrow{\sim} \mathbb{T}_{\tau^\flat}$.
- *For each* r, *let* $\mathbb{T}_\tau$ *act on* $\mathrm{Hom}_{\mathcal{O}}(S^{\mathrm{ord}}_{\kappa'}(K'^{,p}_r,\mathcal{O}),\mathcal{O})_{\mathfrak{m}_\tau}$ *by the natural action twisted by* $\flat$. *Then*

$$\hat{S}^{\mathrm{ord}}_\tau = \mathrm{Hom}_{\mathcal{O}}\Big(\varinjlim_r S^{\mathrm{ord}}_{\kappa'}(K'^{,p}_r,\mathcal{O}),\mathcal{O}\Big)_{\mathfrak{m}_\tau}$$

(*for any sufficiently regular* κ') *is a free* $\mathbb{T}_\tau$*-module.*

The hypothesis in Definition 5.1 follows from Hypothesis 5.2 and Hida's control theorem, for sufficiently regular κ'. More precisely, let

$$\hat{S}^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(S^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O}),\mathcal{O}), \tag{5.3}$$

and define $\hat{S}^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O})_\tau$ analogously. Then Hida's control theorem (see [Eischen et al. 2020, Theorem 7.3.1]) asserts, in the present notation, that

$$\mathbb{T}_\tau \otimes_{\Lambda'} \Lambda'/I_{\kappa'} \xrightarrow{\sim} (\boldsymbol{T}^{\mathrm{ord}}_{K,\kappa,\mathcal{O}})_\tau \tag{5.4}$$

for sufficiently regular κ'. Under Hypothesis 5.2, there is then an isomorphism

$$\hat{S}^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O})_\tau \xrightarrow{\sim} S^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O})_\tau \tag{5.5}$$

of dual free $\mathbb{T}_{\kappa'}$-modules. We introduce compatible bases of these modules in the next section.

5B1. *Bases.* We let

$$\Omega_\tau = \mathrm{Hom}_{\mathbb{T}_\tau}(\hat{\mathbb{T}}_\tau,\mathbb{T}_\tau), \tag{5.6}$$

$$\Omega_{\kappa',\tau} = \mathrm{Hom}_{\mathbb{T}_{\kappa'}}(\hat{\mathbb{T}}_{\kappa'},\mathbb{T}_{\kappa'}). \tag{5.7}$$

Under the Gorenstein hypotheses, Ω_τ is a free rank 1 $\mathbb{T}_\tau$-module, and $\Omega_{\kappa',\tau}$ is a free rank 1 $\mathbb{T}_{\kappa'}$-module. In particular, the set of $\mathbb{T}_\tau$-isomorphisms between $\mathbb{T}_\tau$ and $\hat{\mathbb{T}}_\tau$ is a torsor under $\mathbb{T}_\tau^\times$, and between $\mathbb{T}_{\kappa'}$ and $\hat{\mathbb{T}}_{\kappa'}$ a torsor under $\mathbb{T}_{\kappa'}^\times$.

5C. ***Ramified local components.*** We let π be the automorphic representation of G_V corresponding to the holomorphic modular form F. Let τ be as in the previous section, and let S denote the finite set of finite primes v, not including p, such that either π_v, τ_v, or $\mathcal{K}/\mathbb{Q}$ is ramified. In [Eischen et al. 2020, §7.3.4] we introduce a free $\mathcal{O}$-lattice

$$\hat{I} = \hat{I}_\tau \subset \Big(\bigotimes_{v\in S} \tau_v\Big)^{K^p}$$

with the property that:

(a) For all sufficiently regular κ', there is an isomorphism of $\mathbb{T}_{\kappa',\tau}$-modules

$$\hat{S}^{\text{ord}}_{\kappa'}(K;\mathcal{O})_\tau \xrightarrow{\sim} \mathbb{T}_{\kappa',\tau} \otimes_{\mathcal{O}} \hat{I}.$$

(b) Also, there is an isomorphism of $\mathbb{T}_\tau$ modules $\hat{S}^{\text{ord}}_\tau \xrightarrow{\sim} \mathbb{T}_\tau \otimes_{\mathcal{O}} \hat{I}$, compatible with (a) and with the control isomorphisms (5.4).

(In [Eischen et al. 2020] the group is $\mathrm{GU}(V)$ rather than $U(V')$ and the notation is $\pi^\flat$ rather than τ, but the lattice is defined in the same way.)

Let $\mathbb{L} = \mathrm{Frac}(\mathcal{O})$, $\bar{\mathbb{L}}$ its algebraic closure. For any sufficiently regular κ', there is a decomposition

$$\mathbb{T}_{\kappa',\tau} \otimes_{\mathcal{O}} \bar{\mathbb{L}} \xrightarrow{\sim} \bigoplus_{\pi'\in[\tau]_{\kappa'}} \mathbb{T}_{\pi'}. \tag{5.8}$$

The elements of $[\tau]_{\kappa'}$ are the (antiordinary) antiholomorphic automorphic representations π' of weight κ' whose Hecke eigenvalues at places outside S are congruent to those of τ; the action of $\mathbb{T}_{\kappa',\tau}$ on the vectors in each π' factors through the corresponding component $\mathbb{T}_{\pi'}$.

We let $K'_S = K^p \cap \prod_{v\in S} G_{V'}(\mathbb{Q}_v)$ and assume K'_S admits a factorization $\prod_{v\in S} K'_v$. We have implicitly been assuming that the finite parts of our automorphic representations are defined over $\mathbb{L}$ (see Remark 2.12). Let

$$\hat{I}_{\mathbb{L}} = \hat{I} \otimes_{\mathcal{O}} \mathbb{L} = \bigotimes_{v\in S} (\tau_v)^{K'_v}(\mathbb{L}).$$

This is naturally the tensor product over $v \in S$ of *irreducible* representations of the local (ramified) Hecke algebra $\mathcal{H}_{\mathbb{L}}(G_{V'}(\mathbb{Q}_v), K'_v)$ of compactly supported $\mathbb{L}$-valued K'_v-biinvariant functions on $G_{V'}(\mathbb{Q}_v)$. Let $\mathcal{H}_{\mathcal{O}}(G_{V'}(\mathbb{Q}_v), K'_v) \subset \mathcal{H}_{\mathbb{L}}(G_{V'}(\mathbb{Q}_v), K'_v)$ denote the subalgebra of $\mathcal{O}$-valued functions and let $\mathcal{H}_S = \bigotimes_v \mathcal{H}_{\mathcal{O}}(G_{V'}(\mathbb{Q}_v), K'_v)$. We make the following simplifying hypothesis:

Hypothesis 5.9 (local minimality). *For each $v \in S$, there is an $\mathcal{O}$-lattice $\hat{I}_v \subset (\tau_v)^{K'_v}(\mathbb{L})$, invariant under $\mathcal{H}_{\mathcal{O}}(G_{V'}(\mathbb{Q}_v), K'_v)$, with the property that $\hat{I} \xrightarrow{\sim} \bigotimes_{v\in S} \hat{I}_v$ and*

$$\widehat{M}^0_\tau := \widehat{M}^0_{\tau,\hat{I}} := \mathrm{Hom}_{\mathcal{H}_S}(\hat{I}, \hat{S}^{\text{ord}}_\tau)$$

is a free rank 1 $\mathbb{T}_\tau$-module.

We let

$$\widehat{M}_\tau := \mathrm{Hom}_{\mathbb{T}_\tau}(\widehat{M}^0_\tau, \mathbb{T}_\tau). \tag{5.10}$$

Under Hypothesis 5.9, $\widehat{M}_\tau$ is a free rank 1 $\mathbb{T}_\tau$-module.

Recall that each $v \in S$ is of characteristic prime to p, so we can apply the methods of the mod p representation theory of $G_{V'}(\mathbb{Q}_v)$ [Vignéras 1996]. In particular, one can define the reduction $\bar{\tau}_v$ of each τ_v for $v \in S$ modulo the maximal ideal of $\mathcal{O}$, as a semisimple representation of $G_{V'}(\mathbb{Q}_v)$ of finite length. It is easy to see that Hypothesis 5.9 is automatic if each $\bar{\tau}_v$ is irreducible. In particular, by the theory of [Vignéras 1996], this holds if p is banal for all $G_{V'}(\mathbb{Q}_v)$ with $v \in S$, and in particular if p is sufficiently large. For v split in $\mathcal{K}$, the condition can be read off the p-adic Galois representation attached to τ_v by the local Langlands correspondence, and corresponds to the usual hypothesis in Galois deformation theory that the Galois representations attached to τ' congruent to τ modulo p are minimally ramified at such v. This is probably also the case for v inert or ramified in $\mathcal{K}$, but as far as I know this has not been verified.

The notation $\hat{I}$ is deleted for the sake of legibility. By (5.4), Hypothesis 5.9 implies that, for all sufficiently regular κ',

$$\widehat{M}^0_{\kappa',\tau} := \mathrm{Hom}_{\mathcal{H}_S}(\hat{I}, \hat{S}^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O})_\tau) \quad \text{is a free rank 1 } \mathbb{T}_{\kappa',\tau}\text{-module.} \tag{5.11}$$

An element $f'_S = \sum_j \bigotimes_{v\in S} f'_{v,j} \in \hat{I}$, with $f'_{v,j} \in \tau_v$, is called *primitive* if it generates a $\mathcal{O}$-direct summand of $\hat{I}$. Choose a generator $\widehat{m}$ of $\widehat{M}_\tau$ as $\mathbb{T}_\tau$-module and a primitive $f'_S \in \hat{I}$. Then for every sufficiently regular κ', $\widehat{m}^{-1}(f'_S)$ generates $S^{\mathrm{ord}}_{\kappa'}(K;\mathcal{O})_\tau$ over $\mathbb{T}_{\kappa',\tau}$; it defines a linear combination

$$\widehat{m}^{-1}(f'_S) = \sum_{\pi'\in[\tau]_{\kappa'}} (f_{\pi'}) = f'_S \otimes \Bigl[\sum_{\pi'\in[\tau]_{\kappa'}} f'^{,S}_{\pi'}\Bigr] := f'_S \otimes f'^{,S}_{\widehat{m}} \tag{5.12}$$

with notation as in (5.8), where $f_{\pi'} = f'_S \otimes f'^{,S}_{\pi'} \in \pi'$ with $f'^{,S}_{\pi'}$ a Hecke eigenvector.

Note that each individual $f_{\pi'}$ is not necessarily integral, but the sum $f'^{,S}_{\widehat{m}}$ is a divided congruence and is defined over $\mathcal{O}$. As we have noted, under the Gorenstein hypothesis the set of generators $\widehat{m}$ of $\widehat{M}_\tau$ is a torsor under $\mathbb{T}^\times_\tau$. If $\boldsymbol{t} \in \mathbb{T}^\times_\tau$, then

$$f'^{,S}_{\boldsymbol{t}\cdot\widehat{m}} = \boldsymbol{t} \cdot f'^{,S}_{\widehat{m}}.$$

6. Contraction of p-adic measures with Hida families

We fix a ring $\mathcal{O}$ of integers in a finite extension of $\mathbb{Q}_p$, and we assume $\mathcal{O}$ is a subalgebra of the algebra $\mathcal{V}_V$ of p-adic modular forms. For a p-adic torus A we define $\mathrm{Meas}(A, \mathcal{V}_V)$ as in Section 3A. We choose a collection of congruence subgroups

$$A(\mathbb{Z}_p) \supset A_1 \supset A_2 \cdots \supset A_r \supset \ldots$$

with $\bigcap_i A_i = \{1\}$, and we let

$$C_r(A, \mathcal{O}) \subset C(A(\mathbb{Z}_p), \mathcal{O})$$

be the $\mathcal{O}$-submodule of $\mathcal{O}$-valued functions on $A(\mathbb{Z}_p)/A_r$. For any algebraic character χ of A we consider $C_r(A, \mathcal{O})\chi$ as a finite rank $\mathcal{O}$-submodule of $C(A(\mathbb{Z}_p), \mathcal{O})$.

6A. ***Review of equivariant measures.*** Let α be a character as in Definition 4.5. As in [Eischen et al. 2020, Lemma 7.4.2], we can identify an equivariant measure $\phi \in \mathrm{Meas}(A, \mathcal{V}_V)$ of weight κ

$$\phi(a \cdot f) = \kappa(a) \cdot a \cdot \phi(f), \quad \text{for all } a \in A(\mathbb{Z}_p),\ f \in C(A(\mathbb{Z}_p), \mathcal{O}) \tag{6.1}$$

with a collection

$$(\phi_{r,\chi}) \in \mathrm{Hom}_{\Lambda_{\mathcal{O}}(A)}(C_r(A, \mathcal{O})\kappa \cdot \chi, \mathcal{V}_V), \tag{6.2}$$

satisfying a certain distribution relation, written

$$\eta_r^*(\phi_{r+1,\chi}) = \phi_{r,\chi};$$

we refer the reader to [Eischen et al. 2020] for the definition. (There is a misprint in [Eischen et al. 2020]: the factor corresponding to $a\cdot$ after $\alpha(a)$ in (6.1) is missing.)

In the application in this paper, A is the torus T'_y. The classical weights are Zariski dense in $\mathrm{Spec}(\Lambda')$. Recall that we have defined $[\kappa]'$ to be the restriction of the weight κ to T'_y. We define Λ'_κ (which we could also write $\Lambda_{[\kappa]'}$) to be the quotient of Λ' corresponding to the Zariski closure of highest weights of T'_y of the form $\kappa'_b := [\kappa]' + b(1, 0, \dots, 0)$. As a ring, Λ'_κ is isomorphic to Λ'_0, which is just the Iwasawa algebra $\mathcal{O}[\![T]\!]$. We assume our chosen automorphic representation τ of $U(V')$ is of weight contained in $\mathrm{Spec}(\Lambda'_\kappa)$. In the discussion above, $[\kappa]' \cdot \chi$ is taken to be a character κ'_b, which we henceforth abbreviate κ'. In other words, we don't assume that $\kappa' = [\kappa]'$, but we do want κ' to correspond to a classical point of $\mathrm{Spec}(\Lambda'_\kappa)$.

We choose the filtration $(A_r) = (T'_{y,r})$ to be compatible with the filtration $K'_{p,r}$ of K'_p. In (6.2) we can replace $\Lambda_{\mathcal{O}}(T'_y)$ by its quotient Λ'_κ, which is an Iwasawa algebra in one variable. Let $\mathcal{J} \subset \Lambda_{\mathcal{O}}(T'_y)$ denote the kernel of the homomorphism to Λ'_κ. Let $(\phi_{r,\chi})$ be as above, and define

$$\phi'_{r,\chi} = \mathrm{res}_{V'} \circ \phi_{r,\chi} \in \mathrm{Hom}_{\Lambda'_\kappa}\big(C_r(T'_y, \mathcal{O})[\mathcal{J}]\kappa \cdot \chi, \mathcal{V}_{V'}\big), \quad r \geq 0.$$

Here $C_r(T'_y, \mathcal{O})[\mathcal{J}] \subset C_r(T'_y, \mathcal{O})$ is the $\mathcal{O}$-submodule of functions annihilated by $\mathcal{J}$.

As in [Eischen et al. 2020], we define contraction of p-adic measures with Hida families by fixing a sufficiently regular classical weight $\kappa' = [\kappa]' \cdot \chi$ of T'_y and taking the limit over r of pairings of the $\phi'_{r,\chi}$ with the level K'_r components of a fixed Hida family. We start with the equivariant measure μ_F of weight κ, then, and define an element μ'_F of $\mathrm{Hom}_{\Lambda'_\kappa}(C(\mathbb{Z}_p^\times, \mathcal{O}), \mathcal{V}_{V'})$ by restricting μ_F via the map

$\mathrm{res}_{V'}$; let $(\phi'_{r,\chi}) = (\phi'_{r,\chi,\tau})$ be the corresponding collection of homomorphisms in $\mathrm{Hom}_{\Lambda'_\kappa}(C_r(A,\mathcal{O})[\kappa]'\cdot\chi, \mathcal{V}'_V)$.

Let e'_τ denote the ordinary projector $\mathcal{V}_{V'} \to \mathcal{V}^{\mathrm{ord}}_{V'}$, composed with localization at the maximal ideal $\mathfrak{m}_\tau$. For $\kappa' = [\kappa]'\cdot\chi$ sufficiently regular, it follows from (5.4) that the image of $e'_\tau \circ \phi'_{r,\chi}$ is contained in

$$S^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\hat{S}^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O}),\mathcal{O}). \tag{6.3}$$

More precisely, denoting localization at $\mathfrak{m}_\tau$ by the subscript $_\tau$, the image of $e'_\tau \circ \phi'_{r,\chi}$ lies in

$$S^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\hat{S}^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau,\mathcal{O}), \tag{6.4}$$

where the left-hand side is the image of $S^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})$ in the right-hand side of (6.3) after localization at $\mathfrak{m}_\tau$.

6B. ***Application of the Gorenstein Hypothesis 5.2.*** The right hand side of (6.4) is isomorphic to $\hat{S}^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau$ as finite free module of rank M equal to the $\mathcal{O}$-rank of $\hat{I} = \hat{I}_\tau$ over $\mathbb{T}_\tau$. We choose $\mathbb{T}_\tau$ bases $\widehat{m} \in \widehat{M}_\tau$ and $\omega \in \Omega_\tau$, and a primitive $f'_S \in \hat{I}$, and obtain the following diagram:

$$\begin{aligned}
S^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\hat{S}^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau,\mathcal{O})\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\big(\big[\hat{I}\otimes_{\mathcal{O}}\mathrm{Hom}_{\mathcal{H}_S}(\hat{I},\hat{S}^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau)\big],\mathcal{O}\big)\\
&\xrightarrow{\widehat{m}} \mathrm{Hom}_{\mathcal{O}}(\hat{I}\otimes_{\mathcal{O}}\mathbb{T}_{r,\kappa',\tau},\mathcal{O})\\
&\xrightarrow{\omega} \mathrm{Hom}_{\mathcal{O}}(\hat{I},\mathcal{O})\otimes_{\mathcal{O}}\mathbb{T}_{r,\kappa',\tau}\\
&\xrightarrow{f'_S\otimes\mathrm{id}} \mathbb{T}_{r,\kappa',\tau},
\end{aligned}\tag{6.5}$$

where the last line inserts f'_S in the factor $\mathrm{Hom}_{\mathcal{O}}(\hat{I},\mathcal{O})$.

We now apply the discussion of Section 6A when $(\phi_{r,\chi})$ is attached to the measure μ_F. As in [Eischen et al. 2020, §7.4], the collection

$$(\phi'_{r,\chi,\tau} \in S^{\mathrm{ord}}_{\kappa'}(K'_r,\mathcal{O})_\tau)_{r\geq 0}$$

patches together with an element $L^0(F,\tau)$ of a finite free rank M $\mathbb{T}_\tau$-module. By choosing $\widehat{m}$, ω, and f'_S as above, we identify $L^0(F,\tau)$ with an element

$$L(F,\tau) := L(F, f'_S, \tau, \widehat{m}, \omega) \in \mathbb{T}_\tau.$$

as in [Eischen et al. 2020, Proposition 7.4.10]. Note, however, that this element is not independent of the choices. In particular, the bases $\widehat{m}$ and ω of their respective rank 1 $\mathbb{T}_\tau$-modules are only defined up to multiplication by units in $\mathbb{T}_\tau^\times$. In general there seems to be no canonical choice of these bases, in contrast to the familiar case of new forms for GL(2), when the leading term of the q-expansion defines a canonical choice.

6C. *Pairings and periods.* We summarize the construction in the previous section.

Theorem 6.6. *Suppose F is classical of weight κ. Let $\mathbb{T}_\tau$ be a component of the ordinary Hecke algebra for cusp forms on $U(V')$; we admit Hypotheses 5.2 and 5.9. Let $L(F,\tau)$ denote the element $L(F, f'_S, \tau, \widehat{m}, \omega) \in \mathbb{T}_\tau$ constructed above. Let $x \in \mathrm{Spec}(\mathbb{T}_\tau)$ lie over a weight $\kappa' \in \mathrm{Spec}(\Lambda'_\kappa)$, and suppose it corresponds to an eigenform $f'_x = f'_S \otimes f'^{,S}_{\widehat{m},x}$ as in* (5.12). *Then*

$$L(F,\tau,x) = [\theta^{\kappa,\kappa'}(F), f'_x] = \int_{[U(V')]} \delta^{\kappa,\kappa'}(F)(g') \cdot f'_x(g')\, dg' = P_{U(V')}(\delta^{\kappa,\kappa'}(F), f'_x). \tag{6.7}$$

Proof. Since f'_x is antiordinary, this follows immediately from Proposition 4.8. □

Remark 6.8. The function $L(F,\tau)$ can be described more canonically as an element of $\mathbb{T}_\tau \otimes_{\mathcal{O}} \widehat{M}_\tau \otimes \Omega_\tau$, or alternatively as a section of the line bundle on $\mathrm{Spec}(\mathbb{T}_\tau)$ corresponding to the module $\widehat{M}_\tau \otimes \Omega_\tau$. In cases involving elliptic modular forms (for example, in [Harris and Tilouine 2001]) the theory of the q-expansion provides a canonical everywhere nonvanishing section of this line bundle. I don't know whether or not to expect the corresponding line bundle to be trivial more generally, when the Gorenstein condition is satisfied. The question will be examined more carefully in the sequel to this paper.

7. The Ichino–Ikeda formula and the main theorem

The formula is given by the Ichino–Ikeda–N. Harris conjecture:

Conjecture 7.1. *Let $f \in \pi$, $f' \in \pi'$ be factorizable vectors. Then there is an integer β, depending on the L-packets containing π and π', such that*

$$\mathcal{P}(f,f') = 2^\beta \Delta_H \mathcal{L}^S(\pi,\pi') \prod_{v\in S} I^*_v(f_v, f'_v).$$

Here

$$\mathcal{P}(f,f') := \frac{|P_{U(V')}(f,f')|^2}{\langle f,f\rangle\langle f',f'\rangle}.$$

and

$$\mathcal{L}^S(\pi,\pi') := \frac{L^S\big(\frac{1}{2}, \Pi\otimes\Pi'\big)}{L^S(1,\Pi, As^{(-1)^n})L^S(1,\Pi', As^{(-1)^{n-1}})}, \tag{7.2}$$

where $\Pi = \mathrm{BC}(\pi)$, $\Pi' = \mathrm{BC}(\pi')$. Moreover, the local Euler factors I^*_v are given by the explicit formula

$$I^*_v(f_v, f'_v) = [c_{f_v}(1)c_{f'_v}(1)]^{-1} \cdot \int_{U(V')_v} c_{f_v}(g')c_{f'_v}(g')\, dg', \tag{7.3}$$

where $c_{f_v}(g) = \langle \pi(g) f_v, f_v \rangle_{\pi_v}$, for a fixed inner product $\langle \bullet, \bullet \rangle_{\pi_v}$, and likewise for the matrix coefficient $c_{f'_v}$. Normalizations are explained in the references cited in the following theorem.

Theorem 7.4 [Beuzart-Plessis 2021; Beuzart-Plessis et al. 2020; 2021; Xue 2019; Zhang 2014]. *Suppose π and π' are everywhere tempered. Then Conjecture 7.1 holds.*

We apply this when $f = \delta^{\kappa,\kappa'}(F)$ and $f' = f_x$ in (6.7). If f_x is classical we write it in the form

$$f_x = \frac{\overline{f_x^{\text{hol}}}}{\langle f_x^{\text{hol}}, f_x^{\text{hol}} \rangle},$$

where f_x^{hol} is an holomorphic modular form (of weight κ'_x) rational over $\overline{\mathbb{Q}}$, and the denominator is the Petersson inner product. When π and π' are in the discrete series at archimedean places it is known thanks to a long list of people, ending with Caraiani, that π and π' are necessarily tempered everywhere. Then we have the following theorem.

Theorem 7.5. *Suppose F is classical of weight κ, corresponding to the cuspidal automorphic representation π. Fix an antiordinary antiholomorphic automorphic representation τ of $U(V')$ of weight κ', where κ' is in* $\mathrm{Spec}(\Lambda'_\kappa)$. *Let $f = \delta^{\kappa,\kappa'}(F) \in \pi$ be a vector $\bigotimes'_v f_v$, which is unramified outside a finite set S of places containing p and ∞. Let $x \mapsto f'_x$ be a Hida family over* $\mathrm{Spec}(\Lambda'_\kappa)$. *We assume that, for every classical point x, with f'_x corresponding to the automorphic representation τ_x of $U(V')$, f'_x is a factorizable vector of the form $\bigotimes_{v \notin S} f'_{x,v} \otimes f'_S \in \bigotimes_v \tau_{x,v}$, with $f_{x,v}$ spherical for $v \notin S$, $f_{x,p}$ the antiordinary vector in $\tau_{x,p}$ as in* [Eischen et al. 2020, §8.3].

We admit Hypotheses 5.2 and 5.9 and assume $f'_S = \sum_j \bigotimes_{v \in S} f'_{v,j} \in \hat{I}$ is a primitive vector as in Section 5C. Fix generators $\widehat{m} \in \widehat{M}_\tau$ and $\omega \in \Omega_\tau$ and define $L(F, \tau) = L(F, f'_S, \tau, \widehat{m}, \omega)$ as in Theorem 6.6. Then for every classical point, the function $L(F, \tau, x)$ is an algebraic number that satisfies

$$|L(F, \tau, x)|^2 = 2^\beta \Delta_H |\delta^{\kappa,\kappa'}(F)|^2 \langle f'_x, f'_x \rangle^2 \cdot \mathcal{L}^S(\pi, \tau_x) \cdot Z_S(x),$$

where

$$Z_S(x) = Z_\infty(x) \cdot Z_p(x) \cdot \sum_j \prod_{v \in S \setminus \{p,\infty\}} I_v^*(f_v, f'_{v,j}). \tag{7.6}$$

Next, Z_∞ is the Euler factor attached to $\delta^{\kappa,\kappa'}(F_\infty) \in \pi_\infty$ and $f_{x,\infty} \in \tau_{x,\infty}$. Finally, $Z_p(x)$ is the Euler factor attached to the specified vectors f_p and $f'_{x,p}$.

Proof. By the cuspidality hypotheses on π and τ and Theorem 7.4 the formula in Conjecture 7.1 is valid. The theorem then follows by combining Theorem 6.6 with (7.2) and (7.3). □

Remark 7.7. If we don't insist that f'_S be primitive, we can arrange that Z_S be a product and that the local factors for $v \in S \setminus \{p, \infty\}$. are volume terms.

8. Open questions

8A. *Local factors at p.* The most intriguing open question is the determination of the local factor $I^*_p(f_p, f'_{x,p})$ in Conjecture 7.1. Specifically, the antiordinary vector $f'_{x,p}$ has been identified in [Eischen et al. 2020] as a collection of explicit vectors, of increasing level — bounded below by a constant determined by the level at p of the component τ_x of τ, which belongs to the (possibly ramified) principal series of $\mathrm{GL}(n-1, \mathbb{Q}_p)$. On the other hand, f_p must have the property corresponding to p-depletion in the classical context of elliptic modular forms. We can start by replacing F by $\Theta^1(F)$, where Θ^1 is the operator introduced in Remark 3.12. It will be proved in a subsequent paper that $\Theta^1(F)$ is a classical holomorphic modular form, but of level divisible by p. This probably suffices to determine the vector $f_p \in \pi_p$.

The representation $\pi_p \times \tau_{x,p}$ can be viewed as a representation of

$$\mathrm{GL}(n, \mathbb{Q}_p) \times \mathrm{GL}(n-1, \mathbb{Q}_p),$$

and the Ichino–Ikeda local factors $I^*_p(f_p, f'_{x,p})$ can be computed in terms of Jacquet–Piatetski-Shapiro–Shalika local factors for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$. As Beuzart-Plessis explained to me, this was first observed by Waldspurger; and independently and more explicitly in [Sakellaridis and Venkatesh 2017, §18.4]. Thus it suffices to compute the Jacquet–Piatetski-Shapiro local factors for our chosen vectors, both of which belong to principal series representations. Since these factors behave well with respect to parabolic induction, this may not be as difficult as it appears.

8B. *Local factors at ∞.* The proof of Conjecture 7.1, in the cases in which it is known, is based on a comparison of the local Euler factors (7.3) at all places with corresponding local factors in the Jacquet–Piatetski-Shapiro–Shalika integral representation of the Rankin–Selberg L-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$. In our situation, $\delta^{\kappa,\kappa'}(F_\infty)$ and $f_{x,\infty}$ are vectors in discrete series representations of $U(r,s)$ and $U(r,s-1)$, respectively; the comparison depends on a transfer of test functions on $U(r,s) \times U(r,s-1)$ to $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n-1,\mathbb{C})$. As in the previous section, the local Euler factors in the latter situation can be studied by means of parabolic induction, so the computation of local factors at ∞ mainly depends on understanding the local transfer.

8C. *Maximal dimension.* As b varies among positive integers, the p-adic modular forms $\Theta^b(F)$ can be paired not only with classical forms of weight $[\kappa]' + (b, 0, \dots)$ but with those of more general weights in the decomposition (2.4). The function $L(f, \tau)$ should thus extend to a function on $\min(r,s)$-dimensional Hida families. This will be considered in future work with Ellen Eischen.

8D. ***Extension to coherent cohomology in higher degree.*** Conjecture 7.1 applies to many central values of motivic L-functions that are not realized as pairings of holomorphic and antiholomorphic modular forms. In many cases they can nevertheless be realized as cup products in higher coherent cohomology; some examples are considered in [Grobner et al. 2018]. Pilloni's recent development of *higher Hida theory* shows that higher coherent cohomology classes can also be deformed in ordinary families. Work in progress by Loeffler, Pilloni, Skinner, and Zerbes aims to use this theory to study the p-adic behavior of special values of L-functions of certain automorphic representations, for groups of low dimension, that are known to be related to cup products in coherent cohomology. In future work with Eischen and Pilloni, we hope to make a systematic study of square-root p-adic L-functions for $U(n) \times U(n-1)$, whenever coherent cohomology can be applied.

Many of the period integrals in Conjecture 7.1 involve coherent cohomological representations but are not identified as cup products. There should be square root p-adic L-functions in these cases as well, but it is not clear how they can be defined.

8E. ***Slopes and the Panchishkin condition.*** General conjectures on p-adic L-functions predict that they can be constructed for quite general motives, but that they belong to Iwasawa algebras, or finite extensions thereof, only when the motive satisfies a *Panchishkin condition*, which is the analogue for the p-adic slope filtration of the condition on the Hodge filtration that guarantees that a special value is *critical* in Deligne's sense. No such conjectural restriction has been formulated, as far as I know, for the existence of p-adic analytic functions with values in finite extensions of Hida's ordinary Hecke algebra (itself a finite extension of a multivariable Iwasawa algebra) that interpolate square roots of normalized critical values of L-functions. The method of the present paper presupposes that the forms on $U(V')$ vary in an ordinary family but impose no restriction on the forms in the larger group $U(V)$. Is the construction here consistent with general conjectures?

Appendix

A.1. ***Review of Shimura data for unitary groups.*** The Shimura datum $(U(V), Y_V)$ is introduced in Section 2, following the discussion in [Gan et al. 2012, §27]. We review the definition given there, in the simpler case treated here where $\mathcal{K}$ is an imaginary quadratic field. Let V and (r, s) be as in the beginning of Section 2. Let $\mathrm{GU}(V)$ denote the algebraic group over $\mathbb{Q}$ of unitary similitudes of V; in other words, it is the subgroup of $R_{\mathcal{K}/\mathbb{Q}}(\mathrm{GL}(V))$ of automorphisms of V that preserve the hermitian form up to a scalar. Let $\mathbb{S}$ denote the Serre torus $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. Define a map

$$h_V = \mathbb{S} \to \mathrm{GU}(V)(\mathbb{R})$$

by the formula

$$h_V(z) = \begin{pmatrix} zI_r & 0 \\ 0 & \bar{z}I_s \end{pmatrix}, \tag{A-1}$$

where I_r and I_s are the identity matrices of size r and s, respectively. Denote by X_V the $\mathrm{GU}(V)(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to \mathrm{GU}(V)_{\mathbb{R}}$ containing h_V; thus $(\mathrm{GU}(V), X_V)$ is a Shimura datum

Let V_1 denote the vector space $\mathcal{K}$, endowed with a hermitian form of signature $(0, 1)$ at the chosen complex embedding ι, and define $h_{V_1} = h_\Sigma : \mathbb{S} \to \mathrm{GU}(V_1)(\mathbb{R})$ by analogy with (A-1); as above, we thus have a Shimura datum $(\mathrm{GU}(1), h_{V_1})$ Let $G'_V \subset \mathrm{GU}(1) \times \mathrm{GU}(V)$ be the subgroup consisting of (t, g) with $\nu(t) = \nu(g)$. Then the map

$$h'_V : \mathbb{S} \to [\mathrm{GU}(V_1) \times \mathrm{GU}(V)](\mathbb{R}); \quad h'_V(z) = (h_{V_1}(z), h_V(z))$$

has image contained in $G'_V(\mathbb{R})$; thus we may write $h'_V : \mathbb{S} \to G'_V(\mathbb{R})$. Let X'_V denote the $G'_V(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G'_V(\mathbb{R})$ containing h'_V. Then (G'_V, X'_V) is a Shimura datum, and the inclusion map $G'_V \hookrightarrow \mathrm{GU}(V) \times \mathrm{GU}(1)$ induces a morphism of Shimura data

$$(G'_V, X'_V) \to \big(\mathrm{GU}(V) \times \mathrm{GU}(1), X_V \times X_1\big).$$

There is a natural map

$$u : G'(V) \to U; \quad u(t, g) = t^{-1} g, \qquad \text{for all } (t, g) \in G'(V) \subset \mathrm{GU}(1) \times \mathrm{GU}(V). \quad \text{(A-2)}$$

The map taking $h \in X'_V$ to $u \circ h : \mathbb{S} \to U(V)(\mathbb{R})$ then defines a map of Shimura data

$$(G'_V, X'_V) \to (U(V), Y_V),$$

where Y_V is the $U(V)(\mathbb{R})$-conjugacy class defined by this map. Unlike the other Shimura data introduced in this section, the pair $(U(V), Y_V)$ is not of PEL type. It is of abelian type, however, and arithmetic of the Shimura variety $S(U(V), Y_V)$ has been studied, nevertheless, in [Kisin et al. $\geq$ 2021]. In this paper we write $S(V)$ instead of $S(U(V), Y_V)$, and if $K \subset U(V)(\boldsymbol{A}_f)$ is an open compact subgroup,the corresponding finite level Shimura variety is denoted ${}_K S(V)$.

A.2. *Review of antiholomorphic and antiordinary forms.* The complex structure on the hermitian symmetric space Y_V determines the complex structure on the Shimura variety $\mathrm{Sh}(V)$, and one thus has a well-defined notion of holomorphic sections of the canonical extensions of automorphic vector bundles on toroidal compactifications of $\mathrm{Sh}(V)$. Such a holomorphic section ϕ gives rise by a *canonical trivialization* to an automorphic form F_ϕ on $G(V)(\mathbb{Q})\backslash G(V)(\boldsymbol{A})$, and thus to an automorphic representation $\pi = \pi(\phi)$ of $G(V)(\boldsymbol{A})$. Such a π is called *holomorphic*, and F_ϕ is a highest K_∞-type vector for an appropriately chosen maximal compact subgroup $K_\infty \subset G(V)(\mathbb{R})$. For all this, see [Harris 1997, §§2.4–2.5, especially

(2.4.4 bis)]. If π is holomorphic then its archimedean component π_∞ is isomorphic to a holomorphic discrete series representation $\mathbb{D}_\kappa$, as defined in Section 2A.

Then an antiholomorphic automorphic representation is just the complex conjugate $\bar{\pi}$ of a holomorphic automorphic representation; equivalently, π' is antiholomorphic if π'_∞ is isomorphic to the contragredient of a holomorphic discrete series representation $\mathbb{D}_\kappa$. As functions on $G(V)(\mathbb{Q})\backslash G(V)(\boldsymbol{A})$, antiholomorphic automorphic forms are just the complex conjugates of holomorphic automorphic forms.

We use the term *antiordinary form* as in [Eischen et al. 2020] to denote an element of the dual module $\hat{S}^{\mathrm{ord}}_\kappa(K;\mathcal{O})$ defined in (5.3), for appropriate level K and weight κ. The property of being antiordinary is determined by the valuations of a family of local Hecke operators at primes dividing p that are *dual* to the U-operators used to define Hida's ordinary subspace. The details can be found in [Eischen et al. 2020, §6.6.6 and §8.3.5]. For our purposes here, the main properties of antiordinary forms are the following:

(i) $\hat{S}^{\mathrm{ord}}_\kappa(K;\mathcal{O})_\tau$ is identified with an $\mathcal{O}$-lattice in

$$\bigoplus_{\tau'\in\mathcal{S}(K,\kappa,\tau)} \tau^{\prime,\flat,\text{a-ord}}_{p,r}\otimes\tau^{\prime,\flat,K_S}_S.$$

(ii) The pairing of $S_\kappa(K,\mathcal{O})$ with $\hat{S}^{\mathrm{ord}}(K,\mathcal{O})$ factors through the ordinary projection

$$S_\kappa(K,\mathcal{O})\to S_\kappa(K,\mathcal{O})^{\mathrm{ord}}.$$

Here $\tau^{\prime,\flat,\text{a-ord}}_{p,r}$ is an explicit one-dimensional antiordinary eigenspace (at level $K_{p,r}$, see [Eischen et al. 2020] for details) of the local component $\tau^\flat_p$ of $\tau^\flat$; the antiordinary eigenspace is defined explicitly in the model of $\tau^\flat_p$ as a principal series representation. The set $\mathcal{S}(K,\kappa,\tau)$ is roughly the set of automorphic representations τ' that are congruent modulo p to τ. Property (i) follows from [Eischen et al. 2020, Lemma 6.6.12(ii) and the discussion preceding Lemma 6.6.11], and (ii) follows from [Eischen et al. 2020, Lemma 8.3.4(iii)].

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