INEQUALITIES FOR THE RADON TRANSFORM ON CONVEX SETS

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Abstract. We prove an inequality that unifies previous works of the authors on the properties of the Radon transform on convex bodies including an extension of the Busemann-Petty problem and a slicing inequality for arbitrary functions. Let K and L be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and L, respectively, so that $\|g\|_\infty = g(0) = 1$. Then

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \max_H \frac{\int_{K\cap H} f}{\int_{L\cap H} g},$$

where $|K|$ stands for volume of proper dimension, C is an absolute constant, the maximum is taken over all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $d_{\rm ovr}(K, \mathcal{BP}_k^n)$ is the outer volume ratio distance from K to the class of generalized k-intersection bodies in $\mathbb{R}^n$. Another consequence of this result is a mean value inequality for the Radon transform. We also obtain a generalization of the isomorphic version of the Shephard problem.

1. Introduction

Several years ago the authors started looking at certain problems of convex geometry from a more general point of view, replacing volume by an arbitrary measure. This approach has produced new general properties of the Radon transform on convex sets, including an extension of the Busemann-Petty problem to arbitrary measures and the slicing inequality for arbitrary functions. In this note, we prove a general inequality that serves as an umbrella for all these results. The inequality is as follows.

Theorem 1. *Let K and L be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and L, respectively, so that $\|g\|_\infty = g(0) = 1$. Then*

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \max_{H\in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}. \tag{1}$$

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Here $|K|$ stands for volume of proper dimension. The Grassmanian Gr_{n-k} is defined as the set of all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. The outer volume ratio distance from a star body K to a class Ω of star bodies in $\mathbb{R}^n$ is defined by

$$d_{\rm ovr}(K,\Omega) = \inf\left\{ \left(\frac{|D|}{|K|}\right)^{1/n} \,:\, K \subset D,\ D \in \Omega \right\}. \tag{2}$$

In our case $\Omega = \mathcal{BP}_k^n$ is the class of generalized k-intersection bodies, i.e. bodies for which the Minkowski functional to the power $-k$ can be represented as the spherical Radon transform of a finite measure on the Grassmanian Gr_{n-k} (see the definition in Section 2). It was proved in [37] that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$d_{\rm ovr}(K,\mathcal{BP}_k^n) \le C\sqrt{\frac{n}{k}}\log^{\frac{3}{2}}\left(\frac{en}{k}\right), \tag{3}$$

where C is an absolute constant. Moreover, it was proved in [32] that for unconditional convex bodies K one has $d_{\rm ovr}(K,\mathcal{BP}_k^n) \le e$. If K is the unit ball of an n-dimensional subspace of $L_p,\ p > 2$, the distance is less than $c\sqrt{p}$, where $c > 0$ is an absolute constant, as shown in [43, 35]. The unit balls of subspaces of L_p with $0 < p \le 2$ belong to the classes $\mathcal{BP}_k^n$ for all k, n (see [33, 44]), so the distance for these bodies is equal to 1.

Theorem 1 is closely related to the Busemann-Petty problem. The problem was posed in [8] in 1956 and asks the following question. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and suppose that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller than the corresponding one for L, i.e. $|K \cap \xi^\perp| \le |L \cap \xi^\perp|$ for every $\xi \in S^{n-1}$. Does it necessarily follow that the n-dimensional volume of K is smaller than the volume of L, i.e. $|K| \le |L|$? Here $\xi^\perp = \{x \in \mathbb{R}^n :\ \langle x, \xi\rangle = 0\}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$. The answer is affirmative if the dimension $n \le 4$, and it is negative when $n \ge 5$; see [14, 29] for the solution of the problem and its history.

Since the answer to the Busemann-Petty problem is negative in most dimensions, it is natural to ask whether the inequality for volumes holds up to an absolute constant. This is known as the isomorphic Busemann-Petty problem introduced in [45]. Does there exist an absolute constant C so that for any dimension n and any pair of origin-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying $|K\cap\xi^\perp| \le |L\cap\xi^\perp|$ for all $\xi \in S^{n-1}$, we have $|K| \le C|L|$?

As pointed out in [45], the isomorphic Busemann-Petty problem is equivalent to the slicing problem of Bourgain [4, 5]: Does there exist an absolute constant C so that for any $n \in \mathbb{N}$ and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \le C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp| \ ? \tag{4}$$

In other words, is it true that every origin-symmetric convex body K of volume one in $\mathbb{R}^n$ has a hyperplane section with area greater than an absolute constant, i.e. there exists $\xi \in S^{n-1}$ so that $|K \cap \xi^\perp| > c$, where c does not depend on K and n? The isomorphic Busemann-Petty problem and the slicing problem are still open. Bourgain [6] proved that $C \le O(n^{1/4}\log n)$. Klartag [26] removed the logarithmic term from Bourgain's estimate. In a recent breakthrough result, Chen [11] proved that $C \le o(n^\epsilon)$ for every $\epsilon > 0$, as the dimension goes to infinity.

An extension of the slicing problem to arbitrary functions was considered in [30, 31, 32, 10, 27, 28, 36]. It was proved in [32] that for any $n \in \mathbb{N}$, any star body K in $\mathbb{R}^n$ and any non-negative continuous function f on K,

$$\int_K f \ \le \ 2\ d_{\rm ovr}(K,\mathcal{I}_n)\ |K|^{1/n} \ \max_{\xi\in S^{n-1}} \int_{K\cap\xi^\perp} f. \tag{5}$$

Here $\mathcal{I}_n = \mathcal{BP}_1^k$ is the class of intersection bodies (see the definition in Section 2).

It is interesting to note that we do not need to require the function f to be even, as well as the body K to be symmetric, and additional assumptions are only needed to estimate $d_{\rm ovr}(K,\mathcal{I}_n)$. Since the class of intersection bodies includes ellipsoids, by John's theorem [25], if K is origin-symmetric and convex, then $d_{\rm ovr}(K,\mathcal{I}_n) \le \sqrt{n}$. Thus, there exists a constant $s_n \ge \frac{1}{2\sqrt{n}}$ so that for any origin-symmetric convex body K of volume 1 in $\mathbb{R}^n$ and any integrable non-negative function f on K with $\int_K f = 1$, there exists a direction $\xi \in S^{n-1}$ for which $\int_{K\cap\xi^\perp} f \ge s_n$. In other words, the sup-norm of the Radon transform of any probability density on a convex body of volume one is bounded from below by a positive constant depending only on the dimension. Note that this estimate was extended later to the case of non-symmetric bodies in [10]. An extension to the derivatives of the Radon transform was obtained in [21].

On the other hand, it was proved in [27] that there exists an origin-symmetric convex body M in $\mathbb{R}^n$ and a probability density f on M so that

$$\int_{M\cap H} f \le C\frac{\sqrt{\log\log n}}{\sqrt{n}}|M|^{-1/n},$$

for every affine hyperplane H in $\mathbb{R}^n$, where C is an absolute constant. The logarithmic term was later removed in [28], so $s_n \le C/\sqrt{n}$. Finally, $\frac{c_1}{\sqrt{n}} \le s_n \le \frac{c_2}{\sqrt{n}}$, where $c_1, c_2 > 0$ are absolute constants.

A lower dimensional version of inequality (5) was also proved in [32]. If K is a star body in $\mathbb{R}^n$, f is a continuous non-negative function on K, and $1 \le k < n$, then

$$\int_K f \ \le \ C^k \ (d_{\rm ovr}(K,\mathcal{BP}_k^n))^k \ |K|^{k/n} \ \max_{H\in Gr_{n-k}} \int_{K\cap H} f, \tag{6}$$

where C is an absolute constant. This implies that there exists a constant $c_{n,k} > 0$ such that for any convex body K in $\mathbb{R}^n$ and any probability density f on K, there exists an $(n-k)$-dimensional subspace H in $\mathbb{R}^n$ so that $\int_{K\cap H} f \ge c_{n,k}$. Moreover, applying inequality (3) we get

$$(c_{n,k})^{1/k} \ge \frac{c\sqrt{k}}{\sqrt{n\log^3(\frac{en}{k})}},$$

where $c > 0$ is an absolute constant. It is an open problem whether it is possible to remove the logarithmic term in this estimate or to find the exact upper estimate for $c_{n,k}$. Note that inequality (6) can be obtained as a particular case of Theorem 1 where $L = B_2^n$ is the unit Euclidean ball and $g \equiv 1$.

An extension of the Busemann-Petty problem to arbitrary functions was found in [49, 50]. Suppose that f is an even continuous strictly positive function on $\mathbb{R}^n$, and

K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ so that

$$\int_{K\cap\xi^\perp} f \le \int_{L\cap\xi^\perp} f, \qquad \forall \xi \in S^{n-1}. \tag{7}$$

Does it necessarily follow that $\int_K f \le \int_L f$? The answer is the same as for the volume, affirmative if $n \le 4$ and negative if $n \ge 5$.

An isomorphic version was proved in [38]. Namely, for every dimension n, the validity of (7) for each ξ implies

$$\int_K f \le \sqrt{n}\ \int_L f. \tag{8}$$

It is not known whether $\sqrt{n}$ is the optimal constant in this result.

The estimate (8) was proved in [38] as follows. The validity of (7) for each ξ implies

$$\int_K f \le d_{BM}(K,\mathcal{I}_n) \int_L f,$$

where

$$d_{BM}(K,\mathcal{I}_n) = \inf\{a>0 : \exists D \in \mathcal{I}_n : D \subset K \subset aD\}$$

is the Banach-Mazur distance from K to the class of intersection bodies. Now if K is origin-symmetric and convex, by John's theorem, $d_{BM}(K,\mathcal{I}_n) \le \sqrt{n}$, so the $\sqrt{n}$ estimate follows.

Another version of the isomorphic Busemann-Petty problem was proved in [36]. If star bodies K, L and functions f, g are as in Theorem 1 and

$$\int_{K\cap H} f \le \int_{L\cap H} g, \qquad \forall H \in Gr_{n-k},$$

then

$$\int_K f \le \frac{n}{n-k}\left(d_{\rm ovr}(K, BP_k^n)\right)^k |K|^{\frac{k}{n}} \left(\int_L g\right)^{\frac{n-k}{n}}.$$

Note that this result follows from Theorem 1.

In Section 3 we describe an alternative approach to Theorem 1 which is based on Blaschke-Petkantchin formulas and affine isoperimetric inequalities. This approach was initiated in [10] and leads to a version of (1) which is valid for more general pairs of sets. Below, for any bounded Borel set K in $\mathbb{R}^n$ we denote by ${\rm ovr}(K)$ the outer volume ratio ${\rm ovr}(K) = d_{\rm ovr}(K, L_2) = \inf_{\mathcal{E}} \left(|\mathcal{E}|/|K|\right)^{1/n}$, where the infimum is over all origin symmetric ellipsoids $\mathcal{E}$ in $\mathbb{R}^n$ with $K \subseteq \mathcal{E}$.

Theorem 2. *Let K and L be two bounded Borel sets in $\mathbb{R}^n$. Let f and g be two bounded non-negative measurable functions on K and L, respectively, and assume that $\|g\|_1 > 0$ and $\|g\|_\infty = 1$. For every $1 \le k \le n-1$ we have that*

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le (C\cdot {\rm ovr}(K))^k \max_{H\in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}, \tag{9}$$

where $C>0$ is an absolute constant.

It should be noted that even for origin-symmetric convex bodies K in $\mathbb{R}^n$ the outer volume ratio $\mathrm{ovr}(K)$ can be as large as $\sqrt{n}$. This is a disadvantage of Theorem 2 which does not provide estimates depending on k in contrast to Theorem 1. Regarding this comparison, we mention that besides Blaschke-Petkantchin formulas and affine isoperimetric inequalities, the proof of Theorem 2 exploits a well-known result of Barány and Füredi [3] which may be stated as follows: if $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^m$, $s \geq m+1$ and $w_1, \dots, w_s \in \mathcal{E}$, then

$$\left(\frac{|\mathrm{conv}(w_1, \dots, w_s)|}{|\mathcal{E}|}\right)^{1/m} \leq C\sqrt{\log(1+s/m)/m}$$

where $C > 0$ is an absolute constant. It would be interesting to obtain an optimal estimate for the corresponding result when $w_1, \dots, w_s$ are chosen from a body $L \in \mathcal{BP}^n_k$.

In Section 4 we obtain a generalization of the isomorphic version of the Shephard problem due to Ball [1, 2]. Ball has proved that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$|K|\xi^\perp| \leq |L|\xi^\perp|, \qquad \forall \xi \in S^{n-1},$$

then

$$|K| \leq d_{\mathrm{vr}}(L, \Pi_n)|L|.$$

In this statement, $K|\xi^\perp$ denotes the orthogonal projection of K in the direction of ξ and $d_{\mathrm{vr}}(L, \Pi_n)$ is the volume ratio distance from L to the class Π_n of projection bodies defined by

$$d_{\mathrm{vr}}(K, \Pi_n) = \inf\left\{\left(\frac{|K|}{|D|}\right)^{1/n} : \ D \subset K, \ D \in \Pi_n\right\}. \tag{10}$$

Replacing Π_n by the class $\Pi_{p,n}$ of p-projection bodies (see Section 4 for background information) we obtain the following:

Theorem 3. *Fix $p \geq 1$ and let K and L be convex bodies in $\mathbb{R}^n$, then*

$$\left(\frac{|K|}{|L|}\right)^{\frac{n-p}{pn}} \leq \ d_{\mathrm{vr}}(L, \Pi_{p,n}) \max_{\xi \in S^{n-1}} \frac{h_{\Pi_p K}(\xi)}{h_{\Pi_p L}(\xi)}.$$

In Section 5, we deduce several general properties of the Radon transform from Theorem 1. For example, if we put $K = L$ and $g \equiv 1$ in (1), we get what we call the mean value inequality for the Radon transform:

$$\frac{\int_K f}{|K|} \leq \frac{n}{n-k}\left(d_{\mathrm{ovr}}(K, \mathcal{BP}^n_k)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{|K \cap H|}. \tag{11}$$

Throughout the paper, we write $a \simeq b$ if $c_1 b \leq a \leq c_2 b$, where $c_1, c_2 > 0$ are absolute constants.

2. Proof of Theorem 1

We need several definitions and facts. A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K

at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$$

is a continuous function on $\mathbb{R}^n$. We use the polar formula for the volume $|K|$ of a star body K :

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{12}$$

If f is an integrable function on K, then

$$\int_K f = \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) dr \right) d\theta. \tag{13}$$

For $1 \leq k \leq n-1$, the $(n-k)$*-dimensional spherical Radon transform* $\mathcal{R}_{n-k}$: $C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$\mathcal{R}_{n-k} g(H) = \int_{S^{n-1} \cap H} g(x) \ dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

For every $H \in Gr_{n-k}$, the $(n-k)$-dimensional volume of the section of a star body K by H can be written as

$$|K \cap H| = \frac{1}{n-k} \mathcal{R}_{n-k}(\| \cdot \|_K^{-n+k})(H). \tag{14}$$

More generally, for an integrable function f and any $H \in Gr_{n-k}$,

$$\int_{K \cap H} f = \mathcal{R}_{n-k} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r \ \cdot) \ dr \right)(H). \tag{15}$$

The class of intersection bodies $\mathcal{I}_n$ was introduced by Lutwak [39]. We consider a generalization of this concept due to Zhang [48]. We say that an origin symmetric star body D in $\mathbb{R}^n$ is a *generalized k-intersection body*, and write $D \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure ν_D on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_D^{-k} g(x) \ dx = \int_{Gr_{n-k}} R_{n-k} g(H) \ d\nu_D(H). \tag{16}$$

When $k = 1$ we get the original Lutwak's class of *intersection bodies* $\mathcal{BP}_1^n = \mathcal{I}_n$.

Proof of Theorem 1. For a small $\delta > 0$, let $D \in \mathcal{BP}_k^n$ be a body such that $K \subset D$ and

$$|D|^{\frac{1}{n}} \leq (1+\delta) \ d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \ |K|^{\frac{1}{n}}, \tag{17}$$

and let ν_D be the measure on Gr_{n-k} corresponding to D by the definition (16).

Let ε be such that

$$\int_{K \cap H} f \leq \varepsilon \int_{L \cap H} g, \qquad \forall H \in Gr_{n-k}.$$

By (15), we have

$$\mathcal{R}_{n-k} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r \ \cdot) \ dr \right)(H) \leq \varepsilon \ \mathcal{R}_{n-k} \left(\int_0^{\|\cdot\|_L^{-1}} r^{n-k-1} g(r \ \cdot) \ dr \right)(H)$$

for every $H \in Gr_{n-k}$. Integrating both sides of the latter inequality with respect to ν_D and using the definition (16), we get

$$\int_{S^{n-1}} \|x\|_D^{-k} \left(\int_0^{\|x\|_K^{-1}} r^{n-k-1} f(rx)\ dr \right) dx \tag{18}$$

$$\leq \varepsilon \int_{S^{n-1}} \|x\|_D^{-k} \left(\int_0^{\|x\|_L^{-1}} r^{n-k-1} g(rx)\ dr \right) dx,$$

which is equivalent to

$$\int_K \|x\|_D^{-k} f(x) dx \leq \varepsilon \int_L \|x\|_D^{-k} g(x) dx. \tag{19}$$

Since $K \subset D$, we have $1 \geq \|x\|_K \geq \|x\|_D$ for every $x \in K$. Therefore,

$$\int_K \|x\|_D^{-k} f(x) dx \geq \int_K \|x\|_K^{-k} f(x) dx \geq \int_K f.$$

On the other hand, by [45, Lemma 2.1] (recall that $g(0) = \|g\|_\infty = 1$),

$$\left(\frac{\int_L \|x\|_D^{-k} g(x) dx}{\int_D \|x\|_D^{-k} dx} \right)^{1/(n-k)} \leq \left(\frac{\int_L g(x) dx}{\int_D dx} \right)^{1/n}.$$

Since $\int_D \|x\|_D^{-k} dx = \frac{n}{n-k} |D|$, we can estimate the right-hand side of (19) by

$$\int_L \|x\|_D^{-k} g(x) dx \leq \varepsilon \frac{n}{n-k} \left(\int_L g \right)^{\frac{n-k}{n}} |D|^{\frac{k}{n}}.$$

Applying (17) and sending δ to zero, we see that the latter inequality in conjunction with (19) implies

$$\int_K f \leq \varepsilon \ \frac{n}{n-k} \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \right)^k |K|^{\frac{k}{n}}.$$

Now put $\varepsilon = \max\limits_{H \in Gr_{n-k}} \frac{\int_{K \cap H} f}{\int_{L \cap H} g}$.

□

If $f \equiv 1,\ g \equiv 1$, we can get a slightly sharper (without the factor $\frac{n}{n-k}$) inequality than what Theorem 1 gives in this case.

Theorem 4. *Let K, L be star bodies in $\mathbb{R}^n$ and $0 < k < n$, then*

$$\left(\frac{|K|}{|L|} \right)^{\frac{n-k}{n}} \leq \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \right)^k \max_{H \in Gr_{n-k}} \frac{|K \cap H|}{|L \cap H|}.$$

Proof: Let ε be such that $|K \cap H| \leq \varepsilon\ |L \cap H|$ for all $H \in Gr_{n-k}$, and let D be as in the proof of Theorem 1. By (14), for all H

$$\mathcal{R}_{n-k}(\| \cdot \|_K^{-n+k})(H) \leq \varepsilon\ \mathcal{R}_{n-k}(\| \cdot \|_L^{-n+k})(H).$$

Integrating both sides with respect to ν_D and using the definition (16) we get

$$\int_{S^{n-1}} \|x\|_D^{-k} \|x\|_K^{-n+k} dx \leq \varepsilon \int_{S^{n-1}} \|x\|_D^{-k} \|x\|_L^{-n+k} dx.$$

Since $K \subset D$, we have $1 \geq \|x\|_K \geq \|x\|_D$, and by (12) the left-hand side is greater than $n|K|$. Using this and Hölder's inequality,

$$\begin{aligned} n|K| &\leq \varepsilon \int_{S^{n-1}} \|x\|_D^{-k} \|x\|_L^{-n+k} dx \leq \varepsilon \left(\int_{S^{n-1}} \|x\|_D^{-n} dx \right)^{\frac{k}{n}} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{\frac{n-k}{n}} \\ &= \varepsilon\, n |D|^{\frac{k}{n}} |L|^{\frac{n-k}{n}} \leq \varepsilon\, n\, (1+\delta)^k \, \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \, |K|^{\frac{k}{n}} |L|^{\frac{n-k}{n}}. \end{aligned}$$

Sending δ to zero and setting

$$\varepsilon = \max_{H \in Gr_{n-k}} \frac{|K \cap H|}{|L \cap H|},$$

we get the result.

□

3. Proof of Theorem 2

Our first tool will be a Blaschke-Petkantchin formula (see [47, Chapter 7.2] and [15, Lemma 5.1]).

Lemma 1 (Blaschke-Petkantschin)**.** *Let $1 \leq s \leq n-1$ be an integer. There exists a constant $p(n,s) > 0$ such that, for every non-negative bounded Borel measurable function $F : (\mathbb{R}^n)^s \to \mathbb{R}$,*

$$\begin{aligned} &\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(x_1, \dots, x_s) dx_s \cdots dx_1 \\ &= p(n,s) \int_{G_{n,s}} \int_H \cdots \int_F f(x_1, \dots, x_s) \, |\mathrm{conv}(0, x_1, \dots, x_s)|^{n-s} \\ &\qquad dx_s \cdots dx_1 \, d\nu_{n,s}(H), \end{aligned} \tag{20}$$

where $\nu_{n,s}$ is the Haar probability measure on Gr_s. The exact value of the constant $p(n,s)$ is

$$p(n,s) = (s!)^{n-s} \frac{(n\omega_n) \cdots ((n-s+1)\omega_{n-s+1})}{(s\omega_s) \cdots (2\omega_2)\omega_1}, \tag{21}$$

where ω_n is the volume of the unit Euclidean ball B_2^n.

We shall also use the next inequality, proved independently by Busemann and Straus [9], and Grinberg [22].

Lemma 2 (Busemann-Straus, Grinberg)**.** *Let K be a bounded Borel set of volume 1 in $\mathbb{R}^n$. For any $1 \leq k \leq n-1$ and $T \in SL(n)$ we have*

$$\int_{Gr_{n-k}} |K \cap H|^n d\nu_{n,n-k}(H) \leq \int_{Gr_{n-k}} |\overline{B}_2^n \cap H|^n d\nu_{n,n-k}(H), \tag{22}$$

where $\overline{B}_2^n$ is the Euclidean ball of volume 1.

Our next tool will be a theorem of Dann, Paouris and Pivovarov from [12]; the proof of this fact combines Blaschke-Petkantschin formulas with rearrangement inequalities.

Lemma 3 (Dann-Paouris-Pivovarov)**.** *Let g be a non-negative, bounded integrable function on $\mathbb{R}^n$ with $\|g\|_1 > 0$. For every $1 \leq k \leq n-1$ we have*

$$\int_{Gr_{n-k}} \frac{1}{\|g|_H\|_\infty^k} \left(\int_H g(x)dx\right)^n d\nu_{n,n-k}(H) \leq \gamma_{n,k}^{-n} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{n-k}, \tag{23}$$

where $\gamma_{n,k} = \omega_n^{\frac{n-k}{n}}/\omega_{n-k}$.

It is checked in [10] that for every $1 \leq k \leq n-1$ one has

$$e^{-k/2} < \gamma_{n,k} < 1 \quad \text{and} \quad [\gamma_{n,k}^{-n} p(n,n-k)]^{\frac{1}{k(n-k)}} \simeq \sqrt{n-k}. \tag{24}$$

Finally, we need a well-known theorem of Bárány and Füredi [3]: if $s \geq m+1$ and $w_j \in \mathbb{R}^m$ satisfy $\|w_j\|_2 \leq 1$ for $j = 1, \ldots, s$, then

$$|\mathrm{conv}(w_1, \ldots, w_s)|^{1/m} \leq C \frac{\sqrt{\log(1+s/m)}}{m}.$$

Equivalently, this says that if $w_j \in B_2^m$, $1 \leq j \leq s$, then the volume radius of their convex hull is bounded by $C\sqrt{\log(1+s/m)/m}$. By affine invariance we obtain:

Lemma 4. *There exists an absolute constant $C > 0$ such that if $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^m$, $s \geq m+1$ and $w_1, \ldots, w_s \in \mathcal{E}$, then*

$$\left(\frac{|\mathrm{conv}(w_1, \ldots, w_s)|}{|\mathcal{E}|}\right)^{1/m} \leq C\sqrt{\log(1+s/m)/m}.$$

Proof of Theorem 2. Let $\mathcal{E}$ be a centered ellipsoid such that $K \subseteq \mathcal{E}$ and

$$\mathrm{ovr}(K) = (|\mathcal{E}|/|K|)^{1/n}.$$

We shall use the next consequence of Lemma 4: if $F \in Gr_{n-k}$ and $x_1, \ldots, x_{n-k} \in K \cap H \subseteq \mathcal{E} \cap H$ then $\mathrm{conv}(0, x_1, \ldots, x_{n-k}) \subseteq \mathcal{E} \cap H$, and since $\mathcal{E} \cap H$ is an $(n-k)$-dimensional centered ellipsoid we must have

$$\begin{aligned} |\mathrm{conv}(0, x_1, \ldots, x_{n-k})| &\leq \left(C_1 \frac{\sqrt{\log(1+(n-k+1)/(n-k)}}{\sqrt{n-k}}\right)^k |\mathcal{E} \cap H| \\ &\leq \left(\frac{C_1}{\sqrt{n-k}}\right)^k |\mathcal{E} \cap H|. \end{aligned} \tag{25}$$

Applying Lemma 1 for the function $F(x_1, \ldots, x_{n-k}) = \prod_{i=1}^{n-k} f(x_i)\mathbf{1}_K(x_i)$, with $s = n-k$, we get

$$\begin{aligned} \left(\int_K f(x)\, dx\right)^{n-k} &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(x_1, \ldots, x_{n-k}) dx_{n-k} \cdots dx_1 \\ &= p(n,n-k) \int_{Gr_{n-k}} \int_{K\cap H} \cdots \int_{K \cap H} g(x_1) \cdots g(x_{n-k}) \\ &\qquad \times\ |\mathrm{conv}(0, x_1, \ldots, x_{n-k})|^k dx_{n-k} \cdots dx_1\, d\nu_{n,n-k}(H). \end{aligned}$$

Let $M := \max_{H\in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}$. Then, (25) shows that

$$\begin{aligned}
&\left(\int_K f(x)\,dx\right)^{n-k} \leq p(n,n-k)\left(\frac{C_1}{\sqrt{n-k}}\right)^{k(n-k)} \int_{Gr_{n-k}}\int_{K\cap H}\cdots\int_{K\cap H} |\mathcal{E}\cap H|^k \\
&\times\ f(x_1)\cdots f(x_{n-k})\,dx_{n-k}\cdots dx_1\,d\nu_{n,n-k}(H) \\
&= p(n,n-k)\left(\frac{C_1}{\sqrt{n-k}}\right)^{k(n-k)} \int_{Gr_{n-k}} |\mathcal{E}\cap H|^k \Big(\int_{K\cap H} f(x)\,dx\Big)^{n-k} d\nu_{n,n-k}(H) \\
&\leq p(n,n-k)\left(\frac{C_1}{\sqrt{n-k}}\right)^{k(n-k)} M^{n-k}\int_{Gr_{n-k}} |\mathcal{E}\cap H|^k \\
&\times\Big(\int_{L\cap H} g(x)\,dx\Big)^{n-k} d\nu_{n,n-k}(H).
\end{aligned}$$

Now, by Hölder's inequality and Grinberg's inequality (22) we get

$$\begin{aligned}
&\int_{Gr_{n-k}} |\mathcal{E}\cap H|^k \left(\int_{L\cap H} g(x)\,dx\right)^{n-k} d\nu_{n,n-k}(H) \\
&\leq \left(\int_{Gr_{n-k}} |\mathcal{E}\cap H|^n d\nu_{n,n-k}(H)\right)^{\frac{k}{n}} \left(\int_{Gr_{n-k}} \left(\int_{L\cap H} g(x)\,dx\right)^n d\nu_{n,n-k}(H)\right)^{\frac{n-k}{n}} \\
&\leq \gamma_{n,k}^{-n}|\mathcal{E}|^{\frac{k(n-k)}{n}} \left(\int_{Gr_{n-k}} \left(\int_{L\cap H} g(x)\,dx\right)^n d\nu_{n,n-k}(H)\right)^{\frac{n-k}{n}} \\
&= \gamma_{n,k}^{-n}|K|^{\frac{k(n-k)}{n}}\mathrm{ovr}(K)^{k(n-k)} \left(\int_{Gr_{n-k}} \left(\int_{L\cap H} g(x)\,dx\right)^n d\nu_{n,n-k}(H)\right)^{\frac{n-k}{n}}.
\end{aligned}$$

Finally, since $\|g|_H\|_\infty \leq \|g\|_\infty = 1$ for all $H\in Gr_{n-k}$, we may apply (23) to get

$$\int_{Gr_{n-k}} \left(\int_{L\cap H} g(x)dx\right)^n d\nu_{n,n-k}(H) \leq \gamma_{n,k}^{-n}\left(\int_L g(x)dx\right)^{n-k}. \tag{26}$$

Combining the above we get

$$\begin{aligned}
\left(\int_K f(x)\,dx\right)^{n-k} &\leq \left[\gamma_{n,k}^{-n}p(n,n-k)(C_1/\sqrt{n-k})^{k(n-k)}\right] \\
&\quad\times M^{n-k}|K|^{\frac{k(n-k)}{n}}\mathrm{ovr}(K)^{k(n-k)}\left(\int_L g(x)dx\right)^{\frac{(n-k)^2}{n}}.
\end{aligned}\tag{27}$$

Note that, by (24),

$$\left[\gamma_{n,k}^{-n}p(n,n-k)\right]^{\frac{1}{n-k}}\left(\frac{C_1}{\sqrt{n-k}}\right)^k \leq C^k$$

for some absolute constant $C>0$. Then, the result follows from (27). $\square$

4. Proof of Theorem 3

The proof of Theorem 3 requires several additional definitions and facts from convex geometry. We refer the reader to [46] for details.

The support function of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max_{\xi \in K} \langle x, \xi \rangle, \quad x \in \mathbb{R}^n.$$

If K is origin-symmetric, then h_K is a norm on $\mathbb{R}^n$. One of the crucial properties of the support function is its relation to the Minkowski sum of convex bodies:

$$h_{K+L}(x) = h_K(x) + h_L(x). \tag{28}$$

The surface area measure $S(K, \cdot)$ of a convex body K in $\mathbb{R}^n$ is defined as follows: for every Borel set $E \subset S^{n-1}$, $S(K, E)$ is equal to Lebesgue measure of the part of the boundary of K where normal vectors belong to E. The volume of a convex body can be expressed in terms of its support function and surface area measure:

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K(x) dS(K, x). \tag{29}$$

If K and L are two convex bodies in $\mathbb{R}^n$, the *mixed volume* $V_1(K, L)$ is equal to

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to +0} \frac{|K + \epsilon L| - |K|}{\varepsilon}. \tag{30}$$

We shall use the first Minkowski inequality: for any pair of convex bodies K, L in $\mathbb{R}^n$,

$$V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{1/n}. \tag{31}$$

The mixed volume $V_1(K, L)$ can also be expressed in terms of the support function and surface area measure:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(x) dS(K, x). \tag{32}$$

For a convex body K in $\mathbb{R}^n$ and $\xi \in S^{n-1}$, denote by $K|\xi^\perp$ the orthogonal projection of K to the central hyperplane $\xi^\perp$. The Cauchy formula states that

$$|K|\xi^\perp| = \frac{1}{2} \int_{S^{n-1}} |\langle x, \xi \rangle| \, dS(K, x). \tag{33}$$

Let K be a convex body in $\mathbb{R}^n$. The projection body ΠK of K is defined as an origin-symmetric convex body in $\mathbb{R}^n$ whose support function in every direction is equal to the volume of the orthogonal projection of K to this direction: for every $\theta \in S^{n-1}$,

$$h_{\Pi K}(\xi) = |K|\xi^\perp|. \tag{34}$$

We denote by Π_n the class of projection bodies of convex bodies and if $D \in \Pi_n$ we simply say that D is a projection body. By Cauchy's formula (33), for every projection body D there exists a finite measure ν_D on S^{n-1} such that

$$h_D(x) = \int_{S^{n-1}} |\langle x, \xi \rangle| \, d\nu_D(\xi), \qquad \forall x \in S^{n-1}. \tag{35}$$

Let $\mathcal{K}_0$ denote the class of convex bodies containing the origin in their interior. Firey [13] extended the concept of Minkowski sum (28), and introduced for each real $p \geq 1$, a new linear combination of convex bodies, the so-called p-sum:

$$h^p_{\alpha K +_p \beta L}(x) = \alpha h^p_K(x) + \beta h^p_L(x).$$

Here $K, L \in \mathcal{K}_0$ and α, β are positive real numbers. In a series of papers Lutwak [40, 41] showed that the Firey sums lead to a Brunn-Minkowski theory for each $p \geq 1$. Extending the classical definition of the mixed volume (30) Lutwak introduced the notion of p-mixed volume, $V_p(K, L)$, $p \geq 1$ as

$$V_p(K,L) = \frac{p}{n} \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon L) - V(K)}{\varepsilon},$$

for all $K, L \in \mathcal{K}_0$. Lutwak proved that for each $K \in \mathcal{K}_0$, there exists a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} so that

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K,u)$$

for all $L \in \mathcal{K}_0$. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$, with Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} \;=\; h(K,\cdot)^{1-p}.$$

Lutwak [40] generalized the first Minkowski inequality to the case of p-mixed volumes as follows:

$$V_p(K,L)^n \;\geq\; |K|^{n-p}|L|^p, \qquad p > 1. \tag{36}$$

We will also use the concept of a p-projection body, introduced by Lutwak [41, 42]. Let $\Pi_p K$, $p \geq 1$ denote the compact convex set whose support function is given by

$$h_{\Pi_p K}(\xi)^p = \frac{1}{2n} \int_{S^{n-1}} |\langle x, \xi \rangle|^p d\, S_p(K,x), \qquad \xi \in S^{n-1}. \tag{37}$$

We note that $\Pi_1 K = n\Pi K$. Moreover, for some fixed $p \geq 1$, we say that D is a p-projection body if D is the p-projection body of some convex body. By (37), for every p-projection body D there exists a finite measure ν_D on S^{n-1} such that

$$h_D^p(x) = \int_{S^{n-1}} |\langle x, \xi \rangle|^p \; d\nu_D(\xi), \qquad x \in \mathbb{R}^n. \tag{38}$$

Let us denote by $\Pi_{p,n}$ the class of all p-projection bodies in $\mathbb{R}^n$.

We may pass now to the proof of Theorem 3.

Proof of Theorem 3. Let $\varepsilon > 0$ be such that for every $\xi \in S^{n-1}$

$$h^p_{\Pi_p K}(\xi) \leq \varepsilon h^p_{\Pi_p L}(\xi). \tag{39}$$

By (37), the condition (39) is equivalent to

$$\int_{S^{n-1}} |\langle x, \xi \rangle|^p \; dS_p(K,x) \leq \varepsilon \int_{S^{n-1}} |\langle x, \xi \rangle|^p \; dS_p(L,x), \; \forall \xi \in S^{n-1}. \tag{40}$$

For small $\delta > 0$, let $D \in \Pi_{p,n}$ be such that $D \subset L$ and

$$|L|^{\frac{1}{n}} \leq (1+\delta) d_{\mathrm{vr}}(L, \Pi_{p,n}) \; |D|^{\frac{1}{n}}, \tag{41}$$

and let ν_D be the measure on S^{n-1} corresponding to D by (38). Integrating both sides of (40) with respect to $d\nu_D(\xi)$, we get

$$\int_{S^{n-1}} \int_{S^{n-1}} |\langle x, \xi \rangle|^p \; dS_p(K,x) d\nu_D(\xi) \leq \varepsilon \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, \xi \rangle|^p \; dS_p(L,x) d\nu_D(\xi),$$

for all $\xi \in S^{n-1}$. Applying Fubini's theorem on S^{n-1} together with (37) we get

$$\int_{S^{n-1}} h_D^p(x)\ dS_p(K,x) \leq \varepsilon \int_{S^{n-1}} h_D^p(x)\ dS_p(L,x). \tag{42}$$

Since $D \subset L$, we have $h_D(x) \leq h_L(x)$ for every $x \in S^{n-1}$, so the right-hand side of (42) can be estimated from above by

$$\varepsilon \int_{S^{n-1}} h_D^p(x)\ dS_p(L,x) \leq \varepsilon \int_{S^{n-1}} h_L^p(x)\ dS_p(L,x) = \varepsilon n|L|.$$

By (32), (36), (31) and (41), the left-hand side of (42) can be estimated from below by

$$\begin{aligned}\int_{S^{n-1}} h_D^p(x)\ dS_p(K,x) &= nV_p(K,L) \geq |K|^{\frac{n-p}{n}}|D|^{\frac{p}{n}}\\ &\geq \frac{n}{(1+\delta)^p d_{\mathrm{vr}}^p(L,\Pi_{p,n})}|K|^{\frac{n-p}{n}}|L|^{\frac{p}{n}}.\end{aligned}$$

Combining these estimates we see that

$$\frac{n}{(1+\delta)^p d_{\mathrm{vr}}^p(L,\Pi_{p,n})}|K|^{\frac{n-p}{n}}|L|^{\frac{p}{n}} \leq \varepsilon n|L|.$$

Sending $\delta \to 0$, we get

$$\left(\frac{|K|}{|L|}\right)^{\frac{n-p}{n}} \leq d_{\mathrm{vr}}^p(L,\Pi_{p,n})\ \varepsilon.$$

Now, putting

$$\varepsilon = \max_{\xi\in S^{n-1}} \frac{h_{\Pi_pK}^p(\xi)}{h_{\Pi_pL}^p(\xi)},$$

we get the result. □

A mixed version of the Busemann-Petty and Shephard problems was posed by Milman and solved in [16]. Namely, if K is a convex body in $\mathbb{R}^n$, D is a compact subset of $\mathbb{R}^n$ and $1 \leq k \leq n-1$, then the inequalities $|K|H| \leq |D \cap H|$ for all $H \in Gr_{n-k}$ imply $|K| \leq |D|$. Here $K|H$ is the orthogonal projection of K onto H. One can easily modify the proof from [16] to get a slightly stronger version of this result.

Theorem 5. *Let K be a convex body in $\mathbb{R}^n$, let D be a compact set in $\mathbb{R}^n$, and $1 \leq k \leq n-1$. Then*

$$\left(\frac{|K|}{|L|}\right)^{\frac{n-k}{n}} \leq \max_{H\in Gr_{n-k}} \frac{|K|H|}{|L\cap H|}.$$

5. Applications

5.1. **Comparison theorem for the Radon transform.** We can recover the isomorphic Busemann-Petty theorem for the Radon transform established in [36], as follows. If, in addition to the conditions of Theorem 1, we assume that

$$\int_{K\cap H} f \leq \int_{L\cap H} g, \qquad \forall H \in Gr_{n-k},$$

then we get

$$\int_K f \le \frac{n}{n-k}\,(d_{\rm ovr}(K,BP_k^n))^k\,|K|^{\frac{k}{n}}\left(\int_L g\right)^{\frac{n-k}{n}}.$$

5.2. **A lower estimate for the sup-norm of the Radon transform.** Theorem 1 with $L = B_2^n$ and $g \equiv 1$ is the slicing inequality for arbitrary functions from [32] (see inequality (6) above):

$$\int_K f \le \frac{n}{n-k}\frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|}\,(d_{\rm ovr}(K,BP_k^n))^k\,|K|^{\frac{k}{n}}\max_H \int_{K\cap H} f.$$

Note that the constant $\frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|}$ is less than 1, and $\frac{n}{n-k} \le e^k$.

5.3. **Mean value inequality for the Radon transform.** Let $K = L$, and $g \equiv 1$. Then, as we have mentioned in the Introduction,

$$\frac{\int_K f}{|K|} \le \frac{n}{n-k}\,(d_{\rm ovr}(K,BP_k^n))^k \max_H \frac{\int_{K\cap H} f}{|K\cap H|}.$$

5.4. **The isomorphic Busemann-Petty problem for sections of proportional dimensions.** Theorem 4 and inequality (3) imply the following result from [34], which solves the isomorphic Busemann-Petty problem in affirmative for sections of proportional dimensions. If K, L are origin-symmetric convex bodies in $\mathbb{R}^n$ and $k \ge \lambda n$, where $0 < \lambda < 1$, so that $|K \cap H| \le |L \cap H|$ for every $H \in Gr_{n-k}$, then $|K|^{\frac{n-k}{n}} \le (C(\lambda))|L|^{\frac{n-k}{n}}$, where the constant $C(\lambda)$ depends only on λ.

5.5. **Inequalities for projections.** Theorem 3 implies an isomorphic version of the Shephard problem first established by Ball; it immediately follows from [1, 2].

Corollary 1. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ such that*

$$|K|\xi^\perp| \le |L|\xi^\perp|, \qquad \forall \xi \in S^{n-1},$$

then

$$|K| \le d_{\rm vr}(L,\Pi_n)|L|.$$

Corollary 2. *Let L be an origin-symmetric convex body in $\mathbb{R}^n$. Then*

$$\min_{\xi\in S^{n-1}} |L|\xi^\perp| \le \sqrt{e}\; d_{\rm vr}(L,\Pi_n)\; |L|^{\frac{n-1}{n}}.$$

Proof: Apply Theorem 3 to $K = B_2^n$ and L, and then use the fact that $c_{n,1} = |B_2^{n-1}|/|B_2^n|^{\frac{n-1}{n}} \le \sqrt{e}$.

□

By John's theorem [25] and the fact that ellipsoids are projection bodies (see, for example [29, 46]), we have $d_{\rm vr}(L,\Pi_n) \le \sqrt{n}$ for any origin-symmetric convex body L in $\mathbb{R}^n$. On the other hand, Ball [1] proved that there exists an absolute constant $c > 0$ so that for every n there exists an origin-symmetric convex body L_n of volume 1 in $\mathbb{R}^n$ satisfying $|L|\xi^\perp| \ge c\sqrt{n}$ for all $\xi \in \mathbb{R}^n$. Combined with Corollary 2, these estimates show that

$$c\sqrt{n} \le \max_L d_{\rm vr}(L,\Pi_n) \le \sqrt{n}, \tag{43}$$

where c is an absolute constant, and the maximum is taken over all origin-symmetric convex bodies in $\mathbb{R}^n$. This estimate was first established by Ball; it immediately follows from [2, Example 2].

Note that the distance $d_{\mathrm{vr}}(L,\Pi_n)$ has been studied by several authors. It was introduced in [2] and was proved to be equivalent to the *weak-right-hand-Gordon-Lewis* constant of L. Also it was connected to the random unconditional constant of the dual space (see Theorem 5 and Proposition 6 in [2]). In [17] this distance was called zonoid ratio, and it was proved that it is bounded from above by the projection constant of the space. In the same paper the zonoid ratio was computed for several classical spaces. We refer the interested reader to [17], [19], [18], [20] for more information.

5.6. **Milman's estimate for the isotropic constant.** We say that a compact set K with volume 1 in $\mathbb{R}^n$ is in isotropic position if for each $\xi\in S^{n-1}$

$$\int_K \langle x,\xi\rangle^2 dx = L_K^2$$

where L_K is a constant that is called the isotropic constant of K. In the case where K is origin-symmetric convex, the slicing problem of Bourgain is equivalent to proving that L_K is bounded by an absolute constant.

Hensley [24] has proved that there exist absolute constants $c_1,c_2>0$ so that for any origin-symmetric convex body K in $\mathbb{R}^n$ in isotropic position and any $\xi\in S^{n-1}$

$$\frac{c_1}{L_K}\le |K\cap\xi^\perp|\le \frac{c_2}{L_K}.$$

The following inequality was proved by Milman [43]. We present a simpler proof.

Theorem 6. *There exists an absolute constant C so that for any origin-symmetric isotropic convex body K in $\mathbb{R}^n$*

$$L_K\le C\ d_{\mathrm{ovr}}(K,\mathcal{I}_n).$$

Proof: By Theorem 4 with $k=1$ and Hensley's theorem, for any origin-symmetric isotropic convex bodies K,D in $\mathbb{R}^n$

$$\left(\frac{|K|}{|D|}\right)^{\frac{n-1}{n}}\le d_{\mathrm{ovr}}(K,\mathcal{I}_n)\max_{\xi\in S^{n-1}}\frac{|K\cap\xi^\perp|}{|D\cap\xi^\perp|}\le d_{\mathrm{ovr}}(K,\mathcal{I}_n)\frac{|K|^{\frac{n-1}{n}}\frac{c_2}{L_K}}{|D|^{\frac{n-1}{n}}\frac{c_1}{L_D}},$$

where $c_1,c_2>0$ are absolute constants, so

$$\frac{L_K}{L_D}\le C\ d_{\mathrm{ovr}}(K,\mathcal{I}_n).$$

Now put $D=B_2^n/|B_2^n|^{\frac{1}{n}}$, and use the fact that L_D is bounded by an absolute constant; see [7].

□

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