

Square Peg in a Circular Hole: Choosing the Right Ansatz for Isolated Black Holes in Generic Gravitational Theories

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The metric of a spacetime can be greatly simplified if the spacetime is circular. We prove that in generic effective theories of gravity, the spacetime of a stationary, axisymmetric, and asymptotically flat solution must be circular if the solution can be obtained perturbatively from a solution in the general relativity limit. This result applies to a broad class of gravitational theories that include arbitrary scalars and vectors in their light sector, so long as their nonstandard kinetic terms and nonminimal couplings to gravity are treated perturbatively.

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Introduction.—Despite the complexity and nonlinearity of the Einstein equations, rotating black holes in general relativity (GR) are described by a remarkably simple analytical solution obtained by Kerr [1,2]. A crucial step in finding the Kerr solution is that the ten unknown functions of four coordinate variables in the metric can be reduced to four unknown functions of only two variables. This simplification is only possible because stationary and axisymmetric vacuum solutions in GR belong to a specific class called “circular spacetimes” [3]. However, this is not necessarily the case in generic gravitational theories [4], and one should not expect *a priori* that black hole solutions in such theories will be circular. In particular, one should expect the validity of the circularity assumption to play a role as important as it did in GR to obtain rotating black hole solutions (either numerically or analytically) in such theories. In turn, knowledge of these solutions constitutes the stepping stone on which many tests of strong-field gravity rely [5]. The use of an oversimplified ansatz based on the circularity condition can lead to spacetimes that are inconsistent with a given theory’s field equations. This was recently observed, for instance, in the case of rotating black hole solutions with linearly time-dependent hair in cubic Galileon theories in which the circularity condition is not satisfied [6].

In this Letter, we investigate the circularity of stationary and axisymmetric solutions in generic gravitational theories, paving the way for finding rotating black hole solutions in GR and beyond. In order to remain generic on the gravitational theory, we work within the effective field theory (EFT) framework in which UV modifications of GR manifest as higher dimensional operators in the low-energy EFT and can be treated perturbatively. The EFT framework

works well for isolated astrophysical black holes, which have masses in the $\sim 5\text{--}10^{10} M_\odot$ range thanks to their low energy scale ($\lesssim 10^{-11}$ eV). The framework is also supported by the agreement of GR predictions with gravitational wave detections [7] and other electromagnetic observations [8]. In particular, we focus on gravitational theories whose low-energy EFT represents extensions of GR involving additional (scalar) fields and other operators. These EFTs include $f(R)$ gravity or more general scalar-tensor theories [9,10] and quadratic gravity [11] such as dynamical Chern-Simons gravity [12,13] and Einstein-dilaton-Gauss-Bonnet gravity [14,15], as well as gravitational EFTs without light scalar fields like those studied in [16–18].

As the modifications of GR are small, black hole solutions in the EFTs can be obtained through a perturbative expansion around one (or more) coupling constants of such theories (see [19–33] for examples). We show here that the spacetime of stationary, axisymmetric, and asymptotically flat solutions is circular in these EFTs, hence also in the corresponding high-energy gravitational theories. In principle, there could be other branches of solutions that are not connected perturbatively to their GR counterparts (see [34,35] for example), but these are not the focus of this Letter. We use geometric units ($c = 8\pi G = 1$) and employ the $(-, +, +, +)$ metric signature.

Circular spacetimes in GR.—Consider a stationary and axisymmetric spacetime associated with two Killing vectors ξ^μ and χ^μ that correspond to the two isometries, respectively. Figure 1 gives a schematic illustration of this geometry. Carter [36] showed that the two Killing vectors commute, which means one can choose adapted coordinates (t, r, θ, ϕ) on the spacetime such that $\xi = \partial_t$ and $\chi = \partial_\phi$. The isometries imply

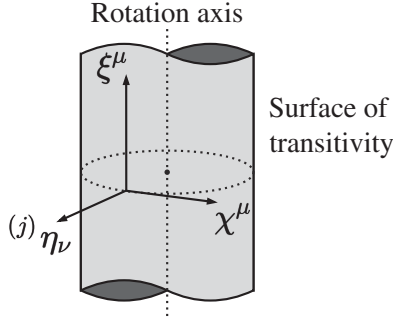


FIG. 1. Geometry of a stationary and axisymmetric spacetime. The Killing vector ξ^μ is associated with time translation and χ^μ is associated with rotations about the symmetry axis. Note that ξ^μ and χ^μ are not necessarily orthogonal. The surface of transitivity is generated by ξ^μ and χ^μ and is degenerate on the rotation axis where χ^μ vanishes. The independent vectors ${}^{(j)}\eta_\nu$ ($j = 3, 4$) are chosen to be orthogonal to the surface of transitivity. Here we only show one of the orthogonal vectors.

$$\partial_t g_{\mu\nu} = 0 = \partial_\phi g_{\mu\nu}. \quad (1)$$

Moreover, there exist privileged 2-dimensional surfaces, called “surfaces of transitivity,” to which the Killing vectors are everywhere tangent except on the rotation axis where χ^μ vanishes. In adapted coordinates, the surfaces of transitivity can be labeled by the values of (r, θ) .

A circular spacetime is a subclass of stationary and axisymmetric spacetimes for which, in addition to Eq. (1), there exists a family of 2-surfaces known as meridional surfaces that are everywhere orthogonal to the surfaces of transitivity. In this case, one can further choose the coordinates r and θ such that

$$g_{tr} = g_{t\theta} = g_{\phi r} = g_{\phi\theta} = 0. \quad (2)$$

Without loss of generality, the metric can then take the following ansatz

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + A^2 (dr^2 + r^2 d\theta^2) + B^2 r^2 \sin^2 \theta (d\phi - \omega dt)^2, \quad (3)$$

in “quasi-isotropic coordinates,” where N , A , B , and ω are functions of r and θ .

Papapetrou [3] (see also [37]) showed that a spacetime is circular if (i) $\xi_{[\mu}\chi_\nu\nabla_\rho\xi_{\sigma]}$ and $\xi_{[\mu}\chi_\nu\nabla_\rho\chi_{\sigma]}$ each vanish at least at one point of the spacetime, and (ii)

$$\xi^\mu R_{\mu}^{[\nu} \xi^{\rho} \chi^{\sigma]} = 0, \quad \chi^\mu R_{\mu}^{[\nu} \xi^{\rho} \chi^{\sigma]} = 0 \quad (4)$$

everywhere in spacetime, where the square brackets denote full antisymmetrization. For asymptotically flat spacetimes, which we focus on, Carter further showed a rotation axis at which $\chi^\mu = 0$ exists [36]; thus, the first condition is satisfied.

The Eq. (4) condition is trivially satisfied if the Ricci tensor vanishes, which means that any stationary, axisymmetric, and asymptotically flat vacuum solution in GR is circular, as well as those Ricci-flat solutions in modified gravity theories (e.g., [24,38]). The Eq. (4) condition can also be recast as a requirement of the Ricci tensor being “invertible” [39–41]. Let ${}_{(i)}\xi^\mu$ ($i = 1, 2$) be the two Killing vectors ξ^μ and χ^μ and ${}^{(j)}\eta_\nu$ ($j = 3, 4$) be two independent vectors everywhere orthogonal to ξ^μ and χ^μ . A tensor is said to be invertible in the isometry group if the scalars obtained by contracting any combinations of the tensor’s indices with any choice of ${}_{(i)}\xi^\mu$ and ${}^{(j)}\eta_\nu$ vanish whenever the number of contracted ${}_{(i)}\xi^\mu$ is odd. In particular, the Ricci tensor is invertible if

$$R_{\mu}^{(\nu} \xi^{\mu(j)} \eta_\nu = 0, \quad i = 1, 2, \quad j = 3, 4. \quad (5)$$

Heuristically, the Eq. (5) condition is equivalent to the Eq. (4) condition because the latter is equivalent to requiring that ${}_{(i)}\xi^\mu R_{\mu}^{\nu}$ be tangent to the surface of transitivity (i.e., proportional to any linear combination of ${}_{(i)}\xi^\mu$) and thus, that any part tangent to the meridional surface [i.e., proportional to any linear combination of ${}^{(j)}\eta^\nu$] vanish. In the following, we shall omit the presub and superscript of ξ^μ and η_ν and bear in mind that each of them represents a vector in a two vector set.

Circularity in generic gravitational theories.—Let us consider a generic gravitational theory potentially containing fields of arbitrary spin and coupling to gravity with the Lagrangian

$$\mathcal{L} = \frac{1}{2}R + \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_{\text{int}}(\nabla_\rho, R_{\rho\sigma\alpha\beta}, \varphi, \psi), \quad (6)$$

where the fields are classified as heavy fields ψ or light fields φ depending on whether their masses are above or below the curvature scale of the solution that we are interested in. Here, $\mathcal{L}_\varphi$ and $\mathcal{L}_\psi$ are the Lagrangians of φ and ψ , while $\mathcal{L}_{\text{int}}$ captures all the interactions between the fields as well as any nonminimal couplings to gravity. In particular, we assume that nonstandard kinetic terms of φ , if there is any in $\mathcal{L}_\varphi$, can be treated perturbatively. At the energy scale of the solution, we can integrate out the heavy fields with masses larger than the curvature of the solution we are interested in,

$$e^i \int d^4x \sqrt{-g} \mathcal{L}_{\text{EFT}} = \int \mathcal{D}\psi e^i \int d^4x \sqrt{-g} \mathcal{L}, \quad (7)$$

and obtain a low-energy EFT with the following Lagrangian (see Refs. [42–44] for explicit examples):

$$\mathcal{L}_{\text{EFT}} = \frac{1}{2}R + \mathcal{L}_0(\varphi, g_{\mu\nu}) + \alpha \mathcal{L}_M(\nabla_\rho, R_{\rho\sigma\alpha\beta}, \varphi), \quad (8)$$

where operators are sorted according to their dimensions. In particular, $\mathcal{L}_0$ are operators constructed by the light fields φ and their covariant derivatives with dimensions equal to or less than 4, while $\mathcal{L}_M$ are higher dimension operators constructed by the Riemann tensor, the light fields, and derivatives of both, which therefore are suppressed by a small parameter α . The heavy fields ψ in Eq. (6) have been integrated out and manifest themselves solely as higher curvatures and derivative corrections in $\mathcal{L}_M$. The curvature scale of isolated astrophysical black holes is expected to be smaller than 10^{-11} eV. Hence, in realistic situations, the heavy fields ψ include all massive particles of the standard model and beyond.

For now, we focus on the case in which the light fields, if any, are all scalar fields. We emphasize that φ denotes all light fields in the EFT, which we shall not distinguish with additional labels, and thus, inner products require an internal space metric, which we will also suppress [45]. This EFT reduces identically and smoothly to GR as $\alpha \rightarrow 0$, i.e., in this limit Eq. (8) reduces to the Einstein-Hilbert action minimally coupled to light scalar fields.

The modified Einstein equations in this theory are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} + \alpha M_{\mu\nu}, \quad (9)$$

where $T_{\mu\nu} \equiv -2\delta(\sqrt{-g}\mathcal{L}_0)/\delta g^{\mu\nu}$ and $M_{\mu\nu} \equiv -2\delta(\sqrt{-g}\mathcal{L}_M)/\delta g^{\mu\nu}$ are the energy-momentum tensors associated with $\mathcal{L}_0$ and $\mathcal{L}_M$, respectively. In particular, terms in $T_{\mu\nu}$ are either proportional to $g_{\mu\nu}$ or proportional to $\partial_\mu\varphi\partial_\nu\varphi$ due to the dimension of the operators in $\mathcal{L}_0$. Given the smallness of α , a solution to Eq. (9) $\{g_{\mu\nu}, \varphi\}$ can be obtained order by order in α . For concreteness, we use $\{g_{\mu\nu}^{(n)}, \varphi^{(n)}\}$ to denote the solution to the n th order in α , i.e., $g_{\mu\nu} = g_{\mu\nu}^{(n)} + \mathcal{O}(\alpha^{n+1})$, with $\mathcal{O}(\alpha^{n+1})$ accounting for all higher-order corrections. We also label a quantity with subscript or superscript (n) , e.g., $T_{\mu\nu}^{(n)}$ or $g_{\mu\nu}^{(n)}$, if it is calculated up to the n th order in α .

The full solution is given by $\{g_{\mu\nu}^{(n)}, \varphi^{(n)}\}$ with n approaching infinity for a sufficiently small α .

In the following, we prove that the spacetime of a stationary, axisymmetric, and asymptotically flat solution is necessarily circular if the solution can be obtained order by order in α . Here we only consider solutions with stationary and axisymmetric scalar fields, which are not necessarily required for the spacetime also be stationary and axisymmetric as we discuss later. The proof can be done in three steps.

First, we prove that the solution is circular at zeroth order in α , i.e., $g_{\mu\nu}^{(0)}$ is circular. At zeroth order, we get back to GR, and $g_{\mu\nu}^{(0)}$ is circular if $T_{\mu\nu}^{(0)}$ is invertible [39]. In order to show the invertibility, let us consider $T_{\mu\nu}^{\zeta^\mu\eta_\nu}$, where ζ^μ are the two Killing vectors and η_ν are the two independent vectors orthogonal to ζ^μ . Since the scalar fields are stationary and

axisymmetric, the vanishing of their Lie derivatives along ζ^μ implies

$$\mathcal{L}_\zeta\varphi \equiv \zeta^\mu\partial_\mu\varphi = 0. \quad (10)$$

Thus, terms in $T_{\mu\nu}$ that are proportional to $\partial_\mu\varphi\partial_\nu\varphi$ vanish after contracting with ζ^μ . The rest of $T_{\mu\nu}$ is proportional to $g_{\mu\nu}$ and do not contribute to $T_{\mu\nu}^{\zeta^\mu\eta_\nu}$ given the orthogonality between ζ^μ and η_ν . Therefore,

$$T_{\mu\nu}^{\zeta^\mu\eta_\nu} = 0, \quad (11)$$

i.e., $T_{\mu\nu}$ is invertible. At zeroth order in α , Eq. (11) means $T_{\mu\nu}^{(0)}$ is invertible, and hence $g_{\mu\nu}^{(0)}$ is circular.

Next, we prove that if $g_{\mu\nu}^{(0)}$ is circular, then $g_{\mu\nu}^{(1)}$ is also circular. This can be proved if the Ricci tensor associated with $g_{\mu\nu}^{(1)}$ is invertible, or equivalently [46],

$$R_{\mu\nu}^{\zeta^\mu\eta_\nu} = 0 + \mathcal{O}(\alpha^2). \quad (12)$$

Contracting Eq. (9) with ζ^μ and η^ν , we find

$$R_{\mu\nu}^{\zeta^\mu\eta_\nu} = \alpha M_{\mu\nu}^{\zeta^\mu\eta_\nu}, \quad (13)$$

where the second term on the left hand side of Eq. (9) does not contribute due to the orthogonality between ζ^μ and η_ν , and the first term on the right hand side of Eq. (9) also vanishes because of the invertibility of $T_{\mu\nu}$.

On the other hand, since $g_{\mu\nu}^{(0)}$ is circular, the Riemann tensor associated with $g_{\mu\nu}^{(0)}$ is invertible (see the Supplemental Material [47] for a proof). Moreover, we show in the Supplemental Material [47] that any tensor constructed from stationary, axisymmetric, and invertible tensors and their covariant derivatives associated with $g_{\mu\nu}^{(0)}$ is also itself invertible. Together with the assumption that the scalar fields φ are stationary and axisymmetric, we conclude that $M_{\mu\nu}$ evaluated at zeroth order in α is invertible, and hence

$$M_{\mu\nu}^{\zeta^\mu\eta_\nu} = 0 + \mathcal{O}(\alpha). \quad (14)$$

Substituting Eq. (14) into Eq. (13), we find $R_{\mu\nu}^{\zeta^\mu\eta_\nu}$ vanishes to first order in α , and therefore, $g_{\mu\nu}^{(1)}$ is circular.

Finally, we assume the solution is circular to the n th order in α and show that the solution to the $(n+1)$ th order is circular. The proof is similar to that in the second step. In this case, $M_{\mu\nu}^{\zeta^\mu\eta_\nu}$ can be evaluated to the n th order in α with $g_{\mu\nu}^{(n)}$ and $\varphi^{(n)}$. The circularity of the n th order solution implies that

$$M_{\mu\nu}^{\zeta^\mu\eta_\nu} = 0 + \mathcal{O}(\alpha^{n+1}). \quad (15)$$

Substituting this into Eq. (13), we find $R_{\mu\nu}^{\zeta^\mu\eta_\nu}$ vanishes to $(n+1)$ th order in α , and hence the solution to the $(n+1)$ th

order is circular. By induction, we conclude that the solution is circular to all orders in α .

Extension to generalized light fields.—Our proof can be further extended to theories with more general light fields, as long as the light fields and their leading-order stress-energy tensor $T_{\mu\nu}$ are invertible.

For light scalar fields, $\mathcal{L}_0$ may also include higher dimension operators that are arbitrary functions of φ , $\nabla_\mu\varphi\nabla^\mu\varphi$, and $\square\varphi$. In this case, the resulting leading order stress-energy tensor is

$$T_{\mu\nu} = -\frac{\partial\mathcal{L}_0}{\partial(\nabla_\lambda\varphi\nabla^\lambda\varphi)}\nabla_\mu\varphi\nabla_\nu\varphi + \nabla_{(\mu}\left(\frac{\partial\mathcal{L}_0}{\partial(\square\varphi)}\right)\nabla_{\nu)}\varphi + \frac{1}{2}g_{\mu\nu}\left\{\mathcal{L}_0 - \nabla_\lambda\left[\frac{\partial\mathcal{L}_0}{\partial(\square\varphi)}\nabla^\lambda\varphi\right]\right\}, \quad (16)$$

where terms proportional to $\nabla_\mu\varphi\nabla_\nu\varphi$ or $g_{\mu\nu}$ are invertible for the same reasons discussed above. Moreover, $\partial\mathcal{L}_0/\partial(\square\varphi)$ inherits the symmetries of φ , so its Lie derivatives along ζ^μ vanish. Thus, the second term on the right hand side of Eq. (16), and hence the aggregated $T_{\mu\nu}$, is invertible, indicating stationary, axisymmetric, and asymptotically flat vacuum solutions in such more general scalar-tensor theories are also circular.

In addition to the light scalars as described above, our proof can also be generalized to gravitational theories that include light vectors, as long as the nonstandard kinetic terms and nonminimal couplings to gravity may be treated perturbatively. In particular, our proof can be extended to include light vectors with the following restrictions: (i) $\mathcal{L}_0$ is totally constructed from the vector fields V_μ and their exterior derivatives $F_{\mu\nu} = 2\nabla_{[\mu}V_{\nu]}$, and (ii) the vector fields V_μ , apart from being stationary and axisymmetric, are invertible. In this case, since the exterior derivative does not depend on the metric, the energy-momentum tensor associated with $\mathcal{L}_0$ is completely constructed from V_μ and $F_{\mu\nu}$. We show in the Supplemental Material [47] that $F_{\mu\nu}$ inherits the vector field's invertibility without assuming circularity. Therefore, $T_{\mu\nu}$ is invertible, and any such vector-tensor theory admits a circular ansatz for stationary and axisymmetric vacuum solutions. In addition, any generalized Proca theory, as introduced in [48–52], would inherit the same properties so long as the higher-order Lagrangians introduced in these theories are treated perturbatively.

Discussions.—Our main result is a proof that the spacetime of stationary, axisymmetric, and asymptotically flat rotating black holes in a broad class of gravitational EFTs is circular. We emphasize that in addition to the light fields we have considered, the theory may also include any heavy field of arbitrary spin and coupling to gravity, as long as the mass of these fields is larger than the curvature scale of the black holes. Our result is of immediate importance to the ongoing effort of testing the strong-field regime of

gravity through gravitational waves [53–57] and electromagnetic observations [58,59] in which black holes play a central role [5]. These tests generally require knowledge of a rotating black hole solution (within a certain EFT) from which observable consequences are then deduced and then ultimately compared to observations. Here, we proved that circularity is shared among a broad class of solutions, justifying the use of this ansatz when searching for analytical and numerical solutions.

What are the implications of our result for some specific theories? Consider, for instance, dynamical Chern-Simons gravity in which a scalar field couples to the Pontryagin density [12,13]. This theory must be treated as an EFT to admit a well-posed initial value problem [60], and, in fact, this theory is captured within the assumption of our proof. Rotating black hole solutions in this theory are known numerically [61,62] and analytically [20,27,63–65] in a perturbative expansion in the coupling strength α and black hole spin $a \ll 1$ to $\mathcal{O}(\alpha^2 a^5)$ [26,30,63] and in the extremal limit [66]. Our results indicate that the spacetime of rotating black holes in this theory is circular, justifying the use of the ansatz [Eq. (3)] in numerical calculations. The same applies to scalar Gauss-Bonnet gravity with shift-symmetric and dilatonic couplings where rotating black hole spacetimes are known analytically [24,25,27–29] and numerically [67–70], including the final state of black holes that results at late times after highly dynamical black hole formation [71–73]. In fact, it applies to any EFT extension of GR, including any low-energy EFT of gravity that includes massive fields of arbitrary spins.

Our results agree with those of [74], which suggested the nonexistence of rotating noncircular black holes in dynamical Chern-Simons gravity and shift-symmetric scalar-Gauss-Bonnet gravity, by working perturbatively to $\mathcal{O}(\alpha^2 a^2)$. Our conclusions extend to all orders in these two parameters. Moreover, our results also apply to non-vacuum solutions in generic gravitational theories of the type discussed in this Letter, as long as the matter fields in the GR solution are stationary, axisymmetric, and possess an invertible stress-energy tensor. That is, our conclusion holds for a gravitational theory minimally coupled to an ordinary matter source, such as a perfect fluid that satisfies the same symmetries as the metric (i.e., stationarity and axisymmetry).

We stress that our results only apply to solutions that reduce to a GR solution in the limit when the perturbative parameter α goes to zero. In general, this does not have to be the case, as other branches of solutions may be entropically favored, as is the case with theories that exhibit spontaneous black hole scalarization [34,35].

The requirement that the fields are stationary and axisymmetric (and invertible if of spin-1) is a sufficient but not a necessary condition for the solution to be circular, and it is not necessarily required by the isometries of the spacetime. There are cases in which the extra fields can be

time- and angle-dependent, yet this dependence does not manifest itself in the gravitational equations. For example, there are hairy, nonlinear black hole solutions and solitonic solutions that arise in GR coupled to complex and massive (scalar) fields [75–78], where the metric is circular while the fields have time- or angle-dependent phases. Other examples are the stealth black holes of [79] in which the scalar field has a linear time dependence, although such black hole solutions usually suffer from a strong coupling problem [80–82].

Our results do imply that if a theory satisfies the conditions of our theorem, then *all* black hole solutions must have a circular spacetime, but the converse is not necessarily true. Imagine one were to find a black hole solution in a modified theory (in which our theorem does not apply) by requiring *a priori* that the spacetime be circular. The existence of this solution does not then mean that other noncircular solutions do not exist. For example, black hole solutions have been found in Einstein-Yang-Mills theories with [83] and without [84] a dilaton field, and in Einstein-ether theory in the slow-rotation approximation [85,86] assuming *a priori* that the spacetime must be circular. In both cases, however, our theorem does not apply because either the Yang-Mills vector gauge field is noninvertible after gauge fixing or the ether field is noninvertible because of its timelike constraint. Thus, the existence of those solutions does not imply that other noncircular black hole solutions do not exist in these theories, which could be explored further.

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