

ON THE ERGODIC WARING–GOLDBACH PROBLEM

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ABSTRACT. We prove an asymptotic formula for the Fourier transform of the arithmetic surface measure associated to the Waring–Goldbach problem and provide several applications, including bounds for discrete spherical maximal functions along the primes and distribution results such as ergodic theorems.

1. INTRODUCTION

In this paper, we study several questions on the interface between harmonic analysis and analytic number theory. Our results are motivated in part by the study of discrete maximal functions in harmonic analysis, in part by applications of those maximal functions in ergodic theory, and in part by connections to classical problems in analytic number theory—in particular, the Waring–Goldbach problem.

The harmonic analytic motivation behind our work comes from celebrated results by Bourgain [4, 5] on ergodic averages over certain sequences of integers and later work of Magyar, Stein and Wainger [24] on discrete spherical maximal functions. Driven by applications in ergodic theory, Bourgain [4] initiated the study of discrete maximal functions. A key feature of Bourgain’s approach is his use of the circle method from analytic number theory. With this in mind, Magyar [20] provided some partial results on discrete maximal functions related to Waring’s problem, leading to Magyar, Stein and Wainger’s consideration of the discrete spherical averages

$$S_\lambda f(\mathbf{x}) := \frac{1}{\#\{\mathbf{y} \in \mathbb{Z}^n : |\mathbf{y}|_2^2 = \lambda\}} \sum_{|\mathbf{y}|_2^2 = \lambda} f(\mathbf{x} - \mathbf{y}),$$

along with their maximal function

$$S_* f(\mathbf{x}) := \sup_{\lambda \in \mathbb{N}} |S_\lambda f(\mathbf{x})|.$$

Here, $f : \mathbb{Z}^n \rightarrow \mathbb{C}$ and $|\cdot|_2$ denotes the Euclidean norm on $\mathbb{R}^n$ (thus, $|\mathbf{y}|_2^2 = y_1^2 + \cdots + y_n^2$).

Magyar, Stein and Wainger [24] provided a complete answer to the question of ℓ^p -boundedness for the maximal operator S_* : they proved that, when $n \geq 5$, S_* is bounded on $\ell^p(\mathbb{Z}^n)$ when $p > n/(n-2)$ and unbounded when $p \leq n/(n-2)$. Furthermore, it is shown that this result cannot hold when $n < 5$. In their work, they took the symbiosis between harmonic analysis and number theory a step further by using a full-fledged application of the circle method to analyze the Fourier transform of the arithmetic surface measure underlying the discrete averages $S_\lambda f$. Particularly, define

$$\widehat{\sigma}_\lambda(\boldsymbol{\xi}) := \frac{1}{\#\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}|_2^2 = \lambda\}} \sum_{|\mathbf{x}|_2^2 = \lambda} e(\mathbf{x} \cdot \boldsymbol{\xi}),$$

where $\boldsymbol{\xi} \in \mathbb{T}^n$ and, as usual, $e(z) := e^{2\pi iz}$. Magyar, Stein and Wainger established the following approximation formula for $\widehat{\sigma}_\lambda(\boldsymbol{\xi})$.

Theorem (Magyar–Stein–Wainger). *When $n \geq 5$, one has the decomposition*

$$\widehat{\sigma}_\lambda(\boldsymbol{\xi}) = \sum_{q=1}^{\infty} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} e(-a\lambda/q) \sum_{\mathbf{b} \in \mathbb{Z}^n} G(a, q; \mathbf{b}) \Psi(q\boldsymbol{\xi} - \mathbf{b}) \widetilde{d\sigma}_\lambda(\boldsymbol{\xi} - q^{-1}\mathbf{b}) + \widehat{E}_\lambda(\boldsymbol{\xi}),$$

where $\widetilde{d\sigma}_\lambda$ is the continuous Fourier transform of the surface measure of the sphere of radius $\sqrt{\lambda}$,

$$G(a, q; \mathbf{b}) = \sum_{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n} e\left(\frac{a|\mathbf{x}|_2^2 + \mathbf{b} \cdot \mathbf{x}}{q}\right)$$

is an n -dimensional Gauss sum, and Ψ is a smooth bump function which is 1 on $[-1/8, 1/8]^n$ and supported in $[-1/4, 1/4]^n$. The convolution operators E_λ associated with the error terms $\widehat{E}_\lambda$ satisfy the maximal inequality

$$\left\| \sup_{\Lambda \leq \lambda \leq 2\Lambda} |E_\lambda| \right\|_{\ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)} \lesssim \Lambda^{1-n/4}$$

for all $\Lambda > 0$.

This theorem has served as a model for several authors [21, 2, 23, 15] who have studied the maximal functions of the discrete surface measures on other arithmetic surfaces over the integers. It is also the inspiration for one of the results of the present paper—see Theorem 1 below. However, in contrast to earlier work on discrete maximal functions, we study the more singular maximal function of the “prime points” on the k -sphere. The goal of this paper is to study the distribution of points with prime coordinates on the algebraic surface

$$\mathbf{f}(\mathbf{x}) := x_1^k + \cdots + x_n^k = \lambda, \tag{1.1}$$

for $\lambda \in \mathbb{N}$. By combining number-theoretic techniques from the study of the Waring–Goldbach problem with ideas from harmonic analysis, we are able to prove several results on the distribution of such points, including: an equidistribution theorem, an L^2 -ergodic theorem, and a pointwise ergodic theorem. A quantitative version of our equidistribution theorem, Theorem 3 below, is another main result: we take the spherical maximal function in a new direction by proving $\ell^p(\mathbb{Z}^n)$ bounds for a discrete variant along the primes.

While restrictions of results about integer sequences to the primes are common in both number theory and ergodic theory, the study of maximal functions related to the primes has been limited to sequential averages (e.g. [6, 32, 27, 25, 26]); this paper appears to be the first work on such restricted problems in harmonic analysis related to prime points on forms in many variables. The new obstacles arising in these problems require further development of the Bourgain–Magyar–Stein–Wainger paradigm of using the circle method to decompose the maximal operator and thus to reduce the problem to estimates for exponential sums and integrals. Earlier works have been able to employ a classical variant of the circle method which uses the Poisson summation formula to estimate the major arc contribution. The restriction to primes forces us to draw on our knowledge about equidistribution of primes in arithmetic progressions and to employ more primitive tools (compared to Poisson summation) to do so. Therefore, in order to be able to obtain any result at all, we blend mean

value theorems of Vinogradov's type into our minor arc analysis. In contrast, in problems over unrestricted integers previous approaches were able to rely merely on L^∞ bounds for the relevant exponential sums. Indeed, insights gained from this present work have already led us to improve (in [1]) on the results of the third author [15, 16] on (unrestricted) integer points on the k -sphere.

The study of prime points $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$ (here $\mathbb{P}$ is the set of primes) on the surface (1.1) is known in number theory as the Waring–Goldbach problem. Classic work by Hua [14] established the asymptotic for the number of representations of a large natural number λ as a sum of n k th powers of primes when k and n are positive integers such that $n > 2^k$ and λ belongs to an appropriate infinite arithmetic progression $\Gamma_{n,k}$. Write $\log \mathbf{x} = (\log x_1) \cdots (\log x_n)$, and let $R(\lambda)$ denote the number of prime solutions of (1.1), counted with logarithmic weights:

$$R(\lambda) = \sum_{\mathfrak{f}(\mathbf{p})=\lambda} \log \mathbf{p},$$

where (and through the remainder of the paper) $\mathbf{p}$ denotes a vector in $\mathbb{P}^n$. Using the Hardy–Littlewood circle method, Hua proved that when $\lambda \rightarrow \infty$, one has the asymptotic

$$R(\lambda) \sim \mathfrak{S}_{n,k}(\lambda) \lambda^{n/k-1}, \quad (1.2)$$

where $\mathfrak{S}_{n,k}(\lambda)$ is a product of local densities:

$$\mathfrak{S}_{n,k}(\lambda) = \prod_{p \leq \infty} \mu_p(\lambda).$$

Here $\mu_p(\lambda)$ with $p < \infty$ is related to the solubility of (1.1) over the p -adic field $\mathbb{Q}_p$, and $\mu_\infty(\lambda)$ to solubility over the reals. In particular, the set $\Gamma_{n,k}$ is determined by the requirement that $\mu_p(\lambda) > 0$ for all primes p . Some examples of progressions $\Gamma_{n,k}$ (see Chapter VIII in Hua [14] for more details, including the full definition of $\Gamma_{n,k}$) include:

- $\Gamma_{n,k}$ is the residue class $\lambda \equiv n \pmod{2}$ when k is odd;
- $\Gamma_{5,2}$ is the residue class $\lambda \equiv 5 \pmod{24}$;
- $\Gamma_{17,4}$ is the residue class $\lambda \equiv 17 \pmod{240}$.

The starting point to our main results lies in extending (1.2) to an approximation formula for the Fourier transform of the arithmetic probability measure

$$\omega_\lambda(\mathbf{x}) := \frac{1}{R(\lambda)} \mathbf{1}_{\{\mathbf{p} \in \mathbb{P}^n : \mathfrak{f}_{n,k}(\mathbf{p}) = \lambda\}}(\mathbf{x}) \log \mathbf{x},$$

defined when $R(\lambda) > 0$. The Fourier transform of this measure is the exponential sum

$$\widehat{\omega}_\lambda(\boldsymbol{\xi}) = \frac{1}{R(\lambda)} \sum_{\mathfrak{f}(\mathbf{p})=\lambda} (\log \mathbf{p}) e(\mathbf{p} \cdot \boldsymbol{\xi}). \quad (1.3)$$

We note that $\widehat{\omega}_\lambda$ is defined only for sufficiently large $\lambda \in \Gamma_{n,k}$ and n sufficiently large in terms of k . Based on the current state of affairs in the Waring–Goldbach problem [17, 18], the latter means that for large k , the value of n must be at least as large as $4k \log k$. In reality, the true size of $R(\lambda)$ is only known for $n \geq k^2 - k + O(\sqrt{k})$, so it only makes sense to study the Fourier transform $\widehat{\omega}_\lambda(\boldsymbol{\xi})$ when $n \geq k^2 - k$.

Our first theorem is a variant of the Magyar–Stein–Wainger theorem above for the Fourier transform (1.3). Before stating the result, we need to introduce some notation. Given an

integer $q \geq 1$, we write $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ and $\mathbb{U}_q = \mathbb{Z}_q^*$, the group of units. If $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n$, with $\mathbf{q} \geq 1$ (by which we mean that $q_i \geq 1$ for all i), we write $\mathbb{U}_{\mathbf{q}} = \mathbb{U}_{q_1} \times \dots \times \mathbb{U}_{q_n}$; it is also convenient to set $\mathbf{a}/\mathbf{q} = (a_1/q_1, \dots, a_n/q_n)$ and $\mathbf{a}\mathbf{q} = (a_1q_1, \dots, a_nq_n)$ if $\mathbf{a} = (a_1, \dots, a_n)$ is another vector in $\mathbb{Z}^n$. Given $\lambda \in \mathbb{Z}$ and $\mathbf{a}, \mathbf{q} \in \mathbb{Z}^n$, with $\mathbf{q} \geq 1$, we now define

$$g(a, q; b, r) = \frac{1}{\varphi([q, r])} \sum_{x \in \mathbb{U}_{[q, r]}} e\left(\frac{ax^k}{q} + \frac{bx}{r}\right),$$

$$\mathfrak{S}(\lambda; \mathbf{a}, \mathbf{q}) = \sum_{q=1}^{\infty} \sum_{a \in \mathbb{U}_q} e(-\lambda a/q) \prod_{i=1}^n g(a, q; a_i, q_i),$$

where φ is Euler's totient function and $[q, r] = \text{lcm}[q, r]$. We fix a smooth bump function ψ such that

$$\mathbf{1}_{\mathcal{Q}}(\mathbf{x}) \leq \psi(\mathbf{x}) \leq \mathbf{1}_{\mathcal{Q}}(\mathbf{x}/2),$$

where $\mathbf{1}_{\mathcal{Q}}$ is the indicator function of the cube $\mathcal{Q} = [-1, 1]^n$, and we write $\psi_h(\mathbf{x}) = \psi(h\mathbf{x})$ for $h > 0$. We also define $n_1(k) := \min(2^k, k^2 + k) + 3$.

Theorem 1 (Approximation Formula). *Let $k \geq 2$ and $n \geq n_1(k)$. Also, let $\lambda \in \Gamma_{n,k}$ be large, and suppose that $\lambda^{1/k} \leq N \lesssim \lambda^{1/k}$. For any fixed $B > 0$, there exists a $C = C(B) > 0$ such that one has the decomposition*

$$\widehat{\omega}_{\lambda}(\boldsymbol{\xi}) = \frac{\lambda^{n/k-1}}{R(\lambda)} \sum_{1 \leq \mathbf{q} \leq Q} \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \mathfrak{S}(\lambda; \mathbf{a}, \mathbf{q}) \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \widetilde{d\sigma}_{\lambda}(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q}) + \widehat{E}_{\lambda}(\boldsymbol{\xi}), \quad (1.4)$$

where $Q = (\log N)^C$, $\widetilde{d\sigma}_{\lambda}$ is the Fourier transform of the k -spherical surface measure on the surface defined by (1.1) in $\mathbb{R}_+^n$ (cf. (3.12)), and the convolution operators E_{λ} associated with the error terms $\widehat{E}_{\lambda}(\boldsymbol{\xi})$ satisfy the maximal inequality

$$\left\| \sup_{\Lambda \leq \lambda \leq 2\Lambda} |E_{\lambda}| \right\|_{\ell^2(\mathbb{Z}^n) \rightarrow \ell^2(\mathbb{Z}^n)} \lesssim (\log \Lambda)^{-B} \quad (1.5)$$

for all $\Lambda > 0$.

Note that (1.5) implies that

$$\left\| \widehat{E}_{\lambda} \right\|_{L^{\infty}(\mathbb{T}^n)} \lesssim (\log \lambda)^{-B}. \quad (1.6)$$

We remark that the proof of Theorem 1 allows us to establish (1.6) in a slightly wider range of dimension n than the theorem does for the stronger bound (1.5). Namely, if $2m$ is any even integer such that one can apply the circle method to establish the asymptotic formula in Waring's problem for $2m$ k th powers, then (1.6) holds for $n \geq 2m + 1$. In particular, using recent advances by Bourgain [7] and Wooley [33], we obtain (1.6) for $n \geq n_0(k)$, where $n_0(k) = 2^k + 1$ when $k = 2, 3$ or 4 , and

$$n_0(k) = k^2 + 3 - \max_{1 \leq j \leq k-1} \left\lceil \frac{kj - \min(2^j, j^2 + j)}{k - j + 1} \right\rceil$$

when $k \geq 5$. These observations are useful in our next result, which describes the decay of $\widehat{\omega}_{\lambda}$ at irrational frequencies.

Theorem 2. *Let $k \geq 2$ and $n \geq n_0(k)$. If $\boldsymbol{\xi} \notin \mathbb{Q}^n$, then $\widehat{\omega}_{\lambda}(\boldsymbol{\xi}) \rightarrow 0$ as $\lambda \rightarrow \infty$ along $\Gamma_{n,k}$.*

Let $r(\lambda)$ denote the number of prime points on the k -sphere (1.1). It follows readily from Theorem 2 that, when $\xi \notin \mathbb{Q}^n$, one has

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Gamma_{n,k}}} \frac{1}{r(\lambda)} \sum_{\mathfrak{f}(\mathbf{p})=\lambda} e(\mathbf{p} \cdot \xi) = 0. \quad (1.7)$$

This gives a pair of interesting corollaries. The first is obtained by noting that (1.7) is precisely the Weyl criterion for uniform distribution on a torus.

Corollary 1. *Let $k \geq 2$, $n \geq n_0(k)$, and $\alpha \in (\mathbb{R} \setminus \mathbb{Q})^n$. The sets*

$$\{(\alpha_1 p_1, \dots, \alpha_n p_n) : \mathfrak{f}(\mathbf{p}) = \lambda\}$$

become uniformly distributed with respect to the Lebesgue measure on the n -dimensional torus $\mathbb{T}^n$ as $\lambda \rightarrow \infty$ along $\Gamma_{n,k}$.

Our second corollary is an L^2 -convergence result regarding certain ergodic averages; as in Section 4 of [21], where the analogous ‘integral’ result is proven, this follows from the spectral theorem for unitary operators. To state this corollary, let (X, μ) denote a probability space with a commuting family of n invertible measure preserving transformations $T = (T_1, \dots, T_n)$. Such a family is referred to as a *fully ergodic family of transformations* if the hypothesis

$$T_1^s f = T_2^s f = \dots = T_n^s f = f,$$

where $s \in \mathbb{N}$ and $f \in L^2(X, \mu)$, implies that f is constant. Here Tf should be interpreted as $f \circ T$. As observed in [21], the notion of full ergodicity is actually a condition on the joint spectrum of the T_i . More precisely, full ergodicity implies that given $f \in L^2(X, \mu)$, if $T_i f = e(\lambda_i) f$ holds with λ_i rational for all $i \leq n$, then f is constant almost everywhere. For a function $f : X \rightarrow \mathbb{C}$, $\lambda \in \Gamma_{n,k}$ and $x \in X$, define the Waring–Goldbach ergodic averages on X with respect to T by

$$\mathcal{A}_\lambda f(x) := \frac{1}{R(\lambda)} \sum_{\mathfrak{f}(\mathbf{p})=\lambda} (\log \mathbf{p}) f(T^{\mathbf{p}} x), \quad (1.8)$$

where $T^{\mathbf{m}} x := T_1^{m_1} \dots T_n^{m_n} x$ for $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$.

Corollary 2 (L^2 -mean ergodic theorem). *Let $k \geq 2$, $n \geq n_0(k)$, and let (X, μ) be a probability space with a fully ergodic family of transformations $T = (T_1, \dots, T_n)$. Then for all $f \in L^2(X, \mu)$, the ergodic averages of f defined by (1.8) converge in $L^2(X, \mu)$ to the space average of f ; that is, one has that*

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Gamma_{n,k}}} \mathcal{A}_\lambda f = \int_X f d\mu$$

in $L^2(X, \mu)$.

Remark 1. As observed in Section 3 of [21], this result does not hold in general if one omits the full ergodicity condition.

To prove the ergodic theorems, we consider the convolution operator A_λ with Fourier multiplier $\widehat{\omega}_\lambda$: for functions $f : \mathbb{Z}^n \rightarrow \mathbb{C}$, we write

$$A_\lambda f := \omega_\lambda \star f. \quad (1.9)$$

This is our discrete spherical averaging operator along the primes. We will use the Approximation Formula to prove a maximal theorem, stated below. In the remaining theorems, define $n_2(k) := k^2(k-1) + 1$ for $k \geq 7$, $n_2(k) := k2^{k-1} + 1$ for $3 \leq k \leq 6$, and $n_2(2) = 5$; also define $p_{k,n} := 1 + \frac{n_2(k)}{2n-n_2(k)} = \frac{2n}{2n-n_2(k)}$.

Theorem 3. *Let $k \geq 2$ and $n \geq \max\{n_1(k), n_2(k)\}$. The maximal function given by*

$$A_* f := \sup_{\lambda \in \Gamma_{n,k}} |A_\lambda f| \quad (1.10)$$

is bounded on $\ell^p(\mathbb{Z}^n)$ for all $p > p_{k,n}$. Moreover, when $k = 2$ and $n \geq 7$, the above maximal function is bounded for all $p > n/(n-2)$.

Remark 2. In sufficiently large dimensions, the maximal function A_* is unbounded on $\ell^p(\mathbb{Z}^n)$ for $p < \frac{n}{n-k}$. This can readily be seen by testing the maximal function on a delta function at the origin and using the asymptotic for $R(\lambda)$ as $\lambda \rightarrow \infty$ in $\Gamma_{n,k}$. With this in mind, we conjecture that A_* should be bounded on $\ell^p(\mathbb{Z}^n)$ for all $p > \frac{n}{n-k}$ in sufficiently large dimensions; this is the same conjectured range of p as for the integral maximal function. We refer the reader to [15] for more information on the conjectured range of $\ell^p(\mathbb{Z}^n)$ -boundedness for the integral maximal function.

Remark 3. In the quadratic case, the Magyar–Stein–Wainger theorem holds for $n \geq 5$ whereas ours only holds for $n \geq 7$. (Theorem 3 matches the Magyar–Stein–Wainger theorem in the range of p , and both ranges are sharp.) An aspect of this work is that improvements to the analogues of $n_2(k)$ and $p_{k,n}$ in the integer setting automatically translate to corresponding improvements to $n_2(k)$ and $p_{k,n}$ in the present setting. Indeed, after the present work was completed, we used some of the techniques developed in this paper to obtain such improvements in the integer setting. That improvement, which has already appeared in print in [1], lets one replace the above choice of $n_2(k)$, for $k \geq 3$, by

$$n_2(k) = k^2 - \max_{2 \leq j \leq k-1} \left\{ \frac{kj - \min(2^j + 2, j^2 + j)}{k - j + 1} \right\}.$$

The value of $p_{n,k}$ can also be somewhat improved. The interested reader can find the details in [1].

We take this moment to describe the proof of our maximal theorem and to compare it with previous works. Throughout the paper we follow the paradigms of [4] as embellished in the integral version of our averages in [24] and [21]. In particular we assume that the reader is familiar with the transference technology of [24]. As in [24], our maximal theorem will exploit the Approximation Formula which decomposes $\widehat{\omega}_\lambda = \widehat{M}_\lambda + \widehat{E}_\lambda$ into the sum of a main term and error term. We will use separate techniques to get good bounds on the suprema over λ of both the main term and error term. As in all previous works, our decomposition requires a major arc/minor arc decomposition of the degree k frequency variable. Unlike previous works we require an additional major arc/minor arc decomposition of the linear frequency variables. For the main term we will use estimates for relevant exponential sums and oscillatory integrals in addition to the transference results of [24] to bound the main term. However, as already mentioned, the methods in previous works such as [24, 15, 16] are insufficient to handle the error term from our circle method approximation in the Approximation Formula. This is due to the logarithmic decay in (1.5) as opposed to power savings that appeared in previous

works. To overcome this obstacle, we introduce a *hybrid sup and mean value bound* to control the relevant exponential sums on our set of minor arcs and consequently bound the error term in ℓ^2 ; this is one of the novel aspects of our paper. From this, the known bounds for the integer case in [24], and the bounds we are able to prove for the main term on ℓ^p , we are able to bound the analogue of the Magyar–Stein–Wainger discrete spherical maximal function along the primes.

Following Magyar [22] and Bourgain [4], we will use our maximal theorem to prove the following pointwise ergodic theorem along the primes.

Theorem 4. *Let $k \geq 2$, $n \geq \max\{n_1(k), n_2(k)\}$, and let (X, μ) be a probability space with a fully ergodic family of transformations $T = (T_1, \dots, T_n)$. Then for all $f \in L^2(X, \mu)$, the ergodic averages of f defined by (1.8) converge almost everywhere to the space average of f ; that is,*

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Gamma_{n,k}}} \mathcal{A}_\lambda f = \int_X f d\mu \quad (1.11)$$

μ -almost everywhere.

Again, a standard argument (see for instance [32]) implies the same result without the logarithmic weights.

Corollary 3. *Suppose that (X, μ) is a probability space with n commuting measure-preserving operators $T_1, \dots, T_n$ satisfying the conditions of Theorem 4. Then, for all $f \in L^2(X, \mu)$, one has*

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Gamma_{n,k}}} \frac{1}{r(\lambda)} \sum_{\mathfrak{f}(\mathbf{p})=\lambda} f(T^{\mathbf{p}}x) = \int_X f d\mu \quad (1.12)$$

μ -almost everywhere.

Combining our pointwise ergodic theorem on ℓ^2 with our maximal function bounds, we immediately obtain, via standard approximation arguments, the following corollary.

Corollary 4. *Suppose that (X, μ) is a probability space with n commuting measure-preserving transformations $T_1, \dots, T_n$ as in Theorem 4. Then, for $p > p_{k,n}$ and for all $f \in L^p(X, \mu)$, one has*

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Gamma_{n,k}}} \mathcal{A}_\lambda f = \int_X f d\mu \quad (1.13)$$

μ -almost everywhere.

The paper is organized as follows. In Section 2, we collect some needed number theoretic facts. Then in Section 3, we use the circle method to decompose $\widehat{\omega}_\lambda$ into a main term and an error term; we also prove ℓ^2 bounds on the error in this section. One key additional technical difficulty here compared with the work in [24] is that the precise shape of our error terms is more complicated than in the integral case; in particular, we need to perform a major and minor arc analysis of the linear phases (in addition to the higher degree phases). In Section 4, we develop a careful analysis and interpolation argument to get ℓ^p bounds on the main term, since we cannot apply the techniques in [24] directly. In Section 5, we compare the averages along the primes to the integral ones to control the error terms and prove Theorem 3. Finally, we prove the ergodic theorems in Section 6.

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2. BOUNDS FOR EXPONENTIAL SUMS AND INTEGRALS

Here we recall and prove some results from analytic number theory.

Lemma 1. *Let a, b, q be integers with $\gcd(a, b, q) = 1$. Then, for any fixed $\varepsilon > 0$, one has*

$$\sum_{x \in \mathbb{U}_q} e\left(\frac{ax^k + bx}{q}\right) \lesssim q^{1/2+\varepsilon}.$$

Proof. This is a special case of Theorem 1 of Shparlinski [28]. $\square$

Lemma 2. *Let $f(x) = \alpha x^k + \dots + \alpha_1 x \in \mathbb{R}[x]$, with $k \geq 2$, and suppose that there exist integers a, q such that $(a, q) = 1$ and $|q\alpha - a| \leq q^{-1}$. Then*

$$\sum_{p \leq N} (\log p) e(f(p)) \lesssim NL^c (q^{-1} + N^{-1/2} + qN^{-k})^{2^{1-2k}},$$

where $L = \log N$ and $c = c_k$ is a constant.

Proof. This is a variant of Theorem 1 in Harman [13], where the exponent of 2^{1-2k} is replaced by 4^{1-k} at the expense of replacing the factor L^c above by N^ε . The present version is well-known to the experts, but since we were unable to locate it in the literature, we will provide a brief sketch of the argument. The proof requires small adjustments to the proofs of Lemmas 2–4 in [13]. Those proofs use the inequality

$$\sum_{x \leq X} \tau_r(x) \min(Y, \|\theta x\|^{-1}) \lesssim X^\varepsilon \sum_{x \leq X} \min(Y, \|\theta x\|^{-1}), \quad (2.1)$$

where $\tau_r(x)$ is the r -fold divisor function. However, in most places the above inequality is used for convenience rather than by necessity. The places where this inequality is really needed occur towards the ends of the proofs of Lemmas 3 and 4 in [13], when one wants to apply a standard estimate (e.g., Lemma 2.2 in Vaughan [31]) to the sum on the right side of (2.1). In those places, we can replace (2.1) with

$$\sum_{x \leq X} \tau_r(x) \min(Y, \|\theta x\|^{-1}) \lesssim (XY)^{1/2} (\log X)^c \left\{ \sum_{x \leq X} \min(Y, \|\theta x\|^{-1}) \right\}^{1/2}.$$

We can then follow the rest of Harman's proof. $\square$

Lemma 3. *Let a, b, q, r , be integers such that $(a, q) = (b, r) = 1$ and $|\alpha - a/q| \leq 2N^{-1}$. Then*

$$\sum_{\substack{p \leq N \\ p \equiv b \pmod{r}}} (\log p) e(\alpha p) \lesssim NL^3 (q^{-1} + N^{-2/5} + qN^{-1})^{1/2}.$$

Proof. This is the main result of Balog and Perelli [3], with some of the terms slightly simplified for use in the present context. $\square$

When $1 \leq Q \leq X$, we define the *set of major arcs* $\mathfrak{M}(X, Q)$ by

$$\mathfrak{M}(X, Q) = \bigcup_{q \leq Q} \bigcup_{a \in \mathbb{U}_q} \{\theta \in \mathbb{T} : |q\theta - a| \leq QX^{-1}\}.$$

The complement of a set of major arcs, $\mathfrak{m}(X, Q) = \mathbb{T} \setminus \mathfrak{M}(X, Q)$, is the respective *set of minor arcs*. When working with a particular choice of major and minor arcs, we may write $\mathfrak{M}_{a/q}$ for the major arc centered at the rational a/q . Note that when $2Q < X$, the set $\mathfrak{M}(X, Q)$ is the disjoint union of closed intervals of total measure $O(QX^{-1})$.

Our analysis of $\widehat{\omega}_\lambda(\xi)$ will depend on the exponential sum

$$S_N(\theta, \xi) = \sum_{p \leq N} (\log p) e(\theta p^k + \xi p), \quad (2.2)$$

where the summation is over the prime numbers $p \leq N$. In particular, we need to approximate $S_N(\theta, \xi)$ when both θ and ξ are near rationals with small denominators. The approximations involve the exponential sum $g(a, q; b, r)$ defined above and the oscillatory integral

$$I_N(\delta, \eta) = \int_0^N e(\delta x^k + \eta x) dx. \quad (2.3)$$

We note that, by Lemma 1,

$$g(a, q; b, r) \lesssim [q, r]^{-1/2+\varepsilon}, \quad (2.4)$$

and that the k th-derivative estimate for oscillatory integrals (Proposition 2 on p. 332 in Stein [29]) yields

$$I_N(\delta, \eta) \lesssim \frac{N}{(1 + N^k |\delta|)^{1/k}}. \quad (2.5)$$

Furthermore, since

$$I_N(\delta, \eta) = k^{-1} N \int_0^1 u^{1/k-1} e(\delta_0 u + \eta_0 u^{1/k}) du,$$

where $\delta_0 = \delta N^k$ and $\eta_0 = \eta N$, we can also apply the second-derivative estimate (the case $k = 2$ of the corollary on p. 334 of [29]) to deduce the bound

$$I_N(\delta, \eta) \lesssim \frac{N}{(1 + N|\eta|)^{1/2}}. \quad (2.6)$$

Our next lemma uses the Siegel–Walfisz theorem to approximate $S_N(\theta, \xi)$.

Lemma 4. *Let $Q, R \leq (\log N)^C$ for some fixed $C > 0$, let $\theta \in \mathfrak{M}_{a/q}$ for some major arc of the set $\mathfrak{M} = \mathfrak{M}(N^k, Q)$, and let $\xi \in \mathfrak{M}_{b/r}$ for some major arc of the set $\mathfrak{N} = \mathfrak{M}(N, R)$. Then*

$$S_N(\theta, \xi) = g(a, q; b, r) I_N(\theta - a/q, \xi - b/r) + O(N(QR)^{-10}).$$

Proof. We write $\delta = \theta - a/q$, $\eta = \xi - b/r$, and $s = [q, r]$. When we partition the exponential sum $S_N(\theta, \xi_j)$ into sums over primes in fixed arithmetic progressions, we find that

$$\begin{aligned} S_N(\theta, \xi) &= \sum_{h \in \mathbb{U}_s} \sum_{\substack{p \leq N \\ p \equiv h \pmod{s}}} (\log p) e\left(\left(\frac{a}{q} + \delta\right)p^k + \left(\frac{b}{r} + \eta\right)p\right) + O(s) \\ &= \sum_{h \in \mathbb{U}_s} e\left(\frac{ah^k}{q} + \frac{bh}{r}\right) \sum_{\substack{p \leq N \\ p \equiv h \pmod{s}}} (\log p) e(\delta p^k + \eta p) + O(QR). \end{aligned} \quad (2.7)$$

Since $s \leq QR \leq (\log N)^{2C}$ and $h \in \mathbb{U}_s$, the Siegel–Walfisz theorem yields

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{s}}} \log p = \frac{x}{\varphi(s)} + O(N(QR)^{-12})$$

for all $x \leq N$. Using this asymptotic formula and partial summation, we obtain

$$\sum_{\substack{p \leq N \\ p \equiv h \pmod{s}}} (\log p) e(\delta p^k + \eta p) = \varphi(s)^{-1} I_N(\delta, \eta) + O(N(QR)^{-11}). \quad (2.8)$$

The lemma follows from (2.7) and (2.8). $\square$

Lemma 5. *Let $k \geq 2$ and $2s \geq \min(2^k, k^2 + k) + 2$. Then*

$$\int_{\mathbb{T}} \sup_{\xi} |S_N(\theta, \xi)|^{2s} d\theta \lesssim N^{2s-k} L^{2s}, \quad (2.9)$$

where $L = \log N$.

Proof. Set $H_j = sN^j$ and define

$$a_h(\theta) = \sum_{\substack{p_1, \dots, p_s \leq N \\ p_1 + \dots + p_s = h}} (\log \mathbf{p}) e(\theta \mathbf{f}_{s,k}(\mathbf{p})),$$

so that

$$S_N(\theta, \xi)^s = \sum_{h \leq H_1} a_h(\theta) e(\xi h).$$

By applying Cauchy's inequality, we deduce that

$$\sup_{\xi} |S_N(\theta, \xi)|^{2s} \leq H_1 \sum_{h \leq H_1} |a_h(\theta)|^2.$$

Hence,

$$\int_{\mathbb{T}} \sup_{\xi} |S_N(\theta, \xi)|^{2s} d\theta \leq H_1 \sum_{h \leq H_1} \int_{\mathbb{T}} a_h(\theta) \overline{a_h(\theta)} d\theta. \quad (2.10)$$

By orthogonality,

$$\int_{\mathbb{T}} a_h(\theta) \overline{a_h(\theta)} d\theta = \sum_{\mathbf{p}, \mathbf{p}' : (2.11)} (\log \mathbf{p})(\log \mathbf{p}'),$$

where $\mathbf{p}, \mathbf{p}' \leq N$ and satisfy the conditions

$$\mathbf{f}_{s,k}(\mathbf{p}) = \mathbf{f}_{s,k}(\mathbf{p}'), \quad \mathbf{f}_{s,1}(\mathbf{p}) = \mathbf{f}_{s,1}(\mathbf{p}') = h. \quad (2.11)$$

Thus,

$$\sum_{h \leq H_1} \int_{\mathbb{T}} a_h(\theta) \overline{a_h(\theta)} d\theta \lesssim L^{2s} I_{s,k}(N), \quad (2.12)$$

where $I_{s,k}(N)$ denotes the number of integer solutions of the system

$$\mathbf{f}_{s,k}(\mathbf{x}) = \mathbf{f}_{s,k}(\mathbf{y}), \quad \mathbf{f}_{s,1}(\mathbf{x}) = \mathbf{f}_{s,1}(\mathbf{y}), \quad (2.13)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq N$. The lemma follows from (2.10), (2.12) and the inequality

$$I_{s,k}(N) \lesssim N^{2s-k-1}, \quad (2.14)$$

which we establish next.

Under the hypothesis $2s \geq 2^k + 2$, the bound (2.14) is a direct consequence of the main result of Brüdern and Robert [9]. On the other hand, by grouping the solutions of (2.13) according to the values of the expressions $\mathbf{f}_{s,j}(\mathbf{x}) - \mathbf{f}_{s,j}(\mathbf{y})$, $1 < j < k$, we find that

$$I_{s,k}(N) \leq \sum_{|h_2| < H_2} \cdots \sum_{|h_{k-1}| < H_{k-1}} J_{s,k}(N; 0, h_2, \dots, h_{k-1}, 0), \quad (2.15)$$

where $J_{s,k}(N; \mathbf{h})$ is the generalized Vinogradov integral

$$J_{s,k}(N; \mathbf{h}) = \int_{\mathbb{T}^k} \left| \sum_{x \leq N} e(\alpha_k x^k + \cdots + \alpha_1 x) \right|^{2s} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) d\boldsymbol{\alpha}.$$

We can now refer to the recent work by Bourgain, Demeter and Guth [8] on the classic Vinogradov integral $J_{s,k}(N) = J_{s,k}(N; \mathbf{0})$ to get

$$J_{s,k}(N; \mathbf{h}) \leq J_{s,k}(N) \lesssim N^{2s-k(k+1)/2},$$

provided that $2s > k(k+1)$ (see §5 in [8]). Inserting this bound into the right side of (2.15) yields (2.14) for $2s > k^2 + k$. $\square$

In §4, we will need some more refined estimates for $g(a, q; b, r)$ and its averages; we establish those in the next lemma. Here, $\mu(n)$ denotes the Möbius function from number theory (see §16.3 in Hardy and Wright [12]).

Lemma 6. *Let a, b, q, r , be integers with $(a, q) = (b, r) = 1$, and write $q_0 = q/(q, r)$ and $r_0 = r/(q, r)$. Then:*

- (i) *if $(r_0, q) > 1$, one has $g(a, q; b, r) = 0$;*
- (ii) *if $(r_0, q) = 1$, one has*

$$g(a, q; b, r) = \frac{\mu(r_0)}{\varphi(r_0)} g(ar_0^k, q; bq_0, q);$$

- (iii) *one has*

$$\sum_{u \in \mathbb{Z}_r} \left| \sum_{b \in \mathbb{U}_r} g(a, q; b, r) e(-ub/r) \right| \leq \frac{\tau(r)r}{\varphi(r_0)}. \quad (2.16)$$

Proof. (i) Suppose that $(r_0, q) > 1$. Then there is a prime number p and positive integers α, β , with $\alpha < \beta$, such that

$$p^\alpha \mid q, \quad p^{\alpha+1} \nmid q, \quad p^\beta \mid r, \quad p^{\beta+1} \nmid r.$$

Let $q = p^\alpha q_1$ and $r = p^\beta r_1$. By a change of the summation variable $x \in \mathbb{U}_{[q,r]}$ in $g(a, q; b, r)$ to $x = p^\beta y + [q_1, r_1]z$, where $y \in \mathbb{U}_{[q_1, r_1]}$ and $z \in \mathbb{U}_{p^\beta}$, we can factor the exponential sum $g(a, q; b, r)$ as

$$g(a, q; b, r) = g(ap^{k\beta-\alpha}, q_1; b, r_1)g(a_1, p^\alpha; b_1, p^\beta), \quad (2.17)$$

where $a_1 = a[q_1, r_1]^k q_1^{-1}$ and $b_1 = b[q_1, r_1] r_1^{-1}$. We note that $(a_1, p) = (b_1, p) = 1$. Next, we write the variable $z \in \mathbb{U}_{p^\beta}$ in $g(a_1, p^\alpha; b_1, p^\beta)$ as $z = u + p^\alpha v$, where $u \in \mathbb{U}_{p^\alpha}$ and $v \in \mathbb{Z}_{p^\gamma}$, $\gamma = \beta - \alpha$. This gives

$$\varphi(p^\beta)g(a_1, p^\alpha; b_1, p^\beta) = \sum_{u \in \mathbb{U}_{p^\alpha}} e\left(\frac{a_1 u^k}{p^\alpha} + \frac{b_1 u}{p^\beta}\right) \sum_{v \in \mathbb{Z}_{p^\gamma}} e\left(\frac{b_1 v}{p^\gamma}\right).$$

Since $(b_1, p) = 1$, the last sum over v vanishes. Together with the factorization (2.17), this proves (i).

(ii) When $(q, r_0) = 1$, we change the summation variable $x \in \mathbb{U}_{[q,r]}$ in $g(a, q; b, r)$ to $x = r_0 y + qz$, where $y \in \mathbb{U}_q$ and $z \in \mathbb{U}_{r_0}$. Similarly to (2.17), we have

$$g(a, q; b, r) = g(ar_0^k, q; b, (q, r))\varphi(r_0)^{-1} \sum_{z \in \mathbb{U}_{r_0}} e\left(\frac{bq_0 z}{r_0}\right).$$

We now note that the last exponential sum is a Ramanujan sum modulo r_0 and $(bq_0, r_0) = 1$. Hence, the claim follows from a classical expression for the Ramanujan sum (see Theorem 272 in Hardy and Wright [12]).

(iii) Let $h(a, q; u, r)$ denote the sum over b on the left side of (2.16). By part (i), we may assume that $(q, r_0) = 1$. We can then use part (ii) to rewrite $h(a, q; u, r)$ as

$$h(a, q; u, r) = \frac{\mu(r_0)}{\varphi(r_0)\varphi(q)} \sum_{x \in \mathbb{U}_q} e\left(\frac{ar_0^k x^k}{q}\right) \sum_{b \in \mathbb{U}_r} e\left(\frac{(r_0 x - u)b}{r}\right).$$

Since the inner sum is a Ramanujan sum, we deduce that

$$|h(a, q; u, r)| \leq \frac{1}{\varphi(r_0)\varphi(q)} \sum_{d|r} d \sum_{\substack{x \in \mathbb{U}_q \\ d|(r_0 x - u)}} 1.$$

We remark that a divisor d of r factors uniquely as $d = d_1 d_2$, where $d_1 \mid (q, r)$ and $d_2 \mid r_0$. When $d_2 \nmid u$, the sum over x vanishes. On the other hand, when $d_2 \mid u$, the condition $d \mid (r_0 x - u)$ restricts h to a single residue class modulo d_1 ; hence, the inner sum is then bounded by $\varphi(q)/d_1$. We conclude that

$$|h(a, q; u, r)| \leq \frac{1}{\varphi(r_0)\varphi(q)} \sum_{d_1 \mid (q, r)} \sum_{d_2 \mid (r_0, u)} d_1 d_2 \left(\frac{\varphi(q)}{d_1}\right) = \frac{\tau((q, r))}{\varphi(r_0)} \sum_{d \mid (r_0, u)} d.$$

Summing the last bound over u , we deduce

$$\sum_{u \in \mathbb{Z}_r} |h(a, q; u, r)| \leq \frac{\tau((q, r))}{\varphi(r_0)} \sum_{u \in \mathbb{Z}_r} \sum_{d \mid (r_0, u)} d = \frac{\tau((q, r))}{\varphi(r_0)} \sum_{d \mid r_0} d \sum_{\substack{u \in \mathbb{Z}_r \\ d \mid u}} 1 = \frac{\tau(r)r}{\varphi(r_0)},$$

where we have used that $\tau((q, r))\tau(r_0) = \tau(r)$. □

3. PROOF OF THE APPROXIMATION FORMULA

In this section, we use the circle method to prove Theorem 1. However, before we proceed with that, we establish a lemma that allows us to leverage our estimates for exponential sums to bound various dyadic maximal functions, including the maximal function of the error term.

Lemma 7. *Let $\mathcal{L}$ be a set of integers. For $\lambda \in \mathcal{L}$, let T_λ be a convolution operator on $\ell^2(\mathbb{Z}^d)$ with Fourier multiplier $\widehat{m}_\lambda(\boldsymbol{\xi})$ given by*

$$\widehat{m}_\lambda(\boldsymbol{\xi}) = \int_X K(\theta; \boldsymbol{\xi}) e(\Phi(\lambda, \theta)) d\mu(\theta),$$

where (X, μ) is a measure space, $\Phi : \mathbb{Z} \times X \rightarrow \mathbb{R}$, and $K(\cdot; \boldsymbol{\xi}) \in L^1(X, \mu)$ is a kernel independent of λ . Let

$$(T_* f)(\mathbf{x}) = \sup_{\lambda \in \mathcal{L}} |(T_\lambda f)(\mathbf{x})|.$$

Then

$$\|T_*\|_{\ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)} \leq \int_X \sup_{\boldsymbol{\xi} \in \mathbb{T}^d} |K(\theta; \boldsymbol{\xi})| d\mu(\theta).$$

Proof. Suppose that $f \in \ell^2(\mathbb{Z}^d)$. We first exchange the order of integration to get

$$\begin{aligned} |(T_\lambda f)(\mathbf{x})| &= \left| \int_{\mathbb{T}^d} \int_X K(\theta; \boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}) e(\Phi(\lambda, \theta) - \mathbf{x} \cdot \boldsymbol{\xi}) d\mu(\theta) d\boldsymbol{\xi} \right| \\ &\leq \int_X \left| \int_{\mathbb{T}^d} K(\theta; \boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}) e(-\mathbf{x} \cdot \boldsymbol{\xi}) d\boldsymbol{\xi} \right| d\theta =: \int_X |g(\theta; \mathbf{x})| d\mu(\theta). \end{aligned}$$

Note that since the last integral is independent of λ , the same bound holds for $(T_* f)(\mathbf{x})$. Consequently,

$$\begin{aligned} \|T_* f\|_{\ell^2(\mathbb{Z}^d)} &\leq \left\| \int_X |g(\theta; \cdot)| d\mu(\theta) \right\|_{\ell^2(\mathbb{Z}^d)} \\ &\leq \int_X \left\{ \sum_{\mathbf{x} \in \mathbb{Z}^d} |g(\theta; \mathbf{x})|^2 \right\}^{1/2} d\mu(\theta) \\ &\leq \int_X \left\{ \int_{\mathbb{T}^d} |K(\theta; \boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right\}^{1/2} d\mu(\theta) \\ &\leq \int_X \sup_{\boldsymbol{\xi} \in \mathbb{T}^d} |K(\theta; \boldsymbol{\xi})| \|\widehat{f}\|_{L^2(\mathbb{T}^d)} d\mu(\theta), \end{aligned}$$

on using Minkowski's and Bessel's inequalities. The lemma follows by applying Plancherel's theorem to f and $\widehat{f}$. $\square$

Let $\lambda \in \Gamma_{n,k} \cap [\Lambda, 2\Lambda]$. Suppose that $N^k \geq \lambda$ and write $L = \log N$. By orthogonality,

$$\begin{aligned} R(\lambda) \widehat{\omega}_\lambda(\boldsymbol{\xi}) &= \sum_{1 \leq \mathbf{p} \leq N} (\log \mathbf{p}) e(\mathbf{p} \cdot \boldsymbol{\xi}) \int_{\mathbb{T}} e([f(\mathbf{p}) - \lambda]\theta) d\theta \\ &= \int_{\mathbb{T}} \left\{ \prod_{j=1}^n S_N(\theta, \xi_j) \right\} e(-\lambda\theta) d\theta =: \int_{\mathbb{T}} F(\theta; \boldsymbol{\xi}) e(-\lambda\theta) d\theta, \end{aligned} \tag{3.1}$$

where $S_N(\theta, \xi)$ is the exponential sum defined in (2.2). To analyze the last integral, we partition the torus into major and minor arcs. Let $Q = L^C$, where $C > 0$ is a sufficiently large constant to be described later. We set $\mathfrak{M} = \mathfrak{M}(N^k, Q)$ and $\mathfrak{m} = \mathfrak{m}(N^k, Q)$.

3.1. The minor arc contribution. The minor arc contribution to the integral (3.1) will be part of the error term in the Approximation Formula. Let

$$\widehat{E}_1(\xi; \lambda) = R(\lambda)^{-1} \int_{\mathfrak{m}} F(\theta; \xi) e(-\lambda\theta) d\theta.$$

Since $R(\lambda) \gtrsim N^{n-k}$ for $\lambda \in \Gamma_{n,k}$, the estimate (1.5) for $\widehat{E}_1$ will follow from Lemma 7 if we show that for any $B > 1$, we have

$$\int_{\mathfrak{m}} \sup_{\xi \in \mathbb{T}^n} |F(\theta; \xi)| d\theta \lesssim_B N^{n-k} L^{-B}. \quad (3.2)$$

When $\theta \in \mathfrak{m}$, it has a rational approximation a/q such that $Q \leq q \leq N^k Q^{-1}$, $(a, q) = 1$ and $|q\theta - a| < q^{-1}$. By Lemma 2 with $f(x) = \theta x^k + \xi x$, we have

$$\sup_{(\theta, \xi) \in \mathfrak{m} \times \mathbb{T}} |S_N(\theta, \xi)| \lesssim N Q^{-2^{1-2k}} L^{c_k}, \quad (3.3)$$

where c_k is the constant in the statement of Lemma 2. Using this bound and Hölder's inequality, we get

$$\int_{\mathfrak{m}} \sup_{\xi \in \mathbb{T}^n} |F(\theta; \xi)| d\theta \lesssim N Q^{-2^{1-2k}} L^{c_k} \int_{\mathbb{T}} \sup_{\xi \in \mathbb{T}} |S_N(\theta; \xi)|^{n-1} d\theta.$$

Hence, when $n \geq n_1(k)$, we obtain from Lemma 5 that

$$\int_{\mathfrak{m}} \sup_{\xi \in \mathbb{T}^n} |F(\theta; \xi)| d\theta \lesssim N^{n-k} Q^{-2^{1-2k}} L^{n+c_k}.$$

We can therefore choose $C_1 = C_1(B, k, n) > 0$ such that when $C \geq C_1$ in the definition of Q , the last inequality yields (3.2).

3.2. The major arc contribution, I. Let $R = Q^3$ and define

$$\mathfrak{R} = \mathfrak{M}(N, R), \quad \mathfrak{N} = \mathfrak{M}(N, Q), \quad \mathfrak{r} = \mathfrak{m}(N, R), \quad \mathfrak{n} = \mathfrak{m}(N, Q).$$

We will show that when $\xi \notin \mathfrak{N}^n$, the contribution of the major arcs $\mathfrak{M}$ to the integral (3.1) can be estimated similarly to the minor arc contribution.

Suppose that $\theta \in \mathfrak{M}_{a/q}$ and write $\delta = \theta - a/q$. Then, by partial summation,

$$\begin{aligned} |S_N(\theta, \xi)| &\leq \sum_{h \in \mathbb{U}_q} \left| \sum_{\substack{p \leq N \\ p \equiv h \pmod{q}}} e(\delta p^k + \xi p) \right| + q \\ &\lesssim q(1 + N^k |\delta|) \sup_{M, h} \left| \sum_{\substack{p \leq M \\ p \equiv h \pmod{q}}} e(\xi p) \right|, \end{aligned} \quad (3.4)$$

where the supremum is over $2 \leq M \leq N$ and $h \in \mathbb{U}_q$. When $\xi \in \mathfrak{r}$, it has a rational approximation b/r such that

$$R \leq r \leq NR^{-1}, \quad (b, r) = 1, \quad |r\xi - b| \leq RN^{-1}. \quad (3.5)$$

Hence, we may use Lemma 3 to show that

$$\sup_{\theta \in \mathfrak{M}} |S_N(\theta, \xi)| \lesssim R^{-1/2} N Q L^3 \lesssim N Q^{-1/3}. \quad (3.6)$$

On the other hand, if $\xi \in \mathfrak{R}_{b/r}$ for some major arc in $\mathfrak{R}$, Lemma 4 yields

$$S_N(\theta, \xi) = g(a, q; b, r) I_N(\delta, \eta) + O(NQ^{-10}),$$

where $\eta = \xi - b/r$. When $\xi \notin \mathfrak{N}$, we have either $r \geq Q$ or $r|\eta| \geq QN^{-1}$. When $r \geq Q$, (2.4) yields

$$g(a, q; b, r) \lesssim Q^{-1/2+\varepsilon},$$

and when $r \leq Q$ and $r|\eta| \geq QN^{-1}$, (2.4) and (2.6) yield

$$g(a, q; b, r) I_N(\delta, \eta) \lesssim r^{-1/2+\varepsilon} (N/|\eta|)^{1/2} \lesssim NQ^{-1/2+\varepsilon}.$$

We conclude that inequality (3.6) holds whenever $\xi \notin \mathfrak{N}$.

Thus, unless $\xi \in \mathfrak{N}^n$, we have the bound (3.6) for some exponential sum $S_N(\theta, \xi_j)$. Using that bound in place of (3.3) in the argument of §3.1, we conclude that when $C \geq C_2(B, n, k)$ in the definition of Q , the estimate (1.5) holds for

$$\widehat{E}_2(\xi; \lambda) = R(\lambda)^{-1} \Psi(\xi) \int_{\mathfrak{M}} F(\theta; \xi) e(-\lambda\theta) d\theta,$$

where $\Psi(\xi)$ is any bounded function that is supported outside $\mathfrak{N}^n$. In particular, the above inequality holds for

$$\Psi(\xi) = 1 - \sum_{1 \leq \mathbf{q} \leq Q} \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \psi_{N/Q}(\mathbf{q}\xi - \mathbf{a}),$$

where ψ is the bump function appearing in the statement of the Approximation Formula.

3.3. The major arc contribution, II. We now proceed to approximate the contribution of the major arcs to (3.1) when ξ lies close to $\mathfrak{N}^n$. For vectors $\mathbf{a}, \mathbf{q}$ with $1 \leq \mathbf{q} \leq Q$ and $\mathbf{a} \in \mathbb{U}_{\mathbf{q}}$, let $\mathfrak{N}_{\mathbf{a}/\mathbf{q}}$ denote the support of $\psi_{N/Q}(\mathbf{q}\xi - \mathbf{a})$, and let $\mathfrak{N}$ denote the union of all the different sets $\mathfrak{N}_{\mathbf{a}/\mathbf{q}}$. Suppose that $\xi = (\xi_1, \dots, \xi_n) \in \mathfrak{N}_{\mathbf{a}/\mathbf{q}}$. When $\theta \in \mathfrak{M}_{\mathbf{a}/\mathbf{q}}$, we write $\delta = \theta - a/q$ and $\eta_j = \xi_j - a_j/q_j$. By Lemma 4,

$$S_N(\theta, \xi_j) = g(a, q; a_j, q_j) I_N(\delta, \eta_j) + O(NQ^{-20}).$$

Since the major arcs are disjoint, we may define the function

$$F^*(\theta; \xi) = \prod_{j=1}^n g(a, q; a_j, q_j) I_N(\delta, \eta_j)$$

on all of $\mathfrak{M} \times \mathfrak{N}$. This function satisfies

$$\sup_{(\theta, \xi) \in \mathfrak{M} \times \mathfrak{N}} |F(\theta; \xi) - F^*(\theta; \xi)| \lesssim N^n Q^{-20}.$$

Since $|\mathfrak{M}| \lesssim QN^{-k}$, we can use the above inequality and Lemma 7 to show that (1.5) holds for the error term

$$\widehat{E}_3(\xi; \lambda) = R(\lambda)^{-1} \sum_{1 \leq \mathbf{q} \leq Q} \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \psi_{N/Q}(\mathbf{q}\xi - \mathbf{a}) \int_{\mathfrak{M}} [F(\theta; \xi) - F^*(\theta; \xi)] e(-\lambda\theta) d\theta.$$

By (3.1) and the above analysis, we have

$$\widehat{\omega}_\lambda(\boldsymbol{\xi}) = R(\lambda)^{-1} \sum_{1 \leq \mathbf{q} \leq Q} \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \int_{\mathfrak{M}} F^*(\theta; \boldsymbol{\xi}) e(-\lambda\theta) d\theta + \widehat{E}_4(\boldsymbol{\xi}; \lambda), \quad (3.7)$$

with an error term $\widehat{E}_4(\boldsymbol{\xi}; \lambda)$ that satisfies (1.5). Next, let

$$\mathfrak{M}' = \bigcup_{q \leq Q} \bigcup_{a \in \mathbb{U}_q} \{\theta \in \mathbb{T} : |\theta - a/q| \leq QN^{-k}\}.$$

We want to extend the integral on the right side of (3.7) to the set $\mathfrak{M}'$. The hypothesis on n implies readily that $n \geq 3k$. We now apply once again Lemma 7 together with the inequality

$$\begin{aligned} \int_{\mathfrak{M}' \setminus \mathfrak{M}} \sup_{\boldsymbol{\xi} \in \mathfrak{M}} |F^*(\theta; \boldsymbol{\xi})| d\theta &\lesssim \sum_{q \leq Q} \sum_{1 \leq a \leq q} q^{-n/2+\varepsilon} \int_{Q/(qN^k)}^{\infty} \frac{N^n d\delta}{(1 + N^k \delta)^{n/k}} \\ &\lesssim Q^{2-n/k+\varepsilon} N^{n-k} \lesssim Q^{-1+\varepsilon} N^{n-k}, \end{aligned}$$

where we have used (2.4) and (2.5). Combining these estimates and (3.7), we obtain

$$\widehat{\omega}_\lambda(\boldsymbol{\xi}) = R(\lambda)^{-1} \sum_{1 \leq \mathbf{q} \leq Q} \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \int_{\mathfrak{M}'} F^*(\theta; \boldsymbol{\xi}) e(-\lambda\theta) d\theta + \widehat{E}_5(\boldsymbol{\xi}; \lambda),$$

with an error term $\widehat{E}_5(\boldsymbol{\xi}; \lambda)$ that satisfies (1.5).

We now identify

$$\int_{\mathfrak{M}'} F^*(\theta; \boldsymbol{\xi}) e(-\lambda\theta) d\theta \quad (3.8)$$

as an integral over a subset of $\mathbb{Q} \times \mathbb{R}$ with respect to the product measure $\mu(r, \delta) = \nu(r) \times d\delta$, where ν is the counting measure on $\mathbb{Q}$ and $d\delta$ is the Lebesgue measure on $\mathbb{R}$. Then one final appeal to Lemma 7 allows us to replace (3.8) by

$$\sum_{q=1}^{\infty} \sum_{a \in \mathbb{U}_q} \int_{\mathbb{R}} \left\{ \prod_{j=1}^n g(a, q; a_j, q_j) I_N(\delta, \eta_j) \right\} e(-\lambda(a/q + \delta)) d\delta. \quad (3.9)$$

This step requires an estimate for the quantity

$$\left\{ \sum_{\substack{a, q \\ q > Q}} \int_{\mathbb{R}} + \sum_{a, q} \int_{|\delta| \geq QN^{-k}} \right\} \sup_{\boldsymbol{\xi} \in \mathfrak{M}} \prod_{j=1}^n |g(a, q; a_j, q_j) I_N(\delta, \eta_j)| d\delta. \quad (3.10)$$

Using (2.4) and (2.5), we can bound the quantity (3.10) by

$$\sum_{q > Q} q^{-2} \int_{\mathbb{R}} \frac{N^n d\delta}{(1 + N^k |\delta|)^{n/k}} + \sum_{q=1}^{\infty} q^{-2} \int_{QN^{-k}}^{\infty} \frac{N^n d\delta}{(1 + N^k \delta)^{n/k}} \lesssim Q^{-1} N^{n-k}.$$

We remark that the integral (3.9) equals $\mathfrak{S}(\lambda; \mathbf{a}, \mathbf{q}) \mathfrak{I}_\lambda(\boldsymbol{\eta})$, where

$$\mathfrak{I}_\lambda(\boldsymbol{\eta}) = \int_{\mathbb{R}} \left\{ \prod_{j=1}^n I_N(\delta, \eta_j) \right\} e(-\lambda\delta) d\delta.$$

Hence,

$$\widehat{\omega}_\lambda(\boldsymbol{\xi}) = R(\lambda)^{-1} \sum_{1 \leq \mathbf{q} \leq Q} \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \mathfrak{S}(\lambda; \mathbf{a}, \mathbf{q}) \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \mathfrak{I}_\lambda(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q}) + \widehat{E}_\lambda(\boldsymbol{\xi}), \quad (3.11)$$

an error term $\widehat{E}_\lambda(\boldsymbol{\xi})$ that satisfies (1.5). To complete the proof of Theorem 1, we note that by the discussion on p. 498 in [29] (see also §3.1 in [15]), one has

$$\begin{aligned}\mathfrak{I}_\lambda(\boldsymbol{\eta}) &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathbf{1}_{[0,N]^n}(\mathbf{x}) e(\boldsymbol{\eta} \cdot \mathbf{x}) e(\delta(\mathbf{f}(\mathbf{x}) - \lambda)) d\mathbf{x} d\delta \\ &= \lambda^{n/k-1} \int_{\mathbb{R}^n} \mathbf{1}_{[0,N]^n}(\mathbf{x}) e(\boldsymbol{\eta} \cdot \mathbf{x}) d\sigma_\lambda(\mathbf{x}) =: \widetilde{d\sigma}_\lambda(\boldsymbol{\eta}),\end{aligned}\tag{3.12}$$

since the surface measure $d\sigma_\lambda$ is supported in the cube $[0, N]^n$.

3.4. Remarks on the proof of (1.6). We now take a moment to substantiate our claim, made in the introduction, that the L^∞ -bound (1.6) holds under a weaker assumption on the dimension. The key observation is that to prove (1.6) one does not need to refer to Lemma 7, and therefore, the inequality

$$\int_{\mathfrak{m}} |F(\theta; \boldsymbol{\xi})| d\theta \lesssim N^{n-k} L^{-B} \tag{3.13}$$

can replace (3.2) in the treatment of the minor arcs. We can now use (3.3) and the trivial majorization

$$\int_{\mathbb{T}} |S_N(\theta, \xi)|^{2s} d\theta \lesssim L^{2s} \int_{\mathbb{T}} \left| \sum_{m \leq N} e(\theta m^k) \right|^{2s} d\theta$$

to deduce (3.13) from the results in §6 of [7], provided that $k \geq 5$ and $n \geq n_0(k)$. When $k \leq 4$, the same conclusion follows a sharp form of Hua's lemma, such as Lemma 1 in Vaughan [30].

4. L^p BOUNDS FOR THE MAIN TERM MULTIPLIER

In this section, we consider the maximal function of the convolution operator whose multiplier is the main term in the approximation formula. Given a sufficiently large $\lambda \in \Gamma_{n,k}$, let j be the unique integer such that $2^{j-1} \leq \lambda < 2^j$. Let M_λ denote the convolution operator with Fourier multiplier

$$\widehat{M}_\lambda(\boldsymbol{\xi}) = \sum_{q=1}^{\infty} \sum_{a \in \mathbb{U}_q} e(-\lambda a/q) \sum_{\mathbf{q} \leq Q} \widehat{M}_\lambda^{a/q; \mathbf{q}}(\boldsymbol{\xi}),$$

where

$$\widehat{M}_\lambda^{a/q; \mathbf{q}}(\boldsymbol{\xi}) := \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \left\{ \prod_{i=1}^n g(a, q; a_i, q_i) \right\} \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \widetilde{d\sigma}_\lambda(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q}),$$

with $N = 2^{j/k}$, $Q = (\log N)^C$ for some large fixed $C > 0$. We write M_* for the maximal operator defined pointwise as

$$M_* f(\mathbf{x}) := \sup_{\lambda \in \Gamma_{n,k}} |M_\lambda f(\mathbf{x})|.$$

Our main objective in this section is to prove the following theorem.

Theorem 5. *Let $k \geq 2$. If $n \geq \max\{5, k/2 + 2\}$ and $p > \frac{n}{n-2}$, then the maximal operator M_* is bounded on $\ell^p(\mathbb{Z}^n)$.*

Remark 4. Note that $n_1(k), n_2(k) \geq k/2 + 2$ so that these restrictions on the dimension n dominate in Theorem 3. In terms of the exponent p , our range of ℓ^p -spaces is independent of the degree $k \geq 2$ and match those of the quadratic case (when $k = 2$) for the integral spherical maximal function of Magyar, Stein and Wainger [24]. In contrast, from [15] we know that the integral k -spherical maximal functions of Magyar [20] are unbounded on $L^p(\mathbb{R}^n)$ for $p \leq \frac{n}{n-k}$ for each $k \geq 3$. The difference is that in our current setup the analytic piece of the operator (see below) is more localized in Fourier space than it is in previous works; this improves its boundedness properties.

If $\mathcal{D} \subset \mathbb{R}_+^n$, we introduce the maximal functions

$$M_*^{a/q; \mathcal{D}} f(\mathbf{x}) := \sup_{\lambda \in \Gamma_{n,k}} \left| \sum_{\substack{\mathbf{q} \leq Q \\ \mathbf{q} \in \mathcal{D}}} M_\lambda^{a/q; \mathbf{q}} f(\mathbf{x}) \right|,$$

so that we have the pointwise inequality

$$M_* f(\mathbf{x}) \leq \sum_{q=1}^{\infty} \sum_{a \in \mathbb{U}_q} \sum_{\mathbf{j} \in \mathbb{Z}_+^n} M_*^{a/q; \mathcal{D}_{\mathbf{j}}} f(\mathbf{x}), \quad (4.1)$$

where $\mathcal{D}_{\mathbf{j}} = \{\mathbf{x} \in \mathbb{R}^n : 2^{j_i-1} \leq x_i < 2^{j_i}, 1 \leq i \leq n\}$. Applying the triangle inequality on $\ell^p(\mathbb{Z}^n)$ in (4.1), we see that

$$\|M_* f\|_{\ell^p(\mathbb{Z}^n)} \leq \sum_{q=1}^{\infty} \sum_{a \in \mathbb{U}_q} \sum_{\mathbf{j} \in \mathbb{Z}_+^n} \|M_*^{a/q; \mathcal{D}_{\mathbf{j}}} f\|_{\ell^p(\mathbb{Z}^n)}. \quad (4.2)$$

Next, we estimate $\|M_*^{a/q; \mathcal{D}} f\|_{\ell^p(\mathbb{Z}^n)}$ for a fixed rational number a/q and a dyadic box $\mathcal{D}$. Suppressing the dependence on a/q , we write $M_\lambda^{\mathbf{q}}$ for the convolution operator $M_\lambda^{a/q; \mathbf{q}}$. Similarly to [4, 24], we first decompose each Fourier multiplier $\widehat{M}_\lambda^{\mathbf{q}}$ into an analytic piece and an arithmetic piece. Let ψ be the bump function from the statement of the Approximation Formula. For $\mathbf{q} \in \mathbb{Z}_+^n$, we define the function $\Psi_{\mathbf{q}}(\boldsymbol{\xi}) = \psi(16\mathbf{q}\boldsymbol{\xi})$ and note that, when λ is large and $\mathbf{q} \leq Q$, one has

$$\psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) = \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})\Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}).$$

We also write

$$G(\mathbf{a}) = G(a, q; \mathbf{a}, \mathbf{q}) := \prod_{i=1}^n g(a, q; a_i, q_i).$$

We now define the Fourier multipliers

$$\begin{aligned} \widehat{S}^{\mathbf{q}}(\boldsymbol{\xi}) &:= \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} G(a, q; \mathbf{a}, \mathbf{q}) \Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}), \\ \widehat{T}_\lambda^{\mathbf{q}}(\boldsymbol{\xi}) &:= \sum_{\mathbf{a} \in \mathbb{Z}^n} \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \widetilde{d\sigma}_\lambda(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q}), \end{aligned}$$

so that

$$\widehat{M}_\lambda^{\mathbf{q}}(\boldsymbol{\xi}) = \widehat{T}_\lambda^{\mathbf{q}}(\boldsymbol{\xi}) \widehat{S}^{\mathbf{q}}(\boldsymbol{\xi}).$$

Hence,

$$\|M_*^{a/q;\mathcal{D}}f\|_{\ell^p(\mathbb{Z}^n)} \leq \sum_{\mathbf{q} \in \mathcal{D}} \|T_*^{\mathbf{q}}(S^{\mathbf{q}}f)\|_{\ell^p(\mathbb{Z}^n)}, \quad (4.3)$$

where the maximal function $T_*^{\mathbf{q}}$ is defined by

$$T_*^{\mathbf{q}}f(\mathbf{x}) := \sup_{\lambda \in \Gamma_{n,k}} |T_{\lambda}^{\mathbf{q}}f(\mathbf{x})|.$$

The estimation of the sum on the right side of (4.3) is broken into three lemmas. First, we note that when $\mathbf{q} \leq Q$, the supports of the functions $\psi_{N/Q}(\mathbf{q}\xi - \mathbf{a})$ are disjoint, which puts the multipliers $T_{\lambda}^{\mathbf{q}}$ and $T_*^{\mathbf{q}}$ into the form considered by Magyar, Stein and Wainger in Section 2 of [24]. In particular, Corollary 2.1 in [24] allows us to transfer the bound in the next lemma to the maximal operators $T_*^{\mathbf{q}}$. (There is a technical difference in that our $\mathbf{q}$ is composed of different q_i ; that is, in Magyar–Stein–Wainger they consider $\mathbf{q} = (q, \dots, q)$ whereas we are considering more general $\mathbf{q}$ where often $q_i \neq q_j$ for $i \neq j$. This however does not present a problem as we apply the Magyar–Stein–Wainger transference principle in each variable separately).

Lemma 8. *If $n \geq k/2 + 2$ and $p > 1$, the maximal operator*

$$T_*f(\mathbf{x}) := \sup_{\lambda \in \mathbb{N}} |f \star (\psi_{\lambda^{1/k}(\log \lambda)^{-C}} \star d\sigma_{\lambda})(\mathbf{x})|$$

is bounded on $L^p(\mathbb{R}^n)$.

The proof of this lemma appears in the appendix.

From this lemma and Corollary 2.1 in [24], we deduce that

$$\|T_*^{\mathbf{q}}(S^{\mathbf{q}}f)\|_{\ell^p(\mathbb{Z}^n)} \lesssim \|S^{\mathbf{q}}f\|_{\ell^p(\mathbb{Z}^n)}.$$

Thus, (4.3) yields

$$\|M_*^{a/q;\mathcal{D}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim \sum_{\mathbf{q} \in \mathcal{D}} \|S^{\mathbf{q}}f\|_{\ell^p(\mathbb{Z}^n)}. \quad (4.4)$$

Note that Corollary 2.1 in [24] requires an appropriate choice of Banach spaces in order to apply it, hence our chosen decomposition of the multiplier and the application of their Corollary 2.1 at this point in the proof.

Lemma 9. *Let $\mathcal{D}$ be either a dyadic box of the form $\mathcal{D}_{\mathbf{j}}$ above or a singleton in $\mathbb{Z}_+^n$. Then for all a, q and $\varepsilon > 0$, one has*

$$\sum_{\mathbf{q} \in \mathcal{D}} \|S^{\mathbf{q}}f\|_{\ell^2(\mathbb{Z}^n)} \lesssim_{\varepsilon} q^{\varepsilon-n/2} \left\{ \sum_{\mathbf{q} \in \mathcal{D}} w_{\mathbf{q}}(\mathbf{q})^{2-\varepsilon} \right\}^{1/2} \|f\|_{\ell^2(\mathbb{Z}^n)}, \quad (4.5)$$

where

$$w_{\mathbf{q}}(\mathbf{q}) = \prod_{i=1}^n \frac{(q, q_i)}{q_i}.$$

Lemma 10. *For all $a, q, \mathbf{q}$ and $\varepsilon > 0$, one has*

$$\|S^{\mathbf{q}}f\|_{\ell^1(\mathbb{Z}^n)} \lesssim_{\varepsilon} q^{\varepsilon} w_{\mathbf{q}}(\mathbf{q})^{1-\varepsilon} \|f\|_{\ell^1(\mathbb{Z}^n)}. \quad (4.6)$$

Now, we will use the lemmas to complete the proof of Theorem 5; we prove Lemmas 9 and 10 later in the section and Lemma 8 in Appendix A. First, we note that when $1 < p < 2$, interpolation between Lemma 10 and the singleton case of Lemma 9 yields

$$\|S^{\mathbf{q}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim_\varepsilon q^{\varepsilon-n/p'} w_q(\mathbf{q})^{1-\varepsilon} \|f\|_{\ell^p(\mathbb{Z}^n)}, \quad (4.7)$$

where p' is the conjugate exponent of p , defined by the relation $1/p + 1/p' = 1$. Using (4.4) and (4.7), we obtain

$$\|M_*^{a/q;\mathcal{D}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim \sum_{\mathbf{q} \in \mathcal{D}} \|S^{\mathbf{q}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim_\varepsilon q^{\varepsilon-n/p'} \left\{ \sum_{\mathbf{q} \in \mathcal{D}} w_q(\mathbf{q})^{1-\varepsilon} \right\} \|f\|_{\ell^p(\mathbb{Z}^n)} \quad (4.8)$$

for all $p > 1$. On the other hand, using (4.4) and Lemma 9, we have

$$\|M_*^{a/q;\mathcal{D}}f\|_{\ell^2(\mathbb{Z}^n)} \lesssim \sum_{\mathbf{q} \in \mathcal{D}} \|S^{\mathbf{q}}f\|_{\ell^2(\mathbb{Z}^n)} \lesssim_\varepsilon q^{\varepsilon-n/2} \left\{ \sum_{\mathbf{q} \in \mathcal{D}} w_q(\mathbf{q})^{2-\varepsilon} \right\}^{1/2} \|f\|_{\ell^2(\mathbb{Z}^n)}. \quad (4.9)$$

When $1 < p < 2$, we can interpolate between (4.9) and (4.8) with $p_1 = (p+1)/2$. If θ is defined so that $1/p = (1-\theta)/p_1 + \theta/2$, we get

$$\|M_*^{a/q;\mathcal{D}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim_\varepsilon q^{\varepsilon-n/p'} \Sigma_1^{1-\theta} \Sigma_2^\theta \|f\|_{\ell^p(\mathbb{Z}^n)}, \quad (4.10)$$

where

$$\Sigma_s = \left\{ \sum_{\mathbf{q} \in \mathcal{D}} w_q(\mathbf{q})^{s-\varepsilon} \right\}^{1/s}.$$

Recall that we are interested in the case when $\mathcal{D}$ is the Cartesian product of intervals $[2^{j_i-1}, 2^{j_i})$, $j_i \in \mathbb{Z}_+$, and write $D_i = 2^{j_i}$. We have

$$\Sigma_s^s \leq \prod_{i=1}^n \left\{ \sum_{d|q} d^{s-\varepsilon} \sum_{\substack{r \sim D_i \\ d|r}} r^{-s+\varepsilon} \right\} \lesssim_\varepsilon \prod_{i=1}^n \left\{ \sum_{\substack{d|q \\ d \leq D_i}} (d/D_i)^{s-1-\varepsilon} \right\}.$$

Hence, by the well-known inequality $\tau(q) \lesssim_\varepsilon q^\varepsilon$,

$$\Sigma_1 \lesssim_\varepsilon (qD_1 \cdots D_n)^\varepsilon$$

and

$$\Sigma_2 \lesssim_\varepsilon (qD_1 \cdots D_n)^\varepsilon \prod_{i=1}^n \left(\frac{q}{q+D_i} \right)^{1/2} =: (qD_1 \cdots D_n)^\varepsilon \Pi(q, \mathcal{D}).$$

Applying these bounds to the right side of (4.10), we finally obtain

$$\|M_*^{a/q;\mathcal{D}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim_\varepsilon q^{2\varepsilon-n/p'} (D_1 \cdots D_n)^\varepsilon \Pi(q, \mathcal{D})^\theta \|f\|_{\ell^p(\mathbb{Z}^n)}, \quad (4.11)$$

provided that $p > 1$.

We now apply (4.11) to all boxes $\mathcal{D}_{\mathbf{j}}$ that appear on the right side of (4.2) and then sum the resulting bounds over $\mathbf{j}$ to find that

$$\sum_{\mathbf{j} \in \mathbb{Z}_+^n} \|M_*^{a/q;\mathcal{D}_{\mathbf{j}}}f\|_{\ell^p(\mathbb{Z}^n)} \lesssim_\varepsilon q^{2\varepsilon-n/p'} \left\{ \sum_{j=1}^\infty \frac{2^{j\varepsilon} q^{\theta/2}}{(q+2^j)^{\theta/2}} \right\}^n \|f\|_{\ell^p(\mathbb{Z}^n)}. \quad (4.12)$$

Let $j_0 = j_0(q)$ be the unique index for which $2^{j_0} \leq q < 2^{j_0+1}$ and note that (4.12) is uniform in $a \in \mathbb{U}_q$. By splitting the series over j at j_0 , we deduce that

$$\begin{aligned} \sum_{a \in \mathbb{U}_q} \sum_{\mathbf{j} \in \mathbb{Z}_+^n} \left\| M_*^{a/q; \mathcal{D}_\mathbf{j}} f \right\|_{\ell^p(\mathbb{Z}^n)} &\lesssim_\varepsilon q^{1-n/p'+2\varepsilon} \left\{ \sum_{j \leq j_0} 2^{j\varepsilon} + q^{\theta/2} \sum_{j > j_0} 2^{j(\varepsilon-\theta/2)} \right\}^n \|f\|_{\ell^p(\mathbb{Z}^n)} \\ &\lesssim_\varepsilon q^{1-n/p'+2\varepsilon} 2^{nj_0\varepsilon} \|f\|_{\ell^p(\mathbb{Z}^n)} \lesssim_\varepsilon q^{1-n/p'+2n\varepsilon} \|f\|_{\ell^p(\mathbb{Z}^n)}, \end{aligned} \quad (4.13)$$

provided that $0 < \varepsilon < \theta/2$. After choosing $\varepsilon > 0$ sufficiently small, Theorem 5 is an immediate consequence of (4.2) and (4.12), provided that $n/p' > 2$, that is, $p > \frac{n}{n-2}$.

4.1. Proofs of the lemmas.

Proof of Lemma 9. Note that the functions $\Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})$ with distinct central points $\mathbf{a}/\mathbf{q}$, where $\mathbf{q} \in \mathcal{D}$, have disjoint supports. Indeed, if $\Psi_{\mathbf{q}'}(\mathbf{q}'\boldsymbol{\xi} - \mathbf{a}')\Psi_{\mathbf{q}''}(\mathbf{q}''\boldsymbol{\xi} - \mathbf{a}'') \neq 0$, with $\mathbf{a}'/\mathbf{q}' \neq \mathbf{a}''/\mathbf{q}''$, then for some index i , $1 \leq i \leq n$, we have

$$\frac{1}{4^{j_i}} < \frac{1}{q'_i q''_i} \leq \left| \frac{a'_i}{q'_i} - \frac{a''_i}{q''_i} \right| \leq \left| \frac{a'_i}{q'_i} - \xi_i \right| + \left| \frac{a''_i}{q''_i} - \xi_i \right| \leq \frac{1}{8(q'_i)^2} + \frac{1}{8(q''_i)^2} \leq \frac{1}{4^{j_i}};$$

a contradiction. Hence, Plancherel's theorem gives

$$\begin{aligned} \|S^{\mathbf{q}} f\|_{\ell^2(\mathbb{Z}^n)}^2 &= \left\| \widehat{S^{\mathbf{q}} f} \right\|_{L^2(\mathbb{T}^n)}^2 = \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} |G(\mathbf{a})|^2 \int_{\mathbb{T}^n} \Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})^2 |\hat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \\ &\lesssim \left(\max_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} |G(\mathbf{a})|^2 \right) \int_{\mathbb{T}^n} \Phi_{\mathbf{q}}(\boldsymbol{\xi}) |\hat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}, \end{aligned} \quad (4.14)$$

where

$$\Phi_{\mathbf{q}}(\boldsymbol{\xi}) = \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} \Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}).$$

Applying Lemmas 1 and 6 to each factor $g(a, q; a_i, q_i)$ in $G(\mathbf{a})$, we find that

$$|G(\mathbf{a})| \lesssim_\varepsilon q^{\varepsilon-n/2} \prod_{i=1}^n \varphi\left(\frac{q_i}{(q, q_i)}\right)^{-1} \lesssim_\varepsilon q^{\varepsilon-n/2} w_q(\mathbf{q})^{1-\varepsilon}, \quad (4.15)$$

where we have used the well-known inequality

$$\varphi(m)^{-1} \lesssim m^{-1} \log \log m. \quad (4.16)$$

Combining (4.14), (4.15) and Cauchy's inequality (in $\mathbf{q}$), we obtain

$$\begin{aligned} \sum_{\mathbf{q} \in \mathcal{D}} \|S^{\mathbf{q}} f\|_{\ell^2(\mathbb{Z}^n)} &\lesssim_\varepsilon q^{\varepsilon-n/2} \left\{ \sum_{\mathbf{q} \in \mathcal{D}} w_q(\mathbf{q})^{2-2\varepsilon} \right\}^{1/2} \left\{ \int_{\mathbb{T}^n} \left(\sum_{\mathbf{q} \in \mathcal{D}} \Phi_{\mathbf{q}}(\boldsymbol{\xi}) \right) |\hat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right\}^{1/2} \\ &\lesssim_\varepsilon q^{\varepsilon-n/2} \left\{ \sum_{\mathbf{q} \in \mathcal{D}} w_q(\mathbf{q})^{2-\varepsilon} \right\}^{1/2} \|\hat{f}\|_{L^2(\mathbb{T}^n)}, \end{aligned}$$

by our earlier observation about the supports of the functions $\Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})$. The lemma follows by another appeal to Plancherel's theorem. $\square$

Proof of Lemma 10. For $\mathbf{b}, \mathbf{q} \in \mathbb{Z}^n$ and $f : \mathbb{Z}^n \rightarrow \mathbb{C}$, let $f_{\mathbf{b}, \mathbf{q}}$ denote the restriction of f to the residue class $\mathbf{b}$ modulo $\mathbf{q}$ in $\mathbb{Z}^n$: i.e., $f_{\mathbf{b}, \mathbf{q}}(\mathbf{x}) = f(\mathbf{b} + \mathbf{q}\mathbf{x})$. We remark that it suffices to prove the lemma for functions $f_{\mathbf{b}, \mathbf{q}}$. Indeed, if the inequality

$$\|S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} \leq M \|f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)}$$

holds for all restrictions $f_{\mathbf{b}, \mathbf{q}}$, then also

$$\|S^{\mathbf{q}} f\|_{\ell^1(\mathbb{Z}^n)} = \sum_{\mathbf{b} \in \mathbb{Z}_{\mathbf{q}}} \|S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} \leq M \sum_{\mathbf{b} \in \mathbb{Z}_{\mathbf{q}}} \|f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} = M \|f\|_{\ell^1(\mathbb{Z}^n)}.$$

We now proceed to establish (4.6) for restrictions $f_{\mathbf{b}, \mathbf{q}}$. Note that

$$\widehat{f_{\mathbf{b}, \mathbf{q}}}(\boldsymbol{\xi} + \mathbf{a}/\mathbf{q}) = e(\mathbf{b} \cdot \mathbf{a}/\mathbf{q}) \cdot \widehat{f_{\mathbf{b}, \mathbf{q}}}(\boldsymbol{\xi}).$$

From this we can deduce that

$$S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}(\mathbf{y}) = H(a, q; \mathbf{y} - \mathbf{b}, \mathbf{q}) (\widetilde{\Psi}_{\mathbf{q}} \star f_{\mathbf{b}, \mathbf{q}})(\mathbf{y}),$$

where $\widetilde{\Psi}_{\mathbf{q}}$ denotes the inverse Fourier transform of $\Psi_{\mathbf{q}}(\mathbf{q}\boldsymbol{\xi})$ and

$$H(a, q; \mathbf{u}, \mathbf{q}) = \sum_{\mathbf{a} \in \mathbb{U}_{\mathbf{q}}} G(a, q; \mathbf{a}, \mathbf{q}) e(-\mathbf{u} \cdot \mathbf{a}/\mathbf{q}).$$

(Note that $H(a, q; \mathbf{u}, \mathbf{q})$ is a multidimensional version of the sum $h(a, q; u, r)$ that appears in the proof of Lemma 6.) We now have

$$\|S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} = \sum_{\mathbf{y} \in \mathbb{Z}^n} |H(a, q; \mathbf{y} - \mathbf{b}, \mathbf{q}) (\widetilde{\Psi}_{\mathbf{q}} \star f_{\mathbf{b}, \mathbf{q}})(\mathbf{y})|.$$

We rearrange the last sum according to the residue class of $\mathbf{y}$ modulo $\mathbf{q}$. Since $H(a, q; \mathbf{y} - \mathbf{b}, \mathbf{q})$ depends only on the residue class of $\mathbf{y}$ modulo $\mathbf{q}$, we get

$$\begin{aligned} \|S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} &= \sum_{\mathbf{r} \in \mathbb{Z}_{\mathbf{q}}} |H(a, q; \mathbf{r} - \mathbf{b}, \mathbf{q})| \sum_{\mathbf{z} \in \mathbb{Z}^n} |(\widetilde{\Psi}_{\mathbf{q}} \star f_{\mathbf{b}, \mathbf{q}})(\mathbf{q}\mathbf{z} + \mathbf{r})| \\ &= \sum_{\mathbf{r} \in \mathbb{Z}_{\mathbf{q}}} |H(a, q; \mathbf{r} - \mathbf{b}, \mathbf{q})| \sum_{\mathbf{z} \in \mathbb{Z}^n} \left| \sum_{\mathbf{x} \in \mathbb{Z}^n} \widetilde{\Psi}_{\mathbf{q}}(\mathbf{q}\mathbf{z} + \mathbf{r} - \mathbf{x}) f_{\mathbf{b}, \mathbf{q}}(\mathbf{x}) \right| \\ &\leq \sum_{\mathbf{r} \in \mathbb{Z}_{\mathbf{q}}} |H(a, q; \mathbf{r} - \mathbf{b}, \mathbf{q})| \sum_{\mathbf{x} \in \mathbb{Z}^n} |f_{\mathbf{b}, \mathbf{q}}(\mathbf{x})| \sum_{\mathbf{z} \in \mathbb{Z}^n} |\widetilde{\Psi}_{\mathbf{q}}(\mathbf{q}\mathbf{z} + \mathbf{r} - \mathbf{x})|. \end{aligned} \quad (4.17)$$

The sum over $\mathbf{z}$ on the right side of (4.17) is $\mathbf{q}$ -periodic in $\mathbf{r} - \mathbf{x}$, so we may assume that $-\frac{1}{2} \leq (\mathbf{r} - \mathbf{x})/\mathbf{q} \leq \frac{1}{2}$. Since $\widetilde{\Psi}_{\mathbf{q}}(\mathbf{m}) = \widehat{\Psi}_{\mathbf{q}}(\mathbf{m})$, we find that

$$\begin{aligned} \sum_{\mathbf{z} \in \mathbb{Z}^n} |\widetilde{\Psi}_{\mathbf{q}}(\mathbf{q}\mathbf{z} + \mathbf{r} - \mathbf{x})| &= \sum_{\mathbf{z} \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} \psi_{\mathbf{q}}(16\mathbf{q}\boldsymbol{\xi}) e((\mathbf{q}\mathbf{z} + \mathbf{r} - \mathbf{x}) \cdot \boldsymbol{\xi}) d\boldsymbol{\xi} \right| \\ &= \sum_{\mathbf{z} \in \mathbb{Z}^n} \frac{1}{q_1^2 \cdots q_n^2} \left| \widehat{\psi}_{16} \left(\frac{\mathbf{z} + (\mathbf{r} - \mathbf{x})/\mathbf{q}}{\mathbf{q}} \right) \right| \\ &\lesssim \frac{1}{q_1^2 \cdots q_n^2} \sum_{\mathbf{z} \in \mathbb{Z}^n} \frac{1}{1 + |(\mathbf{z} + (\mathbf{r} - \mathbf{x})/\mathbf{q})/\mathbf{q}|^{2n}} \lesssim \frac{1}{q_1 \cdots q_n}. \end{aligned}$$

Inserting the last bound into the right side of (4.17), we deduce the estimate

$$\|S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} \lesssim \frac{\|f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)}}{q_1 \cdots q_n} \sum_{\mathbf{r} \in \mathbb{Z}_{\mathbf{q}}} |H(a, q; \mathbf{r} - \mathbf{b}, \mathbf{q})|.$$

Since

$$\sum_{\mathbf{r} \in \mathbb{Z}_{\mathbf{q}}} |H(a, q; \mathbf{r} - \mathbf{b}, \mathbf{q})| = \sum_{\mathbf{u} \in \mathbb{Z}_{\mathbf{q}}} |H(a, q; \mathbf{u}, \mathbf{q})| = \prod_{j=1}^n \left\{ \sum_{u \in \mathbb{Z}_{q_i}} |h(a, q; u, q_i)| \right\},$$

Lemma 6(iii) now yields

$$\|S^{\mathbf{q}} f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} \lesssim \|f_{\mathbf{b}, \mathbf{q}}\|_{\ell^1(\mathbb{Z}^n)} \prod_{j=1}^n \left(\frac{\tau(q_i)}{\varphi(q_i/(q, q_i))} \right).$$

The desired estimate follows from (4.16) and the bound $\tau(m) \lesssim_\varepsilon m^\varepsilon$. $\square$

5. COMPARISON WITH THE INTEGRAL MAXIMAL FUNCTION

In this section, we show that the maximal function of the error term is bounded on $\ell^p(\mathbb{Z}^n)$ for a range of p by comparing the averages A_λ for $\lambda \in \Gamma_{n,k}$ with the bounds for the corresponding integral operators. This combined with the boundedness of the main term shows that the maximal function A_* is bounded on $\ell^p(\mathbb{Z}^n)$. As we will see, our range of ℓ^p -boundedness for the averages A_* matches that of the integral maximal function B_* below, possibly up to endpoints.

For $f : \mathbb{Z}^n \rightarrow \mathbb{C}$ and $\mathbf{x} \in \mathbb{Z}^n$, define the integral averages by

$$B_\lambda f(\mathbf{x}) := (f \star \sigma_\lambda)(\mathbf{x}) = \frac{1}{\#\{\mathbf{y} \in \mathbb{Z}_+^n : \mathbf{f}(\mathbf{y}) = \lambda\}} \sum_{\mathbf{f}(\mathbf{y})=\lambda} f(\mathbf{x} - \mathbf{y}),$$

along with their maximal function

$$B_* f(\mathbf{x}) := \sup_{\lambda \in \mathbb{N}} |B_\lambda f(\mathbf{x})|.$$

The operator B_* is equivalent to Magyar–Stein–Wainger’s discrete spherical maximal function. Our goal is to prove the following comparison between the integral maximal function and the Waring–Goldbach maximal function.

Theorem 6. *Suppose that $1 < p_0 < 2$ and $n \geq n_1(k)$. If B_* and M_* are bounded on $\ell^{p_0}(\mathbb{Z}^n)$, then A_* is bounded on $\ell^p(\mathbb{Z}^n)$ for $p > p_0$.*

Proof. Recall from the Approximation Formula that for each $\lambda \in \Gamma_{n,k}$ we have

$$A_\lambda f(\mathbf{x}) = M_\lambda f(\mathbf{x}) + E_\lambda f(\mathbf{x}).$$

We will use the decay of the dyadic maximal function of the error term on $\ell^2(\mathbb{Z}^n)$. By (1.5), we have

$$\left\| \sup_{\lambda \approx 2^j} |E_\lambda f| \right\|_{\ell^2(\mathbb{Z}^n)} \lesssim j^{-K} \|f\|_{\ell^2(\mathbb{Z}^n)} \quad (5.1)$$

for an arbitrarily large, fixed $K > 0$, provided that the parameter C in Theorem 1 is chosen sufficiently large. Our first order of business is to establish the following matching bound on $\ell^{p_0}(\mathbb{Z}^n)$:

$$\left\| \sup_{\lambda \approx 2^j} |E_\lambda f| \right\|_{\ell^{p_0}(\mathbb{Z}^n)} \lesssim j^n \|f\|_{\ell^{p_0}(\mathbb{Z}^n)}. \quad (5.2)$$

For each $\mathbf{x} \in \mathbb{Z}^n$ we have

$$|A_\lambda f(\mathbf{x})| \lesssim (\log \lambda)^n (B_\lambda |f|)(\mathbf{x}).$$

Thus,

$$|E_\lambda f(\mathbf{x})| \lesssim |M_\lambda f(\mathbf{x})| + (\log \lambda)^n (B_\lambda |f|)(\mathbf{x})$$

for each $\lambda \in \Gamma_{n,k}$ and all $\mathbf{x} \in \mathbb{Z}^n$. In turn,

$$\sup_{\lambda \sim 2^j} |E_\lambda f(\mathbf{x})| \lesssim \sup_{\lambda \sim 2^j} |M_\lambda f(\mathbf{x})| + j^n \sup_{\lambda \sim 2^j} (B_\lambda |f|)(\mathbf{x}).$$

Taking $\ell^{p_0}(\mathbb{Z}^n)$ norms and applying the hypotheses, we deduce (5.2).

For $p_0 < p < 2$, let θ be such that $1/p = (1 - \theta)/p_0 + \theta/2$, and then choose K sufficiently large to ensure that $n(1 - \theta) - K\theta \leq -2$. Then interpolation between (5.1) and (5.2) reveals that

$$\left\| \sup_{\lambda \sim 2^j} |E_\lambda f| \right\|_{\ell^p(\mathbb{Z}^n)} \lesssim j^{-2} \|f\|_{\ell^p(\mathbb{Z}^n)}.$$

Summing over $j \in \mathbb{N}$, we find that

$$\left\| \sup_{\lambda \in \Gamma_{n,k}} |E_\lambda f| \right\|_{\ell^p(\mathbb{Z}^n)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^n)}$$

for all $p_0 < p < 2$. Combining this with our hypothesis that M_* is bounded on $\ell^{p_0}(\mathbb{Z}^n)$ (and hence, also on $\ell^p(\mathbb{Z}^n)$ —by interpolation with the trivial $\ell^\infty(\mathbb{Z}^n)$ bound), we are done. $\square$

Proof of Theorem 3. For $k = 2$, the main theorem of [24] shows that B_* is bounded on $\ell^p(\mathbb{Z}^n)$ for $p > \frac{n}{n-2}$ and $n \geq 5$. For $k \geq 3$, Theorem 1 of [16] we have that B_* is bounded on $\ell^p(\mathbb{Z}^n)$ for $p > \max\{\frac{n}{n-k}, 1 + \frac{k^2}{2(n-k[k+2]) + k^2}, 1 + \frac{k}{k[k-1] - k}\}$ and $n \geq \max\{k(k+2), k^2(k-1)\}$. Thus the theorem is true for $p > 1 + \frac{k^2[k-1]}{2n-k^2[k-1]} = \frac{2n}{2n-k^2[k-1]}$ and $n \geq k^2(k-1)$. $\square$

6. APPLICATIONS

In this section, we prove Theorems 2 and 4. Recall that in what follows, (X, μ) denotes a probability space with a commuting family of invertible measure preserving transformations $T = (T_1, \dots, T_n)$ without any non-zero rational points in their spectrum. For a function $f : X \rightarrow \mathbb{C}$ the Waring–Goldbach ergodic averages on X with respect to T for $\lambda \in \Gamma_{n,k}$ are defined by (1.8).

6.1. Proof of Theorem 2. Fix $\varepsilon > 0$ and let $\delta > 0$ be a parameter to be chosen later (in terms of ε). Since $\boldsymbol{\xi} \notin \mathbb{Q}^n$, we may assume without loss of generality that $\xi_1 \notin \mathbb{Q}$. Then, we can choose a convergent b/r to the continued fraction of ξ_1 with $r > 2\delta^{-1}$.

Now, for a large $\lambda \in \Gamma_{n,k}$, let $N = \lambda^{1/k}$ and $Q = (\log N)^C$, where $C = C(1) > 0$ is the power in the Approximation Formula corresponding to having (1.6) for $B = 1$. We note that for sufficiently large λ , there is at most one rational point $\mathbf{a}/\mathbf{q}$ such that

$$1 \leq \mathbf{q} \leq Q, \quad \mathbf{a} \in \mathbb{U}_{\mathbf{q}}, \quad \psi_{N/Q}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) > 0. \quad (6.1)$$

If such a rational point does not exist, the main term in (1.4) vanishes, and we have

$$\widehat{\omega}_\lambda(\boldsymbol{\xi}) \lesssim (\log \lambda)^{-1}.$$

Otherwise, (1.4) yields

$$\widehat{\omega}_\lambda(\boldsymbol{\xi}) \lesssim \left| \mathfrak{S}(\lambda; \mathbf{a}, \mathbf{q}) \widetilde{d\sigma_1}(N(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q})) \right| + (\log \lambda)^{-1},$$

where $\mathbf{a}/\mathbf{q}$ satisfies (6.1). Using (2.4) with $\varepsilon = 1/(4n)$, we deduce that, for $n \geq 5$,

$$\mathfrak{S}(\lambda; \mathbf{a}, \mathbf{q}) \lesssim q_1^{-9/20} \sum_{q=1}^{\infty} q^{21/20-n/2} (q, q_1)^{1/2} \lesssim q_1^{-9/20} \sum_{d|q_1} d^{1/2} \sum_{\substack{q=1 \\ d|q}}^{\infty} q^{21/20-n/2} \lesssim q_1^{-2/5}.$$

Hence,

$$\widehat{\omega}_{\lambda}(\boldsymbol{\xi}) \lesssim q_1^{-2/5} |\widetilde{d\sigma_1}(N(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q}))| + (\log \lambda)^{-1}.$$

Using the decay of $\widetilde{d\sigma_1}$ (see for example [10]), we may now choose δ so that

$$q_1^{-2/5} \widetilde{d\sigma_1}(N(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q})) \lesssim \varepsilon,$$

unless

$$1 \leq q_1 \leq \delta^{-1} \quad \text{and} \quad |\xi_1 - a_1/q_1| \leq (\delta N)^{-1}. \quad (6.2)$$

Thus, we have

$$\widehat{\omega}_{\lambda}(\boldsymbol{\xi}) \lesssim \varepsilon + (\log \lambda)^{-1},$$

unless a_1, q_1 and ξ_1 satisfy (6.2). To complete the proof of the theorem, we will show that for sufficiently large λ , inequalities (6.2) are inconsistent with the choice of b/r .

Suppose that conditions (6.2) do hold and recall that $|r\xi_1 - b| < r^{-1}$. Then

$$|bq_1 - a_1r| \leq \frac{rq_1}{\delta N} + \frac{q_1}{r} < \frac{r}{\delta^2 N} + \frac{1}{2} < 1,$$

as $N \rightarrow \infty$. Since b/r and a_1/q_1 are reduced fractions, we conclude that $a_1 = b$ and $q_1 = r$. The latter, however, contradicts the inequalities $q_1 \leq \delta^{-1} < r/2$. $\square$

Remark 5. We comment that a shorter proof of Theorem 2 exists by using the decay of the error term in (1.5), but this proof has the advantage of not relying on the bound (1.5) and instead uses (1.6).

6.2. The Pointwise Ergodic Theorem. To prove Theorem 4 we will utilize the Calderón transference principle and in doing so, we need to introduce some notation. Let K be a large natural number and define the discrete cube

$$\mathcal{C}(K) := \{\mathbf{m} \in \mathbb{Z}^n : |m_i| \leq K \text{ for } i = 1, \dots, n\}.$$

For a μ -measurable function $f : X \rightarrow \mathbb{C}$, define its truncated *transfer function*,

$$F(x, \mathbf{m}) = f(T^{\mathbf{m}}x) \mathbf{1}_{\mathcal{C}(N)}(\mathbf{m}).$$

For $\lambda \in \Gamma_{n,k}$, also define the *transferred averages*

$$\mathcal{A}_{\lambda}F(x, \mathbf{m}) := \frac{1}{R(\lambda)} \sum_{\mathfrak{f}(\mathbf{p})=\lambda} \log(\mathbf{p}) F(x, \mathbf{m} + \mathbf{p})$$

and their *tail maximal function*

$$\mathcal{A}_{>R}F(x, \mathbf{m}) := \sup_{\lambda > R} |\mathcal{A}_{\lambda}F(x, \mathbf{m})|.$$

We endow the transfer space $X \times \mathbb{Z}^n$ with the product measure of μ on X and the counting measure on $\mathbb{Z}^n$. As in [15], we deduce Theorem 4 from the tail oscillation inequality below. We refer to [15] for the details of this reduction, which relies on the Calderón transference principle.

Proposition 1 (Transferred Oscillation Inequality). *Let f be a bounded function of mean zero on X and F its transfer function. For each $\varepsilon > 0$, there exists a sufficiently large radius $R = R(\varepsilon, f)$ such that*

$$\|\mathcal{A}_{>R}F\|_{L^2(X \times \mathbb{Z}^n)} < \varepsilon \|F\|_{L^2(X \times \mathbb{Z}^n)}. \quad (6.3)$$

The proof of the transferred oscillation inequality requires a few steps, which we carry out in succession. First, we extend the Approximation Formula to the lifted averages. For $\xi \in \mathbb{T}^n$, define the partial $\mathbb{Z}^n$ -Fourier transform as

$$\widehat{F}(x, \xi) := \sum_{\mathbf{m} \in \mathbb{Z}^n} F(x, \mathbf{m}) e(\mathbf{m} \cdot \xi).$$

The reader may verify that

$$\widehat{\mathcal{A}_\lambda F}(x, \xi) = \widehat{\omega}_\lambda(\xi) \widehat{F}(x, \xi). \quad (6.4)$$

Equation (6.4) allows us to extend the multipliers on $\mathbb{Z}^n$ to multipliers on $X \times \mathbb{Z}^n$. Suppressing the dependence on a, q , we define the convolution operators $\mathcal{M}_\lambda^{\mathbf{q}}$ by the multipliers

$$\widehat{\mathcal{M}_\lambda^{\mathbf{q}} F}(x, \xi) := \widehat{M}_\lambda^{\mathbf{q}}(\xi) \widehat{F}(x, \xi),$$

where $\widehat{M}_\lambda^{\mathbf{q}}$ is the Fourier multiplier from Section 4 with $N = \lambda^{1/k}$ and $Q = (\log \lambda)^C$. Similarly, define the error term by

$$\widehat{\mathcal{E}_\lambda F}(x, \xi) = \widehat{E}_\lambda(\xi) \widehat{F}(x, \xi). \quad (6.5)$$

Also define their tail maximal functions similarly to $\mathcal{A}_{>R}F$.

Our estimates on the error term in Theorem 1 transfer over to show that

$$\left\| \sup_{\lambda > R} |\mathcal{E}_\lambda F| \right\|_{L^2(X \times \mathbb{Z}^n)} \lesssim (\log R)^{-B_1} \|F\|_{L^2(X \times \mathbb{Z}^n)} \quad (6.6)$$

for all large $B_1 > 0$, so that choosing R sufficiently large we may make this arbitrarily small. This shows that the averages are equiconvergent with the main term. Lemmas 8 and 9 and a version of (4.12) for $p = 2$ combine to give

$$\begin{aligned} \left\| \sup_{\lambda > R} \left| \sum_{q,a} \sum_{\mathbf{q} > Q} \mathcal{M}_\lambda^{\mathbf{q}} F \right| \right\|_{L^2(X \times \mathbb{Z}^n)} &\leq \sum_{a,q,\mathbf{a},\mathbf{q} > Q} \|\mathcal{M}_{>R}^{\mathbf{q}} F\|_{L^2(X \times \mathbb{Z}^n)} \\ &\lesssim Q^{-C_2} \|F\|_{L^2(X \times \mathbb{Z}^n)} \end{aligned}$$

for some positive C_2 when $n \geq \max\{n_1(k), n_2(k)\}$.

Our final proposition completes the proof of Theorem 4. This is the only place where the vanishing of the rational spectrum is used.

Proposition 2. *If $\epsilon > 0$, then there exists a radius $R = R(f; \epsilon, Q) \in \Gamma_{n,k}$ sufficiently large such that for all $q, \mathbf{q} \leq Q$, $a \in U_q$ and $\mathbf{a} \in U_{\mathbf{q}}$,*

$$\|\mathcal{M}_{>R}^{\mathbf{q}} F\|_{L^2(X \times \mathbb{Z}^n)} \lesssim \epsilon \|F\|_{L^2(X \times \mathbb{Z}^n)} \quad (6.7)$$

with implicit constants independent of $a, \mathbf{a}; q, \mathbf{q}$.

As this is the essential part, we include the proof. Our proof will follow that of Proposition 9.2 in [15] for the integral k -spherical maximal function. Unlike the integral maximal function where the localizing bump function depends on the modulus q , our current localizing bump function depends on the radius so that the continuous part of the multiplier behaves like a smooth Hardy–Littlewood averaging operator. This simplifies our exposition.

Proof. By Lemma 8, the tail maximal function of the multipliers

$$\psi_{\lambda^{1/k}(\log \lambda)^{-C}}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a}) \widetilde{d\sigma_{\lambda_0}}(N(\boldsymbol{\xi} - \mathbf{a}/\mathbf{q}))$$

is bounded on $L^2(X \times \mathbb{Z}^n)$ with the bound

$$\|\mathcal{M}_{>R}^{\mathbf{a}} F\|_{L^2(X \times \mathbb{Z}^n)} \lesssim \left\| \left(\prod_{i=1}^n g(a, q; a_i, q_i) \right) \widetilde{\psi_{R'}^{\mathbf{a}/\mathbf{q}}} \star F \right\|_{L^2(X \times \mathbb{Z}^n)}$$

where $R' := R(\log R)^{-C}$. To prove Proposition 2 it suffices to show that

$$\left\| \widetilde{\psi_{R'}^{\mathbf{a}/\mathbf{q}}} \star F \right\|_{L^2(X \times \mathbb{Z}^n)} \lesssim \epsilon \|F\|_{L^2(X \times \mathbb{Z}^n)} \quad (6.8)$$

for each $\mathbf{a}, \mathbf{q}$ and sufficiently large R depending on ϵ . Plancherel's Theorem and the Spectral Theorem imply

$$\left\| \widetilde{\psi_{R'}^{\mathbf{a}/\mathbf{q}}} \star F \right\|_{L^2(X \times \mathbb{Z}^n)}^2 = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\psi_{R'}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})|^2 \sum_{\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{C}(K)} e((\mathbf{m}_1 - \mathbf{m}_2)[\boldsymbol{\eta} + \boldsymbol{\xi}]) d\boldsymbol{\xi} d\nu_f(\boldsymbol{\eta}).$$

Once again, see [21] for this derivation. Collecting $\mathbf{m}_1 - \mathbf{m}_2 = \mathbf{m}$, we define the sequence

$$\Delta_N(\mathbf{m}) := \frac{\#\{(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{C}(K) \times \mathcal{C}(K) : \mathbf{m}_1 - \mathbf{m}_2 = \mathbf{m}\}}{\#\mathcal{C}(K)}.$$

The above becomes

$$\left\| \widetilde{\psi_{R'}^{\mathbf{a}/\mathbf{q}}} \star F \right\|_{L^2(X \times \mathbb{Z}^n)}^2 = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\psi_{R'}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})|^2 \sum_{\mathbf{m} \in \mathbb{Z}^n} \#\mathcal{C}(K) \Delta_K(\mathbf{m}) \cdot e(\mathbf{m} \cdot [\boldsymbol{\xi} + \boldsymbol{\eta}]) d\boldsymbol{\xi} d\nu_f(\boldsymbol{\eta}).$$

Note that $\Delta_K \rightarrow 1$ as $K \rightarrow \infty$. This implies that $\widehat{\Delta_K}(\boldsymbol{\xi}) \rightarrow \delta_0(\boldsymbol{\xi})$ tends pointwise to the Dirac delta function on $\mathbb{T}^n$ as $K \rightarrow \infty$. Therefore,

$$\begin{aligned} \#\mathcal{C}(K)^{-1} \left\| \widetilde{\psi_{R'}^{\mathbf{a}/\mathbf{q}}} \star F \right\|_{L^2(X \times \mathbb{Z}^n)}^2 &= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\psi_{R'}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})|^2 \sum_{\mathbf{m} \in \mathbb{Z}^n} \Delta_K(\mathbf{m}) \cdot e(\mathbf{m} \cdot [\boldsymbol{\xi} + \boldsymbol{\eta}]) d\boldsymbol{\xi} d\nu_f(\boldsymbol{\eta}) \\ &= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\psi_{R'}(\mathbf{q}\boldsymbol{\xi} - \mathbf{a})|^2 \cdot \widehat{\Delta_K}(\boldsymbol{\xi} + \boldsymbol{\eta}) d\boldsymbol{\xi} d\nu_f(\boldsymbol{\eta}) \\ &= \int_{\mathbb{T}^n} (|\psi_{\mathbf{q}R}(\cdot - \mathbf{a}/\mathbf{q})|^2 \star \widehat{\Delta_K})(\boldsymbol{\eta}) d\nu_f(\boldsymbol{\eta}) \end{aligned}$$

where the convolution is on the torus. Now we make use of the fact that multiplier is localized to low frequencies. For all $\epsilon > 0$, there exists $K_\epsilon \in \mathbb{N}$ such that $|\widehat{\Delta_K} - \delta_0| < \epsilon$ for all $K > K_\epsilon$ and

$$\begin{aligned} &\int_{\mathbb{T}^n} \left| |\psi_{\mathbf{q}R}(\cdot - \mathbf{a}/\mathbf{q})|^2 \star \widehat{\Delta_K}(\boldsymbol{\eta}) d\nu_f(\boldsymbol{\eta}) \right| \\ &\leq \int_{\mathbb{T}^n} \left| |\psi_{\mathbf{q}R}(\cdot - \mathbf{a}/\mathbf{q})|^2 \star |\widehat{\Delta_K} - \delta_0|(\boldsymbol{\eta}) \right| + \left| |\psi_{\mathbf{q}R}(\cdot - \mathbf{a}/\mathbf{q})|^2 \star \delta_0(\boldsymbol{\eta}) \right| d\nu_f(\boldsymbol{\eta}) \\ &= \int_{\mathbb{T}^n} |\psi_{\mathbf{q}R}(\cdot - \mathbf{a}/\mathbf{q})|^2 \star |\widehat{\Delta_K} - \delta_0|(\boldsymbol{\eta}) d\nu_f(\boldsymbol{\eta}) + \int_{\mathbb{T}^n} |\psi_{\mathbf{q}R}(\mathbf{q}\boldsymbol{\eta} - \mathbf{a})|^2 d\nu_f(\boldsymbol{\eta}) \\ &\lesssim \epsilon \|f\|_{L^2(X)}^2 + \nu_f(|\boldsymbol{\eta} - \mathbf{a}/\mathbf{q}| \lesssim |\mathbf{q}R|^{-1}). \end{aligned}$$

For $\mathbf{a}/\mathbf{q} = 0$, $\nu_f(|\boldsymbol{\eta}| \lesssim |\mathbf{q}R|^{-1}) \rightarrow \nu_f(0)$ as $R \rightarrow \infty$, but $\nu_f(0) = |\int_X f d\mu|^2 = 0$ since f has mean zero. For $\mathbf{a}/\mathbf{q} \neq 0$, $\nu_f(|\boldsymbol{\eta} - \mathbf{a}/\mathbf{q}| \lesssim |\mathbf{q}R|^{-1}) \rightarrow \nu_f(\mathbf{a}/\mathbf{q})$ as $R \rightarrow \infty$, but $\nu_f(\mathbf{a}/\mathbf{q}) = 0$ by our assumption on the rational spectrum. Since there are finitely many $\mathbf{a}/\mathbf{q}$ and $\mathbf{a}/\mathbf{q}$, we can finish by choosing R large enough. Note that our parameter R depends on the spectral measure ν_f and consequently on the function f , in addition to ϵ and Q . $\square$

APPENDIX A. ESTIMATES FOR MOLLIFIED CONTINUOUS k -SPHERICAL AVERAGES

In this appendix, we sketch the $L^p(\mathbb{R}^n)$ -boundedness of the maximal functions

$$T_* f(\mathbf{x}) := \sup_{\lambda \in \mathbb{N}} |T_\lambda f(\mathbf{x})|$$

defined by the averages

$$T_\lambda f = (\widetilde{\psi_{N/Q}} \star d\sigma_\lambda) \star f,$$

where $N = \lambda^{1/k}$ and Q satisfies $1 \leq Q \leq (\log \lambda)^C$ for some constant $C > 0$. In Section 4 we applied the Magyar-Stein-Wainger transference principle to this maximal function in order to obtain $\ell^p(\mathbb{Z}^n)$ -bounds.

We will need the following two propositions in our proof.

Proposition 3. *Let $K \in \mathbb{N}$. For all $\lambda > 1$ and $Q \geq 1$, one has*

$$\widetilde{\psi_{N/Q}} \star d\sigma_\lambda(\mathbf{x}) \lesssim_K \frac{QN^{-n}}{(1 + |\mathbf{x}|/N)^K}. \quad (\text{A.1})$$

Proof. By rescaling, we only need to prove that

$$\widetilde{\psi_{1/Q}} \star d\sigma_1(\mathbf{x}) \lesssim_K \frac{Q}{(1 + |\mathbf{x}|)^K}.$$

This is well-known for the spherical measure (see for example, equation (6.5.12) on page 480 in [11]), but there is essentially no difference in the proof for the remaining k -spherical measures when $k \geq 3$. $\square$

Let P_j denote the smooth Littlewood–Paley projection operator adapted to frequencies of size approximately 2^j ; that is,

$$\widetilde{P_j f}(\boldsymbol{\xi}) := (\psi(\boldsymbol{\xi}/2^{j+1}) - \psi(\boldsymbol{\xi}/2^j))\widetilde{f}(\boldsymbol{\xi}).$$

Proposition 4. *For $n \geq 2$ and $k \geq 2$, we have that*

$$\left\| \sup_{\lambda \in \mathbb{N}} |P_j f \star (\widetilde{\psi_{N/Q}} \star d\sigma_\lambda)| \right\|_{L^2(\mathbb{R}^n)} \lesssim (1 + 2^j)^{\frac{1}{2} - \frac{n-1}{k}} \|f\|_{L^2(\mathbb{R}^n)}. \quad (\text{A.2})$$

Proof. From [15] we obtain the bounds

$$\begin{aligned} \widetilde{d\sigma_\lambda}(\boldsymbol{\xi}) &\lesssim (1 + |\lambda^{1/k} \boldsymbol{\xi}|)^{-\frac{n-1}{k}}, \\ \boldsymbol{\xi} \cdot \nabla(\widetilde{d\sigma_\lambda})(\boldsymbol{\xi}) &\lesssim (1 + |\lambda^{1/k} \boldsymbol{\xi}|)^{1 - \frac{n-1}{k}}. \end{aligned}$$

The frequency localization induced by P_j implies the bounds

$$\begin{aligned} \widetilde{P_j d\sigma_\lambda}(\boldsymbol{\xi}) &\lesssim (1 + |\lambda^{1/k} 2^j|)^{-\frac{n-1}{k}}, \\ \boldsymbol{\xi} \cdot \nabla(\widetilde{P_j d\sigma_\lambda})(\boldsymbol{\xi}) &\lesssim (1 + |\lambda^{1/k} 2^j|)^{1 - \frac{n-1}{k}}. \end{aligned}$$

The maximal function estimate (A.2) now follows from almost orthogonality of the Littlewood–Paley projections and a standard Sobolev embedding argument in the field. For the details of this Sobolev embedding argument, see the proof of Lemma 6.5.2 on page 478 of [11]. One only needs to change the number $(n-1)/2$ therein to $(n-1)/k$. Alternatively, see Lemma 1.3 of [19] for a self-contained statement (where the role of the interval I there is played by the cutoff function Ψ). $\square$

Proof of Lemma 8. Fix $C > 0$. Since T_* is trivially bounded $L^\infty(\mathbb{R}^n)$, we only need to show that it is also bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$.

We introduce a frequency decomposition of our multipliers depending on a fixed parameter $\Lambda \gg 1$ to be chosen later:

$$\widetilde{P^{\text{low}} f}(\xi) := \psi_{\Lambda^{1/k}(\log \Lambda)^{-C}}(\xi) \tilde{f}(\xi),$$

and

$$P^{\text{high}} f := f - P^{\text{low}} f.$$

With this decomposition if $\lambda > \Lambda$, then $(P^{\text{low}} f) \star \widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} = P^{\text{low}} f$. Proposition 3 implies the following pointwise bound:

$$\sup_{\lambda \leq \Lambda} |f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)(x)| \lesssim_C (\log \Lambda)^C Mf(x), \quad (\text{A.3})$$

and we also have for all $\lambda \in \mathbb{N}$ the bound

$$\sup_{\lambda \in \mathbb{N}} |P^{\text{low}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)(x)| \lesssim_C (\log \Lambda)^C Mf(x). \quad (\text{A.4})$$

We first prove a restricted weak-type inequality via interpolation, splitting up $|\{T_* f > \alpha\}|$ into three sets where we use (A.4), (A.3), and (A.2). Let $F \subset \mathbb{R}^n$ and $f := \mathbf{1}_F$ denote its indicator function so that

$$\begin{aligned} |\{T_* f > \alpha\}| &\leq |\{\sup_{\lambda \in \mathbb{N}} |P^{\text{low}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)| > \alpha/2\}| \\ &\quad + |\{\sup_{\lambda \in \mathbb{N}} |P^{\text{high}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)| > \alpha/2\}| \\ &\leq |\{\sup_{\lambda \in \mathbb{N}} |P^{\text{low}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)| > \alpha/2\}| \\ &\quad + |\{\sup_{\lambda \leq \Lambda} |P^{\text{high}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)| > \alpha/4\}| \\ &\quad + |\{\sup_{\lambda > \Lambda} |P^{\text{high}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)| > \alpha/4\}| \\ &\lesssim (\log \Lambda)^C \|f\|_1 \alpha^{-1} + |\{\sup_{\lambda > \Lambda} |P^{\text{high}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)| > \alpha/4\}|. \end{aligned}$$

Observe that for each $x \in \mathbb{R}^n$, we have

$$\sup_{\lambda > \Lambda} |P^{\text{high}} f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)(x)| \leq \sum_{j \geq \log_2 \Lambda} \left(\sup_{\lambda > 2^j} |P_j f \star (\widetilde{\psi_{\lambda^{1/k}(\log \lambda)^{-C}}} \star d\sigma_\lambda)(x)| \right)$$

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so that

$$\begin{aligned} |\{T_*f > \alpha\}| &\lesssim (\log \Lambda)^C \|f\|_1 \alpha^{-1} + |\{\sum_{j \geq \log_2 \Lambda} \left(\sup_{\lambda > 2^j} |P_j f \star (\psi_{\lambda^{1/k}(\log \lambda)^{-C}} \star d\sigma_\lambda)| \right) > \alpha/4\}| \\ &\lesssim (\log \Lambda)^C \|f\|_1 \alpha^{-1} + \alpha^{-2} \sum_{j \geq \log_2 \Lambda} (1 + 2^{j/k})^{\frac{1}{2} - \frac{n-1}{k}} \|f\|_2^2 \end{aligned}$$

by Chebychev's inequality and Proposition 4. Therefore

$$\begin{aligned} |\{T_*f > \alpha\}| &\lesssim (\log \Lambda)^C \|f\|_1 \alpha^{-1} + \Lambda^{\frac{1}{2k} - \frac{n-1}{k^2}} \|f\|_2^2 \alpha^{-2} \\ &= (\log \Lambda)^C |F| \alpha^{-1} + \Lambda^{\frac{1}{2k} - \frac{n-1}{k^2}} |F| \alpha^{-2}. \end{aligned}$$

Here $|F|$ denotes the Lebesgue measure of the set F .

To interpolate between L^1 and L^2 we need $\frac{1}{2k} - \frac{n-1}{k^2} < 0$ which occurs precisely when $n > k/2 + 1$. For any $1 < p < 2$ we choose $\Lambda > 0$ depending on $0 < \alpha \leq 1$ so that both summands are dominated by $|F| \alpha^{-p}$, which yields the restricted weak-type inequality. The Marcinkiewicz interpolation theorem gives the strong-type inequality. $\square$

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