INEQUALITIES FOR THE DERIVATIVES OF THE RADON TRANSFORM ON CONVEX BODIES

WYATT GREGORY AND ALEXANDER KOLDOBSKY

ABSTRACT. It was proved in [22] that the sup-norm of the Radon transform of an arbitrary probability density on an origin-symmetric convex body of volume 1 is bounded from below by a positive constant depending only on the dimension. In this note we extend this result to the derivatives of the Radon transform. We also prove a comparison theorem for these derivatives.

1. A SLICING INEQUALITY FOR FUNCTIONS.

Let K be an origin-symmetric convex body of volume 1 in $\mathbb{R}^n$, and let f be any non-negative measurable function on K with $\int_K f = 1$. Does there exist a constant c_n depending only on n so that for any such K and f there exists a direction $\xi \in S^{n-1}$ with $\int_{K\cap\xi^\perp} f \geq c_n$? Here $\xi^\perp = \{x \in \mathbb{R}^n : (x,\xi) = 0\}$ is the central hyperplane perpendicular to ξ, and integration is with respect to Lebesgue measure on $\xi^\perp$. It was proved in [22] that, in spite of the generality of the question, the answer to this question is positive, and one can take $c_n > \frac{1}{2\sqrt{n}}$. In [5] this result was extended to non-symmetric bodies K. Moreover, it was shown in [13] that this estimate is optimal up to a logarithmic term, and the logarithmic term was removed in [14], so, finally, $c_n \sim \frac{1}{\sqrt{n}}$. We write $a \sim b$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$.

Note that the same question for volume, where $f \equiv 1$, is the matter of the slicing problem of Bourgain [1, 2]. In this case, the best known result is $c_n > cn^{-\frac{1}{4}}$, where $c > 0$ is an absolute constant, and is due to Klartag [12] who removed a logarithmic term from an earlier result of Bourgain [3].

The constant c_n does not depend on the dimension for several classes of bodies K. For example, it was proved in [23] that if K belongs to the class of unconditional convex bodies, the constant $c_n = \frac{1}{2e}$ works for all functions f and all dimensions n. The same happens for intersection bodies [21], and for the unit balls of subspaces of L_p, $p > 2$, where the constant is of the order $p^{-1/2}$ [25].

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Denote by

$$Rf(\xi,t)=\int_{K\cap\{x\in\mathbb{R}^n:(x,\xi)=t\}} f(x)dx,\qquad \xi\in S^{n-1},\ t\in\mathbb{R}$$

the Radon transform of f. The result described above means that the sup-norm of the Radon transform of a probability density on an origin-symmetric convex body in $\mathbb{R}^n$ is bounded from below by a positive constant depending only on the dimension n.

In this note we prove a similar estimate for the derivatives of the Radon transform. Let us define the fractional derivatives. Let $m\in\mathbb{N}\cup\{0\}$ and suppose that h is an even continuous function on $\mathbb{R}$ that is m times continuously differentiable in some neighborhood of zero. For $q\in\mathbb{C},\ -1<\Re(q)<m,\ q\neq 0,1,...,m-1$, the fractional derivative of the order q of the function h at zero is defined as the action of the distribution $t_+^{-1-q}/\Gamma(-q)$ on the function h, as follows:

$$h^{(q)}(0)=\frac{1}{\Gamma(-q)}\int_0^1 t^{-1-q}\big(h(t)-h(0)-...-h^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!}\big)\ dt+$$

$$\frac{1}{\Gamma(-q)}\int_1^\infty t^{-1-q}h(t)\ dt+\frac{1}{\Gamma(-q)}\sum_{k=0}^{m-1}\frac{h^{(k)}(0)}{k!(k-q)}. \tag{1}$$

It can be seen that for a fixed q the definition does not depend on the choice of $m>\Re(q)$, as long as h is m times continuously differentiable. Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q\in\mathbb{C}:\ \Re(q)>-1\}$ not including integers, and has simple poles at integers. The function $\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers, so after the division we get an analytic function in the whole domain $\{q\in\mathbb{C}:\ m>\Re(q)>-1\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q\to k$, where k is a non-negative integer, we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign (when we compute the limit the first two summands in the right-hand side of (1) converge to zero, since $\Gamma(-q)\to\infty$, and the limit in the third summand can be computed using the property $\Gamma(x+1)=x\Gamma(x)$ of the Γ-function):

$$h^{(k)}(0)=(-1)^k\frac{d^k}{dt^k}h(t)|_{t=0}. \tag{2}$$

The sign does not matter, because h is an even function, and its derivatives of odd orders at the origin are equal to zero. Also, in the case where h is even, for $m-2<\Re q<m$ the expression (1) becomes

$$h^{(q)}(0)=\frac{1}{\Gamma(-q)}\int_0^\infty t^{-1-q}\left(h(t)-\sum_{j=0}^{(m-2)/2}\frac{t^{2j}}{(2j)!}h^{(2j)}(0)\right)dt. \tag{3}$$

We also note that if $-1 < q < 0$ then

$$h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} h(t)\ dt. \tag{4}$$

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 :\ x \in aK\}$ is a continuous function on $\mathbb{R}^n$. If $x \in S^{n-1}$, then $\|x\|_K^{-1} = r_K(x)$ is the radius of K in the direction of x. A star body K is *origin-symmetric* if $K = -K$. A star body K is called *convex* if for any $x, y \in K$ and every $0 < \lambda < 1$, $\lambda x + (1-\lambda) y \in K$.

We say that a star body K in $\mathbb{R}^n$ is infinitely smooth if the restriction to the unit sphere of the Minkowski functional of K belongs to the space $C^\infty(S^{n-1})$ of infinitely differentiable functions on the sphere. For an origin-symmetric infinitely smooth convex body K in $\mathbb{R}^n$, an infinitely differentiable function f on K, fixed $\xi \in S^{n-1}$, and $q \in \mathbb{C}$, $\Re q > -1$, we denote the fractional derivative of the order q at zero of the function $t \to Rf(\xi, t),\ t \in \mathbb{R}$, by

$$(Rf(\xi,t))_t^{(q)}(0) = \left(\int_{K \cap \{x:\ (x,\xi)=t\}} f(x) dx \right)_t^{(q)} (0).$$

The estimate that we prove is as follows.

Theorem 1. *There exists an absolute constant $c > 0$ so that for any infinitely smooth origin-symmetric convex body K of volume 1 in $\mathbb{R}^n$, any even infinitely smooth probability density f on K, and any $q \in \mathbb{R},\ 0 \leq q \leq n-2$, which is not an odd integer, there exists a direction $\xi \in S^{n-1}$ so that*

$$\left(\frac{c(q+1)}{\sqrt{n \log^3(\frac{ne}{q+1})}} \right)^{q+1} \leq \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0). \tag{5}$$

If $q = 2k,\ k \in \mathbb{N} \cup \{0\}$, is an even integer, then $(Rf(\xi,t))_t^{(2k)}(0)$ is the usual derivative and

$$\left(\frac{c(2k+1)}{\sqrt{n \log^3(\frac{ne}{2k+1})}} \right)^{2k+1} \leq (-1)^k (Rf(\xi,t))_t^{(2k)}(0).$$

If $q = 2k-1,\ k \in \mathbb{N}$, is an odd integer, then in the right-hand side of (5) we have $\frac{0}{0}$, and computing the limit as $q \to 2k-1$ we get that there exists a direction $\xi \in S^{n-1}$ so that

$$\left(\frac{2kc}{\sqrt{n \log^3(\frac{ne}{2k})}} \right)^{2k}$$

$$\leq (-1)^k (2k-1)! \int_0^\infty t^{-2k} \left(Rf(\xi,t) - \sum_{j=0}^{k-1} \frac{t^{2j}}{(2j)!} (Rf(\xi,t))_t^{(2j)}(0) \right) dt.$$

We deduce Theorem 1 from a more general result. Note that in the case where $q = 0$ the following theorem was proved in [23].

Theorem 2. *Suppose K is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, f is a non-negative even infinitely smooth function on K and $-1 < q < n-1$ is not an odd integer. Then*

$$\int_K f(x)dx \leq \frac{n}{(n-q-1)\ 2^q \pi^{\frac{q-1}{2}} \Gamma(\frac{q+1}{2})} |K|^{\frac{q+1}{n}} \left(d_{\mathrm{ovr}}(K, L_{-1-q}^n)\right)^{q+1}$$

$$\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0). \tag{6}$$

Here $|K|$ stands for the volume of K. By L_{-1-q}^n we denote the class of star bodies D in $\mathbb{R}^n$ for which the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-1-q}, i.e. the function $\|\cdot\|_D^{-1-q}$ represents a positive definite distribution; see Section 3 for details.

If $\mathcal{A}$ is a class of compact sets in $\mathbb{R}^n$, the outer volume ratio distance from K to $\mathcal{A}$ is defined by

$$d_{\mathrm{ovr}}(K, \mathcal{A}) = \inf \left\{ \left(\frac{|D|}{|K|} \right)^{1/n} : \ K \subset D, \ D \in \mathcal{A} \right\}.$$

Let $0 \leq q \leq n-2$ in Theorem 2. Since $n/(n-q-1) < e^{q+1}$ and by the Stirling formula, if $|K| = 1$ and $\int_K f = 1$, the estimate (6) turns into

$$\max_{\xi \in S^{n-1}} \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0) \geq \left(\frac{c(q+1)}{d_{\mathrm{ovr}}(K, L_{-1-q}^n)} \right)^{\frac{q+1}{2}}, \tag{7}$$

where c is an absolute constant. This means that the lower estimate for the derivatives of the Radon transform is completely controlled by the distance $d_{\mathrm{ovr}}(K, L_{-1-q}^n)$. Indeed, if this distance is equal to 1 or is bounded by an absolute constant, then, for every probability density f and every K with volume 1, the right-hand side of (7) depends only on q :

$$\frac{1}{\cos(\frac{\pi q}{2})} Rf(\xi,t))_t^{(q)}(0) \geq (c(q+1))^{\frac{q+1}{2}}.$$

The distance $d_{\mathrm{ovr}}(K, L_{-1-q}^n)$ is known to be bounded by an absolute constant in the following cases.

Proposition 1. *Let $q \in \mathbb{R}$ be any number from the interval $(-1, n-1)$.*

(i) If K is the unit ball of an n-dimensional subspace of L_p, $0 < p \leq 2$, then $K \in L_{-1-q}^n$, so $d_{\mathrm{ovr}}(K, L_{-1-q}^n) = 1$.

(ii) If K is an unconditional convex body in $\mathbb{R}^n$, i.e. for every vector $(x_1, ..., x_n) \in K$ the vectors $(\pm x_1, ..., \pm x_n) \in K$ for all choices of signs, then $d_{\rm ovr}(K, L^n_{-1-q}) \le e$.

(iii) If K is the unit ball of an n-dimensional subspace of $L_p,\ p > 2$, then $d_{\rm ovr}(K, L^n_{-1-q}) \le c\sqrt{p}$, where c is an absolute constant.

Proof : (i) Proved in [17] and [15, Theorem 6.17].

(ii) It follows from (i) that the ℓ_1^n-ball

$$B_1^n = \{x \in \mathbb{R}^n : |x_1| + ... + |x_n| \le 1\}$$

belongs to L^n_{-1-q} for every $q \in (-1, n-1)$. Also, the classes L^n_{-1-q} are invariant with respect to linear transformations of $\mathbb{R}^n$, which follows from the connection between the Fourier transform and linear transformations, so $T(B_1^n) \in L^n_{-1-q}$ for every linear operator T on $\mathbb{R}^n,\ \det(T) \neq 0$.

By a result of Lozanovskii [30] (see the proof in [36, Corollary 3.4]), there exists a linear operator T on $\mathbb{R}^n$ so that $T(B_\infty^n) \subset K \subset nT(B_1^n)$, where B_∞^n is the cube with sidelength 2 in $\mathbb{R}^n$. Let $D = nT(B_1^n) \in L^n_{-1-q}$. Since $|B_1^n| = 2^n/n!$, we have $|D|^{1/n} \le 2e|\det T|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\det T|$, and $T(B_\infty^n) \subset K$, so $|D|^{1/n} \le e\ |K|^{1/n}$.

(iii) It follows from (i) with $p = 2$ that all n-dimensional origin-symmetric ellipsoids belong to the class L^n_{-1-q} for every $q \in (-1, n-1)$. This can also be shown directly using formula (14). Now the result follows from [33] (see also [25]), where it was proved that the outer volume ratio distance from the unit ball of a subspace of $L_p,\ p > 2$ to the class of ellipsoids is bounded by $c\sqrt{p}$, where c is an absolute constant.

$\square$

Let us show how to get the result of Theorem 1 from the estimate of Theorem 2. In general, one cannot expect to get an estimate for the distance $d_{\rm ovr}(K, L^n_{-1-q})$ independent of the dimension n. In fact, it was shown in [13, 14] that in the case $q = 0$ this distance can be of the order $\sqrt{n}$. Since the classes L^n_{-1-q} contain ellipsoids, one can use John's theorem [9] to prove that $d_{\rm ovr}(K, L^n_{-1-q}) \le \sqrt{n}$ for every origin-symmetric convex body K in $\mathbb{R}^n$ and every $q \in (-1, n-1)$. However, the estimate of Theorem 1 is better for large values of q.

To prove this better estimate, we need to introduce another class of bodies. For $p > 0$, the radial p-sum of star bodies K and L in $\mathbb{R}^n$ is defined as a new star body $K \tilde{+}_p L$ whose radius in every direction $\xi \in S^{n-1}$ is given by

$$r^p_{K\tilde{+}_p L}(\xi) = r^p_K(\xi) + r^p_L(\xi), \qquad \forall \xi \in S^{n-1}.$$

The radial metric in the class of origin-symmetric star bodies is defined by

$$\rho(K, L) = \sup_{\xi \in S^{n-1}} |r_K(\xi) - r_L(\xi)|.$$

Definition 1. *Let $0 < p < n$. We define the class of generalized p-intersection bodies $\mathcal{BP}_p^n$ in $\mathbb{R}^n$ as the closure in the radial metric of radial p-sums of finite collections of origin-symmetric ellipsoids in $\mathbb{R}^n$.*

Note that when $p = k$ is an integer, we get the class of generalized k-intersection bodies introduced by Zhang [39].

The following estimate was proved in [27, Th. 1.1.] for integers p, but the proof remains exactly the same for non-integers. Also note that a mistake in the proof in [27] was corrected in [23, Section 5].

Proposition 2. *([27]) For every $p \in [1, n-1]$ and every origin-symmetric convex body K in $\mathbb{R}^n$*

$$d_{\mathrm{ovr}}(K, \mathcal{BP}_p^n) \le C\sqrt{\frac{n\log^3(\frac{ne}{p})}{p}},$$

where C is an absolute constant.

It was proved in [20] (see also [34, 28]) that for any integer k, $1 \le k < n$ every generalized k-intersection body belongs to the class L_{-k}^n. We need an extension of this fact to non-integers, as follows.

Proposition 3. *For every $0 < p < n$, we have $\mathcal{BP}_p^n \subset L_{-p}^n$.*

Proof : We need to prove that for any star body $K \in \mathcal{BP}_p^n$, the function $\|\cdot\|_K^{-p}$ represents a positive definite distribution in $\mathbb{R}^n$. As mentioned in the proof of Proposition 1, the powers of the Euclidean norm $|x|_2^{-p}$, $0 < p < n$ represent positive definite distributions in $\mathbb{R}^n$; see formula (14). Because of the connection between the Fourier transform of distributions and linear transformations, for any origin-symmetric ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$, the function $\|\cdot\|_{\mathcal{E}}^{-p}$ represents a positive definite distribution. Note that for any unit vector $x \in S^{n-1}$ and any star body K, $r_K(x) = \|x\|_K^{-1}$. Therefore, radial p-sums of ellipsoids in $\mathbb{R}^n$ belong to the class L_{-p}^n. The fact that positive definiteness is preserved under limits in the radial metric follows from [15, Lemma 3.11], which proves the result.

□

Deduction of Theorem 1 from Theorem 2. By Propositions 2 and 3, for any $0 \le q \le n-2$,

$$d_{\mathrm{ovr}}(K, L_{-1-q}^n) \le d_{\mathrm{ovr}}(K, \mathcal{BP}_{q+1}^n) \le C\sqrt{\frac{n\log^3(\frac{ne}{q+1})}{q+1}}. \tag{8}$$

Now the first estimate (5) of Theorem 1 follows from Theorem 2, in the form of (7), combined with (8).

In order to prove the case of even integers in Theorem 1, put $q = 2k$ in (5). In the case of odd integers, we use the expression for the fractional derivative (3) to find the limit in (5) as $q \to 2k-1:$

$$\lim_{q\to 2k-1} \frac{(Rf(\xi,t))_t^{(q)}(0)}{\cos(\frac{\pi q}{2})} = \lim_{q\to 2k-1} \frac{1}{\Gamma(-q)\cos(\frac{\pi q}{2})}$$
$$\times \int_0^\infty t^{-2k} \left(Rf(\xi,t) - \sum_{j=0}^{k-1} \frac{t^{2j}}{(2j)!} (Rf(\xi,t))_t^{(2j)}(0) \right) dt.$$

Now use $\Gamma(x+1) = x\Gamma(x)$ to compute

$$\lim_{q\to 2k-1} \Gamma(-q)\cos\left(\frac{\pi q}{2}\right)$$
$$= \lim_{q\to 2k-1} \frac{\Gamma(-q+2k)}{(-q)(1-q)\cdots(2k-1-q)} \sin\left(\frac{(q-2k+1)\pi}{2}\right)(-1)^k$$
$$= \frac{\pi}{2(2k-1)!}(-1)^k$$

□

The proof of Theorem 2 is presented in Section 4.

We conclude this section by showing the place of the classes L^n_{-1-q} in the general theory of convex bodies. These classes are generalizations of the concept of an intersection body introduced by Lutwak in [31]. Intersection bodies are an important component of Lutwak's dual Brunn-Minkowski theory, and they played the crucial role in the solution of the Busemann-Petty problem; see Section 2.

Definition 2. *([31]) For star bodies D, L in $\mathbb{R}^n$, we say that D is the intersection body of L if*

$$r_D(\xi) = |L \cap \xi^\perp|, \qquad \forall \xi \in S^{n-1}.$$

Taking the closure in the radial metric of the class of intersection bodies of star bodies, we define the class of intersection bodies.

A generalization of the concept of an intersection body was introduced in [20].

Definition 3. *For an integer $k,\ 1 \le k < n$ and star bodies D, L in $\mathbb{R}^n$, we say that D is the k-intersection body of L if*

$$|D \cap H^\perp| = |L \cap H|, \qquad \forall H \in Gr_{n-k}.$$

Taking the closure in the radial metric of the class of k-intersection bodies of star bodies, we define the class of k-intersection bodies.

It was proved in [17] for $k = 1$, and in [20] for $k > 1$, that an origin-symmetric star body K in $\mathbb{R}^n$ is a k-intersection body if and only if the function $\|\cdot\|_K^{-k}$ represents a positive definite Schwartz distribution in $\mathbb{R}^n$. This result is related to embeddings in L_p-spaces. By $L_p,\ p > 0$ we mean the L_p-space of functions on $[0,1]$ with Lebesgue measure. It was shown in [16] that an n-dimensional normed space embeds isometrically in L_p, where

$p > 0$ and p is not an even integer, if and only if the Fourier transform in the sense of Schwartz distributions of the function $\Gamma(-p/2)\|\cdot\|^p$ is a non-negative distribution outside of the origin in $\mathbb{R}^n$. The concept of embedding of finite dimensional normed spaces in L_p with negative p was introduced in [19, 20], as an analytic extension of embedding into L_p with $p > 0$.

Definition 4. *For $0 < p < n$, we say that star body D belongs to the class L_{-p}^n, or, in other words, the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-p}, if the function $\|\cdot\|_D^{-p}$ represents a positive definite Schwartz distribution on $\mathbb{R}^n$.*

A connection between k-intersection bodies and embedding in L_p with negative p was found in [20].

Proposition 4. *([20]) Let $1 \le k < n$. An origin symmetric star body D in $\mathbb{R}^n$ is a k-intersection body if and only if $D \in L_{-k}^n$, or, equivalently, the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k}.*

The classes L_{-p}^n were studied by a number of authors. The advantage of Proposition 4 is that one can take any result about the usual L_p-spaces, extend it to L_{-p}, and immediately get a geometric application to intersection bodies. Let us mention one of the results. If $n-3 \le p < n$, the class L_{-p}^n contains all origin-symmetric convex bodies in $\mathbb{R}^n$; see [15, Corollary 4.9]. This result was proved as an extension of the fact that every two-dimensional normed space embeds in L_1. The result implies that every origin-symmetric convex body in $\mathbb{R}^4$ is an intersection body, which provides an affirmative answer to the Busemann-Petty problem in the critical 4-dimensional case. More results about embeddings in L_{-p} can be found in [10, 27, 24, 33, 34, 29, 38], [15, Chapter 6] and [28].

2. A comparison theorem for the derivatives of the Radon transform.

Our next result is related to the Busemann-Petty problem [4] which is the following question. Let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$, and suppose that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller than the same for L, i.e. $|K \cap \xi^\perp| \le |L \cap \xi^\perp|$ for every $\xi \in S^{n-1}$. Does it necessarily follow that the n-dimensional volume of K is smaller than the volume of L, i.e. $|K| \le |L|$? The answer is affirmative if the dimension $n \le 4$, and it is negative when $n \ge 5$; see [6, 15] for the solution and its history.

It was proved in [18] (see also [15, Theorem 5.12]) that the answer to the Busemann-Petty problem becomes affirmative if one compares the derivatives of the parallel section function of high enough orders. Namely, denote by

$$A_{K,\xi}(t) = R(\chi_K)(\xi, t) = |K \cap \{x \in \mathbb{R}^n : \ (x, \xi) = t\}|, \quad t \in \mathbb{R}$$

the parallel section function of K in the direction ξ. If K, L are infinitely smooth origin-symmetric convex bodies in $\mathbb{R}^n$, $n \ge 4$, $q \in [n-4, n-1)$ is not an odd integer, and for every $\xi \in S^{n-1}$ the fractional derivatives of the order q of the parallel section functions at zero satisfy

$$\frac{1}{\cos(\frac{\pi q}{2})} A_{K,\xi}^{(q)}(0) \le \frac{1}{\cos(\frac{\pi q}{2})} A_{L,\xi}^{(q)}(0),$$

then $|K| \le |L|$. For $-1 < q < n-4$ this is no longer true.

Another generalization of the Busemann-Petty problem, known as the isomorphic Busemann-Petty problem, asks whether the inequality for volumes holds up to an absolute constant. Does there exist an absolute constant C so that for any dimension n and any origin-symmetric convex bodies K, L in $\mathbb{R}^n$ satisfying $|K \cap \xi^\perp| \le |L \cap \xi^\perp|$ for all $\xi \in S^{n-1}$, we have $|K| \le C|L|$? As shown in [35], this question is equivalent to the slicing problem of Bourgain mentioned in Section 1.

Zvavitch [40] considered an extension of the Busemann-Petty problem to general Radon transforms, as follows. Suppose that K, L are origin-symmetric convex bodies in $\mathbb{R}^n$, and f is an even continuous strictly positive function on $\mathbb{R}^n$. Suppose that $R(f|_K)(\xi,t)(0) \le R(f|_L)(\xi,t)(0)$ for every $\xi \in S^{n-1}$, where $f|_K$ is the restriction of f to K. Does it necessarily follow that $|K| \le |L|$? The answer is exactly the same as in the case of volume. Isomorphic versions of this result were proved in [29, 26].

In this note we generalize these results to general Radon transforms as follows. In the case $q = 0$ this result was proved in [26].

Theorem 3. *Let K, L be infinitely smooth origin-symmetric convex bodies in $\mathbb{R}^n$, f, g non-negative infinitely differentiable functions on K and L, respectively, $\|g\|_\infty = g(0) = 1$, and $q \in (-1, n-1)$ is not an odd integer. If for every $\xi \in S^{n-1}$*

$$\frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0) \le \frac{1}{\cos(\frac{\pi q}{2})} (Rg(\xi,t))_t^{(q)}(0),$$

then

$$\int_K f(x)dx \le \frac{n}{n-q-1} \left(d_{\mathrm{ovr}}(K, L_{-1-q}^n)\right)^{q+1} \left(\int_L g(x)dx\right)^{\frac{n-q-1}{n}} |K|^{\frac{q+1}{n}}.$$

3. The main tools

We often use integration in polar coordinates $x = r\theta,\ x \in \mathbb{R}^n,\ r \ge 0,\ \theta \in S^{n-1}$; see [32, Ch.6, Th. 5.2]. If f is an integrable function on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(x)dx = \int_{S^{n-1}} \left(\int_0^\infty r^{n-1} f(r\theta)dr\right) d\theta. \tag{9}$$

If K is a star body in $\mathbb{R}^n$, putting $f(x) = \chi_K(x)$, the indicator function of K, we get a formula for volume:

$$|K| = \int_{\mathbb{R}^n} \chi_K(x)dx = \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1}dr\right) d\theta = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{10}$$

Our main tool is the Fourier transform of distributions; see [15, 28] for a comprehensive introduction to the Fourier approach in convex geometry. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. For any even distribution f, we have $(\hat{f})^\wedge = (2\pi)^n f$; see [37, Th. 7.7] for the inversion formula for the Fourier transform.

If K is a star body and $0 < p < n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents a distribution acting on test functions by integration. Suppose that K is infinitely smooth, i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [15, Lemma 3.16], the Fourier transform of $\|\cdot\|_K^{-p}$ is an extension of some function $g \in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n+p$ on $\mathbb{R}^n$. When we write $\left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)$, we mean $g(\xi),\ \xi \in S^{n-1}$. If K, L are infinitely smooth star bodies, the following spherical version of Parseval's formula was proved in [18] (see [15, Lemma 3.22]): for any $p \in (-n, 0)$

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-p}\right)^\wedge(\xi) \left(\|\cdot\|_L^{-n+p}\right)^\wedge(\xi) = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_L^{-n+p}\, dx. \tag{11}$$

A distribution is called *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ.

We need a lemma that can be found in [15, Lemma 3.14]. We include the proof for completeness.

Lemma 1. *Let $-1 < q < 0$. For every even test function ϕ and every fixed vector $\theta \in S^{n-1}$*

$$\int_{\mathbb{R}^n} |(\theta, \xi)|^{-1-q} \phi(\xi) d\xi = \frac{2\Gamma(-q)\cos(\pi q/2)}{\pi} \int_0^\infty t^q \hat{\phi}(t\theta) dt.$$

Proof : A well-known connection between the Fourier and Radon transforms is that, for any test function ϕ, the function $t \to \hat{\phi}(t\theta)$ is the Fourier transform of the function $z \to \int_{(\theta,\xi)=z} \phi(\xi) d\xi$; see for example [15, Lemma 2.11]. Using the Fubini theorem and the formula for the Fourier transform of $|z|^{-1-q}$ (see [15, Lemma 2.23])

$$(|z|^{-1-q})^\wedge(t) = 2\Gamma(-q)\cos(\pi q/2)|t|^q,$$

we get

$$\int_{\mathbb{R}^n} |(\theta, \xi)|^{-1-q} \phi(\xi) d\xi = \int_{\mathbb{R}} |z|^{-1-q} \left(\int_{(\theta,\xi)=z} \phi(\xi) d\xi \right) dz$$

$$= \left\langle |z|^{-1-q}, \int_{(\theta,\xi)=z} \phi(\xi) d\xi \right\rangle = \frac{1}{2\pi} \left\langle 2\Gamma(-q)\cos(\pi q/2)|t|^q, \hat{\phi}(t\theta) \right\rangle$$

$$= \frac{\Gamma(-q)\cos(\pi q/2)}{\pi} \int_{\mathbb{R}} |t|^q \hat{\phi}(t\theta)dt.$$

Finally, recall that ϕ is an even function.

□

Our next lemma generalizes Theorem 1 from [7] (see also [15, Th. 3.18]).

Lemma 2. *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, let f be an even infinitely smooth function on K, and let $q \in (-1, n-1)$. Then for every fixed $\xi \in S^{n-1}$*

$$(Rf(\xi,t))_t^{(q)}(0) \tag{12}$$

$$= \frac{\cos(\pi q/2)}{\pi} \left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f\left(r\frac{x}{|x|_2} \right) dr \right) \right)_x^\wedge (\xi).$$

Proof : Let $-1 < q < 0$. Then, using the definitions of the Radon transform and the fractional derivative (4), the Fubini theorem and integration in polar coordinates (9) with $x = r\theta$, we get

$$(Rf(\xi,t))_t^{(q)}(0) = \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |t|^{-1-q} \left(\int_{K\cap\{x:\ (x,\xi)=t\}} f(x)dx \right) dt$$

$$= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} |(x,\xi)|^{-1-q} f(x)\chi_K(x)dx$$

$$= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |(\theta,\xi)|^{-1-q} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta)dr \right) d\theta.$$

Consider the latter as a homogeneous of degree $-1-q$ function of $\xi \in \mathbb{R}^n \setminus \{0\}$, apply it to an even test function ϕ and use Lemma 1:

$$\langle (Rf(\xi,t))_t^{(q)}(0), \phi \rangle$$

$$= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} \phi(\xi) \left(\int_{S^{n-1}} |(\theta,\xi)|^{-1-q} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta)dr \right) d\theta \right) d\xi$$

$$= \frac{\cos(q\pi/2)}{\pi} \int_{S^{n-1}} \left(\int_0^{\infty} t^q \hat{\phi}(t\theta)dt \right) \left(\int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta)dr \right) d\theta.$$

On the other hand, if we apply the function in the right-hand side of (12) to the test function ϕ we get

$$\left\langle \frac{\cos(\pi q/2)}{\pi} \left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f\left(\frac{rx}{|x|_2} \right) dr \right) \right)_x^\wedge (\xi), \phi(\xi) \right\rangle$$

$$= \frac{\cos(\pi q/2)}{\pi} \int_{\mathbb{R}^n} |x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f(\frac{rx}{|x|_2})dr \right) \hat{\phi}(x)dx$$

$$= \frac{\cos(\pi q/2)}{\pi} \int_{S^{n-1}} \left(\int_0^\infty t^q \hat{\phi}(t\theta) dt \right) \left(\int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta) dr \right) d\theta,$$

where in the last step we use integration in polar coordinates (9) with $x = t\theta$. Comparing the computations, we see that, for any even test function ϕ,

$$\langle (Rf(\xi,t))_t^{(q)}(0), \phi \rangle \tag{13}$$

$$= \left\langle \frac{\cos(\pi q/2)}{\pi} \left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f\left(\frac{rx}{|x|_2}\right) dr \right) \right)_x^\wedge (\xi), \phi(\xi) \right\rangle.$$

Since both distributions are even, this proves the lemma for $-1 < q < 0$. By an argument similar to that in Lemma 2.22 from [15], one can see that both sides of (13) are analytic functions of q in the domain $-1 < \Re q < n-1$. By analytic extension, (13) holds for all $-1 < q < n-1$, which completes the proof.

$\square$

Let us compute the fractional derivatives of the Radon transform in the case where $f \equiv 1$ and $K = B_2^n$, the unit Euclidean ball.

Corollary 1. *For $-1 < q < n-1$ and every $\xi \in S^{n-1}$*

$$(R(\chi_{B_2^n})(\xi,t))_t^{(q)}(0) = \frac{2^{q+1}\pi^{\frac{n-2}{2}}\Gamma(\frac{q+1}{2})\cos(\frac{\pi q}{2})}{(n-q-1)\Gamma(\frac{n-q-1}{2})}.$$

Proof : First, we use Lemma 2 with $f \equiv 1$ and $K = B_2^n$:

$$(R(\chi_{B_2^n})(\xi,t))_t^{(q)}(0) = \frac{\cos(\frac{\pi q}{2})}{\pi(n-q-1)} (|x|_2^{-n+q+1})^\wedge(\xi).$$

Next, we apply the formula for the Fourier transform of powers of the Euclidean norm (see [8]):

$$(|x|_2^{-n+q+1})^\wedge(\xi) = \frac{2^{q+1}\pi^{\frac{n}{2}}\Gamma(\frac{q+1}{2})}{\Gamma(\frac{n-q-1}{2})} |\xi|_2^{-1-q}, \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{14}$$

$\square$

Note that if we replace the unit Euclidean ball by the Euclidean ball of volume 1, the constant in Corollary 1 has to be divided by $|B_2^n|^{\frac{n-q-1}{n}}$, and it matches the estimate of Theorem 2 for bodies K of volume 1; see inequality (7). We leave this computation for the interested reader.

4. Proofs of the main results

Proof of Theorem 3. The assumption of the theorem is that, for every $\xi \in S^{n-1}$,

$$\frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0) \le \frac{1}{\cos(\frac{\pi q}{2})} (Rg(\xi,t))_t^{(q)}(0).$$

By Lemma 2, for every $\xi \in S^{n-1}$, this assumption is equivalent to

$$\left(|x|_2^{-n+q+1}\left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f\left(r\frac{x}{|x|_2}\right) dr\right)\right)_x^\wedge(\xi) \tag{15}$$

$$\leq \left(|x|_2^{-n+q+1}\left(\int_0^{\frac{|x|_2}{\|x\|_L}} r^{n-q-2} g\left(r\frac{x}{|x|_2}\right) dr\right)\right)_x^\wedge(\xi).$$

Let $\delta > 0$, and let $D \in L_{-1-q}^n$ be such that $K \subset D$ and

$$|D|^{1/n} \leq (1+\delta) d_{\mathrm{ovr}}(K, L_{-1-q}^n)|K|^{1/n}. \tag{16}$$

By approximation, we can assume that D is infinitely smooth; see [15, Lemma 4.10]. Then $(\|x\|_D^{-1-q})^\wedge$ is a non-negative function on the sphere. Multiplying both sides of (15) by $(\|x\|_D^{-1-q})^\wedge(\xi)$, integrating over the sphere and using Parseval's formula on the sphere (11) we get

$$\int_{S^{n-1}} \|\theta\|_D^{-1-q}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta) dr\right) d\theta$$

$$\leq \int_{S^{n-1}} \|\theta\|_D^{-1-q}\left(\int_0^{\|\theta\|_L^{-1}} r^{n-q-2} g(r\theta) dr\right) d\theta,$$

or, using the formula for integration in polar coordinates (9) with $x = r\theta$,

$$\int_K \|x\|_D^{-1-q} f(x) dx \leq \int_L \|x\|_D^{-1-q} g(x) dx. \tag{17}$$

Since $K \subset D$, we have $\|x\|_D \leq 1$ for every $x \in K$, and thus

$$\int_K \|x\|_D^{-1-q} f(x) dx \geq \int_K f(x) dx. \tag{18}$$

On the other hand, by the Lemma from section 2.1 from Milman-Pajor [35, p.76],

$$\left(\frac{\int_L \|x\|_D^{-1-q} g(x) dx}{\int_D \|x\|_D^{-1-q} dx}\right)^{1/(n-q-1)} \leq \left(\frac{\int_L g(x) dx}{\int_D dx}\right)^{1/n}. \tag{19}$$

By (9) and (10),

$$\int_D \|x\|_D^{-1-q} dx = \int_{\mathbb{R}^n} \|x\|_D^{-1-q} \chi_D(x) dx$$

$$= \int_{S^{n-1}} \|\theta\|_D^{-1-q}\left(\int_0^{\|\theta\|_D^{-1}} r^{n-q-2} dr\right) d\theta$$

$$= \frac{1}{n-q-1}\int_{S^{n-1}} \|\theta\|_D^{-n} d\theta = \int \frac{n}{n-q-1}|D|.$$

Now we can rewrite (19) as

$$\int_L \|x\|_D^{-1-q} g(x)dx \le \frac{n}{n-q-1}\left(\int_L g(x)dx\right)^{\frac{n-q-1}{n}} |D|^{\frac{q+1}{n}}. \tag{20}$$

Combining estimates (17), (18) and (20) with the definition of D, (16), and sending δ to zero, we get

$$\int_K f(x)dx \le \frac{n}{n-q-1}\left(\int_L g(x)dx\right)^{\frac{n-q-1}{n}} \left(d_{\mathrm{ovr}}(K, L^n_{-1-q})\right)^{q+1} |K|^{\frac{q+1}{n}},$$

which is the conclusion of the theorem.

□

Proof of Theorem 2. Consider a number $\varepsilon > 0$ such that, for every $\xi \in S^{n-1}$,

$$\frac{1}{\cos(\pi q/2)}(Rf(\xi,t))_t^{(q)}(0) \le \frac{\varepsilon}{\cos(\pi q/2)} R(\chi_{B_2^n})(\xi,t))_t^{(q)}(0).$$

By Lemma 2, for every $\xi \in S^{n-1}$,

$$\left(|x|_2^{-n+q+1}\left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f\left(r\frac{x}{|x|_2}\right)dr\right)\right)_x^\wedge(\xi) \le \frac{\varepsilon}{n-q-1}\left(|x|_2^{-n+q+1}\right)^\wedge(\xi)$$

Let $\delta > 0$, and let $D \in L^n_{-1-q}$ be such that $K \subset D$ and

$$|D|^{1/n} \le (1+\delta) d_{\mathrm{ovr}}(K, L^n_{-1-q})|K|^{\frac{1}{n}}. \tag{21}$$

By approximation, we can assume that D is infinitely smooth. Then $(\|x\|_D^{-1-q})^\wedge$ is a non-negative function on the sphere. Multiplying both sides of the latter inequality by $(\|x\|_D^{-1-q})^\wedge(\xi)$, integrating over the sphere and using Parseval's formula on the sphere we get (like in the proof of Theorem 3)

$$\int_K \|x\|_D^{-1-q} f(x)dx \le \frac{\varepsilon}{n-q-1}\int_{S^{n-1}} \|x\|_D^{-1-q}dx. \tag{22}$$

Since $K \subset D$, we have

$$\int_K \|x\|_D^{-1-q} f(x)dx \ge \int_K \|x\|_K^{-1-q} f(x)dx \ge \int_K f(x)dx. \tag{23}$$

On the other hand, by Hölder's inequality with the exponents $\frac{n}{q+1}$ and $\frac{n}{n-q-1}$, and by (10),

$$\int_{S^{n-1}} \|x\|_D^{-1-q}dx \le |S^{n-1}|^{\frac{n-q-1}{n}} n^{\frac{q+1}{n}} |D|^{\frac{q+1}{n}}, \tag{24}$$

where (see for example [15, Corollary 2.20])

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Combining (22), (23), (24), we get

$$\int_K f(x)dx \le \frac{\varepsilon |S^{n-1}|^{\frac{n-q-1}{n}} n^{\frac{q+1}{n}}}{n-q-1} |D|^{\frac{q+1}{n}}.$$

Now replace D by K using (21):

$$\int_K f(x)dx \le \frac{\varepsilon |S^{n-1}|^{\frac{n-q-1}{n}} n^{\frac{q+1}{n}}}{n-q-1} (1+\delta)^{q+1} \left(d_{\mathrm{ovr}}(K, L^n_{-1-q})\right)^{q+1} |K|^{\frac{q+1}{n}},$$

and put

$$\varepsilon = \max_{\xi \in S^{n-1}} \frac{\frac{1}{\cos(\pi q/2)} (Rf(\xi,t))_t^{(q)}(0)}{\frac{1}{\cos(\pi q/2)} R(\chi_{B_2^n})(\xi,t))_t^{(q)}(0)}.$$

Replace the denominator in the expression for ε using Corollary 1, and substitute the formula for $|S^{n-1}|$:

$$\int_K f(x)dx \le c(n,q)(1+\delta)^{q+1} \left(d_{\mathrm{ovr}}(K, L^n_{-1-q})\right)^{q+1} |K|^{\frac{q+1}{n}} \tag{25}$$

$$\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\pi q/2)} (Rf(\xi,t))_t^{(q)}(0),$$

where

$$c(n,q) = \frac{\pi \Gamma(\frac{n-q-1}{2}) n^{\frac{q+1}{n}} 2^{\frac{n-q-1}{n}} \pi^{\frac{n-q-1}{2}}}{2^{q+1} \pi^{\frac{n}{2}} \Gamma(\frac{q+1}{2}) \left(\Gamma(\frac{n}{2})\right)^{\frac{n-q-1}{n}}}.$$

Now use $\Gamma(x+1) = x\Gamma(x)$ and the inequality

$$\frac{\Gamma(\frac{n-q-1}{2}+1)}{\left(\Gamma(\frac{n}{2}+1)\right)^{\frac{n-q-1}{n}}} \le 1,$$

which follows from the log-convexity of the Γ-function (see [15, Lemma 2.14]). We get

$$c(n,q) = \frac{n\ \Gamma(\frac{n-q-1}{2}+1)}{2^q \pi^{\frac{q-1}{2}} \Gamma(\frac{q+1}{2})(n-q-1) \left(\Gamma(\frac{n}{2}+1)\right)^{\frac{n-q-1}{n}}}$$

$$\le \frac{n}{2^q \pi^{\frac{q-1}{2}} \Gamma(\frac{q+1}{2})(n-q-1)},$$

and (25) implies

$$\int_K f(x)dx \le \frac{n}{2^q \pi^{\frac{q-1}{2}} \Gamma(\frac{q+1}{2})(n-q-1)} (1+\delta)^{q+1} \left(d_{\mathrm{ovr}}(K, L^n_{-1-q})\right)^{q+1} |K|^{\frac{q+1}{n}}$$

$$\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\pi q/2)} (Rf(\xi,t))_t^{(q)}(0).$$

Finally, send δ to zero to get the result.

$\square$

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(Wyatt Gregory) Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: wtgmtf@mail.missouri.edu

(Alexander Koldobsky) Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: koldobskiya@missouri.edu