DEFORMATIONS OF VARIETIES OF GENERAL TYPE

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Abstract. We prove that small deformations of a projective variety of general type are also projective varieties of general type, with the same plurigenera.

Our aim is to prove the following.

Theorem 1. *Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Fix a point $0 \in S$ and assume that the fiber X_0 is projective, of general type, and with canonical singularities. Then there is an open neighborhood $0 \in U \subset S$ such that*

(1.1) *the plurigenera of X_s are independent of $s \in U$ for every r, and*

(1.2) *the fibers X_s are projective for every $s \in U$.*

Here the rth plurigenus of X_s is $h^0(Y_s, \omega_{Y_s}^r)$, where $Y_s \to X_s$ is any resolution of X_s. By [Nak04, VI.5.2] (see also (10.2)) X_s has canonical singularities, so this is the same as $h^0(X_s, \omega_{X_s}^{[r]})$, where $\omega_{X_s}^{[r]}$ denotes the double dual of the rth tensor power $\omega_{X_s}^{\otimes r}$.

Comments 1.3. Many cases of this have been proved, but I believe that the general result is new, even for X_0 smooth and S a disc.

For smooth surfaces proofs are given in [KS58, Iit69], and for 3-folds with terminal singularities in [KM92, 12.5.1]. If g is assumed projective, then of course all fibers are projective, and deformation invariance of plurigenera was proved by [Siu98] for X_0 smooth, and by [Nak04, Chap.VI] when X_0 has canonical singularities. However, frequently g is not projective; see Example 4 for some smooth, 2-dimensional examples. Many projective varieties have deformations that are not projective, not even algebraic in any sense; K3 and elliptic surfaces furnish the best known examples.

In Example 3 we construct a deformation of a projective surface with a quotient singularity and ample canonical class, whose general fibers are non-algebraic, smooth surfaces of Kodaira dimension 0. Thus canonical is likely the largest class of singularities where Theorem 1 holds. See also Example 5 for surfaces with simple elliptic singularities.

The projectivity of X_0 is essential in our proof, but (1.1) should hold whenever X_0 is a proper algebraic space of general type with canonical singularities. Such results are proved in [RT20], provided one assumes that either X_0 is smooth and all fibers are Moishezon, or almost all fibers are of general type.

Our main technical result says that the Minimal Model Program works for $g : X \to S$. For $\dim X_0 = 2$ and X_0 smooth, this goes back to [KS58]. For $\dim X_0 = 3$ and terminal singularities, this was proved in [KM92, 12.4.4]. The next result extends these to all dimensions.

Theorem 2. *Let $g : X \to S$ be a flat, proper morphism of reduced, complex analytic spaces. Fix a point $0 \in S$ and assume that X_0 is projective and has canonical singularities. Then every sequence of MMP-steps $X_0 = X_0^0 \dashrightarrow X_0^1 \dashrightarrow X_0^2 \dashrightarrow \cdots$ (see Definition 7) extends to a sequence of MMP-steps*

$$X = X^0 \dashrightarrow X^1 \dashrightarrow X^2 \dashrightarrow \cdots,$$

over some open neighborhood $0 \in U \subset S$.

The proof is given in Paragraph 8 when S is a disc $\mathbb{D}$, and in Paragraph 12 in general. The assumption that X_0 has canonical singularities is necessary, as shown by semistable 3-fold flips [KM92]. Extending MMP steps from divisors with canonical singularities is also studied in [AK19].

If X_0 is of general type, then a suitable MMP for X_0 terminates with a minimal model X_0^{m} by [BCHM10], which then extends to $g^{\mathrm{m}} : X_U^{\mathrm{m}} \to U$ by Theorem 2. For minimal models of varieties of general type, deformation invariance of plurigenera is easy, leading to a proof of (1.1) in Paragraph 13. This also implies that all fibers are bimeromorphic to a projective variety.

If X_0 is smooth, then it is Kähler, and the X_s are also Kähler by [KS58]. A Kähler variety that is bimeromorphic to an algebraic variety is projective by [Moi66].

However, there are families of surfaces with simple elliptic singularities $g : X \to S$ such that K_{X_0} is ample, all fibers are bimeromorphic to an algebraic surface, yet the projective fibers correspond to a countable, dense set on the base; see Example 5.

We use Theorem 14—taken from [Kol21b, Thm.2]—to obtain the projectivity of the fibers and complete the proof of Theorem 1 in Paragraph 13.

1. Examples and consequences

The first example shows that Theorem 1 fails very badly for surfaces with non-canonical quotient singularities.

Example 3. We give an example of a flat, proper morphism of complex analytic spaces $g : X \to \mathbb{D}$, such that

(3.1) X_0 is a projective surface with a quotient singularity and ample canonical class, yet

(3.2) X_s is smooth, non-algebraic, and of Kodaira dimension 0 for very general $s \in \mathbb{D}$.

Let us start with a K3 surface $Y_0 \subset \mathbb{P}^3$ with a hyperplane section $C_0 \subset Y_0$ that is a rational curve with 3 nodes. We blow up the nodes $Y_0' \to Y_0$ and contract the birational transform of C_0 to get a surface $\tau_0 : Y_0' \to X_0$. Let $E_1, E_2, E_3 \subset X_0$ be the images of the 3 exceptional curves of the blow-up.

By explicit computation, we get a quotient singularity of type $\mathbb{C}^2/\frac{1}{8}(1,1)$, $(E_i^2) = -\frac{1}{2}$ and $(E_i \cdot E_j) = \frac{1}{2}$ for $i \neq j$. Furthermore, $E := E_1 + E_2 + E_3 \sim K_{X_0}$ and it is ample by the Nakai-Moishezon criterion. (Note that $(E \cdot E_i) = \frac{1}{2}$ and $X_0 \setminus E \cong Y_0 \setminus C_0$ is affine.)

Take now a deformation $Y \to \mathbb{D}$ of Y_0 whose very general fibers are non-algebraic K3 surfaces that contain no proper curves. Take 3 sections $B_i \subset Y$ that pass through the 3 nodes of C_0. Blow them up and then contract the birational transform of C_0; cf. [MR71]. In general [MR71] says that the normalization of the resulting central fiber is X_0, but in our case the central fiber is isomorphic to X_0 since $R^1(\tau_0)_*\mathcal{O}_{Y_0'} = 0$. The contraction is an isomorphism on very general fibers since there are no curves to contract. We get $g : X \to \mathbb{D}$ whose central fiber is X_0 and all other fibers are K3 surfaces blown up at 3 points.

In general, it is very unclear which complex varieties occur as deformations of projective varieties; see [KLS21] for some of their properties.

Example 4. [Ati58] Let $S_0 := (g = 0) \subset \mathbb{P}^3_{\mathbf{x}}$ and $S_1 := (f = 0) \subset \mathbb{P}^3_{\mathbf{x}}$ be surfaces of the same degree. Assume that S_0 has only ordinary nodes, S_1 is smooth, $\operatorname{Pic}(S_1)$ is generated by the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ and S_1 does not contain any of the singular points of S_0. Fix $m \geq 2$ and consider

$$X_m := (g - t^m f = 0) \subset \mathbb{P}^1_{\mathbf{x}} \times \mathbb{A}^1_t.$$

The singularities are locally analytically of the form $xy + z^2 - t^m = 0$. Thus X_m is locally analytically factorial if m is odd. If m is even then X_m is factorial since the general fiber has Picard number 1, but it is not locally analytically factorial; blowing up $(x = z - t^{m/2} = 0)$ gives a small resolution. Thus we get that

(4.1) X_m is bimeromorphic to a proper, smooth family of projective surfaces iff m is even, but

(4.2) X_m is not bimeromorphic to a smooth, projective family of surfaces.

Example 5. Let $E \subset \mathbb{P}^2$ be a smooth cubic and take r general lines $L_i \subset \mathbb{P}^2$. To get S_0, blow up all singular points of $E + \sum L_i$ and then contract the birational transform of $E + \sum L_i$. A somewhat tedious computation shows that K_{S_0} is ample for $r \geq 6$. It has 1 simple elliptic singularity (coming from E) and r quotient singularities (coming from the L_i).

Deform this example by moving the $3r$ points $E \cap \sum L_i$ into general position $p^1_t, \dots, p^{3r}_t \in E$ and the points $L_i \cap L_j$ into general position on $\mathbb{P}^2$. Blow up these points and then contract the birational transform of E to get the surfaces S_t. It has only 1 simple elliptic singularity (coming from E).

We get a flat family of surfaces with central fiber S_0 and general fibers S_t. Let L denote the restriction of the line class on $\mathbb{P}^2$ to E.

It is easy to see that such a surface S_t is non-projective if the p^i_t and L are linearly independent in $\operatorname{Pic}(E)$. Thus S_t is not projective for very general t and has Kodaira dimension 0.

The next result is the scheme-theoretic version of Theorem 1. Ideally it should be proved by the same argument. However, some of the references we use, especially [Nak04], are worked out for analytic spaces, not for general schemes. So for now we proceed in a somewhat roundabout way.

Corollary 6. *Let S be a noetherian, excellent scheme over a field of characteristic 0. Let $g : X \to S$ be a flat, proper algebraic space. Fix a point $0 \in S$ and assume that X_0 is projective, of general type and with canonical singularities. Then there is an open neighborhood $0 \in S^\circ \subset S$ such that, for every $s \in S^\circ$,*

(6.1) *the plurigenera $h^0(X_s, \omega^{[r]}_{X_s})$ are independent of s for every r, and*

(6.2) *the fiber X_s is projective.*

Proof. A proper algebraic space Y over a field k is projective iff Y_K is projective over K for some field extension $K \supset k$. Noetherian induction then shows that it is enough to prove the claims for the generic points of the completions (at the point $0 \in S$) of irreducible subvarieties $0 \in T \subset S$. Since the defining equations of $\hat{T}$

and of $X \times_S \hat{T}$ involve only countably many coefficients, we may assume that the residue field is $\mathbb{C}$.

Consider now the local universal deformation space $\operatorname{Def}(X_0)$ of X_0 in the complex analytic category; see [Bin87]. It is the germ of a complex analytic space and there is a complex analytic universal family $G : \mathbf{X} \to \operatorname{Def}(X_0)$. Since a deformation over an Artin scheme is automatically complex analytic, we see that the formal completion $\hat{G} : \hat{\mathbf{X}} \to \widehat{\operatorname{Def}}(X_0)$ is the universal formal deformation of X_0. In particular, $X \times_S \hat{T}$ is the pull-back of $\hat{G} : \hat{\mathbf{X}} \to \widehat{\operatorname{Def}}(X_0)$ by a morphism $\hat{T} \to \widehat{\operatorname{Def}}(X_0)$. Thus Theorem 1 implies both claims. □

2. Relative MMP

See [KM98] for a general introduction to the minimal model program.

Definition 7 (MMP-steps and their extensions)**.** Let $X \to S$ be a proper morphism of complex analytic spaces with irreducible fibers. Assume that $K_{X/S}$ is $\mathbb{Q}$-Cartier. By an *MMP-step* for X over S we mean a diagram

$$\begin{array}{ccccc} X & & \overset{\pi}{\dashrightarrow} & & X^+ \\ \phi \searrow & & & & \swarrow \phi^+ \\ & & Z & & \end{array} \tag{7.1}$$

where all morphisms are bimeromorphic and proper over S, $-K_{X/S}$ is ample over Z, $K_{X^+/S}$ is ample over Z and ϕ^+ is small (that is, without exceptional divisors).

If X is $\mathbb{Q}$-factorial and the relative Picard number of X/Z is 1, then there are 2 possible MMP steps:

- Divisorial: ϕ contracts a single divisor and ϕ^+ is the identity.
- Flipping: both ϕ and ϕ^+ are small.

However, in general there is a more complicated possibility:

- Mixed: ϕ contracts (possibly several) divisors and ϕ^+ is small.

For our applications we only need to know that, by [KM98, 3.52], X^+ exists iff $\oplus_{r\geq 0}\, \omega_{Z/S}^{[r]}$ (which is equal to $\oplus_{r\geq 0}\phi_*\omega_{X/S}^{[r]}$) is a finitely generated sheaf of $\mathcal{O}_Z$-algebras, and then

$$X^+ = \operatorname{Proj}_Z \oplus_{r\geq 0}\, \omega_{Z/S}^{[r]}. \tag{7.2}$$

We index a sequence of MMP-steps by setting $X^0 := X$ and $X^{i+1} := (X^i)^+$.

Fix a point $s \in S$ and let X_s denote the fiber over S. We say that a sequence of MMP-steps (over S) $X^0 \dashrightarrow X^1 \dashrightarrow X^2 \dashrightarrow \cdots$ *extends* a sequence of MMP-steps (over s) $X^0_s \dashrightarrow X^1_s \dashrightarrow X^2_s \dashrightarrow \cdots$ if, for every i,

$$\begin{array}{ccccc} X^i_s & \overset{\pi^i_s}{\dashrightarrow} & X^{i+1}_s \\ \phi^i_s \searrow & & \swarrow (\phi^i_s)^+ \\ & Z^i_s & \end{array} \quad \text{is the fiber over } s \text{ of} \quad \begin{array}{ccccc} X^i & \overset{\pi^i}{\dashrightarrow} & X^{i+1} \\ \phi^i \searrow & & \swarrow (\phi^i)^+ \\ & Z^i & \end{array} \tag{7.3}$$

8 (Proof of Theorem 2 for $S = \mathbb{D}$, the disc)**.** Since MMP-steps preserve canonical singularities, by induction it is enough to prove the claim for one MMP step. So we drop the upper index i and identify $K_{X/\mathbb{D}}$ with K_X.

Let $\phi_0 : X_0 \to Z_0$ be an extremal contraction. By [MR71][1], it extends to a contraction $\phi : X \to Z$, where Z is flat over $\mathbb{D}$ with central fiber Z_0 since

[1] This should be changed to [KM92, 11.4]

$R^1(\phi_0)_*\mathcal{O}_{X_0} = 0$. Note that K_X is $\mathbb{Q}$-Cartier by (10.1), and ϕ is projective since $-K_X$ is ϕ-ample.

If ϕ_0 is a divisorial contraction, then K_{Z_0} is $\mathbb{Q}$-Cartier, and so is K_Z by (10.1). Thus $X^+ = Z$.

If ϕ_0 is a flipping or mixed contraction, then K_Z is not $\mathbb{Q}$-Cartier. By (7.2),

$$X^+ = \operatorname{Proj}_Z \oplus_{r\geq 0} \omega_Z^{[r]}, \tag{8.1}$$

provided $\oplus_{r\geq 0} \omega_Z^{[r]}$ is a finitely generated sheaf of $\mathcal{O}_Z$-algebras. (We have identified ω_Z with $\omega_{Z/\mathbb{D}}$.)

Functoriality works better if we twist by the line bundle $\mathcal{O}_Z(Z_0)$ and write it as

$$X^+ = \operatorname{Proj}_Z \oplus_{r\geq 0} \omega_Z^{[r]}(rZ_0).$$

Let $\tau : Y \to X$ be a projective resolution of X (that is, τ is projective) such that Y_0, the bimeromorphic transform of X_0, is also smooth. Set $g := \phi \circ \tau$.

The hardest part of the proof is Nakayama's theorem (9) which gives a surjection

$$\oplus_{r\geq 0} g_*\omega_Y^r(rY_0) \twoheadrightarrow \oplus_{r\geq 0}(g_0)_*\omega_{Y_0}^r. \tag{8.2}$$

Since X_0 has canonical singularities $\tau_*\omega_{Y_0}^r = \omega_{X_0}^{[r]}$, and hence $g_*\omega_{Y_0}^r = \omega_{Z_0}^{[r]}$. We also have a natural inclusion $g_*\omega_Y^r(rY_0) \hookrightarrow \omega_Z^{[r]}(rZ_0)$. Thus pushing forward (8.2) we get a surjection

$$\oplus_{r\geq 0} g_*\omega_Y^r(rY_0) \to \oplus_{r\geq 0} \omega_Z^{[r]}(rZ_0) \twoheadrightarrow \oplus_{r\geq 0} \omega_{Z_0}^{[r]}. \tag{8.3}$$

Note that $\oplus_{r\geq 0} \omega_{Z_0}^{[r]}$ is a finitely generated sheaf of $\mathcal{O}_{Z_0}$-algebras, defining the MMP-step of $X_0 \to Z_0$.

Now (11) says that $\oplus_{r\geq 0} \omega_Z^{[r]}(rZ_0)$ is also a finitely generated sheaf of $\mathcal{O}_Z$-algebras, at least in some neighborhood of the compact Z_0. $\square$

Next we discuss various results used in the proof.

Theorem 9. [Nak04, VI.3.8] *Let $\pi : Y \to S$ be a projective, bimeromorphic morphism of analytic spaces, Y smooth and S normal. Let $D \subset Y$ be a smooth, non-exceptional divisor. Then the restriction map*

$$\pi_*\omega_Y^m(mD) \to \pi_*\omega_D^m \quad \textit{is surjective for } m \geq 1. \qquad \square$$

This is a special case of [Nak04, VI.3.8] applied with $\Delta = 0$ and $L = K_Y + D$.

Warning. The assumptions of [Nak04, VI.3.8] are a little hard to find. They are outlined 11 pages earlier in [Nak04, VI.2.2]. It talks about varieties, which usually suggest algebraic varieties, but [Nak04, p.231, line 13] explicitly states that the proofs work with analytic spaces; see also [Nak04, p.14]. (The statements of [Nak04] allow for a boundary Δ. However, $K_Y + D + \Delta$ should be $\mathbb{Q}$-linearly equivalent to a $\mathbb{Z}$-divisor and $\lfloor \Delta \rfloor = 0$ is assumed on [Nak04, p.231]. There seem to be few cases when both of these can be satisfied.)

Lemma 10. [Nak04, VI.5.2] *Let $g : X \to S$ be a flat morphism of complex analytic spaces. Assume that X_0 has a canonical singularity at a point $x \in X_0$. Then there is an open neighborhood $x \in X^* \subset X$ such that*

(10.1) *$K_{X^*/S}$ is $\mathbb{Q}$-Cartier, and*

(10.2) *all fibers of $g|_{X^*} : X^* \to S$ have canonical singularities.*

Proof. (1) is proved in [Kol83, 3.2.2]; see also [Kol95, 12.7] and [Kol21a, 2.8]. The harder part is (2), proved in [Nak04, VI.5.2]. □

Remark 10.3. If S is smooth then X^* has canonical singularities. By induction, it is enough to prove this when $S = \mathbb{D}$. Then the proof of [Nak04, VI.5.2] shows that even the pair $(X^*, X_0 \cap X^*)$ has canonical singularities.

Lemma 11. *Let $\pi : X \to S$ be a proper morphism of normal, complex spaces. Let L be a line bundle on X and $W \subset S$ a Zariski closed subset. Assume that $\mathcal{O}_W \otimes_S \big(\oplus_{r\geq 0}\pi_* L^r\big)$ is a finitely generated sheaf of $\mathcal{O}_W$-algebras.*

Then every compact subset $W' \subset W$ has an open neighborhood $W' \subset U \subset S$ such that $\mathcal{O}_U \otimes_S \big(\oplus_{r\geq 0}\pi_* L^r\big)$ is a finitely generated sheaf of $\mathcal{O}_U$-algebras.

Proof. The question is local on S, so we may as well assume that W is a single point. We may also assume that $\mathcal{O}_W \otimes_S \big(\oplus_{r\geq 0}\pi_* L^r\big)$ is generated by $\pi_* L$. After suitable blow-ups we are reduced to the case when the base locus of L is a Cartier divisor D. By passing to a smaller neighborhood, we may assume that every irreducible component of D intersects $\pi^{-1}(W)$. By the Nakayama lemma, the base locus of L^r is a subscheme of rD that agrees with it along $rD \cap \pi^{-1}(W)$. Thus rD is the the base locus of L^r for every r. We may thus replace L by $L(-D)$ and assume that L is globally generated.

Thus L defines a morphism $X \to \operatorname{Proj}_S \oplus_{r\geq 0}\pi_* L^r$, let $\pi' : X' \to S$ be its Stein factorization. Then L is the pull-back of a line bundle L' that is ample on $X' \to S$ and $\oplus_{r\geq 0}\pi_* L^r = \oplus_{r\geq 0}\pi'_* {L'}^r$ is finitely generated. □

12 (Proof of Theorem 2 for general S)**.** As in Paragraph 8, it is enough to prove the claim for one MMP step, so let $\phi_0 : X_0 \to Z_0$ be an extremal contraction and $\phi : X \to Z$ its extension. As before, Z is flat over S with central fiber Z_0.

We claim that, for every r,

(12.1) $\omega_{Z/S}^{[r]}$ is flat over S, and

(12.2) $\omega_{Z/S}^{[r]}|_{Z_0} \cong \omega_{Z_0}^{[r]}$.

In the language of [Kol08] or [Kol21a, Chap.9], this says that $\omega_{Z/S}^{[r]}$ is its own relative hull. There is an issue with precise references here, since [Kol21a, Chap.9] is written in the algebraic setting. However, [Kol21a, 9.72] considers hulls over the spectra of complete local rings. Thus we get that there is a unique largest subscheme $\hat{S}^u \subset \hat{S}$ (the formal completion of S at 0) such that (1–2) hold after base change to $\hat{S}^u$.

By Paragraph 8 we know that (1–2) hold after base change to any disc $\mathbb{D} \to S$, which implies that $\hat{S}^u = \hat{S}$. That is, (1–2) hold for $\hat{S}$. Since both properties are invariant under formal completion, we are done.

Now we know that

$$X^+ := \operatorname{Proj}_Z \oplus_{r\geq 0} \omega_{Z/S}^{[r]}, \tag{12.3}$$

is flat over S and its central fiber is X_0^+. Thus it gives the required extension of the flip of $X_0 \to Z_0$. □

3. Proof of Theorem 1

We give a proof using only the $S = \mathbb{D}$ case of Theorem 2.

13. Fix $r \geq 2$ and assume first that $S = \mathbb{D}$. Since X_0 is of general type, a suitable MMP for X_0 ends with a minimal model $X_0^{\rm m}$, and, by Theorem 2, $X_0 \dashrightarrow X_0^{\rm m}$ extends to a fiberwise bimeromorphic map $X \dashrightarrow X^{\rm m}$. We have $g^{\rm m} : X^{\rm m} \to \mathbb{D}$. (From now on, we replace $\mathbb{D}$ with a smaller disc whenever necessary.) Since $K_{X_0^{\rm m}}$ is nef and big, the higher cohomology groups of $\omega_{X_0}^{[r]}$ vanish for $r \geq 2$. Thus $s \mapsto H^0(X_s^{\rm m}, \omega_{X_s^{\rm m}}^{[r]})$ is locally constant at the origin.

By (10.2) X_s and $X_s^{\rm m}$ both have canonical singularities, so they have the same plurigenera. Therefore $s \mapsto H^0(X_s, \omega_{X_s}^{[r]})$ is also locally constant at the origin. By Serre duality, the deformation invariance of $H^0(X_s, \omega_{X_s})$ is equivalent to the deformation invariance of $H^n(X_s, \mathcal{O}_{X_s})$. In fact, all the $H^i(X_s, \mathcal{O}_{X_s})$ are deformation invariant. For this the key idea is in [DJ74], which treats deformations of varieties with normal crossing singularities. The method works for varieties with canonical (even log canonical) singularities; this is worked out in [Kol21a, Sec.2.5].

For arbitrary S, note that $s \mapsto H^0(X_s, \omega_{X_s}^{[r]})$ is a constructible function on S, thus locally constant at $0 \in S$ iff it is locally constant on every disc $\mathbb{D} \to S$. Once $s \mapsto H^0(X_s, \omega_{X_s}^{[r]})$ is locally constant at $0 \in S$, Grauert's theorem guarantees that $g_* \omega_{X/S}^{[r]}$ is locally free at $0 \in S$ and commutes with base changes.

In principle it could happen that for each r we need a smaller and smaller neighborhood, but the same neighborhood works for all $r \geq 1$ by Lemma 11.

Thus the plurigenera are deformation invariant, all fibers are of general type, and g is fiberwise bimeromorphic to the relative canonical model

$$X^{\rm c} := \operatorname{Proj}_S \oplus_{r \geq 0} g_*^{\rm m} \omega_{X^{\rm m}/S}^{[r]},$$

which is projective over S. The projectivity of all fibers now follows from the more precise Theorem 14. □

The following is a special case of [Kol21b, Thm.2].

Theorem 14. *Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces whose fibers have rational singularities only. Assume that g is bimeromorphic to a projective morphism $g^{\rm p} : X^{\rm p} \to S$, and X_0 is projective for some $0 \in S$.*

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $S = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \quad \text{is projective.} \qquad \square$$

4. Open problems

For deformations of varieties of general type, the following should be true.

Conjecture 15. *Let X_0 be a projective variety of general type with canonical singularities. Then its universal deformation space $\operatorname{Def}(X_0)$ has a representative $\mathbf{X} \to S$ where S is a scheme of finite type and $\mathbf{X}$ is an algebraic space.*

For varieties of non-general type, the following is likely true [RT20, 1.10].

Conjecture 16. *Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Assume that X_0 is projective and with canonical singularities. Then the plurigenera $h^0(X_s, \omega_{X_s}^{[r]})$ are independent of $s \in S$ for every r, in some neighborhood of $0 \in S$.*

Comments. One can try to follow the proof of Theorem 1. If X_0 is not of general type, we run into several difficulties in relative dimensions ≥ 4. MMP is not know to terminate and even if we get a minimal model, abundance is not known. If we have a good minimal model, then we run into the following.

Conjecture 17. *Let X be a complex space and $g : X \to S$ a flat, proper morphism. Assume that X_0 is projective, has canonical singularities and $\omega_{X_0}^{[r]}$ is globally generated for some $r > 0$. Then the plurigenera are locally constant at $0 \in S$.*

Comments. More generally, the same may hold if X_0 is Moishezon (that is, bimeromorphic to a projective variety), Kähler or in Fujiki's class $\mathcal{C}$ (that is, bimeromorphic to a compact Kähler manifold; see [Uen83] for an introduction).

A positive answer is known in many cases. [KM92, 12.5.5] proves this if X_0 is projective and has terminal singularities. However, the proof works for the Moishezon and class $\mathcal{C}$ cases as well.

The projective case with canonical singularities is discussed in [Nak04, VI.3.15–16]; I believe that the projectivity assumption is very much built into the proof given there; see [Nak04, VI.3.11].

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