Shimura Varieties for Unitary Groups and the Doubling Method

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Abstract The theory of Galois representations attached to automorphic representations of $GL(n)$ is largely based on the study of the cohomology of Shimura varieties of PEL type attached to unitary similitude groups. The need to keep track of the similitude factor complicates notation while making no difference to the final result. It is more natural to work with Shimura varieties attached to the unitary groups themselves, which do not introduce these unnecessary complications; however, these are of abelian type, not of PEL type, and the Galois representations on their cohomology differ slightly from those obtained from the more familiar Shimura varieties.

Results on the critical values of the L-functions of these Galois representations have been established by studying the PEL type Shimura varieties. It is not immediately obvious that the automorphic periods for these varieties are the same as for those attached to unitary groups, which appear more naturally in applications of relative trace formulas, such as the refined Gan-Gross-Prasad conjecture (conjecture of Ichino-Ikeda and N. Harris). The present article reconsiders these critical values, using the Shimura varieties attached to unitary groups, and obtains results that can be used more simply in applications.

1 Introduction and Overview

The critical values of L-functions of motives over number fields are the subject of many conjectures and a small number of theorems. In [D79], Deligne formulated a conjecture relating these critical values to periods relating rational structures on de Rham and topological (Betti) cohomology. For the motives conjecturally attached to cohomological automorphic representations of $GL(n)$ over CM fields—imaginary quadratic extensions of totally real fields—the L-functions can be realized explicitly

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by the doubling method of Garrett and Piatetski-Shapiro-Rallis [Ga84, PSR], provided the automorphic representations descend to automorphic representations of unitary groups. In this setting, versions of this conjecture have been established in a series of papers, including [H97, Gu16, GL].[1]

Applications of the doubling method to special values of L-functions are based on interpreting special values of the integral representation as cup products in coherent cohomology of Shimura varieties. In the work cited above, the Shimura varieties are of PEL type: they parametrize families of polarized abelian varieties with endomorphisms by (orders in) a CM field F and level structure, and the various structures satisfy natural compatibilities. Combining standard techniques of complex geometry with the theory of relative Lie algebra cohomology, the coherent cohomology classes on these varieties can be identified with automorphic forms on the reductive algebraic groups attached to the Shimura varieties—in the event, they are essentially the groups of similitudes of a hermitian vector space over F, with similitude factor assumed to be rational. The doubling method, however, was originally devised to apply to the *symmetry groups* of bilinear or hermitian forms—in this case, to the unitary groups themselves, rather than to groups of unitary similitudes. While the methods of [Ga84, PSR] can easily be adapted to the similitude groups, this introduces (rather mild) technical complications as well as additional notation that has no clear connection to the original question. The unwelcome similitude factors pop up elsewhere in the theory, notably in the construction of Galois representations attached to automorphic forms on $GL(n)$ (e.g. in [HT01, M10, Sh11, CHLN, HLTT, Sch, B]) as a kind of fee charged for the right to use the theory of Shimura varieties in order to draw arithmetic conclusions.

It has been known for some time, however—though this author only learned about it a few years ago—that Shimura varieties can be attached to the unitary groups themselves, with no need to introduce the parasitic similitude factor. The Shimura data are described in §27 of [GGP] and have since been used in work on generalizations of the Gross-Zagier formula. These are of abelian type but not of PEL type, and the theory of their L-functions has only been established recently [KSZ], in the setting of the Langlands-Kottwitz method. The purpose of the present paper is primarily to work out the analogue of the results of [H97, Gu16, GL] using these Shimura varieties; secondarily, as a form of penance for the author's failure to do so earlier.

There is nothing really new in this paper, but it is hoped that it will serve as a reference for future work on special values, and specifically on p-adic L-functions. The results of [EHLS], for example, are proved using the PEL Shimura varieties attached to unitary similitude groups, but they can just as well be proved in the

[1]Using very similar methods, Shimura obtained versions of these results in [Sh97] and in subsequent papers; his results are limited to scalar valued automorphic forms, but are more precise in a number of respects. There has also been important work by various authors relating the critical values of these L-functions to automorphic periods that have no direct motivic interpretation; the present paper has nothing to say about this.

current framework. Such applications will require at least that lip service be paid to the moduli theory underlying their canonical models.

The paper follows the pattern established in [H97], but no attempt has been made to relate the results obtained here to the motivic periods that arise in Deligne's conjecture. Section 2 introduces the Shimura varieties that are used to relate special values of zeta integrals to motivic periods. Section 3 describes the parameters for coherent cohomology—in fact, holomorphic automorphic forms—of these Shimura varieties, with coefficients in automorphic vector bundles. Section 4 defines the Eisenstein series used in the doubling method, and interprets appropriately normalized holomorphic (and nearly holomorphic) Eisenstein series as coherent cohomology classes. Section 5 reinterprets the differential operators studied in [H97, Gu16] (and elsewhere) in terms of the Shimura data without the similitude factor. Section 6 recalls the theory of the doubling integral of [Ga84, PSR]. As in the earlier papers, the main result on critical values is proved in Sect. 7 by composing a paragraph that mentions all the objects introduced in the earlier sections.

2 Shimura Varieties for Unitary Groups

2.1 Notation and Conventions

We let F be any CM-field of degree $2d = \dim_{\mathbb{Q}} F$ and set of real places $S_\infty = S(F)_\infty$. Each place $v \in S_\infty$ refers to an ordered pair of conjugate complex embeddings $(\iota_v, \bar{\iota}_v)$ of F, where we will drop the subscript "v" if it is clear from the context. This fixes a choice of a CM-type $\Sigma = \{\iota_\sigma : \sigma \in S_\infty\}$. When there is no danger of ambiguity, we write $\sigma \in \Sigma$ instead of ι_σ. The maximal totally real subfield of F is denoted F^+, and we let ε_{F/F^+} denote the quadratic character of the adeles of F^+ attached to this extension. The set of real places of F^+ is identified with S_∞, identifying a place σ with its first component embedding $\iota_\sigma \in \Sigma$ and we let $\mathrm{Gal}(F/F^+) = \{1, c\}$. The ring of adeles over F (resp. over F) is denoted $\mathbf{A}_F$ (resp. $\mathbf{A}_{F^+}$) and $\mathbf{A}_{\mathbb{Q}}$ for $\mathbb{Q}$. We write $\mathcal{O}_F$ (resp. $\mathcal{O}_{F^+}$) for the respective rings of integers and drop the subscript, if no confusion is possible.

Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional non-degenerate c-hermitian space over F, $n \geq 2$. If V is understood, we denote the corresponding unitary group by $G = G_V := U(V)$ over F^+.

We define the *rational similitude group* $\tilde{G} := \tilde{G} := GU$ over $\mathbb{Q}$ as follows: If $GU_{F^+}(V_n)$ is the subgroup of $GL(V)$ that preserves the hermitian form up to a scalar multiple $\nu(g) \in \mathbb{G}_{m,F^+}$,

$$GU_{F^+}(V) := \{g \in GL(V) | \langle gv, gw\rangle = \nu(g) \cdot \langle v, w\rangle\}$$

we let $GU(V)$ denote the fiber product

$$GU(V) := GU_{F^+}(V) \times_{R_{F^+/\mathbb{Q}}\mathbb{G}_{m,F^+}} \mathbb{G}_{m,\mathbb{Q}}$$

where the map $GU_{F^+}(V) \to R_{F^+/\mathbb{Q}}\mathbb{G}_{m,F^+}$ is the similitude map ν and $\mathbb{G}_{m,\mathbb{Q}} \to R_{F^+/\mathbb{Q}}\mathbb{G}_{m,F^+}$ is the natural inclusion.

When working with several hermitian spaces we sometimes index V by its dimension n; thus we write $V = V_n$. If V_k is some non-degenerate subspace of V_n, we view $U(V_k)$ (resp. $GU(V_k)$) as a natural F^+-subgroup of $U(V_n)$ (resp. $\mathbb{Q}$-subgroup of $GU(V_n)$). If $n = 1$ we write $U(1) = U(V_1)$, $GU(1) = GU(V_1)$. As an algebraic group $U(1)$ is isomorphic to the kernel of the norm map N_{F/F^+} : $R_{F/F^+}(\mathbb{G}_m)_F \to (\mathbb{G}_m)_{F^+}$, and is thus independent of V_1.

Although $U(V)$ is viewed as an F^+-group, we will occasionally abuse notation and identify $U(V)$ with $R_{F^+/\mathbb{Q}}U(V)$, and do the same with related groups over F^+. Thus we can write $U(V)(\mathbb{R})$ for $R_{F^+/\mathbb{Q}}U(V)(\mathbb{R})$, for example.

The theory of *rationality* for automorphic vector bundles, understood as in [H86] and [H90], will be used without comment. Conventions for holomorphic and antiholomorphic modules, or highest K-type modules, are as in [EHLS, §4.4.1]. Thus holomorphic (resp. anti-holomorphic) modules are the Archimedean local components of automorphic representations corresponding to sections of automorphic vector bundles (resp. to coherent cohomology of automorphic vector bundles in the top degree.) Some of the author's earlier papers use the opposite convention. The dual of an automorphic vector bundle $\mathcal{E}$ is denoted $\mathcal{E}^\vee$, and the notation $^\vee$ is used more generally for duality.

2.2 Shimura Data

For each $\sigma \in \Sigma$ we let (r_σ, s_σ) denote the signature of the hermitian form induced by $\langle \cdot, \cdot \rangle$ on the complex vector space

$$V_\sigma := V \otimes_{F,\sigma} \mathbb{C}.$$

Thus $r_\sigma + s_\sigma = n$ for all σ. Let $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$, so that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, canonically. Define a map

$$h_V = (h_{V,\sigma}, \sigma \in \Sigma) : \mathbb{S} \to GU(V)(\mathbb{R}) \subset \prod_{\sigma \in \Sigma} GU(V_\sigma)$$

componentwise, so that

$$h_{V,\sigma}(z) = \begin{pmatrix} zI_{r_\sigma} & 0 \\ 0 & \bar{z}I_{s_\sigma} \end{pmatrix}; \tag{2.1}$$

clearly this map to $\prod_{\sigma \in \Sigma} GU(V_\sigma)$ has image in the subgroup $GU(V)(\mathbb{R})$. We write h_V although the definition clearly depends on the CM type Σ as well.

We assume $(V_1)_\sigma$ has signature $(0,1)$ for all $\sigma \in \Sigma$, and define $h_{V_1} = h_\Sigma : \mathbb{S} \to GU(V_1)(\mathbb{R})$ by analogy with (2.1). Let $G''_V \subset GU(1) \times GU(V)$ be the subgroup of (t,g) with $\nu(t) = \nu(g)$. Then the map

$$h''_V : \mathbb{S} \to [GU(V_1) \times GU(V)](\mathbb{R}); h''_V(z) = (h_{V_1}(z), h_V(z))$$

has image contained in $G''_V(\mathbb{R})$; we use the notation h''_V to designate the map $\mathbb{S} \to G''_V(\mathbb{R})$.

Let X''_V (resp. X_V, resp X_1) denote the $G''_V(\mathbb{R})$-conjugacy class (resp. $GU(V)(\mathbb{R})$-conjugacy class, resp. $GU(1)(\mathbb{R})$-conjugacy class) of homomorphisms $h : \mathbb{S} \to G''_V(\mathbb{R})$ containing h''_V (resp. h_V, resp. h_{V_1}). Then (G''_V, X''_V), $(GU(V), X_V)$, and $(GU(1), X_1)$ are all Shimura data, and the inclusion map $G''_V \hookrightarrow GU(V) \times GU(1)$ induces a morphism of Shimura data

$$(G''_V, X''_V) \to (GU(V) \times GU(1), X_V \times X_1).$$

There is a natural map

$$u : G''(V) \to U; u(t,g) = t^{-1}g, \forall (t,g) \in G''(V) \subset GU(1) \times GU(V) \qquad (2.2)$$

The map taking $h \in X''_V$ to $u \circ h : \mathbb{S} \to U(V)(\mathbb{R})$ then defines a map of Shimura data

$$(G''_V, X''_V) \to (U(V), Y_V)$$

where Y_V is the $U(V)(\mathbb{R})$-conjugacy class defined by this map. Explicitly, the base point $y_V = u \circ h''_V \in Y_V$ is given by

$$y_{V,\sigma}(z) = \begin{pmatrix} (z/\bar{z}) I_{r_\sigma} & 0 \\ 0 & I_{s_\sigma} \end{pmatrix} \qquad (2.3)$$

When $\dim V = 1$ we write Y_1 or $Y_{\Sigma(V)}$ for Y_V, where $\Sigma(V)$ is the set of σ such that V_σ has signature $(0,1)$.

The following is then obvious; we record it here in order to define parameters for automorphic vector bundles in the next section.

Lemma 2.3. *Let $y \in Y_V$. The stabilizer $K_y \in U(V)(\mathbb{R})$ is isomorphic to $\prod_{\sigma \in \Sigma} U(r_\sigma) \times U(s_\sigma)$.*

Here, for any m, we denote by $U(m)$ the compact real form of $GL(m)$. Later we will fix a base point $y \in Y_V$ and let $U_\sigma = K_y \cap U(V \otimes_{F,\sigma} \mathbb{C}) \xrightarrow{\sim} U(r_\sigma) \times U(s_\sigma)$ with respect to this base point.

The Shimura varieties attached to $(GU(V), X_V)$ and $(GU(1), X_1)$ belong to one of the families of PEL type Shimura varieties originally studied (in the form now called *connected Shimura varieties*) by Shimura [Sh64]. The Shimura variety attached to $(U(V), Y_V)$, which we denote $Sh(V, \Sigma)$, parametrizes Hodge structures of weight 0—the homomorphisms $y \in Y_V$ are trivial on the subgroup $\mathbb{R}^\times \subset \mathbb{C}^\times$—and are thus of abelian type but not of Hodge type.

The reflex field $E(V, \Sigma) := E(U(V), Y_V)$ is the subfield of the Galois closure of F over $\mathbb{Q}$ determined as the stabilizer of the cocharacter κ_V with σ-component $\kappa_{V,\sigma}(z) = \begin{pmatrix} zI_{r_\sigma} & 0 \\ 0 & I_{s_\sigma} \end{pmatrix}$. In particular, suppose there is $\sigma_0 \in \Sigma$ such that $r_{\sigma_0} > 0$ but $r_\sigma = 0$ for $\sigma \in \Sigma \setminus \sigma_0$. Then $E(V, \Sigma)$ is the subfield $\sigma_0(F) \subset \mathbb{C}$.

Remark 2.4. Starting in Sect. 3.4.1, the period invariants for the Shimura varieties attached to $U(V)$ will be distinguished from those for $GU(V)$ by the subscript U.

2.5 Measures and Discriminants

The group G is viewed as an algebraic group over F^+, and by restriction of scalars as an algebraic group over $\mathbb{Q}$. We choose Haar measures on the local and adelic groups $G(F_v^+)$ and $G(\mathbf{A})$ as in the introduction to [H97]. Thus if v is a finite place of F^+, we choose a local Haar measure dg_v that is rational in the sense that any open compact subgroup of $G(F_v^+)$ has rational measure; if G is unramified at v we also assume that a hyperspecial maximal compact subgroup has measure 1.

We choose a maximal compact subgroup

$$K_\infty = \prod K_\sigma \subset G_\infty = \prod_{\sigma \in S_\infty} G(F_\sigma^+)$$

that stabilizes a CM point x in the locally symmetric space X_V introduced in Sect. 2.2 above. We can thus write

$$X_V = \prod_{\sigma \in S_\infty} G(F_\sigma^+)/K_\sigma := \prod_{\sigma \in S_\infty} X_\sigma.$$

If σ is a real place of F^+, then we write $dg_\sigma = dk_\sigma \cdot dx_\sigma$ with respect to this factorization. Although it is not necessary, we may assume that x is the image of a map of Shimura data

$$\prod_i (U(1), Y_{\Sigma_i})^n \to (G_V, Y_V),$$

where we have written $V = \oplus V_i$, $\dim V_i = 1$, $\Sigma_i = \Sigma_{V_i}$ in the above notation. We let $E_i = E(U(1), Y_{\Sigma_i})$, so that x is defined over the compositum $E_x = E(\prod_i (U(1), Y_{\Sigma_i}))$ of the E_i. Then at each $\sigma \in S_\infty$ (which we identify with the CM type Σ) the Harish-Chandra decomposition

$$\mathfrak{g}_\sigma := Lie(G(F^+_\sigma)) = \mathfrak{p}^+_\sigma \oplus \mathfrak{p}^-_\sigma \oplus Lie(K_\sigma)$$

is rational over the field $F_{\sigma,x} = E_x \cdot \sigma(F) \subset \mathbb{C}$; $F_{\sigma,x}$ is CM and quadratic over its maximal totally real subfield $F^+_{\sigma,x}$. We choose an element $\daleth \in F$ with $Tr_{F/F^+}(\daleth) = 0$, let $\sigma : F \to \mathbb{C}$ denote the element of Σ associated to σ, and define a top differential

$$\omega_\sigma = \sigma(\daleth)^{-\dim X_\sigma} \bigwedge_{i=1}^{\dim X_\sigma} dz_{\sigma,i} \wedge d\bar{z}_{\sigma,i} \in \wedge^{\dim X_\sigma} \mathfrak{p}^+_\sigma \otimes \wedge^{\dim X_\sigma} \mathfrak{p}^-_\sigma, \tag{2.4}$$

where $dz_{\sigma,i}$ is an $F_{\sigma,x}$-basis of $\mathfrak{p}^+_\sigma$ and $d\bar{z}_{\sigma,i}$ is the complex conjugate basis of $\mathfrak{p}^-_\sigma$. Similarly, for dk_σ we take a Haar measure defined by an $F^+_{\sigma,x}$-basis of $\wedge^{\dim K_\sigma} Lie(K_\sigma)$. Finally, we let

$$dx_\sigma = (2\pi)^{-\dim X_\sigma} \omega_\sigma. \tag{2.5}$$

The factorization of the adelic Tamagawa measure dg as a product of local measures introduces an additional normalizing constant that was not present in [H97]. We can write $dg = dg_\infty \cdot dg_f$, with $dg_f = \prod_\sigma dg_\sigma$ (product over finite places). However, dg_∞ is given by a $\mathbb{Q}$-basis of $\wedge^{\dim R_{F^+/\mathbb{Q}} G} Lie(G)$, and is thus a $\mathbb{Q}$-rational differential form, whereas $\prod_{\sigma \in S_\infty} dg_\sigma$ is not $\mathbb{Q}$-rational. Instead, we have

$$dg_V := (2\pi)^{-\dim X_V} \cdot dg_\infty = \sqrt{D_{F^+}}^{-n^2} \prod_\sigma dg_\sigma, \tag{2.6}$$

up to an E_x-rational factor, where now the product is taken over all places of F^+. The E_x-rational factor is inevitable because the reflex field $E(G_V, Y_V)$, which is contained in E_x, is precisely the stabilizer of the set of signatures together with the CM type Σ, whereas we have constructed the ω_σ as forms over E_x. However, letting $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the set of Shimura varieties conjugate to $Sh(V, \Sigma)$—in other words, on the signatures and CM types—the collection (dg_V) can be chosen consistently with the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action.

3 Coherent Cohomology

3.1 Automorphic Vector Bundles

This paper is primarily devoted to the critical values of standard L-functions of automorphic representations of $U(V)$ attached to holomorphic modular forms. These modular forms are viewed as sections of automorphic vector bundles; as in

[Gu16] (see also [H97]), the relevant automorphic vector bundles on $Sh(V, \Sigma)$ are parametrized by irreducible representations of the stabilizer of a chosen base point in Y_V. By Lemma 2.3, we thus parametrize automorphic vector bundles by highest weights of irreducible representations of $\prod_{\sigma\in\Sigma} U(r_\sigma) \times U(s_\sigma)$. In §3.3 of [Gu16] the parametrization of automorphic vector bundles on the Shimura variety attached to $(GU(V), X_V)$ is given by

$$((a_{\sigma,1}, \dots, a_{\sigma,n})_{\sigma\in\Sigma}; a_0) \text{ such that } \forall\sigma\ a_{\sigma,1} \geq \cdots \geq a_{\sigma,r_\sigma};\quad a_{\sigma,r_\sigma+1} \geq \cdots \geq a_{\sigma,n}. \tag{3.1}$$

The parameters for automorphic vector bundles on $Sh(V, \Sigma)$ are the same as those for (3.1) except that the parameter a_0 is absent:

$$(\kappa_{\sigma,1}, \dots, \kappa_{\sigma,n})_{\sigma\in\Sigma} \text{ such that } \forall\sigma\ \kappa_{\sigma,1} \geq \cdots \geq \kappa_{\sigma,r_\sigma};\quad \kappa_{\sigma,r_\sigma+1} \geq \cdots \geq \kappa_{\sigma,n}. \tag{3.2}$$

Let κ denote a parameter in (3.2), and let $\mathcal{E}_\kappa$ denote the corresponding automorphic vector bundle on $Sh(V, \Sigma)$. Over $\mathbb{C}$, we have

$$\mathcal{E}_\kappa = \varprojlim_{K_f}[U(V)(F^+)\backslash U(V)(\mathbf{A}_{F^+}) \times W_\kappa/(K_{y_V} \times K_f)]$$

where the terms on the right hand side are vector bundles over the finite-level Shimura variety

$${}_{K_f}Sh(V, \Sigma)(\mathbb{C}) = U(V)(F^+)\backslash U(V)(\mathbf{A}_{F^+})/(K_{y_V} \times K_f).$$

Here $K_f \subset U(V)(\mathbf{A}_f)$ is a compact open subgroup (that is small enough to guarantee that $\mathcal{E}_\kappa$ is in fact a vector bundle) and $W_\kappa = \otimes_{\sigma\in\Sigma} W_{\kappa,\sigma}$ is the representation of $\prod_\sigma U(r_\sigma) \times U(s_\sigma)$ with highest weight $(\kappa_{\sigma,1} \geq \cdots \geq \kappa_{\sigma,r_\sigma}; \kappa_{\sigma,r_\sigma+1} \geq \cdots \geq \kappa_{\sigma,n})$, and the group K_{y_V} acts diagonally on $U(V)(\mathbb{R}) \times W_\kappa$.

3.2 Coherent Cohomology and Period Invariants

We fix a level subgroup K_f as above, and let ${}_{K_f}Sh(V, \Sigma) \hookrightarrow {}_{K_f}\ Sh(V, \Sigma)^{tor}$ denote a toroidal compactification. We may and do assume the compactification is smooth and projective, and the boundary divisor D has normal crossings. The automorphic vector bundle $\mathcal{E}_\kappa$ has two natural extensions to vector bundles over ${}_{K_f}Sh(V, \Sigma)^{tor}$: the canonical extension $\mathcal{E}_\kappa^{can}$ and the subcanonical extension $\mathcal{E}_\kappa^{sub}$, defined as in [H90]. We write

$$H_!^*(Sh(V,\sigma), \mathcal{E}_\kappa) := \\ \varinjlim_{K_f} Im[H^*({}_{K_f}Sh(V, \Sigma)^{tor}, \mathcal{E}_\kappa^{sub}) \ \to\ H^*({}_{K_f}Sh(V, \Sigma)^{tor}, \mathcal{E}_\kappa^{can})], \tag{3.3}$$

where for each K_f the toroidal compactification is chosen to be adapted to the level. The space $H^*_!(Sh(V,\Sigma),\mathcal{E}_\kappa)$ defines an admissible and semisimple representation of $U(V)(\mathbf{A}_f)$.

Let π_f be an irreducible representation of $U(V)(\mathbf{A}_f)$ that can be completed to a cuspidal automorphic representation π of $U(V)(\mathbf{A})$ whose base change to $GL(n)_F$ is cuspidal and cohomological. It thus follows in particular (from a long series of partial results, culminating in Theorem 1.2 of [Ca]) that π is everywhere tempered. Fix a degree q and let $H^q_!(Sh(V,\Sigma),\mathcal{E}_\kappa)[\pi]$ denote the π_f-isotypic component of $H^q_!(Sh(V,\Sigma),\mathcal{E}_\kappa)$. If this π_f-isotypic component is non-trivial, it follows from [H90] (in particular Theorem 4.6.2) and [KMSW] that

Theorem 3.4.

(a) The representation π_f occurs with multiplicity one in $H^q_!(Sh(V,\Sigma),\mathcal{E}_\kappa)$, and π is uniquely determined by π_f and the bundle $\mathcal{E}_\kappa$.
(b) In particular, $H^*_!(Sh(V,\Sigma),\mathcal{E}_\kappa)[\pi]$ determines a rational structure on π_f over some number field $E(\pi)$.

Remark 3.3. We always choose $E(\pi)$ to contain an extension $E(\kappa)$ of the reflex field $E(V,\Sigma)$ over which the bundle $\mathcal{E}_\kappa$ has a rational model. As in §1.1 of the Erratum to [H13], $E(\kappa)$ in general strictly contains the fixed field E_κ of the stabilizer in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of the isomorphism class of $\mathcal{E}_\kappa$, and moreover is not uniquely determined; it is chosen to eliminate a Brauer obstruction to realizing the bundle over E_κ.

The main Theorem 7.1 relating periods to critical values of L-functions takes the form

$$\text{Critical value of } L(s,\pi,\alpha,St) \sim_{E(\pi,\alpha)} \text{Normalized period invariant.} \tag{3.4}$$

In Deligne's conjecture the place of $E(\pi,\alpha)$ is taken by the field of coefficients $E(M(\pi,\alpha))$ of the motive whose L-function is $L(s,\pi,\alpha,St)$. The previous paragraph indicates that $E(\pi,\alpha)$ is in general a non-trivial extension of the hypothetical $E(M(\pi,\alpha))$. However, the relation (3.4) is equivariant with respect to $Gal(E(\kappa)/E_\kappa)$ (this can be chosen to be an abelian extension) so the extension of the coefficient field is harmless.

Remark 3.4. The results of [KMSW] are conditional on results that have been claimed, not only by the authors, but not published. Careful readers may therefore prefer to view the results that make use of Theorem 3.4 as conditional. In [H97] and [Gu16] the period invariants introduced in the following paragraph are denoted $Q(\pi,\beta)$, where β is a marker that accounts for possible multiplicity greater than one. It can then be proved, using the main identity relating critical values of L-functions to periods, that the period is independent of β, up to appropriate algebraic factors, assuming $L(s,\pi,\alpha)$ has non-vanishing critical values. Thus the multiplicity one hypothesis is only used for convenience.

Returning to Theorem 3.4, the L_2-inner product on the space of cusp forms on $U(V)(\mathbf{A})$ restricts to a non-degenerate hermitian form on π_f. As in [H13], §3.4.2, one obtains a period invariant, denoted $Q(\pi)$ in *loc. cit.*, in $(E(\pi)\otimes_{\mathbb{Q}}\mathbb{C})^{\times}/E(\pi)^{\times}$. Each embedding $\tau : E(\pi)\to\mathbb{C}$ then determines a period invariant $Q(\pi,\tau)\in\mathbb{C}^{\times}/\tau(E(\pi))^{\times}$. The period invariant is characterized by Proposition 3.19 of [H13]: suppose $v_1, v_2\in\pi$, identified with classes in $H^q_!(Sh(V,\Sigma),\mathcal{E}_\kappa)[\pi]$ by the trivialization as discussed in §3.4 of [H13]. Define

$$\langle v_1, v_2\rangle = (\langle v_1, v_2\rangle_\tau)_{\tau:E(\pi)\to\mathbb{C}} = \int_{[U(V)]} v_1(g)\bar{v}_2(g)dg \in E(\pi)\otimes_{\mathbb{Q}}\mathbb{C} \tag{3.5}$$

as a vector in $E(\pi)\otimes_{\mathbb{Q}}\mathbb{C} = \prod_{\tau:E(\pi)\to\mathbb{C}}\mathbb{C}$, as in [H13, §3.4.2]. Then v_1 corresponds to an $E(\pi)$-rational class if and only if, for all $E(\pi)$-rational v_2, we have

$$Q(\pi,\tau)^{-1}\cdot\langle v_1, v_2\rangle_\tau \in \tau(E(\pi)) \tag{3.6}$$

Here rationality of coherent cohomology is understood as in Theorem 3.4 (b) and [H90, H13].

For the purposes of applications in [GHL], we assume $E(\pi)$ is given as a subfield $\tau(E(\pi))$ of $\mathbb{C}$ and write $P(\pi, V, \Sigma)$ for $Q(\pi,\tau)$. In this paper we will only consider π contributing to $H^0_!(Sh(V,\Sigma),\mathcal{E}_\kappa)$ or $H^{\dim Y_V}_!(Sh(V,\Sigma),\mathcal{E}_\kappa)$; in both cases, the cohomology space is entirely represented by cusp forms [H90, Proposition 5.4.2].

If κ is as above, we define κ^D as in [EHLS, §6.1.3]. Then complex conjugation of functions on the adele group defines an antilinear isomorphism

$$c_B : H^0_!(Sh(V,\Sigma),\mathcal{E}_\kappa) \xrightarrow{\sim} H^{\dim Y_V}_!(Sh(V,\Sigma),\mathcal{E}_{\kappa^D})$$

[EHLS, (6.2.3)]. Moreover, the isomorphism is compatible with the Serre duality pairing

$$\begin{aligned}[\bullet,\bullet]_{Ser} : H^0_!(Sh(V,\Sigma),\mathcal{E}_\kappa)\otimes H^{\dim Y_V}_!(Sh(V,\Sigma),\mathcal{E}_{\kappa^D}) \\ \to H^{\dim Y_V}(Sh(V,\Sigma),\Omega^{\dim Y_V}_{Sh(V,\Sigma)})\end{aligned} \tag{3.7}$$

[EHLS, (6.3.1)]. Note that the line bundle $L(\kappa)$ in *loc. cit.* is trivial for us because it only depends on the similitude factor); in particular we see that we can simply write

$$\mathcal{E}_{\kappa^D} \xrightarrow{\sim} \Omega^{\dim Y_V}_{Sh(V,\Sigma)}\otimes\mathcal{E}^\vee_\kappa.$$

Setting $q = 0$ and bearing in mind that $E(\pi)$ is a CM field, we can take the complex conjugation of both sides of (3.5), we see that, if v_1 corresponds to an $E(\pi)$ rational class in $H^0_!(Sh(V,\Sigma),\mathcal{E}_\kappa)[\pi]$, then, with c denoting complex conjugation,

$$(Q(\pi, c\tau)^{-1}\overline{v}_1|_{\tau:E(\pi)\to\mathbb{C}}) \in H^{\dim Y_V}(Sh(V,\Sigma), \mathcal{E}_\kappa^D)[\pi] \otimes_{E(\pi)} \mathbb{C} \tag{3.8}$$

comes from an $E(\pi)$-rational class.

3.4.1 Change of Polarization

The doubling method naturally gives rise to a different period invariant, which comes from the pairing of a form on $Sh(V,\Sigma)$ with a form on $Sh(-V,\Sigma)$. Namely, with the notation of [EHLS], §6.2, (but with ω replaced by $\mathcal{E}$ as notation for automorphic vector bundles), there is an $E(V,\Sigma) = E(-V,\Sigma)$ linear, $U(V)(\mathbf{A}_f) \xrightarrow{\sim} U(-V)(\mathbf{A}_f)$-equivariant, isomorphism

$$F_\infty : H_!^{\dim Y_V}(Sh(-V,\Sigma), \mathcal{E}_{\kappa^\flat}) \xrightarrow{\sim} H_!^0(Sh(V,\Sigma), \mathcal{E}_{\kappa^D}) \tag{3.9}$$

This isomorphism expresses the $U(V)(\mathbb{R})$-equivariant identity of Y_{-V} with Y_V, endowed with the complex conjugate structure. Now suppose $f \in H_!^{\dim Y_V}(Sh(V,\Sigma), \mathcal{E}_\kappa)[\pi]$ and $f' \in H_!^{\dim Y_V}(Sh(-V,\Sigma), \mathcal{E}_{\kappa^\flat})[\pi^\vee]$ are $E(\pi)$-rational cohomology classes in π and $\pi^\vee$, respectively, for some extension $L \supset E(\pi)$. (In particular, π_∞ is an antiholomorphic discrete series representation, as in [EHLS, §4.5].) Define

$$\begin{aligned} B(\pi)_{f,f'} = [f, F_\infty(f')]_{Ser} &\in H^{\dim Y_V}(Sh(V,\Sigma), \Omega^{\dim Y_V}_{Sh(V,\Sigma)}) \\ &\xrightarrow{\sim} H^0(Sh(V,\Sigma), \mathcal{O}_{Sh(V,\Sigma)}), \end{aligned} \tag{3.10}$$

where the final isomorphism is given by the trace map. It follows from Theorem 3.4 that, for any τ as before, there is a period factor

$$B(\pi) = (B(\pi,\tau), \tau \in \Sigma) \in (E(\pi)\otimes\mathbb{C})^\times$$

such that

$$B(\pi,\tau)^{-1}B(\pi)_{f,f',\tau} \in \tau(E(\pi)),\ \forall f, f', \tau. \tag{3.11}$$

Here we view $B(\pi)_{f,f'}$ as an element of $H^0(Sh(V,\Sigma), \mathcal{O}_{Sh(V,\Sigma)})(E(\pi)) \otimes \mathbb{C} \xrightarrow{\sim} E(\pi)\otimes_{\mathbb{Q}}\mathbb{C}$ and let $B(\pi)_{f,f',\tau}$ denote its projection on the τ-component.

Suppose α is a motivic Hecke character of $\mathbf{A}_F^\times$, with restriction α^U to $U(1)(\mathbf{A})$. Then with f' as above, there is an automorphic line bundle $\Lambda_{\alpha^U_\infty}$ on $Sh(-V,\Sigma)$ such that

$$f' \otimes \alpha^{U,-1}\circ\det \in H_!^{\dim Y_V}(Sh(-V,\Sigma), \mathcal{E}_{\kappa^\flat}\otimes\Lambda_{\alpha^U_\infty})[\pi^\vee\otimes\alpha^{U,-1}\circ\det].$$

There is then a constant $p_U(\alpha, -V) \in (E(\alpha_f) \otimes \mathbb{C})^\times$ such that, letting $E(\pi, \alpha) = E(\pi) \cdot E(\alpha)$, $p_U(\alpha, -V)^{-1} \cdot f' \otimes \alpha^{U,-1} \circ \det$ is an $E(\pi, \alpha)$-rational element of $H_!^{\dim Y_V}(Sh(-V, \Sigma), \mathcal{E}_{\kappa^\flat} \otimes \Lambda_{\alpha^U_\infty})[\pi^\vee \otimes \alpha^{U,-1} \circ \det]$. Put another way, let f'_α be an $E(\pi, \alpha)$-rational element of $H_!^{\dim Y_V}(Sh(-V, \Sigma), \mathcal{E}_{\kappa^\flat} \otimes \Lambda_{\alpha^U_\infty})[\pi^\vee \otimes \alpha^{U,-1} \circ \det]$, and let

$$B(\pi)_{\alpha, f, f'} = [f, F_\infty(f'_\alpha \otimes \alpha^U \circ \det)]_{Ser}.$$

Then for any $\tau : E(\pi, \alpha) \hookrightarrow \mathbb{C}$, there is a constant $B(\pi, \tau)_\alpha \in \mathbb{C}^\times$ such that, with notation as in (3.11)

$$B(\pi, \tau)_\alpha^{-1} B(\pi)_{\alpha, f, f', \tau} \in \tau(E(\pi, \alpha)), \ \forall f, f' \tag{3.12}$$

The formula for $B(\pi)_{\alpha, f, f'}$ may appear artificial but it is what shows up in the analysis of the zeta integral. Of course we have

$$B(\pi)_\alpha \sim p_U(\alpha, -V) \cdot B(\pi). \tag{3.13}$$

We can carry out the same construction with $Sh(GU(-V), X_{-V})$, the Shimura variety attached to the similitude group. If $\tilde{\pi}'$ is an automorphic representation of $GU(-V)(\mathbf{A})$ such that

$$H_!^{\dim Y_V}(Sh(GU(-V), X_{-V}), \mathcal{E}_{\kappa^\flat})[\tilde{\pi}'] \neq 0$$

and if α is as above, there is an automorphic line bundle Λ_{α_∞} on $Sh(GU(-V), X_{-V})$ and a constant $p(\alpha, -V)$ such that, with the analogues of the definitions above (taking into account the similitude factor in the usual way)

$$B(\tilde{\pi})_\alpha \sim p(\alpha, -V) \cdot B(\tilde{\pi}). \tag{3.14}$$

An expression for $p(\alpha, -V)$ can be found in §2.9 of [H97] (the top of p. 138) when $F^+ = \mathbb{Q}$. The relation between $p(\alpha, -V)$ and $p_U(\alpha, -V)$ is given in formula (3.15).

Remark 3.5. The notation F_∞ is not altogether appropriate, because the complex conjugate of $Sh(V, \Sigma)$ is not $Sh(-V, \Sigma)$ but rather $Sh(-V, c\Sigma)$. The difference is reflected in the period invariant. Suppose for simplicity of exposition that $F^+ = \mathbb{Q}$. Because the motive that appears in the π-isotypic component cohomology of $Sh(V, \Sigma)$ is (up to a Tate twist) an exterior power of the motive attached to π, the invariant $B(\pi)$ should be a period of this exterior power. But it is off by an abelian period (denoted $q(M)$ in [GH]), which exactly corresponds to the abelian twist needed to relate rational structures on $Sh(-V, \Sigma)$ and $Sh(-V, c\Sigma)$.

3.5.1 Period Invariants, $n = 1$

We recall the Shimura varieties attached to $H_F := R_{F/\mathbb{Q}}(\mathbb{G}_m)_F$ defined in [H93, §1.1]. For any subset $\Psi \subset \Sigma$ we can define $h_\Psi : \mathbb{S} \to H_{F,\mathbb{R}}$ by the rule that, in the induced Hodge structure on (the $\mathbb{Q}$-vector space) F, the subspace $F_\sigma \subset F \otimes \mathbb{C}$ is of type $(-1, 0)$ (resp. $(0, -1)$, resp. $(0, 0)$) if $\sigma \in \Psi$ (resp. $\sigma \in c\Psi$, where c denotes complex conjugation, resp. if $\sigma \notin \Psi \coprod c\Psi$). If $\Psi = \{\sigma\}$ is a singleton we write h_σ. We define $h_{c\Psi} : \mathbb{S} \to H_{F,\mathbb{R}}$ by exchanging Ψ with $c\Psi$.

For any motivic Hecke character ω of H_F we can thus define the CM periods $p_F(\omega, \Psi)$ and $p_F(\omega, c\Psi)$ as in [H93, Lemma 1.3], with $(H, h) = (H_F, h_\Psi)$ or $(H, h) = (H_F, h_{c\Psi})$ (but see [H97], p. 82 for an explanation of a sign error).

Lemma 3.6. *For any motivic Hecke character ω of H_F we have*

$$p_U(\omega, -V) = p_F(\omega \circ \det, c\Sigma)^{-1} \cdot p(\omega, -V) = p_F(\omega, c\Sigma)^{-n} \cdot p(\omega, -V). \tag{3.15}$$

Proof. By Shimura's product relations ([H93, Corollary 1.5], see [GL], §3.3 for a Galois-equivariant version), this follows by pulling back from $U(-V)$ to $G''(-V) \subset GU(1) \times GU(-V)$ by the map u of (2.2). The ratio between the two invariants is entirely determined by the image in $GU(1)$ of the pullback to $G''(-V)$. □

4 Holomorphic Eisenstein Series

4.1 The Doubled Group and Its Variety

We fix V as in the previous section, with hermitian form $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_V$, and write $-V$ to denote the F-vector space V with hermitian form $-\langle\cdot,\cdot\rangle_V$. Let W denote the $2n$-dimensional hermitian space $V \oplus (-V)$, and let $H = H_V = U(W)$ be the corresponding unitary group. Then H is always quasi-split. More precisely, let $V^d = \{(x, x) \in W : x \in V\}$ and $V_d = \{(x, -x) \in W : x \in V\}$, so $W = V_d \oplus V^d$ is a polarization of $\langle\ ,\ \rangle_W$, which is thus a maximally isotropic hermitian form. Projection to the first summand fixes identifications of V^d and V_d with V. Let $P \subset H$ be the stabilizer of V^d; this is a maximal $\mathbb{Q}$-parabolic, the Siegel parabolic. Let $M \subset P$ be the stabilizer of the polarization $W = V_d \oplus V^d$ and $N \subset P$ the group fixing both V^d and W/V^d, so M is a Levi subgroup and N the unipotent radical of P.

Since $\langle\ ,\ \rangle_W$ is maximally isotropic, we know that $H(F_\sigma) \xrightarrow{\sim} U(n, n)$ for any $\sigma : F \hookrightarrow \mathbb{C}$. Let (H, Y_W) be the Shimura datum attached to H and Σ by the procedure described in the previous section. Then $E(W, \Sigma)$ is the reflex field of the CM type Σ. The isomorphism $W = V \oplus (-V)$ determines a morphism of Shimura data

$$(G_V, Y_V) \times (G_{-V}, Y_{-V}) \hookrightarrow (H, Y_W) \tag{4.1}$$

and thus a map of Shimura varieties

$$Sh(V, \Sigma) \times Sh(-V, \Sigma) \rightarrow Sh(W, \Sigma).$$

4.1.1 Tube Domains

As a complex variety, the Shimura variety $Sh(W, \Sigma)$ is a union of arithmetic quotients of the tube domain $\mathfrak{X}_\Sigma$ attached to the group $SU(n,n)^\Sigma$. The tube domain $\mathfrak{X}_\Sigma$, which is isomorphic to a connected component of Y_W, itself factors as a product $\prod_{\sigma\in\Sigma}$ of $|\Sigma|$ copies of the classical tube domain $X^+_{n,n} \subset Herm_n(\mathbb{C})$, where $Herm_n \subset M(n, \mathbb{C})$ is the space of $n \times n$-hermitian matrices. More precisely, denote by X^+_σ the copy of $X^+_{n,n}$ corresponding to $\sigma \in \Sigma$, and choose a base point $\daleth_\sigma \in X^+_\sigma$; let $U_\sigma \subset H_\sigma := H(F^+ \times_\sigma \mathbb{R})$ be its stabilizer. Then U_σ is isomorphic to the compact group $U(n) \times U(n)$. Without loss of generality, we may choose $\daleth \in M(n, F)$ to be a diagonal matrix whose entries have trace zero down to F^+, and let $\daleth_\sigma = \sigma(\daleth)$ for $\sigma \in \Sigma$. Then X^+_σ is identified with the standard tube domain

$$X_{n,n} := \left\{z \in M_n(\mathbb{C}) \mid \daleth_\sigma \left({}^t\bar{z} - z\right) > 0\right\}.$$

With respect to this identification, any $h_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in H_\sigma$ acts by

$$h_\sigma(z) = (a_\sigma z + b_\sigma)(c_\sigma z + d_\sigma)^{-1}.$$

Here $a_\sigma, b_\sigma, c_\sigma$, and d_σ are $n \times n$ matrices. We let $y_\daleth = (\sigma(\daleth))_{\sigma\in\Sigma} \in Y_W$.

In particular, the maximal parabolic subgroup $P \subset H$ stabilizes a point boundary component F_P of $\mathfrak{X}_\Sigma$. As we see below in Sects. 4.9 and 4.11, the rationality properties of holomorphic Eisenstein series on H are determined by their restriction to the boundary, which is the $H(\mathbf{A}_f)$-orbit of the Shimura variety attached to the pair (G_P, F_P) for a certain reductive subgroup $G_P \subset P$; the point F_P then corresponds to a homomorphism, also denoted $F_P : \mathbb{S} \rightarrow G_P(\mathbb{R})$.

Remark 4.2. The group U_σ is defined over $\sigma(F)$ and, for all $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\gamma(U_\sigma) = U_{\gamma(\sigma)}.$$

We thus have a way of comparing rational representations of the maximal compact subgroups U_σ, and thus the fibers of automorphic vector bundles, as σ varies. Note that, if γ does not fix $E(H, Y_W)$, then

$$U_{\gamma(\sigma)} \subset {}^\gamma H(\mathbb{R}),$$

where ${}^{\gamma}H$ is an inner form of H determined up to isomorphism by Langlands's formula for conjugation of Shimura varieties; see [H13, L79].

4.3 Induced Representations and Eisenstein Series

Denote by Δ the canonical map $\Delta : P \to GL_F(V^d) = GL_F(V)$. Then $M \xrightarrow{\sim} GL_F(V), m \mapsto (\Delta(m))$; the inverse map is

$$A \mapsto m(A) = diag(\left(A^*\right)^{-1}, A).$$

where $A^* = {}^tA^c$ is the transpose of the conjugate under the action of c. Define $\delta_P(\cdot) = |\det \circ \Delta(\cdot)|^n$.

Let $\chi = \otimes \chi_w$ be a character of $F^\times \backslash \mathbf{A}_F^\times$. For $s \in \mathbb{C}$ let

$$I(\chi, s) = Ind_{P(\mathbf{A})}^{H(\mathbf{A})}\left(\chi(\det \circ \Delta(\cdot)) \cdot \delta_P^{-s/n}(\cdot)\right)$$

with the induction smooth and unitarily normalized. This factors as a restricted tensor product

$$I(\chi, s) = \otimes_v I_v(\chi_v, s),$$

with v running over the places of $\mathbb{Q}$, $I_v(\chi_v, s)$ the analogous local induction from $P(\mathbb{Q}_v)$ to $H(\mathbb{Q}_v)$, and $\chi_v = \otimes_{w|v}\chi_w$.

Let $s \mapsto \phi_s \in I(\chi, s)$ be a section, in the sense in which that word is used in the theory of Eisenstein series. We form the *standard (non-normalized)* Eisenstein series,

$$E(\phi_s, h) = \sum_{\gamma \in P(F^+)\backslash H(F^+)} \phi_s(\gamma h). \tag{4.2}$$

If χ is unitary, this series is absolutely convergent, uniformly on compact subsets, for $\mathrm{Re}(s) > \frac{n}{2}$, and defines an automorphic form on $H(\mathbf{A})$. We always assume $\phi_s \in I(\chi, s)$ to be K-finite for a maximal compact subgroup $K \subset H(\mathbf{A})$; then in particular the Eisenstein series $E(\phi_s, h)$ admit a meromorphic continuation in s.

Let $m \geq n$ be a positive integer, which we view as the dimension of a positive-definite hermitian vector space V'. Assume

$$\chi|_{\mathbf{A}_{F^+}^\times} = \varepsilon_{\mathcal{K}}^m \tag{4.3}$$

Then the main result of [Tan] states that the possible poles of $E(\phi_s, h)$ are all simple. Moreover, those poles in the right half plane $Re(s) \geq 0$ can only occur at the points in the set

$$\frac{n-\delta-2r}{2}, \qquad r=0,\ldots,[\frac{n-\delta-1}{2}] \tag{4.4}$$

where $\delta = 0$ if m is even and $\delta = 1$ if m is odd.

Using the theta correspondence between $U(V)$ and $U(V')$, we consider the *Siegel-Weil sections* $\phi_s \in I(\chi, s)$ as in [H08]: these are the sections defined by the functions of the form ϕ_Φ introduced in §2.2.3 of [H08], where V is used to denote what we are now calling V' (and where the presence of the similitude factor introduces an additional complication, irrelevant here). Given a unitary character χ and a Siegel-Weil section $f \in I(\chi, s)$, we put

$$\phi_s := \phi_{\chi,s} := \phi$$
$$E_\phi(s,h) := E(\phi_s, h).$$

4.4 Automorphic Line Bundles on the Doubled Group

Automorphic line bundles on $Sh(W,\Sigma)$ are determined by two sets of parameters $(m_\sigma, k_\sigma)_{\sigma\in\Sigma}$. If we note that, for $U(W)$, $r_\sigma = s_\sigma = n$ for all $\sigma \in \Sigma$, then the parameter $(m_\sigma, k_\sigma)_{\sigma\in\Sigma}$ corresponds to the representations with parameters

$$(a_{1,\sigma} = \cdots = a_{n,\sigma} = m_\sigma; a_{n+1,\sigma} = \cdots = a_{2n,\sigma} = k_\sigma) \tag{4.5}$$

in the notation of (3.2). This corresponds to the representation $\otimes_\sigma \det^{m_\sigma} \otimes \otimes_\sigma \det^{k_\sigma}$ of the maximal compact subgroup $U(n)^\Sigma \times U(n)^\Sigma = \prod_{\sigma\in\Sigma} U_\sigma$ of $U(W)(\mathbb{R})$. In applications we only need to consider parameters in which $m_\sigma = m - a_\sigma$ and $k_\sigma = m + b_\sigma$ for all σ, for some Σ-tuple of pairs of integers (a_σ, b_σ) to be introduced below.

4.5 Automorphic Forms on the Point Boundary Shimura Variety

In [Gu16], Guerberoff identifies the Shimura datum attached to the point boundary component of the Shimura variety attached to $GU(W)$. He denotes the datum (G_P, F_P) but we will call it (G'_P, F'_P); then $G'_P = G_h \cdot A_P$, where, for any $\mathbb{Q}$-algebra R,

$$\begin{aligned} G_h(R) &= \{\beta \cdot I_{2n} \mid \beta \in (F\otimes R)^\times, N_{F/F^+}(\beta) \in R^\times\}; \\ A_P &= \{d(\mathbf{a},\mathbf{d}) := \begin{pmatrix} \mathbf{a}I_n & 0 \\ 0 & \mathbf{d}I_n \end{pmatrix}, \mathbf{a},\mathbf{d} \in R^\times\}. \end{aligned} \tag{4.6}$$

Here N_{F/F^+} is shorthand for the natural map $(F \otimes R)^\times \to (F^+ \otimes R)^\times$. In other words, $G_h = GU(1)$, in the previous notation, and $A_P \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$. Moreover,

$$F'_P(z) = \begin{pmatrix} z\bar{z}I_n & 0 \\ 0 & I_n \end{pmatrix},$$

diagonally embedded in $GU(n,n)^\Sigma \cap GU(W)(\mathbb{R})$.

The factor $\mathbf{a}$ in (4.6) is superfluous, because $G_h \cap A_P$ contains all elements of the form $d(\mathbf{a}, \mathbf{a})$. Let $[\beta, \mathbf{a} = 1, \mathbf{d}]$ denote a typical element of G'_P in the coordinates of (4.6). Then $G_P = G'_P \cap U(W)$, in other words is the group of triples $[\beta, 1, \mathbf{d}]$ with $N_{F/F^+}(\beta) \cdot \mathbf{d} = 1$. Since the coordinate $\mathbf{d} \in \mathbb{G}_m$ is thus superfluous, we have an isomorphism

$$GU(1) \xrightarrow{\sim} G_P; \beta \mapsto [\beta, 1, N_{F/F^+}(\beta)^{-1}]. \tag{4.7}$$

Thus we write $[\beta]$ for the typical element of $G_P \xrightarrow{\sim} GU(1)$.

It follows easily from (2.3) and (2.2) that

$$F_P(z) = \begin{pmatrix} zI_n & 0 \\ 0 & \bar{z}^{-1}I_n \end{pmatrix} = [z] \in G_P \subset U(n,n)^\Sigma.$$

Then

Definition 4.5.1. *Let $\alpha : G_P(F^+)\backslash G_P(\mathbf{A}) \to \mathbb{C}^\times$ be a Hecke character of G_P. Let $(a, b) = (a_\sigma, b_\sigma)_{\sigma\in\Sigma})$ be a $|\Sigma|$-tuple of pairs of integers satisfying $a_\sigma + b_\sigma = -\nu$ for some fixed integer ν, called the weight of (a, b). We say α is of type $(m, (a, b)) = (m, (a_\sigma, b_\sigma)_{\sigma\in\Sigma})$ if it is of the form*

$$\alpha = ||\bullet||^m \cdot \alpha_0$$

where the restriction α_∞ of α to $G_P(F^+_\infty)$ satisfies

$$\alpha_{0,\sigma}([\beta]) = \beta^{-a_\sigma} c(\beta)^{-b_\sigma}$$

(see [EHLS], §4.4). For $h_\infty = (h_\sigma)_{\sigma\in\Sigma} \in U(n,n)^\Sigma$, we define

$$\begin{aligned} J_{\alpha,\sigma} &:= J_{m,(a_\sigma,b_\sigma)}(h_\sigma) = \det(J(h_\sigma))^{-m+a_\sigma} \cdot \det(J'(h_\sigma))^{-m+b_\sigma}; \\ J_\alpha(h_\infty) &:= J_{m,(a,b)}(h_\infty) = \prod_{\sigma\in\Sigma} J_{\alpha,\sigma}(h_\sigma) \end{aligned} \tag{4.8}$$

Here the automorphy factors $J(\bullet)$ and $J'(\bullet)$ are defined as in [EHLS], §4.4.2, relative to the chosen base point $y_\daleth \in Y_W$ (corresponding to the element $\daleth \in F$ in *loc. cit.*).

Let $M^{der} \subset M$ denote the derived subgroup of M. A function

$$\phi : (P(F^+) \cdot N(\mathbf{A})M^{der}(\mathbf{A}))\backslash H(\mathbf{A}) \ \rightarrow \ \mathbb{C}$$

is said to be of type $(m, (a, b))$ if it is of the form

$$(h_\infty, h_f) \mapsto J_{m,(a,b)}(h_\infty) \otimes \phi_f(h_f)$$

for some ϕ_f on $(N(\mathbf{A}_f)M^{der}(\mathbf{A}_f))\backslash H(\mathbf{A}_f)$.

Note that we have

$$\alpha \circ \det(F_P(z)) = (z\bar{z})^{[F^+:\mathbb{Q}]nm} \cdot \prod_\sigma z^{-a_\sigma} \bar{z}^{-b_\sigma} \tag{4.9}$$

Lemma 4.6. *The space $I(m, (a, b))$ of functions of type $(m, (a, b)))$ decomposes as the direct sum of subspaces*

$$I(\alpha) = \{J_{m,(a,b))}(h_\infty) \otimes \phi_f(h_f)\}$$

where $\phi_f(h_f) \in I_{P(\mathbf{A}_f)}^{H(\mathbf{A}_f)} \alpha_f \circ \det$.

This follows from the decomposition of $I(m, (a, b))$ under the right action of $M(\mathbf{A}_f)$. We let $I(m, (a, b))(\mathbf{A}_f)$ be the space of $\phi_f(h_f)$ such that $J_{m,(a,b))}(h_\infty) \otimes \phi_f(h_f) \in I(m, (a, b))$, viewed as a space of functions on $(N(\mathbf{A}_f)M^{der}(\mathbf{A}_f))\backslash H(\mathbf{A}_f)$.

For $\sigma \in \Sigma$, let $\sigma^+ : F^+ \rightarrow \mathbb{R}$ denote the restriction of σ to F^+; let

$$H_\sigma = H(F^+ \otimes_{\sigma^+} \mathbb{R}), \quad \mathfrak{h}_\sigma = Lie(H_\sigma), \quad U(V)_\sigma = U(V \otimes_{F^+,\sigma} \mathbb{R}).$$

Let $U_\sigma = U(r_\sigma) \times U(s_\sigma)$ be the maximal compact subgroup of $U(V)_\sigma$ (with respect to the chosen base point $y \in Y_V$, see the discussion following Lemma 2.3). Let $\alpha = \alpha_0 \cdot || \bullet ||^m$ be an algebraic Hecke character as above, with $m \geq 0$. Define $\mathbb{D}(\alpha_\sigma) = \mathbb{D}(m, \alpha_{0,\sigma})$ to be the holomorphic $(\mathfrak{h}_\sigma, U_\sigma)$-module with highest U_σ-type

$$\begin{aligned}\Lambda(\alpha_\sigma) &= \Lambda(-m, (a_\sigma, b_\sigma)) \\ &= (m - b_\sigma, m - b_\sigma, \ldots, m - b_\sigma; -m + a_\sigma, \ldots, -m + a_\sigma)\end{aligned}$$

in the notation of [H97, (3.3.2)] (with the character on the $\mathbb{R}$-split center omitted). We define a map of $(U(\mathfrak{h}_\sigma), U_\sigma)$-modules

$$\iota(\alpha_\sigma) : \mathbb{D}(\alpha_\sigma) \ \rightarrow \ C^\infty(H_\sigma) \tag{4.10}$$

as follows. Let $v(\alpha_\sigma)$ be the tautological generator of the $\Lambda(m, \alpha_{0,\sigma})$-isotypic subspace (highest U_σ-type subspace) of $\mathbb{D}(m, \alpha_{0,\sigma})$. Let

$$\iota(\alpha_\sigma)(v(\alpha_\sigma)) = J_{\alpha,\sigma},$$

defined as in (4.8), and extend this to a map of $U(\mathfrak{h}_\sigma)$-modules. Let $C(H_\sigma, \alpha_\sigma)$ denote the image of $\iota(\alpha_\sigma)$.

We let $Sh(W, \Sigma)_{G_P}$ denote the point boundary stratum of the (adèlic) Shimura variety $Sh(W, \Sigma)$, corresponding to the maximal parabolic subgroup $P \subset U(W)$. This is a totally disconnected 0-dimensional proscheme over $E(W, \Sigma)$. As in [H86], §8, there is a line bundle $\mathcal{E}_{m,(a,b)}$ on $Sh(W, \Sigma)_{G_P}$ and an isomorphism (*canonical trivialization*)

$$\begin{aligned} Triv_{m,(a,b)}: \quad & H^0(Sh(W,\Sigma)_{G_P}, \mathcal{E}_{m,(a,b)}) \xrightarrow{\sim} \\ & \{J_{m,(a,b)}(h_\infty) \otimes \phi_f(h_f), \phi_f \in I(m,(a,b))(\mathbf{A}_f)\} \end{aligned} \tag{4.11}$$

normalized as in the discussion in [H08, (2.4)].[2] If $\xi \in H^0(Sh(W, \Sigma)_{G_P}, \mathcal{E}_{m,(a,b)})$ we include ξ in the subscript on the right-hand side:

$$Triv_{m,(a,b)}(\xi) = J_{m,(a,b)}(h_\infty) \otimes \phi_{\xi,f}(h_f).$$

There is a number field $E(m, (a, b)) \supset E(G_P, F_P)$ such that the vector bundle $\mathcal{E}_{m,(a,b)}$ has a canonical model over $E(m, (a, b))$, compatible with the canonical model of $Sh(W, \Sigma)_{G_P}$ over $E(G_P, F_P)$. Moreover, the group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of pairs $(Sh(W, \Sigma)_{G_P}, \mathcal{E}_{m,(a,b)})$ through its natural action on the set of CM types Σ and parameters $(m, ((a, b)_\sigma)_{\sigma\in\Sigma})$ (the m is invariant under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$; see [H13], §4.1 for the action).

Proposition 4.7. *Fix a Σ-tuple of pairs of integers $(a, b) = (a_\sigma, b_\sigma)$ of weight v, and consider characters $\alpha : G_P(F^+)\backslash G_P(\mathbf{A}) \to \mathbb{C}^\times$ of type $(m, (a, b))$. For each such α let $E(\alpha) \supset E(m, (a, b))$ be the field of coefficients of α; it is the field generated by the values of α on $G_P(\mathbf{A}_f)$. Let*

$$H^0(Sh(W,\Sigma)_{G_P}, \mathcal{E}_{m,(a,b)})[\alpha] \subset H^0(Sh(W,\Sigma)_{G_P}, \mathcal{E}_{m,(a,b)})$$

be the subspace corresponding to $I(\alpha)$ with respect to the decomposition of Lemma 4.6 and the isomorphism $Triv_{m,(a,b)}$ (4.11). There is a constant $p_U(\alpha, \Sigma, W) \in \mathbb{C}^\times$ with the property that, for any extension $L/E(\alpha)$, the section $\xi \in H^0(Sh(W, \Sigma)_{G_P}, \mathcal{E}_{m,(a,b)})[\alpha]$ is rational over L if and only if

$$Triv_{m,(a,b)}(\xi) = J_{m,(a,b)}(h_\infty) \otimes \phi_{\xi,f}(h_f) \tag{4.12}$$

where $p_U(\alpha, \Sigma, W)^{-1}\phi_{\xi,f}(h_f)$ takes values in L.

[2] The ϕ_f should be understood as the constant terms of Fourier expansions of the Eisenstein series attached to the functions $J_{m,(a,b)}(h_\infty) \otimes \phi_f(h_f)$.

Proof. This is the analogue of Proposition 4.3.1 of [Gu16] in the setting of the Shimura variety attached to $U(W)$. We may restrict $\mathcal{E}_{m,(a,b)}$ to any $G_P(\mathbf{A}_f)$-orbit in $Sh(W, \Sigma)_{G_P}$ and identify the orbit with an $H(\mathbf{A}_f)$-translate of the Shimura variety $Sh(G_P, F_P)$; then the restriction of $\mathcal{E}_{m,(a,b)}$ to the orbit is an automorphic vector bundle on the toric Shimura variety $Sh(G_P, F_P)$. As such, it has a canonical period invariant $p_U(\alpha, \Sigma, W) \in \mathbb{C}/E(m,(a,b))$ that satisfies the analogue of (4.12) for sections of the bundle over $H(\mathbf{A}_f)$-translate. Since the action of $H(\mathbf{A}_f)$ respects both the property of (4.12) and the rational structure of $\mathcal{E}_{m,(a,b)}$, the Proposition follows from the analogous statement for $Sh(G_P, F_P)$. □

The constant $p_U(\alpha, \Sigma, W)$ is a period attached to automorphic forms on the CM Shimura variety $Sh(G_P, F_P)$, and can thus be related to standard CM periods $p_U(\alpha, \sigma, W)$, defined as follows. We identify G_P with $GU(1)$, as in (4.7). For any $\sigma \in \Sigma$ we let

$$h_\sigma : \mathbb{C}^\times \to GU(1)_\infty = \prod_{\sigma \in \Sigma} \mathbb{C}^\times$$

be the inclusion of $\mathbb{C}^\times$ as the σ-factor of the above product. The pair $(GU(1), h_\sigma)$ is thus a Shimura datum.

On the other hand, as in [Gu16, Prop. 4.3.1] (for α trivial) or [H97, Lemma 3.3.5.3] (for $F^+ = \mathbb{Q}$), there is an analogous invariant $p(\alpha, \Sigma, W)$ attached to the point boundary component of the Shimura variety $Sh(GU(W), \Sigma)$. As in the proof of Lemma 3.6, the ratio of the factors comes from the pullback of $\alpha \circ \det$ to the factor $GU(1)$ of the map u of (2.2)—we use the fact that $u(\mathbb{G}_m \cdot [U(1) \times \{1\}]) \subset G_P$. Here the determinant is taken on the Levi factor $GL(n)$ of $P \subset U(W)$, whereas in Lemma 3.6, the determinant is the map $\det : U(-V) \to U(1)$.

Then Shimura's product relations again imply

Proposition 4.8. *For any motivic Hecke character ω of H_F we have*

$$p_U(\omega, \Sigma, W) = p(\omega \circ \det, c\Sigma)^{-1} \cdot p(\omega, \Sigma, W) = p(\omega, c\Sigma)^{-n} \cdot p(\omega, \Sigma, W). \tag{4.13}$$

4.9 Holomorphic Eisenstein Series: Absolutely Convergent Case

Fix a Σ-tuple of pairs of integers $(a, b) = (a_\sigma, b_\sigma)$ of weight ν. We now fix $\alpha = \alpha_0$ to be a Hecke character of type $(0, (a, b))$, $\alpha_m = || \bullet ||^{m-\frac{n}{2}} \cdot \alpha$.

Let $\phi \in I(\alpha_m) \subset I(m - \frac{n}{2}, (a, b))$. Extend $\phi = \phi_0$ to a section $s \mapsto \phi_s \in I(\alpha, s + m - \frac{n}{2})$, in the sense used in Sect. 4.3, and define $E(\phi, s) := E(\phi_s)$ as in (4.2).

Proposition 4.10. *If $m > n - \frac{\nu}{2}$ then $E(\phi, 0)$ converges absolutely to a holomorphic automorphic form which defines a section*

$$E(\phi, 0) \in H^0(Sh(W, \Sigma), \mathcal{E}_m^{can}).$$

This proposition follows from the analogous statement for the Shimura variety attached to $GU(W)$. Guerberoff ([Gu16], §4.3) treats the case when α is trivial, but the general case is no more difficult. It then follows from Proposition 4.7 and the results of §8 of [H86] that

Corollary 4.10.1. *Let L be a field containing $E(\alpha)$. Then the section $E(\phi, 0) \in H^0(Sh(W, \Sigma), \mathcal{E}_m^{can})$ is rational over L if and only if $p(\alpha, \Sigma)^{-1}\phi_f$ takes values in L.*

Here and in what follows we are thinking of $\phi_f = \phi_{0,f} \in I(\alpha, 0)$ as a complex valued function on $H(\mathbf{A}_f)$, and the condition in the corollary is that the values be L-rational multiples of the period invariant $p(\alpha, \Sigma)$.

4.11 Holomorphic Eisenstein Series: Application of the Siegel-Weil Formula

We continue with the hypotheses of the previous section, but now assume $n \geq m \geq \frac{n-\nu}{2}$. We assume $\chi = \alpha$ satisfies the hypothesis (4.3), with m replaced by $m_0 := 2m + \nu$ (cf. Theorem 4.3 of [H08], where s_0 is what we are calling m, and m is what we are calling m_0, and $\kappa = -\nu$). and we take ϕ to be a *Siegel-Weil section*, in the sense of [H08].

Theorem 4.14. *Suppose $n \geq m \geq \frac{n-\nu}{2}$. Then the function $E(\phi, s)$ has a meromorphic continuation whose value at $s = 0$ is holomorphic and defines a section $E(\phi, 0) \in H^0(Sh(W, \Sigma), \mathcal{E}_m^{can})$. Moreover, with L as in the previous Corollary, the section $E(\phi, 0) \in H^0(Sh(W, \Sigma), \mathcal{E}_m^{can})$ is rational over L if and only if $p(\alpha, \Sigma)^{-1}\phi_f$ takes values in L.*

Proof. This is essentially the main theorem of [H08]. In fact, Corollary 3.3.3 of [H08] is proved in the setting of similitude groups, and in general is only valid on a subgroup of index 2 of $GU(W)(\mathbf{A})$. For the unitary group the method of [H08] works without restriction. □

5 Differential Operators

The method of [H97] and [Gu16] is based on constructing a family of differential operators that take holomorphic Eisenstein series on $Sh(W, \Sigma)$ to holomorphic automorphic forms on $Sh(V, \Sigma) \times Sh(-V, \Sigma)$. The parameters for these differential operators were worked out in §4.2 of [Gu16] and, with different notation, in [EHLS], §4.4. In both cases this was done in the setting of similitude groups, rather than unitary groups, but the differential operators are insensitive to the similitude factor and can be indexed by the same parameters. We follow [EHLS] because it is slightly more general. If κ is a parameter for $Sh(V, \sigma)$, then $\kappa^\flat$ is defined in [EHLS] (6.2.5).

5.1 Parameters for Differential Operators

Theorem 5.1 (Gu16, §4.2). *Let κ be a parameter in (3.2), and let $\mathcal{E}_\kappa$ be the corresponding automorphic vector bundle on $Sh(V, \Sigma)$. Let $\kappa^\flat$ be the corresponding (dual) parameter for $Sh(-V, \Sigma)$ (see [EHLS, (6.2.5)]) and define*

$$\mathcal{E}_{\kappa^\flat}((a,b)) = \mathcal{E}_{\kappa^\flat(a,b)}$$

where, if $\kappa^\flat = (\kappa_\sigma^\flat, \sigma \in \Sigma)$, then $\kappa^\flat(a,b) = \kappa_\sigma^\flat \otimes \alpha_{0.\sigma}$ in the notation of [EHLS, (4.4.6)]. *There exists a holomorphic differential operator*

$$\Delta(m, (a,b), \kappa) : \mathcal{E}_{m,(a,b)}|_{Sh(V,\Sigma)\times Sh(-V,\Sigma)} \ \rightarrow \ \mathcal{E}_\kappa \boxtimes \mathcal{E}_{\kappa^\flat}((a,b)),$$

defined over the reflex field $E(\kappa, (a,b))$ and equivariant with respect to $G_V(\mathbf{A}_f) \times G_{-V}(\mathbf{A}_f)$, if and only if κ has the form $(\kappa_{\sigma,i}, \sigma \in \Sigma, 1 \le i \le n)$ where, for all $\sigma \in \Sigma$,

$$(\kappa_{\sigma,1}, \ldots, \kappa_{\sigma,r_\sigma}) = (-m + b_\sigma - c_{r_\sigma}, \ldots, -m + b_\sigma - c_1);$$

$$(\kappa_{\sigma,r_\sigma+1}, \ldots, \kappa_{\sigma,n}) = (m - a_\sigma + d_1, \ldots, m - a_\sigma + d_{s_\sigma}),$$

with

$$c_1 \ge \cdots \ge c_{r_\sigma} \ge 0; d_1 \ge \cdots \ge d_{s_\sigma} \ge 0.$$

Moreover, the space of such differential operators is of dimension 1 over $E(\kappa, (a,b))$.

The differential operator $\Delta(m, (a,b), \kappa)$ is obtained by applying a non-holomorphic (Maass) operator to a section of $\mathcal{E}_{m,(a,b)}$, viewed as a C^∞ automorphic form on $U(W)(\mathbf{A})$, and then restricting the result to the subvariety $Sh(V, \Sigma) \times Sh(-V, \Sigma)$. For future use, we denote the Maass operator $D^\infty(m, (a,b), \kappa)$, and we define $G_3 = G_V \times G_{-V}$, as in [EHLS].

Definition 5.2. *Let (a,b) and κ be as in the statement of Theorem 5.1. We say that $m \ge \frac{n-\nu}{2}$ is critical for κ and (a,b) if the above inequalities are satisfied for all σ.*

5.2 Parameters for Nearly Holomorphic Eisenstein Series

Let m be critical for κ and (a, b). Let $\phi \in I(\alpha_m)$ be as in Proposition 4.10 or Theorem 4.14 depending on m. We write $E(m, \phi) = E(\phi, 0, h)$, emphasizing that ϕ is defined relative to the parameter m (although the m is superfluous in the notation). We assume L is a field as in one of those statements, so that

Hypothesis 5.3. *The function $\phi \in I(\alpha_m)$ has the property that the section $E(m, \phi) \in H^0(Sh(W, \Sigma), \mathcal{E}_m^{can})$ is rational over L.*

We also define

$$E^W(m, \phi) = p(\alpha, \Sigma)^{-1} E(m, \phi);$$

this is of course attached to the L-rational function ϕ_f on the finite adèles.

We write

$$\begin{aligned} E(m, \phi, \kappa) &= \Delta(m, (a, b), \kappa)(E(m, \phi)) \\ &\in H^0(Sh(V, \Sigma) \times Sh(-V, \Sigma), \mathcal{E}_\kappa \boxtimes \mathcal{E}_{\kappa^\flat}((a, b))); \end{aligned} \tag{5.1}$$

$$\begin{aligned} E^W(m, \phi, \kappa) &= \Delta(m, (a, b), \kappa)(E^W(m, \phi)) \\ &\in H^0(Sh(V, \Sigma) \times Sh(-V, \Sigma), \mathcal{E}_\kappa \boxtimes \mathcal{E}_{\kappa^\flat}((a, b))). \end{aligned} \tag{5.2}$$

Corollary 5.2.1. *Under Hypothesis 5.3, the sections $E(m, \phi, \kappa)$ are rational over L.*

We also write $E(m, \phi, h)$, $E^W(m, \phi, \kappa, h)$, etc. with $h \in U(W)(\mathbf{A})$, when it is necessary to emphasize that the Eisenstein series are functions on the adèle group. To distinguish the Eisenstein series on $U(W)(\mathbf{A})$ from its restriction to the subgroup $G_3(\mathbf{A})$, we write

$$E^\infty(m, \phi, \kappa, h) = D^\infty(m, (a, b), \kappa)(E(m, \phi))(h).$$

6 The Doubling Integral

6.1 Zeta Integrals

In this section, we briefly summarize key details of the doubling method, which we use to obtain zeta integrals. The doubling method holds for general classes of cuspidal automorphic representations π of $G_V(\mathbf{A})$, but we assume the local factors at Archimedean primes are discrete series representations. By Theorem 3.4, this implies that, for any finite prime v, the component part π_v of π has a model over a number field $E(\pi)$.

Denote by $\mathcal{O}^+$ the ring of integers of F^+. We write $G_V(\mathbf{A}) = \prod_v' G_{V,v}$, with the (restricted) products over all the places of F^+ and $G_{V,v} = G_V(F_v^+)$. Similarly, we write $H(\mathbf{A}) = H_\infty \times \prod_v' H_v$ and $P(\mathbf{A}) = P_\infty \times \prod_v' P_v$.

Let π be an irreducible cuspidal anti-holomorphic automorphic representation of $G_V(\mathbf{A})$, and let $\pi^\vee$ be its contragredient, and set

$$\pi' := \pi^\vee; \;\; \pi'_\alpha = \pi^\vee_\alpha = \pi' \otimes (\alpha \circ \det)^{-1}$$

with α an algebraic Hecke character of weight ν as above. We view $\pi^\vee_\alpha$ as an anti-holomorphic automorphic representation of $G_{-V}(\mathbf{A})$ and as a *holomorphic* automorphic representation of $G_V(\mathbf{A})$. Let S_π be the set of finite primes v of $\mathcal{O}^+$ for which π_v is ramified. Before introducing the zeta integral for π, we need to explain what it means for a function in π to be factorizable over places in F^+. We fix non-zero unramified vectors $\varphi_{w,0}$ and $\varphi'_{w,0}$ in π_w and $\pi^\vee_w$, respectively, for all finite places w outside S_π, and choose factorizations compatible with the unramified choices:

$$\pi \stackrel{\sim}{\longrightarrow} \pi_\infty \otimes \pi_f; \;\; \pi_f \stackrel{\sim}{\longrightarrow} \pi^{S_\pi} \otimes \pi_{S_\pi}; \tag{6.1}$$

and analogous factorizations for $\pi^\vee$ and $\pi^\vee_\alpha$. Let $\varphi \in \pi^{K^{S_\pi}}$, $\varphi' \in \pi^{\vee, K^{S_\pi}}$; we think of φ and φ' as forms on G_V and G_{-V}, respectively. We suppose they decompose as tensor products with respect to the above factorizations:

$$\varphi = \otimes_v \varphi_v; \;\; \varphi' = \otimes_v \varphi'_v \tag{6.2}$$

with φ_v and φ'_v equal to the chosen $\varphi_{v,0}$ and $\varphi'_{v,0}$ when $v \notin S_\pi$. We write equalities but the formulas we write below depend on the factorizations in (6.1) and its counterpart for $\pi^\vee$ and $\pi^\vee_\alpha$. We write

$$\varphi'_\alpha = \varphi' \otimes \alpha^{-1} \circ \det \in \pi^\vee_\alpha.$$

If $L \supset E(\pi)$, we will call a vector $\varphi_v \in \pi_v$ rational over L if it is rational with respect to the factorization (6.2), with respect to the $E(\pi)$-rational model introduced above.

Remark 6.2. Our conventions are as in [EHLS], except that we are writing $\pi' = \pi^\vee_\alpha$ instead of $\pi^\flat$. The test vectors in π and π' are denoted φ and φ', respectively, whereas the section for the Eisenstein series is denoted $\phi = \phi_s$. In [H97] different choices were made: φ was the datum for the Eisenstein series, whereas f and f' were cusp forms on G_V and G_{-V}, respectively.

We also fix local $G_V\left(F_v^+\right)$-invariant pairings $\langle\ ,\ \rangle_{\pi_v} : \pi_v \times \pi_v^\vee \to \mathbb{C}$ for all v such that $\langle \varphi_{v,0}, \varphi'_{v,0}\rangle_{\pi_v} = 1$ for all $v \notin S_\pi$. We assume these pairings are $E(\pi)$-rational for all v.

Normalizations are now as in Sect. 4. Let $\phi = \phi_s(\bullet) \in I(\alpha, s)$. We write $G_3 = G_V \times G_{-V}$ as above. Let $\varphi \in \pi$ and $\varphi' \in \pi'$, be factorizable vectors as above. The zeta integral for ϕ,φ, and φ' is

$$I(\varphi, \varphi', \phi, s) = \int_{G_3(\mathbb{Q})\backslash G_3(\mathbf{A})} E(\phi, s, (g_1, g_2))\varphi(g_1)\varphi'_\alpha(g_2)d(g_1, g_2).$$

Here the measure on G_3 is the Tamagawa measure $dg_V \times dg_{-V}$ discussed in Sect. 2.5. By the cuspidality of φ and φ' this converges absolutely for those values of s at which $E(\phi, s, h)$ is defined and defines a meromorphic function in s (holomorphic wherever $E(\phi, s, h)$ is). Moreover, it follows from the unfolding in [PSR] that $(\varphi, \varphi') \mapsto I(\varphi, \varphi', \phi, s)$ defines a $G_V(\mathbf{A})$-invariant pairing between π and π'. By the multiplicity one property Theorem 3.4, this implies that:

$$\text{If } \langle \varphi, \varphi' \rangle := \int_{G_V(\mathbb{Q})\backslash G_V(\mathbf{A})} \varphi(g)\varphi'(g)dg = 0 \text{ then } I(\varphi, \varphi', \phi, s) = 0 \text{ for all } s.$$

So we suppose

$$\langle \varphi, \varphi' \rangle \neq 0.$$

Then $\langle \varphi_v \otimes \varphi'_v \rangle_{\pi_v} \neq 0$ for all v. For $Re(s)$ sufficiently large, 'unfolding' the Eisenstein series then yields

$$I(\varphi, \varphi', \phi, s) = \int_{G_V(\mathbf{A})} \phi_s(g_V, 1)\langle \pi(g_V)\varphi, \varphi' \rangle_\pi dg_V.$$

Henceforward we assume $\phi(h) = \otimes_v \phi_v(h_v)$ with

$$\phi_v = \phi_{v,s} \in I_v(\alpha_v, s), \alpha_v = \bigotimes_{w|v} \alpha_w.$$

Then bearing in mind the formula (2.6) for the factorization of the measure dg_V, the last expression for $I(\varphi, \varphi', \phi, s)$ factors as

$$\begin{aligned} I(\varphi, \varphi', \phi, s) &= \sqrt{D_{F^+}}^{-n^2} \prod_v I_v(\varphi_v, \varphi_v, \phi_v, s) \cdot \langle \varphi, \varphi' \rangle, \quad \text{where} \\ I_v(\varphi_v, \varphi'_v, \phi_v, s) &= \frac{\int_{G_V(F_v^+)} \phi_{v,s}(g_v, 1)\langle \pi_v(g_v)\varphi_v, \varphi'_v \rangle_{\pi_v} dg_v}{\langle \varphi_v, \varphi'_v \rangle_{\pi_v}}. \end{aligned} \tag{6.3}$$

By hypothesis, the denominator of the above fraction equals 1 whenever $v \notin S_\pi$. We denote the integral in the numerator by $Z_v(s, \varphi_v, \varphi'_v, \phi_v)$.

Theorem 6.4 ([PSR, Li92, LR05]). *We have the identity*

$$I(\varphi, \varphi', \phi, s-\frac{n}{2})$$
$$=\sqrt{D_{F^+}}^{-n^2} B(\pi)_{\alpha,\varphi,\varphi'} \cdot \prod_{v\in S_\pi} Z_v(s-\frac{n}{2}, \varphi_v, \varphi'_v, \phi_v) d^{S_\pi}(s-\frac{n}{2}, \alpha)^{-1} L^{\mathrm{mot},S}(s, \pi, \alpha, St)$$

Here

$$d^{S_\pi}(s, \alpha) = \prod_{0\leq j<n-1} L^{S_\pi}(2s + n - j, \alpha^+ \cdot \varepsilon_{F/F^+}) \tag{6.4}$$

is the product of partial Hecke L-functions of F^+, as in [Gu16, §4.4], and $L^{\mathrm{mot}}(s, \pi, \alpha, St)$ denotes the L-function attached to the standard representation of the L-group of G_V, in the motivic normalization. We have written α^+ for the restriction of α to the idèles of F^+. Thus $\alpha^+_\infty(t_\sigma) = t^\nu_\sigma$ for any σ, so that

$$\xi_{\alpha,0} := ||\bullet||^{-\nu} \cdot \xi_\alpha$$

is a Hecke character of F^+ of finite order. We then have (replacing the partial L-function by the complete L-function, since they are equal up to scalars in the field of coefficients)

$$\begin{aligned} d(s_0, \alpha) &= \prod_{0\leq j<n-1} L(2s_0 + n - j - \nu, \xi_{\alpha,0} \cdot \varepsilon_{F/F^+}) \\ &\sim \prod_{0\leq j<n-1} (2\pi i)^{(2s_0+n-j-\nu)[F^+:\mathbb{Q}]} \cdot \delta([\xi_{\alpha,0} \cdot \varepsilon_{F/F^+}])^{[\frac{n+1}{2}]} \cdot |D^{\frac{1}{2}}_{F^+}|^{[\frac{n}{2}]}. \end{aligned} \tag{6.5}$$

Here $\delta(([\xi_{\alpha,0} \cdot \varepsilon_{F/F^+}])$ is defined as in [Gu16] (2.2.2) (essentially a version of the Gauss sum multiplied by a power of the square root of the discriminant) and $[\bullet]$ denotes the greatest integer function.

Now suppose φ and φ' correspond to elements of

$$[\varphi] \in H^{\dim Y_V}_!(Sh(V, \Sigma), \mathcal{E}_\kappa)[\pi]$$

and

$$[\varphi'] \in H^{\dim Y_V}_!(Sh(-V, \Sigma), \mathcal{E}_{\kappa^\flat} \otimes \Lambda_{\alpha^U_\infty})[\pi^\vee \otimes (\alpha \circ \det)^{-1}].$$

Corollary 6.2.1. *Suppose m is critical for κ and (a, b). Suppose*

$$E(m, \phi, \kappa, h) = \Delta(m, (a, b), \kappa)E(m, \phi)(h)$$

where $\phi \in I(m, \alpha)$. Then

$$I(\varphi, \varphi', \phi, m - \frac{n}{2})$$
$$= \sqrt{D_{F^+}}^{-n^2} B(\pi)_{\alpha,\varphi,\varphi'} \cdot \prod_{v \in S_\pi} Z_v(m - \frac{n}{2}, \varphi_v, \varphi'_v, \phi_v) \frac{L^{\text{mot},S}(m, \pi, \alpha, St)}{d^{S_\pi}(m - \frac{n}{2}, \alpha)}$$

6.3 Review of the Local Theory

Notation is as in the previous section. We assume φ_v and φ'_v define $\overline{\mathbb{Q}}$-rational vectors for all v (equivalently, we assume φ defines a $\overline{\mathbb{Q}}$-rational section of an appropriate automorphic vector bundle). The following is then proved as in [H97, Lemma 3.5.6] or [Gu16, Lemma 4.5.2].

Lemma 6.4. *For every finite place v, there exists a finite section $\phi_{v,s}(\bullet) \in I_v(\alpha, s)$, rational over $E(\alpha)$ such that*

$$Z_{S_\pi}(m - \frac{n}{2}, \varphi, \varphi', \phi) := \prod_{v \in S_\pi} Z_v(m - \frac{n}{2}, \varphi_v, \varphi'_v, \phi_v) \in \overline{\mathbb{Q}}^\times.$$

Moreover, for any $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, we have

$$\gamma(Z_{S_\pi}(m - \frac{n}{2}, \varphi, \varphi', \phi)) = Z_{S_{\gamma(\pi)}}(m - \frac{n}{2}, \gamma(\varphi), \gamma(\varphi'), \gamma(\phi)).$$

If γ does not fix $E(\pi) \cdot E(\alpha)$ then $\gamma(\pi)$ is not generally an automorphic representation of G_V, but rather of some inner form, as in [H13], which is isomorphic to G_V at all finite places. Nevertheless, $\gamma(\pi)$ is coherent cohomological and the rationality of $\gamma(\varphi)$ and $\gamma(\varphi')$ are determined relative to that structure.

6.5 The Archimedean Theory

The following result of Garrett [H08] has already been used in §3.5 of [H97] and in Lemma 4.5.1 of [Gu16]; we record it here for future reference. Note that Garrett's computation is native to the unitary group and makes no reference to the similitude factor. We recall that the local factors are defined relative to the choice of Haar measure made in [H97].

Lemma 6.6. *For every archimedean place σ, with $\phi_{\sigma,s}(\bullet)$ defined as in Lemma 4.6, the local factor*

$$Z_\sigma(m - \frac{n}{2}, \varphi_\sigma, \varphi'_\sigma, \phi_\sigma) \in \sigma(F)^\times.$$

We denote this factor $\mathfrak{z}_\sigma$.

Moreover, for every $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and every $\sigma : F \hookrightarrow \mathbb{C}$, we have

$$\gamma(\mathfrak{z}_\sigma) = \mathfrak{z}_{\gamma\circ\sigma},$$

where the latter is viewed as a local zeta integral for the conjugate Shimura variety $\gamma(Sh(V, \Sigma)) = Sh(V, \gamma(\Sigma))$.

Proof. We clarify the last assertion. Write $Z_\sigma = Z_\sigma(m - \frac{n}{2}, \varphi_\sigma, \varphi'_\sigma, \phi_\sigma)$. Theorem 2.1 of [Ga08] asserts that a zeta integral Z'_σ is a $\sigma(F)$-rational multiple of a constant C, which in turn is a $\sigma(F)$-rational multiple of π^{pq}. As explained in [H08], taking into account the difference between Garrett's normalization of measures and the one given in (2.5), we can conclude that

$$\mathfrak{z}_\sigma := Z_\sigma(m - \frac{n}{2}, \varphi_\sigma, \varphi'_\sigma, \phi_\sigma) \in \sigma(F).$$

It remains to analyze the behavior under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of the $\sigma(F)$-rational constants $\mathfrak{z}_\sigma$ that arise in the computation. Following the argument in §2 of [Ga08], one observes that $\mathfrak{z}_\sigma$ is a product of two factors $\mathfrak{z}'_\sigma \cdot \mathfrak{z}''_\sigma$, determined by a $\sigma(F)$-rational structure on the representation W_κ of U_σ and on $Lie(\mathfrak{g}_\sigma)$. The factor $\mathfrak{z}'_\sigma$ is an element of $End_{\sigma(F)}(W_\kappa)$. Note that (bearing in mind Remarks 3.3 and 4.2) the group U_σ becomes isomorphic to $GL(r_\sigma) \times GL(s_\sigma)$ over F, and the representations of U_σ are thus defined over $\sigma(F)$. The factor $\mathfrak{z}''_\sigma$ is a normalization of the measure which comes down to the choice of ω_σ in Sect. 2.5.

Now it follows from (2.4) that, for $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\begin{aligned}
\gamma(\omega_\sigma) &= \gamma[\sigma(\daleth)^{-\dim X_\sigma} \bigwedge_{i=1}^{\dim X_\sigma} dz_{\sigma,i} \wedge d\bar{z}_{\sigma,i}] \\
&= \gamma[\sigma(\daleth)^{-\dim X_\sigma} \bigwedge_{i=1}^{\dim X_\sigma} dz_{\sigma,i} \wedge dz_{c\circ\sigma,i}] \\
&= (\gamma\circ\sigma)(\daleth)^{-\dim X_\sigma} \bigwedge_{i=1}^{\dim X_\sigma} dz_{\gamma\circ\sigma,i} \wedge d\bar{z}_{c\circ\gamma\circ\sigma,i} \\
&= \omega_{\gamma\circ\sigma}
\end{aligned}$$

Thus

$$\gamma(\bigwedge_{\sigma\in\Sigma}\omega_\sigma) = \bigwedge_{\sigma'\in\gamma(\Sigma)}\omega_{\sigma'}.$$

Thus the collection of factors $\mathfrak{z}''_\sigma$ is equivariant with respect to $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Noting that a similar rule applies to $\mathfrak{z}'_\sigma$, the equivariance property follows. □

7 Main Results

Theorem 7.1. *Let π be an automorphic representation of G_V of type κ, and let α be a Hecke character of F of type $(0,(a,b))$, of weight ν. Suppose $m \geq \frac{n-\nu}{2}$ is critical, in the sense of Definition 5.2. Define*

$$P(V,\alpha) = \prod_{\sigma\in\Sigma} p_F(\alpha,\sigma)^{-s_\sigma} p_F(\alpha, c\sigma)^{-r_\sigma}.$$

Then

(i) *If $m > n - \frac{\nu}{2}$, then*

$$L(m,\pi,St,\alpha) \sim_{E(\pi,\alpha,\Sigma)} \\ |D_{F^+}|^{\frac{\lfloor\frac{n+1}{2}\rfloor - n^2}{2}} (2\pi i)^{|\Sigma|(mn-\frac{n(n-1)}{2})}\cdot\delta(\varepsilon_F)^{\lfloor\frac{n+1}{2}\rfloor}\cdot P(V,\alpha)\cdot B(\pi)^{-1}. \tag{7.1}$$

Here $\delta(\varepsilon_F)$ is the period of the Artin motive attached to the quadratic character for the extension F/F^+, as in the introduction to [GL].

(ii) *Suppose (a) $n-\frac{\nu}{2}\geq m\geq\frac{n-\nu}{2}$, and (b) $m\neq\frac{n-\nu+1}{2}$. Then* (7.1) *remains true, provided α satisfies* (4.3)*, i.e.*

$$\alpha|_{\mathbf{A}^\times_{F^+}} = \varepsilon^\nu_{\mathcal{K}}.$$

(iii) *Suppose $m = \frac{n-\nu+1}{2}$ (the near central point). Suppose that α satisfies* (4.3) *and, for some finite place v of F^+ that does not split in F, α_v does not belong to the finite set A_v of characters introduced in Theorem 3.4 of* [H07]. *Then* (7.1) *is valid for π and α.*

Proof. This is simply an adaptation of the proofs in [H97, H07, H08, Gu16] based on unitary similitude groups. When $F=\mathcal{K}$ is imaginary quadratic then the formula for $P(V,\alpha)$ specializes to the expression $p_{\mathcal{K}}(\alpha,1)^{-s}p_{\mathcal{K}}(\alpha^\iota,1)^{-r}$ at the end of the third line on p. 138 of [H97]. The subsequent lines of that calculation, when $(a,b) = (-\kappa,0)$, yield the expression $(2\pi i)^{\kappa s}g(\alpha_0)^s p_{\mathcal{K}}(\check{\alpha},1)^{r-s}$ that is the contribution of α to the final formula of [H97, Theorem 3.5.13]. In the general case, α's contribution in the computation made using the GU-periods is the expression $P(V,\alpha)$ given above. We thus just need to explain (a) why this doesn't change when we work with the Shimura variety $Sh(V, \Sigma)$; (b) the appearance of the power of the discriminant, and (c) why the period $B(\pi)^{-1}$ is the appropriate replacement for $P^{(s)}(\pi, *, \beta)$.

We interpret Corollary 6.2.1 in terms of our notation. We assume the Eisenstein series that occurs in the integral $I(m, \varphi, \varphi', \phi)$ is the function $E^{\infty}(m, \phi, \kappa, h)$, where

- ϕ_∞ satisfies the conditions of Lemma 6.6, and
- $p_U(\alpha, \Sigma, W)^{-1}\phi_f$ takes values in $E(\pi, \alpha)$.

Suppose moreover that φ and φ' define L-rational classes in coherent cohomology. Then $I(m, \varphi, \varphi', \phi)$ is the cup product between the L-rational coherent cohomology classes, and thus

$$I(m, \varphi, \varphi', \phi) \in L. \tag{7.2}$$

On the other hand, it follows from Lemmas 6.4 and 6.6, and our hypothesis on ϕ_f, that

$$p(\alpha, \Sigma)^{-1} \prod_{v \in S_\pi} Z_v(s - \frac{n}{2}, \varphi_v, \varphi'_v, \phi_v) \in L. \tag{7.3}$$

Completing the partial Euler products, Theorem 6.4 and (3.13) then yields

$$\begin{aligned} L^{\text{mot}}(s, \pi, \alpha, St) &\sim p_U(\alpha, \Sigma, W) p_U(\alpha, -V)^{-1} B(\pi)^{-1} d(s - \frac{n}{2}, \alpha)^{-1} \sqrt{D_{F^+}}^{n^2} \\ &\sim p(\alpha, \Sigma, W) p(\alpha, -V)^{-1} B(\pi)^{-1} d(s - \frac{n}{2}, \alpha)^{-1} \sqrt{D_{F^+}}^{n^2} \end{aligned} \tag{7.4}$$

where we have used Lemma 3.6 and Proposition 4.8 to simplify the local factors. This explains point (a) above, because the factors related to α are now the same as in the similitude calculation. The discriminant factor in the final formula is the product of the factor $\sqrt{D_{F^+}}^{n^2}$ in (7.4) and the discriminant factor in the expression (6.5) for $d(s - \frac{n}{2}, \alpha)^{-1}$. Finally the power of π is the same as in Theorem 3.6.1 of [GL]. $\square$

Remark 7.1. As far as I can tell, Shimura's proofs in [Sh97], which apply to holomorphic automorphic forms of scalar weight, do not require that α satisfy the hypothesis of (ii) when treating values below the range of absolute convergence. The proofs in [H08] use the Siegel-Weil formula for critical values in this range, whereas Shimura uses an explicit evaluation of the Fourier coefficients of holomorphic Eisenstein series below the range of absolute convergence. The treatment of these values in [EHLS]—where less attention is paid to rationality questions—is based on Shimura's calculations, but it applies as well to vector-valued forms. In particular, [EHLS] implicitly contains a proof of Theorem 7.1 (ii) and (iii) without the indicated restrictions on α.

The reasons for the restriction on α at the near central point are developed at length in [H07]. The point is roughly the following: for each finite place v of F^+ that is not split in F, there are two hermitian spaces $\mathcal{W}_v^+$ and $\mathcal{W}_v^-$ of dimension m. For given π_v and splitting character α_v, the theta lift of π_v to at least one of the $U(\mathcal{W}_v^{\pm})$ is non-trivial. At the central value $m = \frac{n-\nu}{2}$, exactly one of these lifts is non-trivial. Moreover, if v is archimedean then the theta lift of π_v is non-trivial to exactly one of the corresponding unitary groups (a totally definite one). The collection of local $\mathcal{W}_v^{\pm}$ is called *coherent* (following Kudla and Rallis) if and only if there is a global hermitian space $\mathcal{W}$ with the given localizations. Thus at the central value, there is exactly one global unitary group $U(\mathcal{W})$ to which the theta lift of π is non-trivial if and only if the central L-value does not vanish. On the other hand, if $m > \frac{n-\nu+1}{2}$, then it is proved in [H07] that the theta lifts to both $U(\mathcal{W}_v^{\pm})$ are non-trivial as long as π_v is tempered, which is known to be the case in our applications. However, at the near central point $m = \frac{n-\nu+1}{2}$, it is possible that π_v has a theta lift—for the splitting characters α_v—to only one of the $U(\mathcal{W}_v^{\pm})$. If the resulting collection of local hermitian spaces is incoherent then the Rallis inner product formula used in [H08] does not provide an interpretation of the near central value. It is shown in [H07] that this only happens at most for a finite set of α_v at each v; if α_v belongs to this finite set for all v, then π_v is locally a theta lift at all non-split finite v from a $U(\mathcal{W}_v)$ with $\dim \mathcal{W}_v = n - 1$. There seems to be no way a priori to exclude this possibility.

The most comprehensive analysis of the Rallis inner product formula, and of its implications for theta lifts, is contained in [GQT], which contains a complete proof of the second term identity.

8 Unitary and Similitude Periods

The map u of (2.2) is surjective on points over any field; indeed, the restriction of u to $\{1\} \times U(V)$ is just the identity on points. The image of $U(V)(\mathbb{Q})\backslash U(\mathbf{A})$ in $Sh(GU(V), X_V)$ is an open and closed variety that we denote $Sh^U(V)$. It follows that the integral defining $B(\pi)_{f,f'}$ in (3.10) can be viewed as an integral over $Sh^U(V)$. However, the forms f and f' chosen to be rational on $Sh(V, \Sigma)$ over some field L are not rational on $Sh^U(V)$. We assume $f \in \pi$ and $f' \in \pi^\vee$, as before; then there are constants

$$b_U(\pi, V) = (b_U(\pi, V, \tau), \tau \in \Sigma) \in (E(\pi) \otimes \mathbb{C})^\times;$$

$$b_U(\pi^\vee, V^\vee) = b_U(\pi^\vee, -V, \tau), \tau \in \Sigma) \in (E(\pi) \otimes \mathbb{C})^\times$$

such that $b_U(\pi, V)f$ and $b_U(\pi^\vee, V^\vee)f'$ are L-rational elements of

$$H^{\dim Y_V}(Sh^U(V), \mathcal{E}_\kappa) \text{ and } H^{\dim Y_V}(Sh^U(-V), \mathcal{E}_{\kappa^\flat})$$

respectively. Here I use the same notation to denote the automorphic vector bundles on $Sh(V, \Sigma)$ and on $Sh^U(V)$, although this is abusive because the former is attached to (for example) a representation of the stabilizer of $y \in Y_V$ whereas the latter is attached to a representation of the stabilizer of y in X_V.

Define

$$B(\pi)^U_{f,f'}=[b_U(\pi,V)f, F_\infty(b_U(\pi^\vee),V^\vee)f')]_{Ser}=b_U(\pi,V)\cdot b_U(\pi^\vee,V^\vee)B(\pi)_{f,f'},$$

where the second equality is clear because F_∞ and Serre duality are linear.

Lemma 8.1. *With notation as above, we have*

$$b_U(\pi,V)\cdot b_U(\pi^\vee,V^\vee)\sim 1.$$

In particular, $B(\pi)\sim_{E(\pi)} B(\pi)^U$.

Proof. Let ξ_π denote the central character of π; it is an automorphic representation of $U(1)$. Let $\pi_u = \pi\circ u$ be the pullback to G''_V of π, and let $\tilde{\pi}_u$ denote some extension of π_u to $GU(V)\times GU(1)$. Then we can factor $\tilde{\pi}_u$ as $\tilde{\pi}\otimes\tilde{\xi}_\pi^{-1}$, where $\tilde{\pi}$ is an extension of π to $GU(V)$ and $\tilde{\xi}_\pi$ is an extension of ξ_π from $U(1)$ to $GU(1)$. Define $\tilde{\pi}_u^\vee$, $\tilde{\pi}^\vee$, and $\tilde{\xi}_{\pi^\vee}^{-1}$ likewise. We can and do assume that $\tilde{\xi}_\pi$ is a motivic Hecke character and that

$$\tilde{\xi}_{\pi^\vee}^{-1}=\tilde{\xi}_\pi.$$

Now the statement is clear because the constants $b_U(\pi,V)$ and $b_U(\pi^\vee,V^\vee)$ depend only on the restrictions of $\tilde{\pi}_u$ and $\tilde{\pi^\vee}_u$ to the factor $(GU(1),X_1)$ of π and $\pi^\vee$, and these are $\tilde{\xi}_\pi^{-1}$ and $\tilde{\xi}_\pi$, respectively. More precisely, we view the Shimura variety $Sh(G''_V,X''_V)$ as an open and closed subvariety of the product $Sh(GU(V),X_V)\times Sh(GU(1),X_1)$ and one checks that the constants $b_U(\pi,V)$ and $b_U(\pi^\vee,V^\vee)$ are determined by viewing the pullbacks via u of f and f' as cohomology classes of this open and closed subvariety of the product $Sh(GU(V),X_V)\times Sh(GU(1),X_1)$, with coefficients in the bundles $\mathcal{E}_\kappa$ and $\mathcal{E}_{\kappa^\flat}$. Using the product structure, the result follows. □

Remark 8.2. Note that the coefficient field $E(\pi)$ in Lemma 8.1 is defined relative to the rational structure on $Sh(V,\Sigma)$, and in particular is an extension of the reflex field $E(V,\Sigma)$. This is in general strictly larger than the reflex field $E(GU(V),X_V)$. Thus if π extends to an automorphic representation $\tilde{\pi}$ of $GU(V)$, the fields $E(\pi)$ and $E(\tilde{\pi})$ are in general not the same. We use the same $E(\pi)$ in Proposition 8.3 below.

Proposition 8.3. *Suppose there is a character α and a critical value m of $L(s,\pi,\alpha,St)$ such that $L(m,\pi,\alpha,St)\neq 0$. Then*

$$B(\pi)^U\sim_{E(\pi)} Q^*(\pi)$$

where $Q^*(\pi)$ is the period that occurs in Corollary 3.7.2 of [GL].

Proof. We compare two formulas for the non-vanishing critical value $L(m, \pi, \alpha, St)$. The one in [GL] is of the form

$$L(m, \pi, \alpha, St) \sim_{E(\pi,\alpha)} A(\alpha, m) \cdot Q^*(\pi)^{-1}$$

and the one in Theorem 7.1 is of the form

$$L(m, \pi, \alpha, St) \sim_{E(\pi,\alpha)} A(\alpha, m) \cdot B(\pi)^{-1},$$

for the same non-vanishing $A(\alpha, m)$. Thus, letting α vary among its Galois conjugates, we have

$$B(\pi) \sim_{E(\pi)} Q^*(\pi),$$

and the Proposition follows from Lemma 8.1. □

Acknowledgments The immediate motivation for this paper was to provide a convenient comparison between periods computed on similitude groups, as in [H97, Gu16, GL], and periods computed on unitary groups, which appear in the Ichino-Ikeda and N. Harris conjectures. The comparison is applied to factorization of automorphic periods in [GHL]. I thank my co-authors Harald Grobner and Lin Jie for discussions of this point that made clear the need for a paper like this one. I also thank Wee Teck Gan for an exchange that helped to clarify the limits of the approach to special values based on the Rallis inner product formula. Finally, I thank the organizers of the Simons Symposium on Relative Trace Formulas—Werner Müller, Sug Woo Shin, and Nicolas Templier—for their invitation, for their eclectic choice of topics, and for encouraging me to contribute to the proceedings.

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